Variance cancellation proves rank-one compensation on the six-row conjugate-disjoint locus

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Luminary Oddities

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Abstract

We prove a uniform positive bound for an actual compensated permanent derivative on the entire rank-one conjugate-disjoint locus of centered complex positive semidefinite matrices of order six. With trace r, the bound is $[1248-(1672/3)\sqrt{2}]r^4 > 459r^4$ for nonzero matrices, without equal-modulus or phase restrictions. The proof derives an exact paired-moment identity and cancels its difficult signed cubic moment against ten times the same ordinary omission variance. Centered projection and moment bounds then pay the remainder. We also prove the sharp bound $(14688-208\sqrt{2})r^4$ on the full two-direction cube-root family, including unequal norms, arbitrary complex overlaps and singular endpoints, together with a sharp spectrum-resolved variance tradeoff. The earlier explicit Lipschitz estimate gives full-dimensional neighborhoods and quantitative exclusion of the rank-one boundary from the remaining rank-two problem.

1 The discovery and its scope

Let $J = \mathbf{1}\mathbf{1}^*$ and $W = 6I - J$. All matrices in this paper have order six, and positive semidefinite means Hermitian positive semidefinite. A matrix K is *centered* when $K\mathbf{1} = 0$. A bar denotes entrywise conjugation and $\circ$ denotes entrywise multiplication. Put

$$f_K(v) = \operatorname{per}(K + vJ), \qquad p(K) = f_K(1) + f_K'(1), \qquad D(K) = p(K)^2 - \operatorname{per}|K + \sqrt{2}J|^2.$$

The absolute square is entrywise. Define

$$E_j(K) = [x^j]\, \partial_v D(xK + vW)|_{v=0}\,, \qquad F_4(K) = E_4(K) + \frac{\operatorname{tr} K}{2} E_3(K). \tag{1}$$

The direction vW is additive after xK; an additional x in that direction changes the coefficient. The functional F_4 is homogeneous of degree four over real nonnegative scalars.

The weighted expression $f(1) + f'(1)$ appears in the border identity of [7, Proposition3.2(iii)]. Their complete-polynomial comparison through order six does not yield the signed homogeneous derivative bound studied here.

Our predecessor [1] classified the forty normalized zeros of an ordinary omission variance and evaluated F_4 sharply on a flat, equal-spectrum rank-two stratum. Classifying those variance zeros does not control the signed paired derivative on the continuous rank-one locus. The present theorem supplies that missing uniform boundary control. The contribution is the exact cancellation in the actual permanent-derived expression, rather than a new frame projection or a change in an independently optimized numerical envelope.

Theorem 1.1 (Uniform rank-one compensation)**.** *Let $K \succeq 0$ be centered,* $\operatorname{rank} K \le 1$*, and $K\overline{K} = 0$. Writing $r = \operatorname{tr} K$, one has*

$$F_4(K) \ge \left(1248 - \frac{1672}{3}\sqrt{2}\right) r^4. \tag{2}$$

For $K \neq 0$ this is strictly greater than $459r^4$. Arbitrary complex phases, unequal diagonal entries and zero coordinates are allowed.

The main mechanism will be the exact identity

$$\frac{T_4}{8\sqrt{2}} = 10\delta + R, \tag{3}$$

where T_4 is the actual paired derivative and $\delta \ge 0$ is the ordinary omission variance. The troublesome mixed cubic moment cancels in R. Crucially, the ordinary side still contains $(20160 - 80\sqrt{2})\delta$ after this payment. Independent worst-case bounds on the mixed moment and variance would lose this relation.

The sharp supporting theorem concerns the six Gram forms

$$(u, \omega u, \omega^2 u, v, \omega v, \omega^2 v), \qquad \omega^3 = 1, \quad \omega \neq 1, \tag{4}$$

where u, v are arbitrary vectors in a complex Hilbert space.

Theorem 1.2 (Sharp two-direction compensation)**.** *For the Gram matrix K of* (4)*,*

$$F_4(K) \ge (14688 - 208\sqrt{2})(\operatorname{tr} K)^4.$$

For nonzero K, equality holds precisely at the equal-weight doubled cube-root rank-one configuration, up to row permutation and a common Gram-factor unitary. The complete spectral tradeoff and equality conditions are given in Theorem 5.1.

These are statements about the auxiliary derivative (1). The occupation theorem already covers the original permanent inequality on the two-direction family [2]. Positivity of a full permanent defect does not imply positivity of each homogeneous derivative coefficient. The quadratic permanent comparisons of [6] and the raw cofactor operator estimates of [3] concern different statements.

2 The actual ordinary and paired coefficients

For a centered matrix define

$$\alpha_j(K) = [t^j] \operatorname{per}(J + tK), \qquad \delta(K) = \partial_v \alpha_5(K + vW)|_{v=0},$$
$$T_4(K) = [t^4]\, \partial_v \operatorname{per} |\sqrt{2}J + tK + vW|^2\Big|_{v=0}.$$

We use the invariants

$$r = \operatorname{tr} K, \qquad d_j = \sum_i K_{ii}^j, \qquad S_j = \sum_{i,k} K_{ik}^j,$$
$$B_2 = \operatorname{tr} K^2, \qquad A_0 = 6d_2 - S_2, \qquad S_{21} = \sum_{i,k} K_{ik}|K_{ik}|^2,$$
$$Z_0 = \operatorname{Re} \sum_{i,k} K_{ii} K_{ik}^2, \qquad U = \sum_{i,k} K_{ii}|K_{ik}|^2, \qquad T_{21} = \operatorname{tr}(K^2\overline{K}), \quad T_3 = \operatorname{tr} K^3.$$

Hermitian symmetry makes these expressions real. In particular $S_2 = \operatorname{tr}(K\overline{K})$ differs from B_2. The actual coefficient identities are

$$R_4 = 20160\delta + 23040rS_3 + 5760S_2A_0, \qquad E_4 = R_4 - T_4, \tag{5}$$

$$\begin{aligned} E_3 = {} & 2135808d_3 - 361728S_3 - 2163456Z_0 + 265920rS_2 \\ & + 360576T_{21} + 5760S_{21} + 32256U - 8832rB_2 - 768T_3. \end{aligned} \tag{6}$$

These are the coefficient identities of [1, Section2 and AppendixA]. For convenience Appendix A recalls their centered-slot reduction, so that their normalizations can be checked against (1).

Choose centered Gram linear forms g_i with $\langle g_i, g_j\rangle = K_{ij}$. Use the Fischer inner product, $\|z^\beta\|^2 = \beta!$; the inner product of products of linear forms is the permanent of their Gram matrix. This is the classical polynomial pairing used, for example, in [4]. Write $p_j = \sum_i g_i^j$. Then the omission states are

$$\xi_i = g_i^4 - \frac{1}{2}g_i^2 p_2 - \frac{1}{3}g_i p_3,$$

and

$$\delta = 6\sum_i \|\xi_i\|^2 - \left\|\sum_i \xi_i\right\|^2 = \sum_{i<j}\|\xi_i - \xi_j\|^2 \geq 0. \tag{7}$$

Indeed $\alpha_5 = \|e_5(g_1,\dots,g_6)\|^2$. Its Gram derivative contracts W with the Gram matrix of omitted elementary products $e_4(g_1,\dots,\widehat{g}_i,\dots,g_6)$. Newton's identities and $p_1 = 0$ express these as $e_4 + \xi_i$. The common e_4 cancels because $W\mathbf{1} = 0$, proving (7).

Since $\|p_2\|^2 = 2S_2$, the condition $K\overline{K} = 0$ makes $p_2 = 0$. On this locus the full contraction identity is

$$\delta = 144d_4 + 4rS_3 + 12\operatorname{tr}\big(\overline{K}^2(K\circ K)\big) - 96\operatorname{Re}\sum_{i,j} K_{ii}K_{ij}^3 - 24S_4. \tag{8}$$

It follows by expanding $\xi_i = g_i^4 - g_i p_3/3$: $\|g_i^4\|^2 = 24K_{ii}^4$, $\langle g_i^4, g_i p_3\rangle = 24K_{ii}\sum_j K_{ij}^3$, and

$$\sum_i \|g_i p_3\|^2 = 6rS_3 + 18\operatorname{tr}\big(\overline{K}^2(K\circ K)\big).$$

The last permanent has six matchings pairing the singled-out factors together and eighteen pairing them separately. Also $\sum_i \xi_i = p_4$ and $\|p_4\|^2 = 24S_4$, which gives (8).

3 A universal paired-moment identity

3.1 The rank-one real projection

For Theorem 1.1, $K = 0$ is immediate. By homogeneity normalize $\operatorname{tr} K = 1$ and write $K = zz^*$. Centering and conjugate disjointness give

$$\sum_i z_i = \sum_i z_i^2 = 0, \qquad \sum_i |z_i|^2 = 1. \tag{9}$$

Put $t_i = |z_i|^2$, $d_j = \sum_i t_i^j$, and define scalar moments

$$\begin{gathered} p_3 = \sum_i z_i^3, \quad p_4 = \sum_i z_i^4, \quad m = \sum_i t_i z_i, \quad m_2 = \sum_i t_i^2 z_i, \\ m_3 = \sum_i t_i z_i^3, \qquad n = \sum_i t_i z_i^2. \end{gathered} \tag{10}$$

In this section p_3, p_4 are scalar moments, distinct from the weighted permanent $p(K)$ in the definition of D.

Let $A = K + \overline{K}$ and $B = K \circ \overline{K} = tt^T$. If $z = x + iy$ with real x, y, (9) makes x, y orthogonal, centered, and of squared norm $1/2$. Thus

$$A = 2(xx^T + yy^T), \qquad A^2 = A, \qquad A\mathbf{1} = 0, \qquad A_{ii} = 2t_i. \tag{11}$$

In particular $A \preceq I - J/6$. This is elementary orthogonal-projection geometry; the related centered frame complements are classical [5, Proposition6.1].

Proposition 3.1 (Actual paired-moment identity)**.** *Under* (9)*,*

$$\begin{aligned}\frac{T_4}{8\sqrt{2}} &= 8208d_4 - 5184d_3 + 1512d_2 - 576d_2^2 - 672|n|^2 \\ &\quad - 960\,\mathrm{Re}(m_3\overline{p_3}) - 4608\,\mathrm{Re}(m_2\overline{m}) - 120|p_4|^2 + 208|p_3|^2 + 2256|m|^2 - 112.\end{aligned} \tag{12}$$

We prove this identity by exact partition and contraction calculations.

3.2 Mixed permanent coefficients before specialization

For a centered real symmetric A and symmetric B, set

$$b = B\mathbf{1}, \quad q = (A \circ A)\mathbf{1}, \quad c = (A \circ A \circ A)\mathbf{1}, \quad h = (A \circ B)\mathbf{1},$$

and $s_j = \sum_{i,k} A_{ik}^j$, $H = \sum_{i,k} A_{ik}B_{ik}$. The notation A^3 below denotes matrix multiplication, whereas A_{ik}^3 denotes an entrywise power. Put

$$V_4 = [x^3y]\alpha_4(xA + yB), \qquad V_3 = [xy^2]\alpha_3(xA + yB).$$

One has the exact identities

$$\begin{aligned}V_4 &= 12\sum_{i,k} A_{ik}^3B_{ik} - 8c^Tb - 12q^Th + \tfrac{4}{3}s_3\sum B_{ik} \\ &\quad + 4q^TAb + s_2H + 2\sum_{i,k} B_{ik}(A^3)_{ik},\end{aligned} \tag{13}$$

$$V_3 = 12\sum_{i,k} A_{ik}B_{ik}^2 - 24h^Tb + 6H\sum B_{ik} + 6b^TAb. \tag{14}$$

Here B need not be centered.

To verify the formulas, express an ordered residual-slot coefficient as the distinct-row/distinct-column sum with factor $(6-j)!/j!$ and use partition-lattice inclusion–exclusion. A block of size b has Möbius coefficient $(-1)^{b-1}(b-1)!$. A singleton in an A slot vanishes by centering; a B singleton need not vanish. For the distinguished B slot in A^3B, the surviving row partitions are U (one four-block, weight -6), V (the three A slots together and the B singleton, weight 2), and the three pairings P in which B is paired with an A (weight 1). The column partitions are the same. Their grouped contractions, before the factor $4 \cdot 2!/4! = 1/3$, are

(U,U)	$36\sum A_{ik}^3B_{ik}$
$(U,V) + (V,U)$	$-24c^Tb$
$(U,P) + (P,U)$	$-36q^Th$
(V,V)	$4s_3\sum B_{ik}$
$(V,P) + (P,V)$	$12q^TAb$
(P,P), same A partner	$3s_2H$
(P,P), different partners	$6\sum B_{ik}(A^3)_{ik}$.

For AB^2, the surviving partitions are the triple block (weight 2) and the two pairings of A with a B (weight -1). The grouped contractions are respectively $4\sum A_{ik}B_{ik}^2$, $-8h^Tb$, $2H\sum B_{ik}$, and $2b^TAb$. Multiplication by $3\cdot 3!/3! = 3$ proves (14).

The degree-five homogeneous part of the actual paired permanent is

$$[\text{degree } 5]\,\operatorname{per}|\sqrt{2}J + K|^2 = 8\sqrt{2}\,[\alpha_5(A) + V_4 + V_3].$$

The residual-slot patterns are five A, three A and one B, or one A and two B; each gives the displayed coefficient after the constant slots are counted. Under the actual perturbation $K + vW$,

$$\dot{A} = 2W = 12I - 2J, \qquad \dot{B} = W \circ A = 12\,\operatorname{diag}(t) - A. \tag{15}$$

Consequently

$$\frac{T_4}{8\sqrt{2}} = 2\delta(A) + \dot{V}_4 + \dot{V}_3. \tag{16}$$

3.3 The Fischer variance of the real projection

For a general real centered Gram matrix A, let $a_i = A_{ii}$. The omitted-state permanent contractions give

$$\begin{aligned}\|\xi_i\|^2 &= 24a_i^4 + q_i^2 + 4a_i(A^3)_{ii} + a_i^2 s_2 + \tfrac{2}{3}a_i s_3 + 2(A(A\circ A)A)_{ii}\\ &\quad - 24a_i^2 q_i - 16a_i c_i + 4\sum_{j,k} A_{ik}^2 A_{ji}A_{jk} + 4a_i(Aq)_i,\\ \left\|\sum_i \xi_i\right\|^2 &= 24s_4 - 24\sum_i q_i^2 + 2s_2^2 + 4\,\operatorname{tr}(A^4).\end{aligned}$$

These formulas follow by matching the four factors in each Fischer product. More explicitly, the squared $g_i^2 p_2/2$ term contributes $q_i^2 + 4a_i(A^3)_{ii} + a_i^2 s_2$; the squared $g_i p_3/3$ term contributes $\frac{2}{3}a_i s_3 + 2(A(A\circ A)A)_{ii}$. The cross terms with g_i^4 contribute $-24a_i^2 q_i$ and $-16a_i c_i$, and the remaining cross term contributes the last two terms. Also $\sum_i \xi_i = p_4 - p_2^2/2$, proving the second formula.

Specialize now to (11) and use

$$q = 2t,\ b = t,\ h = t\circ(At),\ s_2 = 2, \quad H = t^TAt,\ G = t^T(A\circ A)t,\ L = \sum_i t_i^2(At)_i,\ C = t^Tc.$$

One has $a_i = q_i = 2t_i$, $A^3 = A$, $\operatorname{tr}A^4 = 2$, $\sum_i (A(A\circ A)A)_{ii} = s_3$, and $\sum_{j,k} A_{ik}^2 A_{ji}A_{jk} = c_i$. Substitution gives

$$\begin{aligned}\sum_i \|\xi_i\|^2 &= 384d_4 - 192d_3 + 28d_2 - 32C + 16H + \tfrac{22}{3}s_3,\\ \left\|\sum_i \xi_i\right\|^2 &= 24s_4 - 96d_2 + 16.\end{aligned}$$

Thus

$$2\delta(A) = 4608d_4 - 2304d_3 + 528d_2 - 384C + 192H + 88s_3 - 48s_4 - 32. \tag{17}$$

3.4 Differentiating the actual mixed terms

Differentiate (13)–(14) before using the projection relations. From (15),

$$\dot b = 12t, \quad \dot q = 48t, \quad \dot c = 144t^{\circ 2} - 12t, \quad \dot h = 36t^{\circ 2} - 4t,$$
$$\dot s_2 = 48, \quad \dot s_3 = 144d_2 - 12, \quad (\sum B_{ik})^{\cdot} = 12, \quad \dot H = 36d_2 - 4, \quad (A^3)^{\cdot} = 36A.$$

For clarity the full termwise derivative table is

term of V_4	its derivative
$12\sum A_{ik}^3 B_{ik}$	$2880d_4 - 72G - 12s_4$
$-8c^T b$	$-1152d_3 + 96d_2 - 96C$
$-12q^T h$	$-576L - 864d_3 + 96d_2$
$\frac{4}{3}s_3 \sum B_{ik}$	$192d_2 - 16 + 16s_3$
$4q^T Ab$	$288H + 96d_2 - 16$
$s_2 H$	$48H + 72d_2 - 8$
$2\sum B_{ik}(A^3)_{ik}$	$48d_2 - 4 + 72H$

and

term of V_3	its derivative
$12\sum A_{ik}B_{ik}^2$	$720d_4 - 24d_2^2 - 24G$
$-24h^T b$	$-864d_3 + 96d_2 - 288L$
$6H\sum B_{ik}$	$216d_2 - 24 + 72H$
$6b^T Ab$	$144H + 72d_2 - 12.$

For example, the first row of the first table is $12\sum(3A_{ik}^2\dot A_{ik}B_{ik} + A_{ik}^3\dot B_{ik})$; the diagonal terms give $2880d_4$, and the others give $-72G - 12s_4$. Summing both tables with (17) in (16) gives

$$\begin{aligned}\frac{T_4}{8\sqrt{2}} &= 8208d_4 - 5184d_3 + 1512d_2 - 24d_2^2 - 96G - 480C - 864L \\ &\quad - 60s_4 + 104s_3 + 816H - 112.\end{aligned} \tag{18}$$

Finally expand $A_{ik} = z_i\overline{z}_k + \overline{z}_i z_k$:

$$\begin{aligned} G &= 2|n|^2 + 2d_2^2, & L &= 2\,\mathrm{Re}(m_2\overline{m}), \\ C &= 2\,\mathrm{Re}(m_3\overline{p_3}) + 6\,\mathrm{Re}(m_2\overline{m}), & s_3 &= 2|p_3|^2 + 6|m|^2, \\ s_4 &= 2|p_4|^2 + 8|n|^2 + 6d_2^2, & H &= 2|m|^2. \end{aligned}$$

Substitution in (18) proves Proposition 3.1.

4 Variance cancellation and uniform boundary positivity

Under (9), the actual ordinary variance simplifies to

$$\delta = 144d_4 - 96\,\mathrm{Re}(m_3\overline{p_3}) + 16|p_3|^2 - 24|p_4|^2 \geq 0. \tag{19}$$

This is also immediate from (8), since a rank-one trace-one Gram matrix has $\mathrm{tr}(\overline{K}^2(K \circ K)) = S_3 = |p_3|^2$. Subtracting ten times (19) from (12) gives (3), with

$$\begin{aligned} R &= 6768d_4 - 5184d_3 + 1512d_2 - 576d_2^2 - 672|n|^2 \\ &\quad - 4608\,\mathrm{Re}(m_2\overline{m}) + 48|p_3|^2 + 2256|m|^2 + 120|p_4|^2 - 112. \end{aligned} \tag{20}$$

The signed $m_3\overline{p_3}$ term is now absent.

The projection (11) yields $2t_i \le 5/6$. Set $M = 5/12$ and $v_0 = d_2 - 1/6 \ge 0$. Since $A\mathbf{1} = 0$,

$$2|m|^2 = t^T A t \le \left\| t - \tfrac{1}{6}\mathbf{1} \right\|^2 = v_0.$$

Centering of z also gives

$$|m_2| = \left| \sum_i (t_i^2 - 1/36) z_i \right| \le \|t^{\circ 2} - \tfrac{1}{36}\mathbf{1}\| \le (M + 1/6)\sqrt{v_0} = \tfrac{7}{12}\sqrt{v_0}.$$

Hence $|m_2\overline{m}| \le 7v_0/(12\sqrt{2}) \le 7v_0/12$. The other elementary bounds are

$$|p_3|^2 \le d_2, \quad |p_4|^2 \le d_2^2, \quad d_4 \le M d_3, \quad d_3 \ge d_2^2, \quad d_2 \ge 1/6. \tag{21}$$

For the first use Cauchy–Schwarz on $\sum z_i^2 z_i$; for the second use $|\sum z_i^4| \le \sum |z_i|^4$. The last two are the ordinary moment inequalities with $\sum t_i = 1$. Applying these bounds to (20), while preserving the negative d_3 coefficient, gives

$$\begin{aligned} R &\le -2364 d_3 + 1560 d_2 - 456 d_2^2 + 3816 v_0 - 112 \\ &\le G(d_2), \qquad G(t) = -2820 t^2 + 5376 t - 748. \end{aligned} \tag{22}$$

In (6) the rank-one invariants are $S_2 = Z_0 = T_{21} = 0$, $B_2 = T_3 = 1$, $U = d_2$, $S_3 = |p_3|^2$, and $S_{21} = |m|^2$. Thus

$$\begin{aligned} R_4 + E_3/2 &= 20160\delta + 1067904 d_3 - 157824 |p_3|^2 + 2880 |m|^2 + 16128 d_2 - 4800 \\ &\ge 20160\delta + f(d_2), \qquad f(t) = 1067904 t^2 - 141696 t - 4800. \end{aligned}$$

Combining with (3) and (22),

$$F_4 \ge (20160 - 80\sqrt{2})\delta + f(d_2) - 8\sqrt{2} G(d_2). \tag{23}$$

The coefficient of δ is positive. The derivative of $f(t) - 8\sqrt{2}G(t)$ is increasing and at $t = 1/6$ equals $214272 - 35488\sqrt{2} > 0$. Therefore its minimum for $t \ge 1/6$ is at $1/6$. Since $f(1/6) = 1248$ and $G(1/6) = 209/3$, (23) proves (2). Finally $\sqrt{2} < 99/70$ gives $1248 - (1672/3)\sqrt{2} > 459$. All estimates allow zero coordinates. Scaling gives the general-trace statement and $K = 0$ is immediate. This completes the proof of Theorem 1.1.

5 Sharp compensation with unequal trine directions

Let K be the Gram matrix of (4), and write

$$a = \|u\|^2, \quad b = \|v\|^2, \quad h_0 = \langle u, v \rangle, \quad c_0 = |h_0|^2, \quad z_0 = \mathrm{Re}(h_0^3), \quad r = 3(a + b).$$

Thus $a, b \ge 0$, $0 \le c_0 \le ab$, $|z_0| \le c_0^{3/2}$. The Gram matrix is centered and $K\overline{K} = 0$, since $\sum g_i = \sum g_i^2 = 0$. Let $p = 9(ab - c_0)$, the product of its two eigenvalues, allowing a zero eigenvalue.

Theorem 5.1. *The exact spectral tradeoff is*

$$\delta + 2 r d_3 - \left[\frac{2r^4}{9} + \frac{2}{3} p (r^2 - 4p) \right] = 54 c_0 (7a^2 - 6ab + 7b^2 + 4c_0) \ge 0. \tag{24}$$

For nonzero K, equality holds precisely when $c_0 = 0$. The sharp actual compensation bound is

$$F_4(K) \ge (14688 - 208\sqrt{2}) r^4. \tag{25}$$

For nonzero K, equality in (25) holds precisely when $a = b$, $c_0 = ab$ and $h_0^3 = a^3$.

Proof. Here $p_3 = 3(u^3 + v^3)$. The first three omission states are $-\omega^k uv^3$ and the last three are $-\omega^k vu^3$; their sum is zero. The permanent contraction $\|uv^3\|^2 = 6ab^3 + 18c_0b^2$ gives

$$\delta = 108(a^2 + b^2)(ab + 3c_0), \qquad d_3 = 3(a^3 + b^3). \tag{26}$$

Substitution proves (24). The factor in parentheses on its right is $4(a^2 + b^2) + 3(a - b)^2 + 4c_0$, so equality for a nonzero matrix is exactly $c_0 = 0$, including a zero-vector endpoint.

The actual cubic and ordinary coefficients are

$$E_3 = 10368[307(a^3 + b^3) - 23ab(a + b) - 24c_0(a + b) - 628z_0], \tag{27}$$

$$R_4 = 20160\delta + 23040r[9(a^3 + b^3) + 18z_0]. \tag{28}$$

For example, (6) reduces using

$$S_3 = 9(a^3 + b^3) + 18z_0, \quad U = 9(a^3 + b^3 + c_0(a + b)),$$
$$B_2 = 9(a^2 + b^2 + 2c_0), \quad T_3 = 27(a^3 + b^3 + 3c_0(a + b)),$$
$$S_{21} = Z_0 = T_{21} = S_2 = 0.$$

The paired coefficient is

$$\begin{aligned} T_4/\sqrt{2} = {} & 12096(a^4 + b^4) + 13824(a^3b + ab^3) + 129600a^2b^2 \\ & + 20736c_0(a^2 + b^2) + 51840abc_0 - 46656c_0^2 + 20736(a + b)z_0. \end{aligned} \tag{29}$$

For a universal verification, T_4 is homogeneous of degree four in $a, b, h_0, \overline{h}_0$. Cycling one triple sends h_0 to ωh_0 and leaves the permanent derivative invariant; its monomials have h_0 and $\overline{h}_0$ exponents differing by a multiple of three. Conjugation and interchange of the triples leave precisely the seven combinations in (29). Their coefficients can therefore be determined by seven exact linearly independent evaluations. Appendix B supplies the evaluations and an actual permanent-coefficient rule. The degree and symmetry argument proves the identity on the whole family.

The coefficient of z_0 in $F_4 = R_4 - T_4 + rE_3/2$ is $-(8522496 + 20736\sqrt{2})(a + b)$. For $r > 0$ the minimum over its phase therefore occurs at $z_0 = c_0^{3/2}$, attained by $h_0 \geq 0$ real. Put $q_0 = ab/(a+b)^2$ and $y = \sqrt{c_0}/(a + b)$; then $0 \leq y^2 \leq q_0 \leq 1/4$. With this phase choice,

$$F_4/(a + b)^4 = N(q_0, y) - \sqrt{2}\,T(q_0, y),$$

where

$$\begin{aligned} N = {} & -4354560q_0^2 - 13063680q_0y^2 - 14370048q_0 - 8522496y^3 + 6158592y^2 + 5396544, \\ T = {} & 126144q_0^2 + 10368q_0y^2 - 34560q_0 - 46656y^4 + 20736y^3 + 20736y^2 + 12096. \end{aligned}$$

Their derivatives are $N_{q_0} = -435456(20q_0 + 30y^2 + 33)$ and $T_{q_0} = 3456(73q_0 + 3y^2 - 10)$. Hence

$$N_{q_0} - \sqrt{2}T_{q_0} \leq -435456 \cdot 33 + 34560\sqrt{2} < 0.$$

At fixed y, the minimum is at $q_0 = 1/4$, or $a = b$. After dividing by $r^4 = 81(a + b)^4$ and setting $t = 2y \in [0, 1]$, the remaining polynomial is

$$F(t) = 18912 + 8928t^2 - 13152t^3 - \sqrt{2}(140 + 72t^2 + 32t^3 - 36t^4).$$

Its exact factorization is

$$F(t) - F(1) = (1 - t)\big[(4224 + 68\sqrt{2})(1 + t) + t^2(13152 - 4\sqrt{2} - 36\sqrt{2}t)\big].$$

The bracket is strictly positive on $[0,1]$. Thus the minimum is $F(1) = 14688 - 208\sqrt{2}$, proving (25). Equality requires all three minimizations to be equalities: $z_0 = c_0^{3/2}$, $a = b$, and $t = 1$. Equivalently $a = b$, $c_0 = ab$, $h_0^3 = a^3$. Cauchy–Schwarz then makes the two vectors identical up to a cube-root phase; permutation gives the equal-weight doubled cube-root configuration. The zero matrix follows separately by homogeneity. □

6 Full-rank neighborhoods and the remaining rank-two problem

Let $\mathcal{D} = \{K \succeq 0 : K\mathbf{1} = 0,\ \operatorname{tr} K = 1\}$ and $\mathcal{Z} = \{K \in \mathcal{D} : K\overline{K} = 0\}$. The set $\mathcal{Z}$ is a nonlinear locus, not a convex face of the positive semidefinite cone. Its elements have rank at most two: the orthogonal conjugate ranges lie in the same five-dimensional centered complex space.

We use the inherited estimate [1, Lemma5.1]

$$|F_4(K) - F_4(Q)| \le C_0 \|K - Q\|_{\mathrm{F}}, \qquad C_0 = 3190000000, \quad K, Q \in \mathcal{D}. \tag{30}$$

That estimate follows by telescoping actual permanent monomials on the trace-one PSD section; it has no inverse or condition-number hypothesis. Write $\mathcal{Z}_1 = \{Q \in \mathcal{Z} : \operatorname{rank} Q = 1\}$ and let $\mathcal{U}$ denote the normalized two-direction trine family.

Corollary 6.1. *For $K \in \mathcal{D}$,*

$$\operatorname{dist}_{\mathrm{F}}(K, \mathcal{Z}_1) \le 10^{-8} \quad \Longrightarrow \quad F_4(K) > 427,$$

and

$$\operatorname{dist}_{\mathrm{F}}(K, \mathcal{U}) \le 10^{-6} \quad \Longrightarrow \quad F_4(K) > 11203.$$

Both regions contain relatively open 24-real-dimensional families of centered rank-five matrices; their positive J anchors have rank six.

Proof. The normalized sets are compact, so distance is attained. Apply (30) and Theorems 1.1,5.1. The first loss is 31.9, and the second 3190; the margins follow from $1248 - (1672/3)\sqrt{2} > 459$ and $14688 - 208\sqrt{2} > 14393$.

For any center $K_0 \in \mathcal{D}$ in the relevant set, put $K_\varepsilon = (1-\varepsilon)K_0 + \varepsilon W/30$. The minimum eigenvalue on $\mathbf{1}^\perp$ is at least $\varepsilon/5$, and $\|W/30 - K_0\|_{\mathrm{F}}^2 = \operatorname{tr} K_0^2 - 1/5 \le 4/5$. Every centered Hermitian trace-zero perturbation H with $\|H\|_{\mathrm{F}} \le \varepsilon/10$ has minimum centered eigenvalue at least $\varepsilon/10$, and distance from K_0 at most $\varepsilon\sqrt{4/5} + \varepsilon/10 < \varepsilon$. Choose $\varepsilon = 10^{-8}$ or 10^{-6} respectively. The centered trace-one Hermitian affine space has real dimension $5^2 - 1 = 24$. The anchored matrix $K_\varepsilon + H + sJ$ has rank six for every $s > 0$. This allows arbitrary complex Hermitian perturbations. □

There is also explicit exclusion of nearly rank-one points within $\mathcal{Z}$. Write $K = (1-\mu)uu^* + \mu vv^*$ with orthonormal centered eigenvectors, $0 < \mu \le 1/2$. Conjugate disjointness gives $u^T u = v^T v = u^T v = 0$, so $Q = uu^* \in \mathcal{Z}_1$ and $\|K - Q\|_{\mathrm{F}} = \sqrt{2}\mu$. Thus

$$F_4(K) \ge 1248 - (1672/3)\sqrt{2} - C_0\sqrt{2}\mu. \tag{31}$$

In particular the closed strip $\mu \le 10^{-8}/\sqrt{2}$ has $F_4 > 427$. This quantitative reduction is a consequence of the boundary theorem, not a solution of the whole rank-two locus.

The remaining nonflat, unequal-spectrum interior requires a coupled payment with both Gram directions retained. If $K = \lambda uu^* + \mu vv^* \in \mathcal{Z}$, $\lambda + \mu = 1$, then $A = K + \overline{K}$ has spectrum $\lambda, \lambda, \mu, \mu$ on its real rank-four support projection P. Hence $A - A^2 = \lambda\mu P$. The rank-one proof used $A^2 = A$; its

extension must keep the resulting correction terms in the same variance/paired budget. Positivity of this spectral defect alone does not pay its signed contractions. A positive uniform bound on all of $\mathcal{Z}$ would combine with the canonical positive-part reduction of [1, Theorem6.1] to cover sufficiently small conjugate overlap. Moderate overlap and the other blocks remain separate obligations.

A Linking the ordinary identities to the actual permanent

For ordered residual slots the coefficient α_j has prefactor $(6-j)!/j!$ before the distinct-row and distinct-column sum. Partition inclusion–exclusion and centering give $\alpha_0 = 720$, $\alpha_1 = 0$, $\alpha_2 = 12S_2$, $\alpha_3 = 4S_3$, and

$$\alpha_4(K) = 3\sum K_{ik}^4 - 3\,\mathrm{Re}\sum_i(\sum_k K_{ik}^2)^2 + \tfrac{1}{4}S_2^2 + \tfrac{1}{2}\,\mathrm{tr}((K\overline{K})^2).$$

The surviving four-slot partition types are 4 and $2+2$. Differentiating yields $X = \partial_W\alpha_4 = 72d_3 - 12S_3 - 72Z_0 + 6rS_2 + 12T_{21}$. Since $p(K) = \sum_j(7-j)\alpha_j(K)$, the ordinary cubic derivative is $30240X + 86400rS_2$, and its quartic derivative is R_4 in (5); use $\partial_W S_2 = 12r$ and $\partial_W S_3 = 3A_0$.

For $A = K + \overline{K}$, $B = K \circ \overline{K}$, the degree-four paired coefficient before differentiation is

$$\begin{aligned}16\alpha_4(A) + 192\sum A_{ik}^2B_{ik} - 192\sum_i(\sum_k A_{ik}^2)(\sum_k B_{ik})\\ + 48(\sum A_{ik}^2)(\sum B_{ik}) + 192[(\sum B_{ik})^2 - 2\sum_i(\sum_k B_{ik})^2 + \sum B_{ik}^2].\end{aligned}$$

Its differentiated value along $\dot{A} = 2W$, $\dot{B} = W \circ A$ is

$$\begin{aligned}Y = 41472d_3 - 1152S_3 - 5760S_{21} - 13824Z_0 - 32256U\\ + 1920rS_2 + 8832rB_2 + 768T_3 + 2304T_{21}.\end{aligned}$$

Consequently $E_3 = 30240X + 86400rS_2 - Y$, which is (6). These formulas retain matrix products and entrywise products separately; replacing S_2 by B_2 would invalidate the complex-phase calculation.

B Exact determination of the seven trine coefficients

For any actual K, put $A = K + \overline{K}$, $B = |K|^2$ and let $N(t)_{ij} = 2 + \sqrt{2}tA_{ij} + t^2B_{ij}$. Then

$$T_4 = [t^4]\sum_{\pi\in S_6}\sum_{i=1}^{6}(2\sqrt{2}W_{i,\pi(i)} + tW_{i,\pi(i)}A_{i,\pi(i)})\prod_{j\neq i}N(t)_{j,\pi(j)}. \tag{32}$$

This is direct permanent differentiation; it provides an exact finite rule over rational algebraic fields. In the trine family use $\omega = (-1+\sqrt{-3})/2$ and the block entries $K_{ij} = \omega^{i-j}a$, $\omega^{i-j}b$, or $\omega^{i-j}h_0$ and its conjugate, as appropriate within or across the two triples. The seven invariant combinations are

$$a^4 + b^4,\ a^3b + ab^3,\ a^2b^2,\ c_0(a^2+b^2),\ abc_0,\ c_0^2,\ (a+b)z_0.$$

The following exact evaluations of (32) determine them:

a	b	h_0	$T_4/\sqrt{2}$
0	0	$\sqrt{-3}$	-419904
0	1	0	12096
0	1	$\sqrt{-3}$	-345600
0	1	1	6912
1	1	0	181440
1	1	$\sqrt{-3}$	41472
1	2	0	862272

The associated seven-by-seven rational evaluation matrix has rank seven, as may be checked by elimination. Its solution is $(12096, 13824, 129600, 20736, 51840, -46656, 20736)$, proving (29) after the invariant-space argument in the text. Some identification points are not positive semidefinite; this is legitimate polynomial coefficient determination, not a sign argument at those points. Feasible exact controls are separate corroboration.

The supplementary standard-library Python verifier supplies exact permanent coefficient evaluation and checks the reported algebraic constants. Checks use explicit exceptions and are unchanged by Python optimization.

Computational assistance

AI assistance was used in research, drafting and checking. The author is responsible for the mathematical statements and presentation.

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