Weyl Laws and Angular Stability for Rapidly Twisted Tubes

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Abstract

We determine exact eigenvalue-counting asymptotics for infinite-volume Dirichlet tubes confined by rotation. Continuous regularly varying reference speeds of positive index require neither monotonicity nor derivative bounds; a second class includes smooth rapidly varying speeds such as exponential and double-exponential growth. The two axial ends may have different types. Finite-band cross-sections allow continuous angular endpoints, pinching widths, multiple components and finitely many topology transitions. The law remains unchanged under angle perturbations with finite limits, without control of their derivatives, and therefore permits infinitely many actual speed reversals and zeros. In the positive-index regime the coefficient is a nonlinear angular-width moment; in the rapid regime it depends only on area. We give full box and localization proofs, logarithmic trace thresholds, width-distribution recovery from a family of coefficients and a finite-data obstruction.

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1 Introduction and main theorem

Twisting can confine an infinite-volume Dirichlet tube without shrinking its cross-sectional area. Krejčiřík established a class with compact resolvent and asked for eigenvalue asymptotics [7]. Barseghyan and Khrabustovskyi proved eigenvalue-moment upper bounds [2]; their Proposition 4.1 evaluates the asymptotic of a bound's right-hand side, rather than the actual count. Open Problem 8 of [8] retains the broader counting question.

The author's earlier paper [12] establishes an exact law for a positive-width C^1 polar strip and the particular speed $c(1+s^2)^{p/2}$. This paper substantially extends that result. The frozen-box comparison is inherited from [12] and reproved here. The extensions concern general regularly varying speeds, a rapidly varying regime, independent end behavior, finite-band cross-sections, and angular perturbations with unrestricted derivatives. The result remains a concrete-class answer, not a general resolution of the twisting problem.

Nonclassical Weyl laws retaining transverse modes, including spectral-zeta coefficients and non-power growth, have classical antecedents in horn analysis [13, 5] and the operator-valued viewpoint [6, Section 3.2.1]. The uniform comparison for the twisted geometry and its angle stability are the relevant contributions here. Finite-threshold twisted-waveguide asymptotics [3, 10] concern another regime. Convergent-angle geometry also appears in sheared-tube essential-spectrum analysis [11]; it is not an exact high-energy counting law for the present confining geometry.

Definition 1.1 (Admissible cross-section). Let $0 < a < b < \infty$. A nonempty bounded Lipschitz open $\omega \subset \mathbb{R}^2$ is admissible if its closure lies in $a \le r \le b$ and a finite partition $a = r_0 < \cdots <$

$r_M = b$ has the following property. On $I_m = (r_{m-1}, r_m)$ the angular fiber is a finite disjoint union

$$\omega_r = \bigcup_{j=1}^{k_m} (l_{mj}(r), u_{mj}(r)) \pmod{2\pi}.$$

The number k_m may be zero. The endpoint lifts extend continuously to $\overline{I}_m$ and, when present, have positive widths and positive intercomponent gaps on I_m. All their closed-band values lie in one fixed real interval J_m of length $A_m < 2\pi$. Widths and gaps may vanish at the band endpoints. Fibers at partition radii are subject only to the openness and Lipschitz hypothesis on ω itself.

Put $\alpha_{mj} = u_{mj} - l_{mj}$ and

$$M_q(\omega) = \sum_m \int_{I_m} r \sum_j \alpha_{mj}(r)^q \, \mathrm{d}r, \qquad q > 0. \tag{1.1}$$

Thus $M_1(\omega) = |\omega|$. Individual widths are raised to the power before they are summed.

A positive function b belongs to RV_p if $b(tx)/b(x) \to t^p$ for every $t > 0$. It is rapidly varying if this ratio tends to zero for $0 < t < 1$ and to infinity for $t > 1$. We use classical uniform convergence, integration, inversion and Potter bounds from [1].

Let $\Theta \in C^1(\mathbb{R})$ be a reference angle with eventually fixed nonzero derivative sign at each end. Write $b_\pm(x) = |\Theta'(\pm x)|$ for large x. Each end is permitted independently to have either type:

(P) b_e is continuous and belongs to RV_{p_e} with $p_e > 0$, with asymptotic inverse R_e.

(Q) b_e is eventually positive, C^1, strictly increasing and rapidly varying, with $b_e'/b_e^2 \to 0$. Its eventual inverse is R_e.

For type (P), $R_e \in \mathrm{RV}_{1/p_e}$; for type (Q), R_e is slowly varying and tends to infinity. Define

$$A_e(\omega) = \begin{cases} \dfrac{p_e \zeta(1 + 1/p_e)}{2\pi^{2+1/p_e}(p_e + 1)(3p_e + 1)} M_{1+1/p_e}(\omega), & \text{type (P)}, \\ \dfrac{|\omega|}{6\pi^2}, & \text{type (Q)}, \end{cases} \tag{1.2}$$

and

$$F_\omega(\Lambda) = \Lambda^{3/2} \sum_{e=+,-} A_e(\omega) R_e(\sqrt{\Lambda}). \tag{1.3}$$

For the actual angle $\theta \in C^1(\mathbb{R})$ put

$$\Omega_\theta = \{(R_{\theta(s)} y, s) : y \in \omega, \ s \in \mathbb{R}\},$$

where R_γ is planar rotation. Let H_θ be the Friedrichs Dirichlet Laplacian and count strictly below energy, with multiplicity.

Theorem 1.2. *For an admissible cross-section and reference angle as above, suppose $\theta(s) - \Theta(s) \to \gamma_\pm \in \mathbb{R}$ as $s \to \pm\infty$. Then H_θ has compact resolvent and*

$$N_\theta(\Lambda) := \operatorname{Tr} \mathbf{1}_{(-\infty,\Lambda)}(H_\theta) \sim F_\omega(\Lambda). \tag{1.4}$$

No asymptotic derivative control or eventual sign is required of the actual speed θ'.

The equivalence concerns the entire positive sum, not a two-term expansion if the end orders differ. Taking $\theta = \Theta$ gives the reference-speed law. For symmetric tails $b_\pm(x) \sim cx^p$, this yields

$$N(\Lambda) \sim C_{p,c,\omega} \Lambda^{d_p}, \quad d_p = \frac{3}{2} + \frac{1}{2p}, \quad C_{p,c,\omega} = \frac{p\zeta(1 + 1/p)c^{-1/p}}{\pi^{2+1/p}(p + 1)(3p + 1)} M_{1+1/p}(\omega). \tag{1.5}$$

2 The form, confinement and bounded cores

For a sector $a < r < b$, $0 < t < \alpha < 2\pi$, pullback uses $L^2(r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s)$ and gives

$$q_\beta[v] = \int \left(|v_r|^2 + r^{-2}|v_t|^2 + |X_\beta v|^2\right) r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s, \qquad X_\beta = \partial_s - \beta(s)\partial_t. \tag{2.1}$$

The physical axial coordinate and sector opening make the map injective. Original Dirichlet domains are form closures of compact tests. Auxiliary radial-Neumann domains permit free radial traces and retain angular Dirichlet traces. Finite axial caps are either Dirichlet or free. Locally bounded β controls ordinary H^1 on bounded axial pieces.

Lemma 2.1 (Half-tail estimate)**.** *Suppose $\beta \geq B > 0$ on $s > S$. For either radial boundary convention, restrictions of form vectors satisfy*

$$\int_{s>S} |Xv|^2 r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s \geq \frac{\pi^2 B^2}{4\alpha^2} \int_{s>S} |v|^2 r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s. \tag{2.2}$$

There is an analogous estimate on the negative half-tail when its twisting speed has the same positive lower bound.

Proof. At fixed physical polar angle $\varphi = t + \theta(s)$, X becomes differentiation in s. The change from (t, s) to (φ, s) has Jacobian one, with φ interpreted modulo 2π. Since $\theta' = \beta > 0$, each connected permitted vertical-line component in the half-tail corresponds to a θ-interval of width at most α. Its length in s is at most α/B. The genuine lateral endpoints are Dirichlet. At most one component can meet the artificial cap $s = S$; that component has at least one Dirichlet endpoint and may have a free trace at the cap. The one-dimensional Dirichlet–Neumann Poincaré inequality on an interval of length L has constant $\pi^2/(4L^2)$. Apply it on each component and integrate in r, φ. The argument first applies to locally smooth vectors with angular zero trace and then passes to the form domain by approximation. The radial boundary convention does not enter the estimate. The negative tail is handled in the same way. □

The same bound applies to a fixed negative sign after reversing the angular orientation. In particular diverging reference speed supplies uniformly vanishing tail mass for form-bounded families.

Lemma 2.2 (Cores and fixed caps)**.** *A bounded axial piece of a sector, or of a tube with bounded Lipschitz cross-section and locally C^1 angle, has compact resolvent and count $O(\Lambda^{3/2})$ with either axial cap convention and original lateral Dirichlet boundary. If a one-ended sector with a Dirichlet cap has an equivalent $G \in \mathrm{RV}_d$ such that $G(\Lambda)/\Lambda^{3/2} \to \infty$, then a free cap and any fixed change of cap position have the same law.*

Proof. On a fixed piece the form and norm are uniformly comparable to ordinary gradient and volume norms. Transverse Dirichlet zero extension into a bounded annulus times the axial interval makes no extension through free caps. Polar coordinates, followed by relaxing angular periodicity and radial conditions to Neumann conditions, bound the count by a rectangular Neumann lattice count at a fixed energy multiple. This is $O(\Lambda^{3/2})$ and also proves compactness.

For cap comparison choose a fixed squared IMS partition on the free-cap half-tube, one part in a bounded cap collar and one vanishing near the cap. The latter extends by zero into the original Dirichlet half-tube. Cutoffs depend only on s, so $X_\beta \chi = \chi'$ and the energy error is a fixed constant C. Isometric localization and min–max give

$$N_D(\Lambda) \leq N_N(\Lambda) \leq N_D(\Lambda + C) + O(\Lambda^{3/2}).$$

Compactness follows first from the same localization into the compact comparison pieces. A fixed energy shift is negligible for G. Moving a cap by a fixed amount uses this same collar argument and inclusion. □

For a full tube, a fixed squared partition into a bounded core and remote tails gives

$$\sum_j q_\beta[\chi_j v] = q_\beta[v] + \int \sum_j |\chi_j'|^2 |v|^2 r \,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s.$$

The full count is bounded above by the two Dirichlet-tail counts at $\Lambda + C$ plus a bounded core count, and below by their unshifted sum. This combines independently obtained one-end laws without assuming confinement in short free-cap boxes.

3 An exact frozen-box comparison

The following comparison, used in [12], is proved here in full.

Fix ρ in a compact positive interval, a constant $B > 0$, radial length $h > 0$ and longitudinal length $L > 0$. On the rectangular box use Lebesgue measure and the constant-coefficient form

$$q_{\rho,B}[v] = \int \left(|v_r|^2 + \rho^{-2}|v_t|^2 + |v_s - Bv_t|^2 \right) \mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s. \tag{3.1}$$

Angular faces are always Dirichlet. The radial condition is either Dirichlet on both faces, denoted D, or Neumann on both, denoted N. Longitudinal faces are Dirichlet. Denote the corresponding count by $D_{\mathsf{b}}(h, L, \rho, B; E)$. Put

$$\mathcal{J}_{\mathsf{D}} = \{1, 2, \ldots\}, \qquad \mathcal{J}_{\mathsf{N}} = \{0, 1, \ldots\}, \qquad \nu_n = \frac{\pi n}{\alpha}, \tag{3.2}$$

and define

$$J_{\mathsf{b}}(h, \rho, B; E) = \frac{\sqrt{1 + B^2\rho^2}}{\pi} \sum_{\substack{j \in \mathcal{J}_{\mathsf{b}} \\ n \geq 1}} \left[E - (\pi j/h)^2 - (B^2 + \rho^{-2})\nu_n^2 \right]_+^{1/2}. \tag{3.3}$$

All sums at a fixed energy are finite. For negative energy, all counts and positive-part sums in this section are defined to be zero.

Lemma 3.1 (Uniform longitudinal boundary comparison)**.** *For $\eta \in (0, 1)$ there is $C_\eta > 0$, independent of B, ρ, h, L, E and the radial condition, such that*

$$\frac{L}{1+\eta} J_{\mathsf{b}}\left(h, \rho, B; E - \frac{C_\eta}{L^2}\right) \leq D_{\mathsf{b}}(h, L, \rho, B; E) \leq L J_{\mathsf{b}}(h, \rho, B; E). \tag{3.4}$$

Proof. First replace the longitudinal boundary condition by $v(L) = e^{i\vartheta} v(0)$. The longitudinal frequencies are $\xi = (2\pi k + \vartheta)/L$, $k \in \mathbb{Z}$. At a fixed frequency, completion of the angular square gives

$$\rho^{-2}|w_t|^2 + |i\xi w - Bw_t|^2 = (B^2 + \rho^{-2}) \left| w_t - \frac{iB\xi}{B^2 + \rho^{-2}} w \right|^2 + \frac{\xi^2}{1 + B^2\rho^2} |w|^2. \tag{3.5}$$

The constant-modulus angular gauge preserves the two angular Dirichlet conditions. Its phase is independent of r, since ρ is frozen. The exact quasiperiodic eigenvalues are therefore

$$(\pi j/h)^2 + (B^2 + \rho^{-2})\nu_n^2 + \frac{\xi^2}{1 + B^2\rho^2}. \tag{3.6}$$

The longitudinal Dirichlet form domain is contained in every quasiperiodic form domain. Its count is at most the count for each ϑ. Averaging the latter over $[0, 2\pi)$ converts the frequency lattice exactly into $L\,\mathrm{d}\xi/(2\pi)$. Integrating the intervals allowed by (3.6) proves the upper bound in (3.4).

For the lower bound set $d = L/(1+\eta)$. On a circle of length Kd, for K sufficiently large, choose K translates of a smooth square partition of unity, with spacing d, supports of length L, and

$$\sum_{k=1}^{K} \chi_k^2 = 1, \qquad \sum_{k=1}^{K} |\chi_k'|^2 \le C_\eta L^{-2}. \tag{3.7}$$

Such a partition follows by normalizing translated nonnegative bumps which are positive on the spacing intervals and supported in their slightly enlarged intervals. The IMS identity is

$$\sum_k q_{\rho,B}[\chi_k v] = q_{\rho,B}[v] + \int \sum_k |\chi_k'|^2 |v|^2. \tag{3.8}$$

Indeed $(\partial_s - B\partial_t)(\chi_k v) = \chi_k X_B v + \chi_k' v$; the localization error has no factor depending on B.

Let $N_{\mathrm{per}}(Kd, E_0)$ be the periodic count on that circle. The isometric localization $v \mapsto (\chi_1 v, \dots, \chi_K v)$ and min–max give

$$N_{\mathrm{per}}(Kd, E_0) \le K D_{\mathsf{b}}(h, L, \rho, B; E_0 + C_\eta L^{-2}). \tag{3.9}$$

For completeness, if the left spectral subspace had greater dimension than the direct sum of the right spectral subspaces, some nonzero vector would localize orthogonally to every right low-energy subspace. Its summed localized energy would be at least the right threshold times its squared norm, contradicting (3.8) and the strict energy bound on the left finite-dimensional spectral subspace. At fixed E_0, h, ρ, B, d, only finitely many bands in (3.6) are active. The count for each band is a lattice count in one bounded frequency interval, with an error at most two. Consequently

$$\lim_{K\to\infty} \frac{1}{K} N_{\mathrm{per}}(Kd, E_0) = d J_{\mathsf{b}}(h, \rho, B; E_0). \tag{3.10}$$

Divide (3.9) by K, pass to this limit, and take $E_0 = E - C_\eta L^{-2}$. This proves the lower bound. □

Lemma 3.2 (Uniform radial sum)**.** *Let ρ range in a fixed compact positive interval. Uniformly in $B > 0$, $h > 0$, $E \ge 0$ and $\mathsf{b} \in \{\mathsf{D}, \mathsf{N}\}$,*

$$J_{\mathsf{b}}(h, \rho, B; E) = \frac{h}{4\pi} \sqrt{1 + B^2\rho^2} \sum_{n\ge 1} \left[E - (B^2 + \rho^{-2})\nu_n^2\right]_+ + O(E). \tag{3.11}$$

The implied constant is independent of the growing twist B.

Proof. For $A \ge 0$, decreasing-sum comparison gives

$$\sum_{j\ge 1} \left[A - (\pi j/h)^2\right]_+^{1/2} = \frac{h}{4} A + O(\sqrt{A}). \tag{3.12}$$

The integral is $hA/4$ and the sum-integral error is bounded by $\sqrt{A}$. The inclusion of $j = 0$ changes only that error bound. Put $a_B = B^2 + \rho^{-2}$. A further decreasing-sum comparison yields

$$\sum_{n\ge 1} \left[E - a_B \nu_n^2\right]_+^{1/2} \le \int_0^\infty \left[E - a_B(\pi x/\alpha)^2\right]_+^{1/2} \mathrm{d}x = \frac{\alpha E}{4\sqrt{a_B}}. \tag{3.13}$$

The identity $\sqrt{1 + B^2\rho^2} = \rho\sqrt{a_B}$ cancels the possible large coefficient in the summed error of (3.12). Substitution into (3.3) proves the result. □

4 Angular clocks and reference speeds

Lemma 4.1 (Global clock and smoothing)**.** *A positive continuous $b \in \mathrm{RV}_p$, $p > 0$, has a smooth eventually increasing equivalent b_0 with $xb_0'(x)/b_0(x) \to p$. If positive continuous $a, b \in \mathrm{RV}_p$ satisfy $a/b \to 1$, their sector tail counts have the same regularly varying equivalent whenever that equivalent dominates $\Lambda^{3/2}$.*

Proof. Choose a nonnegative smooth nonzero ϕ supported compactly in $(1, 2)$ and put

$$C_\phi = \int_1^2 t^p \phi(t)\,\mathrm{d}t, \qquad b_0(x) = C_\phi^{-1} \int_1^2 b(xt)\phi(t)\,\mathrm{d}t.$$

Uniform convergence gives $b_0/b \to 1$. Changing variable before differentiating proves smoothness without differentiating b, and yields

$$xb_0'(x) = -C_\phi^{-1} \int_1^2 b(xt)(\phi + t\phi')\,\mathrm{d}t.$$

Uniform convergence and integration by parts give the limit p after division by $b(x)$, hence eventual increase.

For the second statement take primitives $A = \int a$, $B = \int b$ with fixed initial points and $f = A^{-1} \circ B$. Karamata integration gives $A(x) \sim xa(x)/(p + 1) \sim B(x)$. Comparison at $(1 \pm \epsilon)x$ gives $f(x)/x \to 1$. Uniform convergence then gives $g(x) = f'(x) = b(x)/a(f(x)) \to 1$. For $u(r, t, x) = v(r, t, f(x))$, exact substitution gives

$$\|v\|^2 = \int g|u|^2 r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}x, \tag{4.1}$$
$$q_a[v] = \int \{g(|u_r|^2 + r^{-2}|u_t|^2) + g^{-1}|u_x - b(x)u_t|^2\} r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}x.$$

There is no unitary square-root weight to differentiate. If $1 - \epsilon \le g \le 1 + \epsilon$, Rayleigh quotients obey

$$\frac{1 - \epsilon}{1 + \epsilon} Q_b \le Q_a \le (1 - \epsilon)^{-2} Q_b.$$

The tail map is bi-Lipschitz, preserves traces and gives a form-domain isomorphism by completion from compact tests. Apply min–max at a sufficiently remote fixed cap, discard bounded pieces by Lemma 2.2, use regular variation for fixed energy factors, then send $\epsilon \downarrow 0$. No derivative of a or g is used. □

Lemma 4.2 (Local angular clock)**.** *On an axial interval I of length L let $\beta > 0$ and $B = L^{-1} \int_I \beta$. If $\sup_I |\beta/B - 1| \le \epsilon$, then its form is within relative $O(\epsilon)$ of the constant-speed B form, uniformly in B, with the same interval length and Dirichlet axial traces. Radial freezing adds a relative $O(h)$ on a radial cell of length at most h.*

Proof. Set $x(s) = B^{-1} \int_{\inf I}^s \beta$. This maps I onto $(0, L)$ and $g = ds/dx = B/\beta = 1 + O(\epsilon)$. Identity (4.1), now with the constant B, proves the quotient comparison, independent of B. The density r and r^{-2} vary relatively by $O(h)$. These changes affect only positive terms of the form and the norm. No derivative of g or absolute speed oscillation enters. □

After Lemma 4.1, a type (P) end may be replaced by a smooth increasing equivalent with $b'/b^2 \to 0$. Type (Q) already has this property. For such reference speeds define R to be the eventual true inverse. Its index is $\gamma = 1/p > 0$ in type (P) and $\gamma = 0$ in type (Q). Inversion of rapid variation gives $R(cT)/R(T) \to 1$ for every $c > 0$ in the latter case [1, Section 2.4].

5 Phase integrals and the one-ended law

For a sector of opening $\alpha < 2\pi$ put $a_s = b(s)^2 + r^{-2}$ and $\nu_n = \pi n/\alpha$. The one-end phase integral is

$$\mathcal{I}_\alpha(\Lambda) = \frac{1}{4\pi} \int_{\text{end}} \int_a^b \sqrt{1 + b(s)^2 r^2} \sum_{n \geq 1} \left[\Lambda - a_s \nu_n^2\right]_+ \, \mathrm{d}r \, \mathrm{d}s. \tag{5.1}$$

Lemma 5.1. *For type (P),*

$$\mathcal{I}_\alpha(\Lambda) \sim \frac{p\zeta(1+1/p)}{2\pi^{2+1/p}(p+1)(3p+1)} \alpha^{1+1/p} \left(\int_a^b r \, \mathrm{d}r \right) \Lambda^{3/2} R(\sqrt{\Lambda}).$$

For type (Q),

$$\mathcal{I}_\alpha(\Lambda) \sim \frac{\alpha \int_a^b r \, \mathrm{d}r}{6\pi^2} \Lambda^{3/2} R(\sqrt{\Lambda}).$$

In either case its integrand is bounded by $C\Lambda^{3/2}$ uniformly in r, s.

Proof. At most $\alpha\sqrt{\Lambda}/(\pi\sqrt{a_s})$ angular modes are active, and $\sqrt{1 + b^2 r^2} = r\sqrt{a_s}$. This proves the bound. Let $T = \sqrt{\Lambda}$ and rescale $s = R(T)y$.

In type (P), $b(R(T)y)/T \to y^p$ for $y > 0$. Monotonicity bounds the rescaled support in a fixed compact interval. Dominated convergence and Tonelli give the limit

$$\frac{1}{4\pi} \int_a^b r \, \mathrm{d}r \sum_{n \geq 1} \int_0^\infty y^p [1 - \nu_n^2 y^{2p}]_+ \, \mathrm{d}y.$$

The inner integral is $2p\, \nu_n^{-1-1/p}/[(p+1)(3p+1)]$. The convergent series over n gives the stated zeta factor.

In type (Q), rapid variation gives $b(R(T)y)/T \to 0$ for $0 < y < 1$ and infinity for $y > 1$. For $y < 1$ put $z = \sqrt{a_s}/T \to 0$. The normalized integrand is

$$\frac{r}{4\pi} z \sum_{n \geq 1} [1 - z^2(\pi n/\alpha)^2]_+ \longrightarrow \frac{r\alpha}{6\pi^2},$$

by a decreasing Riemann-sum comparison with $\int_0^{\alpha/\pi} [1 - (\pi x/\alpha)^2] \, \mathrm{d}x = 2\alpha/(3\pi)$. For $y > 1$ it vanishes eventually. Since $R(CT)/R(T) \to 1$, the rescaled support remains bounded. The same uniform bound proves dominated convergence and the area coefficient. The exceptional values $y = 0, 1$ do not affect the integral. □

Proposition 5.2 (One-ended sector law)**.** *The sector count, with either radial convention, angular Dirichlet walls and either fixed axial cap, is asymptotic to* (5.1) *in both reference classes.*

Proof. First take a Dirichlet cap in the eventual increasing range of a smooth reference. Compactness follows from Lemma 2.1 and local compactness. Fix $\eta \in (0, 1)$ and choose a fixed $C > 1$ so large that the half-tail bound beyond $S = R(CT)$ exceeds 2Λ. Both classes have $S = O(R(T))$. Set

$$D_T = \sup_{\text{cap} \leq s \leq R(2CT)} |(\log b)'(s)|.$$

Then $D_T/T \to 0$: on a fixed initial segment it is bounded, and beyond any fixed M the hypothesis $b'/b^2 \to 0$ gives $b'/b \leq \epsilon_M b$ with $\epsilon_M \to 0$, while $b \leq 2CT$ on this interval. Choose

$$\rho_T = \min\{T^{1/4}, [T/(1 + D_T)]^{1/2}\}, \qquad \ell = T^{-1}\rho_T, \qquad h = T^{-1/2}. \tag{5.2}$$

Thus $\rho_T \to \infty$, $\ell \to 0$, $\ell^{-2} = o(\Lambda)$, $D_T \ell = o(1)$ and $hT \to \infty$.

Use radial Neumann interfaces with cell lengths at most h and an axial squared partition with spacing ℓ, supports of length $(1+\eta)\ell$, and squared derivative sum at most $C_\eta \ell^{-2}$. Only supports meeting the range before S contribute. The relevant support margins remain below $R(2CT)$: otherwise a distance $O_\eta(\ell)$ would span a logarithmic speed change at least $\log 2$, contradicting $D_T \ell = o(1)$. Hence the relative oscillation of b on all contributing supports is $o(1)$.

Apply Lemma 4.2, freezing the speed at its box average, and freeze r at a point in its cell. The IMS energy error and all quotient distortions are uniform $o(\Lambda)$ and $o(1)$, respectively. The localization map is isometric; the same spectral dimension argument used in Lemma 3.1 bounds the count by the sum of frozen-box counts at $(1+o(1))\Lambda$. Apply Lemmas 3.1 and 3.2. There are $O_\eta(S/(h\ell))$ boxes. Their total radial error is

$$O_\eta(S\Lambda/h) = o(R(T)\Lambda^{3/2}).$$

The axial support-to-spacing factor is $1+\eta$.

To compare the principal sums with (5.1), put $a = b^2 + r^{-2}$ and $w = \sqrt{1+b^2 r^2}$. Both vary relatively by $o(1)$ on these boxes. If $a_0 = (1+O(\delta))a$, every positive part $[E - a_0 \nu_n^2]_+$ is squeezed between the corresponding positive parts at $E/(1\pm\delta)$ multiplied by $1\pm\delta$. Bounds for w then preserve the complete sum over n, including threshold crossings. The phase equivalent in Lemma 5.1 is regularly varying in energy, so perturbations $(1+o(1))\Lambda$ do not alter it. This proves the upper limit with factor $1+\eta$. Fixed initial pieces cost $O(\Lambda^{3/2})$ by a fixed core IMS partition and Lemma 2.2.

For the lower bound insert disjoint radial and axial Dirichlet boxes of lengths h, ℓ in the active range. Local clocks and radial freezing apply as before. The lower bound of Lemma 3.1 costs $C_\eta \ell^{-2} = o(\Lambda)$ and the factor $1/(1+\eta)$. The summed radial error is unchanged. Discarded radial layers cost $O(hS\Lambda^{3/2})$ and end axial layers $O(\ell\Lambda^{3/2})$, using the uniform phase bound. Both are lower order. The same positive-part squeeze gives the lower limit divided by $1+\eta$. Let $\eta \downarrow 0$.

Free caps and other fixed cap positions follow from Lemma 2.2. For original type (P) speeds, Lemma 4.1 transfers the law from the smooth equivalent back to the continuous speed without derivative assumptions. Reversing the axial coordinate or reflecting $t \mapsto \alpha - t$ handles either end and either eventual sign. No radius-dependent reflection is used. □

6 Finite-band geometry and angle stability

Lemma 6.1 (Lateral zero extension)**.** *Restrict an original form vector to a radial slab and an axial half-line. If its physical support there lies in a comparison sector tube, its lateral zero extension preserves norm and energy and permits free radial and axial cap traces.*

Proof. On bounded axial pieces the original transverse Dirichlet zero extension is H^1, first for compact smooth tests and then by completion. Restrict this extension to the slab and half-line; no extension is made through artificial caps. Its gradient is the original gradient on the support and zero elsewhere, with no distributional Dirichlet-wall term. This argument in physical coordinates does not differentiate continuous polar endpoints. Exhaustion passes to finite-energy vectors on a half-line. If a union of disjoint outer sectors contains the support with open gaps, restriction to those sectors gives the corresponding direct-sum injection. □

Proof of Theorem 1.2. At each end absorb γ_e into a rigid rotation of the reference comparison sectors. Sector counts are unchanged by this rotation, so we describe an end where the remaining angle error tends to zero.

First prove compactness. Cut at the finitely many radial band boundaries and at fixed axial caps, allowing free cap traces. On sufficiently remote tails, the actual support in band m lies in the reference envelope J_m, enlarged by a small fixed angular amount on either side. All enlarged openings remain below 2π. Lemma 6.1 embeds the restrictions into finitely many reference sector tails with free radial faces and axial caps. Their embeddings are compact by Proposition 5.2 and its tail estimate. The bounded actual core has compact embedding by Lemma 2.2. The finite direct-sum comparison therefore proves actual compactness before counting.

For the coefficient fix $\epsilon > 0$ and trim radial neighborhoods of the partition radii of total length at most ϵ, subdividing at the partition radii. On each retained compact subband all widths and gaps have positive minima. Uniform continuity permits a fine fixed radial mesh whose inner intervals are

$$(\max_C l_{mj}, \min_C u_{mj})$$

and whose outer intervals are

$$(\min_C l_{mj}, \max_C u_{mj}).$$

The inner intervals are nonempty, the outer intervals are disjoint, and all openings remain below 2π. Denote the lengths by $\alpha^-_{mj,C}$ and $\alpha^+_{mj,C}$.

Keep trimming and mesh fixed. For small fixed $\delta > 0$, contract the inner intervals by δ at each endpoint and enlarge the outer intervals by δ. Positivity and gaps persist. Choose fixed axial caps so that the actual angle error has absolute value less than δ on each tail. These are literal physical-domain inclusions between contracted inner reference sectors, the actual domain, and enlarged outer reference sectors. On the trimmed radial neighborhoods use only enlarged envelope sectors.

The lower comparison inserts Dirichlet radial and axial faces and discards the trimmed layers and core. The upper comparison allows free radial and axial traces and applies Lemma 6.1. Its bounded core costs $O(\Lambda^{3/2})$. For a type (P) end, the comparison coefficients relative to its fixed prefactor in (1.2) without M_q are

$$\sum_{m,j,C} (\alpha^-_{mj,C} - 2\delta)^q \int_C r\,\mathrm{d}r, \qquad \sum_{m,j,C} (\alpha^+_{mj,C} + 2\delta)^q \int_C r\,\mathrm{d}r, \quad q = 1 + 1/p_e.$$

For a type (Q) end the same formula holds with $q = 1$ and prefactor $1/(6\pi^2)$. Each upper envelope adds at most $C_e\epsilon$, uniformly for small δ, because there are finitely many bounded openings and $r \le b$.

Take the energy limit first; then let $\delta \downarrow 0$, refine the mesh, and finally let $\epsilon \downarrow 0$. Uniform continuity identifies each retained width-moment integral. The omitted target moment is $O(\epsilon)$ since component number and widths are bounded. This gives (1.2) separately for each end.

Each scale $\Lambda^{3/2} R_e(\sqrt{\Lambda})$ dominates the core count. Each is regularly varying in energy, so fixed energy shifts from core localization are harmless. For unequal end orders, the relative coefficient error of their positive sum is bounded by the larger of the two individual relative errors. No comparison limit for R_+/R_- is assumed. The upper and lower bounds yield (1.4). □

7 Examples and standard spectral consequences

7.1 Disks, irregular speeds and changes of regime

A disk of radius R whose center is distance $d > R$ from the rotation axis has

$$d - R < r < d + R, \qquad \alpha(r) = 2\arccos\frac{r^2 + d^2 - R^2}{2dr}. \tag{7.1}$$

Its polar width pinches at both ends and has unbounded endpoint derivative. The physical boundary is smooth and the disk lies in a wedge of opening $2\arcsin(R/d) < \pi$. Thus it is admissible.

For type (P), the speed $b(x) = cx^p[1 + x^{-1/4}\sin(e^{x^2})]$ eventually has the same law as cx^p, despite derivatives which can grow faster than every power. For actual speed reversals choose a reference $\Theta' = c(1+s^2)^{p/2}$ and set

$$\theta(s) = \Theta(s) + (1+s^2)^{-A/2}\sin((1+s^2)^{B/2}), \qquad A > 0, \quad B > A + p + 1. \tag{7.2}$$

The angle error tends to zero while its derivative has leading oscillatory size $B|s|^{B-A-1}$. At successive cosine extrema it dominates the reference derivative with opposite signs. Continuity gives infinitely many speed zeros, yet Theorem 1.2 still applies to every admissible cross-section.

Type (Q) includes $b(x) = c\exp(\kappa x^r)$ for all $c, \kappa, r > 0$. Its inverse is $R(T) \sim \kappa^{-1/r}(\log T)^{1/r}$. For two exponential ends, $r = 1$,

$$N(\Lambda) \sim \frac{|\omega|}{12\pi^2}(\kappa_+^{-1} + \kappa_-^{-1})\Lambda^{3/2}\log\Lambda. \tag{7.3}$$

For $b_e(x) = c_e\exp(\exp(\kappa_e x))$,

$$N(\Lambda) \sim \frac{|\omega|}{6\pi^2}(\kappa_+^{-1} + \kappa_-^{-1})\Lambda^{3/2}\log\log\Lambda.$$

Direct differentiation verifies $b_e'/b_e^2 \to 0$ in all these examples.

The angular coefficient changes between the regimes: type (P) involves $M_{1+1/p}$, whereas type (Q) involves $M_1 = |\omega|$. This is the bulk angular-mode Riemann sum in Lemma 5.1, not a finite-volume three-dimensional Weyl formula: the tube itself always has infinite volume. Logarithmic and other slowly varying Weyl factors are known in other geometries [4]; the assertion here is for these Euclidean twisted tubes.

7.2 Logarithmic type (P) tails and trace ideals

Corollary 7.1. *For symmetric type (P) reference tails $b_\pm(x) \sim cx^p(\log x)^q$, where $c, p > 0$ and $q \in \mathbb{R}$, put $d = d_p$ from (1.5) and $C = C_{p,c,\omega}(2p)^{q/p}$. Then*

$$\begin{aligned} N(\Lambda) &\sim C\Lambda^d(\log\Lambda)^{-q/p}, \\ \operatorname{Tr}(e^{-tH}) &\sim C\Gamma(d+1)t^{-d}(\log(1/t))^{-q/p}. \end{aligned} \tag{7.4}$$

For $z > 0$, $\operatorname{Tr}(H^{-z})$ is finite for $z > d$ and infinite for $z < d$. At the endpoint,

$$\operatorname{Tr}(H^{-d}) < \infty \quad \Longleftrightarrow \quad q > p.$$

Equivalently, $H^{-1} \in \mathcal{S}_d$ exactly when $q > p$.

Proof. Substitution gives $R(T) \sim c^{-1/p}p^{q/p}T^{1/p}(\log T)^{-q/p}$. The factor $2^{q/p}$ comes from $\log\sqrt{\Lambda} = \frac{1}{2}\log\Lambda$. The transverse Dirichlet Poincare inequality gives a positive spectral bottom. Tonelli yields

$$\operatorname{Tr}(H^{-z}) = z\int_0^\infty N(\Lambda)\Lambda^{-z-1}\,\mathrm{d}\Lambda.$$

The integral test proves the assertions, including divergence at $q = p$. Schatten membership is exactly this eigenvalue summability.

For heat, integrate the counting measure: $\operatorname{Tr}(e^{-tH}) = t\int_0^\infty e^{-t\Lambda}N(\Lambda)\,\mathrm{d}\Lambda$. After $x = t\Lambda$, Potter bounds [1, Theorem 1.5.6] give a majorant $Ce^{-x}\max(x^{d-\epsilon}, x^{d+\epsilon})$ on the high-energy part, with $0 < \epsilon < d$. The bounded-energy part is negligible. Dominated convergence gives $\Gamma(d+1)$. These deductions are classical once the counting law is known. □

7.3 Scope restrictions

A centered annular cross-section avoids the axis but stays an infinite straight cylinder under every rotation. It has essential spectrum. A contained annular subdomain likewise supplies disjoint translated bounded-energy trial functions, contradicting compact embedding. Persistent Dirichlet slits cannot be discarded as sets of area zero: separated angular intervals contribute powers of individual widths rather than the power of their sum.

The type (Q) derivative condition is a sufficient condition for the uniform local-clock proof, not a necessity classification. Nor are results for Neumann, Maxwell, nonlinear, or ruled-surface operators implied by analogy; the last model has a different spectral mechanism [9].

8 Inverse width information and shape bounds

These statements concern leading coefficients from a family of different operators.

Theorem 8.1. *For admissible ω define the finite measure*

$$\mu_\omega(E) = \sum_m \int_{I_m} r \sum_j \alpha_{mj}(r) \mathbf{1}_E(\alpha_{mj}(r)) \, \mathrm{d}r \quad \text{on } [0, 2\pi].$$

Its mass is $|\omega|$ and its atom at zero vanishes. The leading coefficients in (1.5)*, with known c and $p = 1/n$ for $n = 1, 2, \ldots$, determine μ_ω uniquely.*

Proof. $M_q = \int x^{q-1} \, \mathrm{d}\mu_\omega$. The normalized coefficients give all positive integer moments of μ_ω. The measure $\nu = x\mu_\omega$ therefore has all nonnegative integer moments known. Polynomial density in $C([0, 2\pi])$ determines ν; division by x recovers μ_ω on $(0, 2\pi]$, and its zero atom is zero. This is compact moment uniqueness. □

The recovered distribution does not determine angular centerlines or radial ordering that preserves its weighted measure. This is not full shape recovery from one spectrum.

Theorem 8.2 (Finite-data obstruction)**.** *Fix a, b and $0 < A < 2\pi V$, where $V = (b^2 - a^2)/2$, and a finite list $p_1, \ldots, p_k > 0$. There are two connected admissible Lipschitz cross-sections in this annulus with area A and distinct measures μ_ω, but equal leading coefficients in* (1.5) *for every listed exponent and common c.*

Proof. Put $\bar{\alpha} = A/V$. Choose $k+3$ distinct $x_j \in (0, L)$ on both sides of $\bar{\alpha}$, where $\bar{\alpha} < L < 2\pi$. Choose positive weights w_j with $\sum w_j = V$, $\sum w_j x_j = A$; their existence follows because $\bar{\alpha}$ is interior to the convex hull of the x_j. Set $q_i = 1 + 1/p_i$. The matrix with rows $1, x_j, x_j^{q_i}$ has $k+2$ rows and $k+3$ columns, so has a nonzero real kernel vector z. For small $\delta > 0$ all $w_j^\pm = w_j \pm \delta z_j$ are positive. The two atomic width distributions have identical mass, area and selected moments, but differ.

Realize them by radial interfaces $r_j^\pm = (a^2 + 2\sum_{i \le j} w_i^\pm)^{1/2}$ and centered intervals $(-x_j/2, x_j/2)$ on the successive bands. Take the interior of the closure of the sector union. Adjacent centered intervals are nested and share a nonempty interval, so the result is connected. Its polar boundary is a finite step polygon with ordinary right or reentrant corners. The polar map on the compact annulus and wedge of opening $L < 2\pi$ is bi-Lipschitz, so this domain is Lipschitz and admissible. Interfaces at finitely many radii do not change its moments. Multiplication by distinct positive x_j leaves its two area-weighted measures distinct. Formula (1.5) proves the selected coefficient equalities. □

Proposition 8.3. *If there are at most m components at each radius and all widths are at most L, then for $q > 1$,*

$$\frac{|\omega|^q}{(mV)^{q-1}} \le M_q(\omega) \le L^{q-1}|\omega|.$$

For one component, constant width attains the lower bound when admissible. Equal splitting of a total width into m components changes its local moment by $m^{1-q} = m^{-1/p}$ when $q = 1+1/p$.

Proof. Pad missing widths by zero and apply Jensen on m copies of $r\,\mathrm{d}r$. The upper bound is $\alpha^q \le L^{q-1}\alpha$; the splitting identity is $m(\alpha/m)^q = m^{1-q}\alpha^q$. These are elementary convexity consequences, not a general spectral shape-optimization theorem. □

9 Further questions

The result remains a concrete-class answer to the diverging-twist counting question. Effective remainders under quantitative growth and regularity assumptions would yield finite-energy estimates. Removing finite fiber topology or envelope gaps requires care with small gaps and genuine Dirichlet cracks. Slowly varying speeds of index zero need different active-region scaling. The necessity of the sufficient type (Q) condition is also left open.

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