Zero geometry and sharp compensated permanent derivatives on a complex Gram stratum

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Luminary Oddities

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Abstract

We study a homogeneous compensated derivative arising from permanent inequalities for centered complex positive semidefinite matrices of order six. On the conjugate-disjoint locus $K\overline{K} = 0$, the ordinary omission variance has exactly forty trace-normalized zeros, all sparse rank-one cube-root configurations. On the entire flat, equal-spectrum rank-two stratum, including all compatible complex phases, we determine the compensated derivative explicitly and prove its sharp lower bound $18912 - 140\sqrt{2}$. The proof retains the common Gram structure: a real balanced-frame complement gives a complete complex-structure chart, and threefold permutation symmetry eliminates the remaining phase from the paired permanent derivative. An explicit Frobenius tube of radius 10^{-6} has a positive uniform margin and contains full-dimensional families of centered rank-five matrices, whose positive rank-one anchors have rank six. A conditioning-free positive-part construction also reduces small conjugate overlap to the exact locus.

1 The functional and the contribution

Throughout, positive semidefinite means Hermitian positive semidefinite. Let $J = \mathbf{1}\mathbf{1}^*$ and $W = 6I - J$, both of order six. A matrix is *centered* if $K\mathbf{1} = 0$. Conjugation is entrywise; in particular $\overline{K}$ is distinct from $K^* = K$. Put

$$f_K(v) = \operatorname{per}(K + vJ), \qquad p(K) = f_K(1) + f_K'(1), \qquad D(K) = p(K)^2 - \operatorname{per}|K + \sqrt{2}J|^2.$$

Absolute squares are entrywise. Define homogeneous real polynomials

$$E_j(K) = [x^j]\ \frac{\partial}{\partial v} D(xK + vW)\bigg|_{v=0}. \tag{1}$$

Our target is the degree-four combination

$$F_4(K) = E_4(K) + \frac{\operatorname{tr} K}{2} E_3(K). \tag{2}$$

The perturbation in (1) is additive vW after xK; inserting an additional x in that perturbation changes the coefficient.

The occupation and certificate-degree predecessors [1, 2] study different sufficient estimates and certificate mechanisms for permanent inequalities. The cofactor bounds [3] concern a permanent-relative operator norm, rather than the signed derivative (2). Here the new conclusions are an exact singular zero classification and a sharp phase-complete evaluation of this derivative on a

geometric stratum. The underlying Fischer norm, positive-part decomposition and balanced-frame complement are classical tools; see [4, 5, 6].

Write $r = \operatorname{tr} K$ and

$$\mathcal{D} = \{K \succeq 0 : K\mathbf{1} = 0,\ \operatorname{tr} K = 1\}, \qquad \mathcal{Z} = \{K \in \mathcal{D} : K\overline{K} = 0\}.$$

The distinguished compact stratum is

$$\mathcal{T} = \{K \in \mathcal{Z} : K^2 = K/2,\ K_{ii} = 1/6 \text{ for all } i\}. \tag{3}$$

The equation $K^2 = K/2$ and the trace impose exactly two nonzero eigenvalues, both $1/2$.

Theorem 1.1. *For every $K \in \mathcal{T}$, there is $x \in [0,1]$ such that*

$$F_4(K) = 31356 - 7404x - 5040x^2 - \sqrt{2}(150 - 8x - 2x^2).$$

Consequently

$$F_4(K) \geq 18912 - 140\sqrt{2}.$$

Equality holds precisely at the disjoint cube-root trines, up to row permutation and conjugation of either trine. Moreover every $K \in \mathcal{D}$ with $\operatorname{dist}_{\mathrm{F}}(K, \mathcal{T}) \leq 10^{-6}$ satisfies $F_4(K) > 15524$.

This is a theorem about the compensated derivative target (2). Positivity of the whole permanent defect does not imply positivity of each homogeneous derivative block. Homogeneity extends the theorem to trace $r > 0$ by applying it to K/r and multiplying the bounds by r^4.

2 Ordinary omission variance and exact identities

For centered K, let $\alpha_j(K) = [t^j] \operatorname{per}(J + tK)$. Define

$$S_j = \sum_{i,k} K_{ik}^j, \qquad d_j = \sum_i K_{ii}^j, \qquad B_2 = \operatorname{tr} K^2,$$
$$A_0 = 6d_2 - S_2, \qquad Z = \Re \sum_{i,k} K_{ii} K_{ik}^2, \qquad U = \sum_{i,k} K_{ii} |K_{ik}|^2,$$
$$S_{21} = \sum_{i,k} K_{ik} |K_{ik}|^2, \qquad T_{21} = \operatorname{tr}(K^2 \overline{K}), \qquad T_3 = \operatorname{tr} K^3.$$

The sums S_j and S_{21} are real by Hermitian symmetry. Crucially, $S_2 = \operatorname{tr}(K\overline{K})$ is not B_2. Set

$$\delta(K) = \partial_v \alpha_5(K + vW)|_{v=0}\,, \tag{4}$$
$$R_4 = 20160\delta + 23040 r S_3 + 5760 S_2 A_0, \tag{5}$$
$$T_4 = [t^4]\ \partial_v \operatorname{per} |\sqrt{2}J + tK + vW|^2 \Big|_{v=0}. \tag{6}$$

Then direct centered-slot expansion gives

$$\begin{aligned} E_3 = {} & 2135808 d_3 - 361728 S_3 - 2163456 Z + 265920 r S_2 \\ & + 360576 T_{21} + 5760 S_{21} + 32256 U - 8832 r B_2 - 768 T_3, \end{aligned} \tag{7}$$

and $E_4 = R_4 - T_4$. We give the coefficient reduction in Appendix A so these identities are linked to the actual permanent definition rather than an independently relaxed model.

Choose centered Gram linear forms g_i with $\langle g_i, g_j \rangle = K_{ij}$. Give their polynomial algebra the Fischer inner product, in which $\|z^\beta\|^2 = \beta!$; hence the inner product of products of linear forms is the permanent of their Gram matrix. Write $p_j = \sum_i g_i^j$ and

$$\xi_i = g_i^4 - \frac{1}{2} g_i^2 p_2 - \frac{1}{3} g_i p_3.$$

Lemma 2.1. *For centered positive semidefinite K,*

$$\delta = 6\sum_i \|\xi_i\|^2 - \Big\|\sum_i \xi_i\Big\|^2 = \sum_{i<j} \|\xi_i - \xi_j\|^2.$$

If $K\overline{K} = 0$, then $p_2 = 0$ and

$$\delta = 144d_4 + 4rS_3 + 12\,\mathrm{tr}\big(\overline{K}^2(K \circ K)\big) - 96\Re\sum_{i,j} K_{ii}K_{ij}^3 - 24S_4. \tag{8}$$

Proof. The coefficient α_5 is the squared Fischer norm of the elementary symmetric polynomial $e_5(g_1, \ldots, g_6)$. Differentiating its Gram entries gives the contraction of W with the Gram matrix of the omitted products $e_4(g_1, \ldots, \widehat{g}_i, \ldots, g_6)$. Newton's identities and $p_1 = 0$ give these products as $e_4 + \xi_i$. Since $W\mathbf{1} = 0$, the common e_4 cancels. Contracting $W = 6I - J$ gives the variance formula.

Also $\|p_2\|^2 = 2S_2$. The trace $S_2 = \mathrm{tr}(K\overline{K})$ is nonnegative; it vanishes exactly when $K\overline{K} = 0$. On that locus, $\xi_i = g_i^4 - g_i p_3/3$ and $\sum_i \xi_i = p_4$. The contractions are

$$\sum_i \|g_i^4\|^2 = 24d_4, \qquad \|p_4\|^2 = 24S_4,$$
$$\sum_i \langle g_i^4, g_i p_3\rangle = 24\sum_{i,j} K_{ii}K_{ij}^3,$$
$$\sum_i \|g_i p_3\|^2 = 6rS_3 + 18\,\mathrm{tr}\big(\overline{K}^2(K \circ K)\big).$$

For the last equality, the permanent between (g_i, g_j, g_j, g_j) and (g_i, g_k, g_k, g_k) is

$$6K_{ii}K_{jk}^3 + 18K_{ik}K_{ji}K_{jk}^2.$$

Sum first over i. Substitution proves (8). □

3 The complete zero set on the conjugate-disjoint locus

Theorem 3.1. *For $K \in \mathcal{Z}$, the equality $\delta(K) = 0$ holds exactly at the forty matrices obtained by choosing a three-element support and taking the Gram vector $(1, \omega, \omega^2)/\sqrt{3}$ on that support, in either conjugate orientation; all other coordinates are zero. Here ω is a primitive cube root of unity.*

Proof. Lemma 2.1 implies that all $g_i^4 - g_i p_3/3$ agree. Over the fraction field of the auxiliary polynomial ring, the forms g_i are roots of one monic quartic $X^4 - (p_3/3)X - c$. Thus there are $q \le 4$ distinct forms h_j, with positive multiplicities m_j totaling six. The Gram matrix G of $\sqrt{m_j}h_j$ annihilates the real vector $(\sqrt{m_j})$ and satisfies $G\overline{G} = 0$. The mutually orthogonal ranges of G and $\overline{G}$ lie in a complex space of dimension $q - 1$. Hence $2\,\mathrm{rank}\,K \le q - 1 \le 3$, and K has rank one.

Write $g_i = a_i g$. The distinct coefficients $a, b, \ldots$ satisfy

$$\sum m_j a_j = \sum m_j a_j^2 = 0,$$

and are roots of $X^4 - (M_3/3)X - c_0$, where $M_3 = \sum m_j a_j^3$. Its four polynomial roots, counted with multiplicity, have first and second unweighted power sums zero.

If there are four distinct actual values, the extra multiplicities $m_j - 1$ total two. Two distinct extra values would obey $a + b = a^2 + b^2 = 0$, forcing both zero, a contradiction. Thus one value

has multiplicity three and is zero. The three other roots have first and second sums zero, so they are the roots of $X^3 - u$, $u \neq 0$. Their cubic moment is $3u$, giving exactly the required quartic $X(X^3 - u)$.

If there are three actual values, their multiplicities, up to order, are $(4,1,1), (3,2,1)$ or $(2,2,2)$. In the first case, $b+c = -4a$, $b^2+c^2 = -4a^2$, and the fourth polynomial root is $3a$. Its square-sum relation gives $6a^2 = 0$, forcing all values zero. In the second case $c = -3a - 2b$, $b^2 + 2ab + 2a^2 = 0$, and the fourth root is $2a + b$. Its square-sum relation reduces to $2a(a + 2b) = 0$. Either possibility again forces all values zero. These arguments allow the fourth polynomial root to repeat an actual value. In the third case the three values are the roots of $X^3 - u$ and the fourth root is zero. Their weighted cubic moment is $6u$, so the required linear coefficient $-2u$ contradicts the actual coefficient $-u$.

One or two distinct values are impossible except for the zero matrix: from $ma+nb = ma^2+nb^2 = 0$ one obtains $a = b = 0$. The remaining sparse cube-root configurations satisfy $p_1 = p_2 = 0$ and every $\xi_i = 0$, which proves the converse. For each of the $\binom{6}{3} = 20$ supports, the six phase assignments yield two Gram matrices after quotienting by the three common cube-root phase shifts. This proves the count forty. □

In particular there is no strictly positive uniform lower bound for δ on $\mathcal{Z}$. At such a sparse point direct substitution gives $R_4 = 7680$, $E_3 = 117888$, $T_4 = 448\sqrt{2}/3$, and

$$F_4 = 66624 - 448\sqrt{2}/3 > 0.$$

Thus a vanishing ordinary variance is not a compensation counterexample.

4 A complete chart and the sharp formula

4.1 Real complement and compatible complex structure

Let $K \in \mathcal{T}$. The matrix $P = 2(K + \overline{K}) = 4\Re K$ is a real orthogonal projection of rank four. It is centered and has diagonal $2/3$. In the real centered five-space its complement has rank one, so $P = I - J/6 - vv^T$ with $\|v\| = 1$ and $v_i^2 = 1/6$. Centering makes $v = \sigma/\sqrt{6}$, where σ has three plus and three minus signs. After a row permutation,

$$P = \operatorname{diag}(P_3, P_3), \qquad P_3 = I_3 - J_3/3.$$

This is the balanced-frame complement construction [6, Proposition 6.1]; the consequence here is elementary because the complement is one-dimensional.

Write $K = (P + iL)/4$. The real skew-symmetric L satisfies $PL = LP = L$ and $L^2 = -P$. In orthonormal bases for the two triangle planes,

$$L = \begin{pmatrix} tJ_0 & B \\ -B^T & dJ_0 \end{pmatrix}, \qquad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The equation $L^2 = -I$ gives $BB^T = (1-t^2)I$, $B^TB = (1-d^2)I$ and $tJ_0B + dBJ_0 = 0$. For $|t| < 1$ this is precisely

$$B = \sqrt{1-t^2}\,O, \quad O \in O(2), \quad d = -t \det O. \tag{9}$$

At $|t| = 1$, $B = 0$ and $d = \pm 1$, both covered by choosing the determinant in (9). Thus this chart covers both orientation components and every endpoint. Set $x = t^2$.

4.2 The ordinary contractions

Choose a 3×2 real isometry Q onto the triangle plane. Then $QQ^T = P_3$ and $QJ_0Q^T = C_3/\sqrt{3}$, where C_3 has zero diagonal and cyclic off-diagonal entries $-1, 1$. Within either triple the off-diagonal entries of K are $(-1 \pm i\sqrt{3}t)/12$, with t replaced by d in the second triple. Across the triples they are $i\sqrt{1-t^2}(QOQ^T)_{ij}/4$.

For a rotation, $(QOQ^T)_{ij} = (2/3)\cos(\theta + \phi_j - \phi_i)$, where the ϕ_i are spaced by $2\pi/3$; a reflection replaces a phase difference by a phase sum. The identity $\cos^4 u = (3 + 4\cos 2u + \cos 4u)/8$ yields $\sum_{ij}(QOQ^T)_{ij}^4 = 2/3$, independently of the angle. Consequently

$$S_3 = \frac{1}{48} + \frac{x}{16}, \qquad S_4 = \frac{(1-x)^2}{96}, \qquad S_{21} = \frac{1-x}{48}. \tag{10}$$

Odd cross-block powers cancel in conjugate pairs. For S_4 the within- triple and cross-triple contributions are each $(1-x)^2/192$. The other contractions are

$$d_3 = 1/36,\ d_4 = 1/216,\ B_2 = 1/2,\ T_3 = 1/4,\ U = 1/12, \qquad S_2 = Z = T_{21} = 0.$$

Since $\overline{K}^2 = \overline{K}/2$, the trace contraction in (8) is $S_3/2$, and its diagonal-weighted cubic sum is $S_3/6$. Equations (8)–(10) give

$$\delta = \frac{7}{24} + \frac{x}{8} - \frac{x^2}{4}, \tag{11}$$

$$E_3 = 49992 - 22728x, \qquad R_4 = 6360 + 3960x - 5040x^2. \tag{12}$$

They also prove the sharp variance–diagonal tradeoff on this stratum:

$$\delta + 2d_3 - \frac{2}{9} = \frac{(1-x)(1+2x)}{8} \geq 0. \tag{13}$$

4.3 Retaining the paired phases

Put $A = K + \overline{K} = P/2$ and $B = K \circ \overline{K} = |K|^2$. An untouched slot of the permanent in (6) is $2 + \sqrt{2}tA_{ij} + t^2B_{ij}$; its differentiated slot is $2\sqrt{2}W_{ij} + tW_{ij}A_{ij}$. Thus extracting t^4 uses at most two B factors. The fixed A has zero cross blocks. The variable B has angle frequencies $0, \pm 2$, so T_4 has only frequencies $0, \pm 2, \pm 4$. A cyclic permutation of one triangle shifts the angle by $2\pi/3$ and preserves T_4. Only frequency zero survives. A transposition within one triangle exchanges the two determinant components. Therefore T_4 is independent of the angle and orientation component.

It remains a polynomial of degree at most two in x. Indeed with $O = I$, the entries are rational affine functions of x:

$$A_{\text{within}} = P_3/2, \qquad A_{\text{cross}} = 0, \qquad B_{ii} = 1/36,$$
$$B_{\text{within,off}} = (1+3x)/144, \qquad B_{\text{cross,ij}} = (1-x)(P_3)_{ij}^2/16.$$

Exact expansion at $x = 0, 1/2, 1$ gives respectively $T_4/\sqrt{2} = 150, 291/2, 140$. The complete degree bound then proves

$$T_4 = \sqrt{2}(150 - 8x - 2x^2). \tag{14}$$

Appendix B provides an explicit rational finite-sum rule for these evaluations; the accompanying exact verifier implements that rule without floating-point arithmetic.

Combining (12) and (14) proves the formula in Theorem 1.1. Its derivative is $-7404 - 10080x + \sqrt{2}(8 + 4x) < 0$ on $[0, 1]$. The minimum occurs exactly at $x = 1$, when B in the complex-structure chart vanishes and each triangle becomes an independent complex cube-root line. This proves the sharp bound and equality statement. The same endpoint is the equality case of (13).

4.4 The stratum is not confined to circulant models

For $0 < |t| < 1$ the matrices above cannot be made circulant by row phases and permutations. To see this, normalize their Gram rows to unit spinors in $\mathbb{C}^2$, with unit Bloch vectors $b_i \in \mathbb{R}^3$. The elementary Pauli identity gives $b_i \cdot b_j = 2|\langle f_i, f_j\rangle|^2 - 1$. Within either triangle this dot product is $(3t^2-1)/2$. If b is the sum of the first triangle, then $\|b\|^2 = 9t^2$ and $b_i \cdot b = 3t^2$ there. Thus this noncollinear triangle lies in a plane at distance $|t|$ from the origin. Tightness gives total Bloch sum zero, so the second noncollinear triangle lies in the distinct parallel plane on the opposite side. The six points are not coplanar.

A circulant rank-two Gram matrix with equal eigenvalues has a factor whose two Fourier coordinates have equal absolute value in every row. Its normalized Bloch states lie on a great circle. Common changes of factor rotate the Bloch sphere; row phases leave each state unchanged, and permutations reorder them. Coplanarity is therefore necessary for such a gauge-circulant representation, contradicting the two-plane configuration. This distinguishes the phase-complete theorem from an evaluation on a two-character circulant model.

5 A uniform full-rank neighborhood

Lemma 5.1. *On $\mathcal{D}$ the functional F_4 is Frobenius-Lipschitz with constant*

$$C = 2972838528 + 152064000\sqrt{2} + 412128\sqrt{6} < 3190000000 =: C_0.$$

Proof. Every matrix in $\mathcal{D}$ has Frobenius norm and entry magnitudes at most one. A degree-d monomial in entries and their conjugates changes by at most $d\|K-Q\|_{\mathrm{F}}$ between two such matrices, by telescoping. For δ, the differentiated-slot weight is $\sum_{\pi,i} |W_{i,\pi(i)}| = 5! \sum_{ij} |W_{ij}| = 7200$. Selecting four of five untouched slots gives coefficient budget 36000. The fixed-trace term $rS_3 = S_3$ has budget 36 and degree three. The product S_2A_0 has budget $36(36+36) = 2592$ and degree four. Consequently the Lipschitz contribution of R_4 is

$$4(20160)(36000) + 3(23040)(36) + 4(5760)(2592) = 2965248000.$$

The actual paired derivative is majorized coefficientwise in absolute value by $14400(\sqrt{2}+t)^{11}$. Its degree-four coefficient budget is $38016000\sqrt{2}$; telescoping contributes $152064000\sqrt{2}$.

For (7), each of the seven three-slot contractions $d_3, S_3, Z, T_{21}, S_{21}, U, T_3$ has real-trilinear norm at most one in Frobenius norm. For entry contractions, bound one entry or diagonal entry by its Frobenius norm and apply Cauchy–Schwarz to the remaining two slots. For trace contractions use $|\operatorname{tr} ABC| \le \|A\|_{\mathrm{F}}\|B\|_{\mathrm{F}}\|C\|_{\mathrm{F}}$. The terms rS_2, rB_2 have norm at most $\sqrt{6}$, since $|\operatorname{tr} A| \le \sqrt{6}\|A\|_{\mathrm{F}}$. Summing their coefficients gives $L_3 = 5060352 + 274752\sqrt{6}$. Telescoping the three slots of $E_3/2$ contributes $3L_3/2 = 7590528 + 412128\sqrt{6}$. Adding proves the displayed constant. The rational bounds $\sqrt{2} < 99/70$, $\sqrt{6} < 49/20$ give $C < 111611856456/35 < C_0$. □

The compactness of $\mathcal{T}$ makes distance to it attained. Its sharp minimum $m = 18912 - 140\sqrt{2}$ is greater than 18714. Lemma 5.1 therefore gives, on the closed 10^{-6} tube, $F_4 \ge m - C10^{-6} > 18714 - 3190 = 15524$. This finishes Theorem 1.1, including its closed endpoint.

This tube has full-dimensional interior in $\mathcal{D}$. For any $K_0 \in \mathcal{T}$ put

$$\varepsilon = 10^{-6}, \qquad K_\varepsilon = (1-\varepsilon)K_0 + \varepsilon W/30.$$

On $\mathbf{1}^\perp$ its eigenvalues are $1/2 - 3\varepsilon/10$ twice and $\varepsilon/5$ three times. Also $\|W/30 - K_0\|_{\mathrm{F}}^2 = 3/10$. Every centered Hermitian trace-zero perturbation H with $\|H\|_{\mathrm{F}} \le 10^{-7}$ preserves the smallest centered

eigenvalue at least 10^{-7} and has

$$\|K_\varepsilon + H - K_0\|_{\mathrm{F}} \le \varepsilon\sqrt{3/10} + 10^{-7} < 7 \cdot 10^{-7}.$$

Thus this entire ball is certified. Its ambient affine space has real dimension $5^2 - 1 = 24$. These matrices have centered rank five, and $K_\varepsilon + H + sJ$ has rank six for every $s > 0$. The argument imposes no circulant constraint, phase restriction or repeated-spectrum condition on the perturbations. The original tube theorem includes singular residuals as well.

6 A conditioning-free reduction to the exact locus

Theorem 6.1. *For centered $K \succeq 0$ of order six set $L = (K - \overline{K})_+$, $\ell = \operatorname{tr} L$, $r = \operatorname{tr} K$. Then $L \succeq 0$, $L\mathbf{1} = 0$, $L\overline{L} = 0$, $\operatorname{rank} L \le 2$, and*

$$\|K - L\|_{\mathrm{F}}^2 \le S_2, \qquad 0 \le r - \ell \le \sqrt{5S_2}.$$

Both unnormalized constants are sharp. If $r > 0$ and $S_2/r^2 < 1/5$, then $\ell > 0$ and

$$\|K/r - L/\ell\|_{\mathrm{F}} \le \sqrt{6S_2}/r.$$

Proof. For arbitrary positive semidefinite A, B put $X = A - B = P - N$, with its orthogonal positive and negative parts. Expanding squares gives

$$\operatorname{tr}(AB) - \|A - P\|_{\mathrm{F}}^2 = \operatorname{tr}(AN) + \operatorname{tr}(BP) \ge 0, \qquad \operatorname{tr}((A - P)P) = \operatorname{tr}(BP) \ge 0.$$

Apply this classical positive-part algebra to $A = K, B = \overline{K}$. Conjugation reverses X, so $\overline{L} = N$. The two ranges are orthogonal and contained in the common centered five-space, proving $\operatorname{rank} L \le 2$. Let $E = K - L$. It is real symmetric and centered, but need not be positive semidefinite. Trace-norm triangle inequality gives $\ell = \|K - \overline{K}\|_1/2 \le r$. Cauchy–Schwarz on the centered five-space gives $r - \ell = \operatorname{tr} E \le \sqrt{5}\|E\|_{\mathrm{F}}$.

For $\Delta = r - \ell \ge 0$, the first identities and $\|L\|_{\mathrm{F}} \le \ell$ give

$$\|K - (r/\ell)L\|_{\mathrm{F}}^2 = \|E\|_{\mathrm{F}}^2 + (\Delta/\ell)^2\|L\|_{\mathrm{F}}^2 - 2(\Delta/\ell)\operatorname{tr}(EL) \le S_2 + \Delta^2 \le 6S_2.$$

The assumed threshold ensures $\ell > 0$. Real nonzero centered K has $L = 0$ and $S_2 = \|K\|_{\mathrm{F}}^2$, proving the first sharpness claim. Taking $K = c(I - J/6)$ proves the trace constant $\sqrt{5}$ sharp. □

In general order n, the same argument gives rank at most $\lfloor (n-1)/2 \rfloor$, trace-loss constant $\sqrt{n-1}$, and normalized-distance constant $\sqrt{n}$ under the corresponding strict threshold. This is an application of spectral positive-part geometry.

For $K \in \mathcal{D}$ with $S_2 < 1/5$, put $Q = L/\ell \in \mathcal{Z}$. Then

$$F_4(K) \ge F_4(Q) - C_0\sqrt{6S_2}.$$

If a future argument proves $F_4(Q) \ge m_0 > 0$ on all of $\mathcal{Z}$, this would give positivity whenever $S_2 < \min\{1/5, m_0^2/(6C_0^2)\}$. Theorem 1.1 covers (3). The remaining nonuniform diagonals and unequal eigenvalues require additional coupled estimates. The forty-point classification rules out a variance-only positive floor and identifies the boundary that any such estimate must accommodate.

A Coefficient identities and the actual derivative

The centered-slot formulas used above can be obtained without numerical fitting. For j ordered residual slots, the coefficient α_j is $(6-j)!/j!$ times the sum of products with distinct row indices and distinct column indices. Eliminate distinctness by the partition-lattice Möbius rule: a partition block of size b contributes $(-1)^{b-1}(b-1)!$. Centered row and column sums kill singleton blocks. This gives

$$\alpha_0 = 720, \quad \alpha_1 = 0, \quad \alpha_2 = 12S_2, \quad \alpha_3 = 4S_3,$$

and, for any centered Hermitian H,

$$\alpha_4(H) = 3\sum_{ij} H_{ij}^4 - 3\Re\sum_i \left(\sum_j H_{ij}^2\right)^2 + \frac{1}{4}\left(\sum_{ij} H_{ij}^2\right)^2 + \frac{1}{2}\operatorname{tr}((H\overline{H})^2).$$

For four slots the only surviving partition types are 4 and $2+2$, with block Möbius weights -6 and 1. Differentiation along W yields

$$X := \partial_W \alpha_4 = 72d_3 - 12S_3 - 72Z + 6rS_2 + 12T_{21}.$$

Since $p(K) = \sum_j (7-j)\alpha_j(K)$, the ordinary degree-three derivative is $30240X + 86400rS_2$. The ordinary degree-four derivative is $20160\delta + 23040rS_3 + 5760S_2A_0$, using $\partial_W S_2 = 12r$ and $\partial_W S_3 = 3A_0$.

For the paired coefficient of degree four before differentiation set $A = H + \overline{H}$, $B = H \circ \overline{H}$. Selection of four A slots, two A and one B, or two B slots gives

$$\begin{aligned} &16\alpha_4(A) + 192\sum_{ij} A_{ij}^2 B_{ij} - 192\sum_i \left(\sum_j A_{ij}^2\right)\left(\sum_j B_{ij}\right) \\ &\quad + 48\left(\sum_{ij} A_{ij}^2\right)\left(\sum_{ij} B_{ij}\right) + 192\left[\left(\sum_{ij} B_{ij}\right)^2 - 2\sum_i \left(\sum_j B_{ij}\right)^2 + \sum_{ij} B_{ij}^2\right]. \end{aligned}$$

Differentiating $A' = 2W$, $B' = W \circ A$ and using the centered row sums gives

$$Y = 41472d_3 - 1152S_3 - 5760S_{21} - 13824Z - 32256U + 1920rS_2 + 8832rB_2 + 768T_3 + 2304T_{21}.$$

Thus $E_3 = 30240X + 86400rS_2 - Y$, which is (7). This calculation preserves matrix products and entrywise products separately; replacing S_2 by B_2 is invalid for complex matrices.

B A rational rule for the paired quartic

Here is a finite expression sufficient to reproduce (14). For a permutation π and a distinguished slot i, let $a_j = A_{j,\pi(j)}$, $b_j = B_{j,\pi(j)}$ for $j \neq i$. For the five remaining slots define

$$A_k = \sum_{|S|=k} \prod_{j\in S} a_j, \quad B_2' = \sum_{j<k} b_j b_k,$$

and

$$M_{21} = \sum_k b_k \sum_{\substack{j<l \\ j,l\neq k}} a_j a_l, \qquad M_{11} = \sum_{j\neq k} a_j b_k.$$

The exact coefficient rule is

$$\frac{T_4}{\sqrt{2}} = \sum_{\pi\in S_6}\sum_{i=1}^{6} W_{i,\pi(i)}\left[16(A_4 + M_{21} + B_2') + 8A_{i,\pi(i)}(A_3 + M_{11})\right].$$

The factors 16 and 8 follow by counting constant slots in the two differentiated-slot cases. With the rational $O = I$ entries displayed in the text, every term is a polynomial in x of degree at most two. The rule yields $150, -8, -2$ as its constant, linear and quadratic coefficients. The accompanying verifier extracts this complete symbolic polynomial by subset recursion and independently checks its determining values by summing all 720 permutations. A verifier may sum this rule directly, or extract the same coefficient by exact polynomial permanent differentiation. The universal angular conclusion uses the chart and Fourier selection, not a finite list of phase samples.

Computational assistance

Exact computations use Python's standard-library rational arithmetic; the supplementary program states its finite verification scope. AI assistance was used in research, drafting and checking. The author is responsible for the mathematical statements and presentation.

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