Unrestricted Half-Power Obstruction and Polynomial Certificates for the Vámos Quartic

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Luminary Oddities

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Abstract

Complete monotonicity of an inverse polynomial power is equivalent to a positive Laplace representation. We prove failure at the critical half power for the specialized four-variable Vámos quartic and every interior stable deformation $h_\lambda = 4e_1e_3 + \lambda a^2b^2$, $0 < \lambda < 64/27$. A negative correction to the inverse transform is isolated outside the support of its positive leading term; uniform estimates on complex inversion planes justify its return to the full four-variable inverse. For the original quartic the fixed-test remainder is $O(\sqrt{q}(1 + |\log q|))$ as the inverse-section coordinate q tends to zero. The obstruction persists for a strict exponent interval above $1/2$, uniformly on compact interior parameter sets. Polynomial squares also detect the finite correction measure at every interior parameter, with locally bounded detection degree. For the original quartic this degree is exactly 54: every nonzero polynomial of degree at most 53 has positive square integral, while an explicit integer polynomial of degree 54 has negative square integral. The associated 55-dimensional moment matrix has inertia $(54, 1, 0)$, and the exact degree 54 persists on an open parameter interval about the original quartic. Rational moments, fraction-free elimination and independent formal-series reconstruction give portable certificates. At the sharp stable parameter $64/27$, an exact formula describes the nonnegative first correction outside the leading support.

Keywords. Vámos quartic; complete monotonicity; inverse Laplace transform; signed measures; exact moment certificates.
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1 Introduction and statements

Let e_j denote the elementary symmetric polynomial of degree j in a, b, c, d, and put

$$h_\lambda = 4(a+b+c+d)(abc+abd+acd+bcd) + \lambda a^2b^2, \qquad L = \frac{64}{27}. \tag{1}$$

The polynomial h_1 is the specialized Vámos quartic. It appears in the study of hyperbolicity, spectrahedral cones and determinantal representations [1, 2, 3, 4]. Its cone is spectrahedral, although no positive integer power of the polynomial has a definite symmetric determinantal representation [5, 1]. Neither fact determines complete monotonicity of its inverse square root: the determinantal construction is sufficient, and its failure is not a converse positivity theorem.

A smooth function f on the positive orthant is *completely monotone* if

$$(-1)^{|\alpha|}\partial^\alpha f(x) \geq 0 \quad (x \in \mathbb{R}^4_{>0},\ \alpha \in \mathbb{Z}^4_{\geq 0}).$$

By the multivariate Bernstein–Hausdorff–Widder theorem, this is equivalent to a representation

$$f(x) = \int_{\mathbb{R}^4_{\geq 0}} e^{-x\cdot Y}\, d\mu(Y) \tag{2}$$

with a positive measure, the integral being finite on the orthant; see [6, Section 2] and [3, Section 2]. The corresponding definition on an open convex cone uses derivatives in directions from that cone. For inverse powers of a hyperbolic polynomial, complete monotonicity on its full hyperbolicity cone is equivalent to complete monotonicity on any open convex subcone [3, Proposition 4.15].

The real-stability interval of (1) is $[0, L]$ [7, Theorem 2.2]. Thus, for $0 < \lambda \leq L$, h_λ is positive and hyperbolic in every positive direction. For $\lambda = 1$ these facts also follow directly from the classical Vámos construction recorded in [1]. The stability theorem supplies the hyperbolicity input for the inversion arguments below; [7] also treats the associated geometry.

Theorem 1.1. *For every $0 < \lambda < L$, the function $h_\lambda^{-1/2}$ is not completely monotone on $\mathbb{R}^4_{>0}$, or on its full hyperbolicity cone. In particular this holds for the specialized Vámos quartic h_1.*

Theorem 1.1 concerns the unrestricted four-variable inverse. For comparison, the full eight-variable Vámos basis polynomial has a $U_{2,4}$ minor, so the minor-closure and quadratic criteria of [6, Propositions 7.6 and 7.9] exclude its half power. The four-variable specialization requires its own inverse-sign argument: identifying variables can change complete monotonicity. Classical Lorentz examples similarly distinguish real stability from half-power positivity [6, Theorem 1.9].

Our method gives the following consequence.

Corollary 1.2. *For each compact set $J \subset (0, L)$, there exists $\epsilon_J > 0$ such that $h_\lambda^{-\alpha}$ is not completely monotone whenever $\lambda \in J$ and $0 < \alpha < 1/2 + \epsilon_J$.*

The compact-set gap follows from continuity of signed derivatives and a Lorentz-quadratic scaling limit. In the large-exponent range, [8, Section 5] gives complete monotonicity for the specialized quartic when $\alpha \geq 65536$. These results locate two distinct parts of the admissible-exponent problem.

A second main result gives an exact finite certificate for the correction at $\lambda = 1$. With κ_1 defined by the Cauchy inverse in Lemma 4.1, the finite signed measure $\nu = y\kappa_1$ satisfies

$$\int P^2\, d\nu > 0 \quad (0 \neq P \in \mathbb{R}[y],\ \deg P \leq 53), \qquad \int P_{54}^2\, d\nu < 0$$

for an explicit integer polynomial P_{54} of degree 54. Theorem 7.2 gives the complete assertion, including moment-matrix inertia. This ordinary-square test probes the fixed one-variable correction measure. The analytic obstruction instead uses a compact test supported outside the positive leading wedge.

Proposition 3.4 makes the original-quartic section limit quantitative, with a $\sqrt{q}(1+|\log q|)$ error controlled by a weighted Fourier seminorm of the chosen test. It gives a sufficient small-section condition in terms of the actual negative margin, complementing the exact finite arithmetic.

The analytic and polynomial tests answer different questions about the same inverse correction. The first proves failure of a positive four-variable representation; the second determines how long ordinary polynomial squares fail to detect its signed correction measure. Corollary 10.1 connects the tests across the whole interior stable family: detection always occurs at finite degree, and the exact degree 54 is stable near $\lambda = 1$. The shared inversion and branch definitions are given once, building on the earlier analytic and certificate arguments [9, 10].

The proof has three analytic ingredients. First, an oscillatory Fourier inverse is reduced to a K_0 integral. Subtracting its leading model leaves a symbol of degree zero. Second, estimates on complex Bromwich planes justify convergence after inverse Laplace transformation. Third, a Cauchy jump computes a negative part of the correction outside the support of the leading term. The complex domination and section regularity connect the correction sign to the full inverse. Sections 2–4 establish these shared analytic tools; Sections 5–6 prove interior failure and the exponent gap; Sections 7–10 derive the exact degree threshold; Section 11 treats the sharp parameter correction.

2 Coordinates and the partial Fourier inverse

Use the primal coordinates

$$s = a + b, \quad \delta = a - b, \quad U = c + d, \quad V = c - d, \qquad P = ab = \frac{s^2 - \delta^2}{4}.$$

The dual coordinates are

$$w = \frac{Y_1 + Y_2}{2}, \quad \rho = \frac{Y_1 - Y_2}{2}, \quad p = \frac{Y_3 + Y_4}{2}, \quad q = \frac{Y_3 - Y_4}{2}, \tag{3}$$

so that $x \cdot Y = sw + \delta\rho + Up + Vq$. An invertible real coordinate change preserves positivity of a measure under push-forward. We use inverse Fourier factors $1/(2\pi)$ and inverse Laplace transforms on fixed vertical lines.

A direct expansion gives

$$\begin{aligned} h_\lambda &= A_\lambda - BV^2, \\ A_\lambda &= \lambda P^2 + 4sPU + (4P + s^2)U^2 + sU^3, \qquad B = s(s+U). \end{aligned} \tag{4}$$

On $\delta = i\xi$, define

$$L_\lambda = 2s^2 + mU(s+U), \qquad C_\lambda = s^4 + msU(s+U)^2, \qquad m = 16/\lambda. \tag{5}$$

Then

$$16A_\lambda = \lambda(\xi^4 + L_\lambda \xi^2 + C_\lambda). \tag{6}$$

We always fix $\operatorname{Re} s = \sigma > 0$ and $\operatorname{Re} U = \upsilon > 0$. Branches are continued from the positive real quadrant; in particular the square roots of B and A_λ/B have positive real part. These assertions, including quantitative bounds, are proved below.

For $q > 0$, Fourier inversion in $V = i\eta$ gives

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{iq\eta} (A_\lambda + B\eta^2)^{-1/2} \, d\eta = \frac{K_0(q\sqrt{A_\lambda/B})}{\pi\sqrt{B}}. \tag{7}$$

The integral is oscillatory. For positive real parameters the identity follows from the gamma integral and a Gaussian Fourier integral. For complex parameters, subtract $B^{-1/2}(1+\eta^2)^{-1/2}$ from the integrand. The difference is $O(|\eta|^{-3})$, locally uniformly in the parameters, and the subtracted model has the known inverse $K_0(q)/(\pi\sqrt{B})$. This proves holomorphic distributional continuation of the identity; nonvanishing on the tube fixes its branch. This interpretation avoids claiming absolute convergence of the unsubtracted integral. Classical Bessel formulas used here and below can be found in [11, Sections 10.29–10.32 and 10.43].

At $\rho = 0$ the double partial inverse has Laplace symbol

$$T_{\lambda,q}(s,U) = \frac{1}{2\pi^2\sqrt{B}} \int_{\mathbb{R}} K_0(q f_\lambda(\xi)) \, d\xi, \qquad f_\lambda = \sqrt{A_\lambda/B}. \tag{8}$$

For a general ρ, insert $e^{i\rho\xi}$ in this integral.

3 Uniform complex subtraction

Lemma 3.1 (Tube bounds)**.** *For λ in a compact subset of $(0, L]$, let $R = 1 + |\operatorname{Im} s| + |\operatorname{Im} U|$ on the fixed Bromwich planes. There are positive constants, uniform in λ, such that*

$$c \le |B| \le CR^2, \qquad \operatorname{Re}(1/\sqrt{B}) \ge c/R^2, \tag{9}$$

$$|f_\lambda| \le C(R^2 + \xi^2), \qquad \operatorname{Re} f_\lambda \ge \frac{c}{R^2(R^2+\xi^2)}. \tag{10}$$

Put $g_\lambda = \sqrt{\lambda}\,\xi^2/(4\sqrt{B})$. For a sufficiently large fixed C_0 and $X_0 = C_0R^3$,

$$|f_\lambda - g_\lambda| \le CR^4, \qquad \operatorname{Re} f_\lambda, \operatorname{Re} g_\lambda \ge c\xi^2/R^2 \qquad (|\xi| \ge X_0), \tag{11}$$

$$|f_\lambda - g_\lambda| \le CR^6, \qquad \operatorname{Re} f_\lambda \ge cR^{-8} \qquad (|\xi| \le X_0). \tag{12}$$

Proof. For a positive real base point x, hyperbolicity and homogeneity give $h_\lambda(x+ty) = h_\lambda(x)\prod_j(1+t\gamma_j)$ with real γ_j, allowing zero factors for a reduced degree. Consequently

$$|h_\lambda(x+iy)| \ge h_\lambda(x) > 0.$$

Apply this to $x = (\sigma/2, \sigma/2, \upsilon/2, \upsilon/2)$. Varying the imaginary coordinates gives

$$|A_\lambda + B\eta^2| \ge c > 0 \quad (\eta \in \mathbb{R}), \qquad \operatorname{dist}(A_\lambda/B, (-\infty, 0]) \ge c/|B|.$$

The constant is uniform on compact positive parameter sets. Also, for $\operatorname{Re} z, \operatorname{Re} t > 0$,

$$\operatorname{Re}\sqrt{zt} \ge \sqrt{\operatorname{Re} z\, \operatorname{Re} t},$$

because $|z||t| \ge \operatorname{Re} z \operatorname{Re} t + \operatorname{Im} z \operatorname{Im} t$. Thus $\operatorname{Re}\sqrt{B} \ge \sqrt{\sigma(\sigma+\upsilon)}$, which proves (9). Polynomial upper bounds on A_λ, together with

$$\operatorname{Re}\sqrt{z} \ge \frac{\operatorname{dist}(z, (-\infty, 0])}{2\sqrt{|z|}},$$

prove (10).

The exact identity

$$(f_\lambda - g_\lambda)(f_\lambda + g_\lambda) = \frac{\lambda(L_\lambda\xi^2 + C_\lambda)}{16B}$$

and $|f_\lambda + g_\lambda| \ge \operatorname{Re} g_\lambda \ge c\xi^2/R^2$ give $|f_\lambda - g_\lambda| \le C(R^4 + R^6/\xi^2)$. At $|\xi| \ge C_0R^3$, choosing C_0 large makes the error less than half the lower bound for $\operatorname{Re} g_\lambda$. This gives (11). The remaining estimates follow from (10) on $|\xi| \le C_0R^3$. □

Lemma 3.2 (Subtraction bound and limit)**.** *Uniformly for $0 < q \le 1$,*

$$\int_{\mathbb{R}} |K_0(qf_\lambda) - K_0(qg_\lambda)|\, d\xi \le CR^{18}. \tag{13}$$

On each fixed pair (s, U), the difference integral converges to $-\pi(r_+ + r_-)$, where

$$r_+^2 + r_-^2 = L_\lambda, \qquad r_+^2 r_-^2 = C_\lambda, \tag{14}$$

and $r_\pm > 0$ initially for $s, U > 0$.

Proof. The integral representation of K_1 yields $|K_1(z)| \le K_1(\operatorname{Re} z)$ in the right half-plane. The identity $(xK_1(x))' = -xK_0(x)$ and its limit at zero give $K_1(x) \le 1/x$. Integrating $K_0' = -K_1$ along the straight segment from g_λ to f_λ therefore gives

$$|K_0(qf_\lambda) - K_0(qg_\lambda)| \le |f_\lambda - g_\lambda| \int_0^1 \frac{dt}{(1-t)\operatorname{Re} g_\lambda + t \operatorname{Re} f_\lambda}. \tag{15}$$

The factor q cancels. For positive a, b, this last integral is $(\log a - \log b)/(a-b)$, with its continuous value at equality, and is at most $2(1 + |\log(a/b)|)/\max(a,b)$. On the tail, (11) bounds (15) by CR^6/ξ^2. On the inner region it is at most

$$CR^{14}(1 + \log R + |\log|\xi||).$$

Here the lower bound cR^{-8} for $\operatorname{Re} f_\lambda$ controls the maximum in the denominator, and the quadratic zero of $\operatorname{Re} g_\lambda$ contributes only the integrable logarithm. Integrating over the inner interval of length $O(R^3)$, and then the tail, proves (13).

For fixed nonzero ξ, the small-argument expansion of K_0 gives the limit $-\log f_\lambda + \log g_\lambda$, with logarithms on the right half-plane. The preceding majorants permit dominated convergence. For real positive s, U,

$$L_\lambda^2 - 4C_\lambda = mU^2(s+U)\big((m-4)s + mU\big) > 0,$$

and $L_\lambda, C_\lambda > 0$, so $r_\pm$ exist as positive roots. Using

$$\int_{\mathbb{R}} \log \frac{\xi^2 + r^2}{\xi^2}\, d\xi = 2\pi r \quad (r > 0)$$

gives the stated limit. The identity extends holomorphically to the fixed complex tube. Outside the positive quadrant, the root sum means this continuation, not a new selection of individual principal roots. □

Define the degree-zero symbol

$$F_\lambda(x) = -\frac{\sqrt{2 + mx(1+x) + 2\sqrt{1 + mx(1+x)^2}}}{\sqrt{1+x}}, \qquad m = 16/\lambda, \tag{16}$$

continued from $x > 0$, so $F_\lambda(0) = -2$. Equations (14) give $F_\lambda(U/s) = -(r_+ + r_-)/\sqrt{B}$. The degree-zero property follows either from this expression or by scaling (s, U, ξ) in the logarithmic difference integral. It holds holomorphically, so the symbol depends only on U/s. Each ratio off the nonpositive real axis has a representative with $\operatorname{Re} s, \operatorname{Re} U > 0$; the set of such representatives for a fixed ratio is connected. Thus F_λ is holomorphic on this slit plane.

Proposition 3.3 (Inverse-distribution limit)**.** *Let $K_\lambda = \mathcal{L}_{s,U}^{-1}[F_\lambda(U/s)]$. Then, in distributions in (w,p), as $q \downarrow 0$,*

$$\mathcal{L}_{s,U}^{-1} T_{\lambda,q} = c_\lambda q^{-1/2} \mathcal{B} + \frac{K_\lambda}{2\pi} + o(1), \qquad c_\lambda = \frac{\Gamma(1/4)^2 \lambda^{-1/4}}{2\sqrt{2}\pi^2}, \tag{17}$$

where

$$\mathcal{B}(w,p) = \frac{\mathbf{1}_{\{0<p<w\}}(w-p)^{-3/4} p^{-3/4}}{\Gamma(1/4)^2}. \tag{18}$$

In particular $\operatorname{supp} \mathcal{B} \subset \{0 \le p \le w\}$.

Proof. The Mellin integral of K_0 gives exactly

$$\int_{\mathbb{R}} K_0(qg_\lambda)\,d\xi = \frac{\Gamma(1/4)^2\lambda^{-1/4}B^{1/4}}{\sqrt{2}\sqrt{q}}.$$

Initially this is a real positive identity; holomorphy gives its complex continuation. Lemma 3.2, with the prefactor in (8), therefore gives the symbol expansion with regular term $F_\lambda(U/s)/(2\pi)$.

For a smooth compactly supported test $\phi(w,p)$, Fourier inversion on the fixed Bromwich planes pairs the symbol with the Fourier transform of $e^{\sigma w+\upsilon p}\phi(w,p)$, up to the fixed Fourier normalization and sign. This transform is Schwartz. The uniform polynomial majorant (13) and $|B|^{-1/2}\le C$ justify dominated convergence in the two remaining Fourier variables. This proves (17), rather than only pointwise convergence of real Laplace transforms. Formula (18) follows by convolving the one-dimensional inverse gamma densities of $s^{-1/4}$ and $(s+U)^{-1/4}$. □

Proposition 3.4 (Quantitative remainder for the original quartic)**.** *Fix $\lambda=1$ and the Bromwich lines used above. For $R=1+|\operatorname{Im}s|+|\operatorname{Im}U|$, write the exact symbol as*

$$T_{1,q} = c_1 q^{-1/2}B^{-1/4} + F_1(U/s)/(2\pi) + E_q(s,U).$$

There is a finite line-dependent constant C such that

$$|E_q(s,U)| \le C\sqrt{q}(1+|\log q|)R^{22}, \qquad 0<q\le 1. \tag{19}$$

For every fixed compact smooth test $\phi(w,p)$, the remainder in (17) therefore has pairing bounded by $C_\phi\sqrt{q}(1+|\log q|)$. One may take

$$C_\phi = \frac{C}{(2\pi)^2}\int_{\mathbb{R}^2}(1+|t|+|v|)^{22}\left|\widehat{e^{\sigma w+\upsilon p}\phi}(t,v)\right|\,dt\,dv,$$

with the Fourier convention $\widehat{\psi}(t,v)=\int e^{-i(tw+vp)}\psi(w,p)\,dw\,dp$.

Proof. Suppress the parameter 1 in f,g. Define

$$H(z) = K_0(z)+\log(z/2)+\gamma, \qquad \operatorname{Re}z>0.$$

The convergent small-argument series for K_0 [11, Section 10.31] gives, for any fixed $z_0<1$,

$$|H'(z)| \le C|z|(1+|\log|z||), \qquad 0<|z|\le z_0, \quad \operatorname{Re}z>0. \tag{20}$$

Indeed the logarithmic term of K_0 is $-[\log(z/2)+\gamma]I_0(z)$; the remaining convergent even series has derivative $O(|z|)$, while $I_0-1=O(|z|^2)$, $I_0'=O(|z|)$ and $|\arg z|<\pi/2$.

The exact error is

$$E_q = \frac{1}{2\pi^2\sqrt{B}}\int_{\mathbb{R}}[H(qf)-H(qg)]\,d\xi. \tag{21}$$

The value at $\xi=0$ is interpreted by its integrable limit. Lemma 3.1 gives the global estimate $|f-g|\le CR^6$. Fix a small constant $\kappa>0$ and set $Z=\kappa q^{-1/2}$. First suppose $Z\ge C_0R^3$, enlarging C_0 so that $Z\ge 1$. Since $qR^2\le(\kappa/C_0)^{2/3}q^{2/3}$, a sufficiently small fixed κ ensures that qf, qg and their joining segment lie in the disk $|z|\le z_0$ when $|\xi|\le Z$. For $\xi\ne 0$, the segment lies in the right half-plane. Its points $z_t=(1-t)g+tf$ satisfy

$$\operatorname{Re}z_t \ge c\min\left\{\frac{1}{R^2(R^2+\xi^2)}, \frac{\xi^2}{R^2}\right\}, \qquad |z_t|\le C(R^2+\xi^2).$$

Thus their logarithms are bounded by a constant times $1 + \log R + \log(1 + |\xi|) + \mathbf{1}_{|\xi|<1}|\log|\xi||$. Integrating (20) along the segment bounds the inner part of the error integral by

$$Cq^2R^6(R^2Z + Z^3)(1 + |\log q| + \log R) \le C\sqrt{q}R^7(1 + |\log q|).$$

Here the logarithmic singularity at zero is integrable, and the last bound uses $qR^2 \le C$, $R \ge 1$ and $\log R \le R$.

For $|\xi| \ge Z$, the tail estimates and $K_0' = -K_1$, $|K_1(z)| \le 1/\operatorname{Re} z$, give

$$|K_0(qf) - K_0(qg)| + |\log f - \log g| \le CR^6/\xi^2.$$

The tail of (21) is therefore $O(R^6/Z) = O(\sqrt{q}R^6)$. This proves the claimed estimate in the first regime, with R^7 in place of R^{22}.

For the complementary regime $Z < C_0R^3$, Lemma 3.2 bounds the integral of the Bessel difference by CR^{18}. Its pointwise logarithmic limit and Fatou's lemma give the same bound for $\int |\log f - \log g|\, d\xi$. Consequently

$$|E_q| \le CR^{18} \le C(C_0/\kappa)\sqrt{q}R^{21},$$

where $|B|^{-1/2} \le C$ has been absorbed. The conservative power 22 proves (19) in both regimes. Finally, the exact two-dimensional Bromwich pairing of E_q with the weighted test has Fourier factor $(2\pi)^{-2}$. Its Schwartz transform pays the displayed R^{22} weight and proves the fixed-test assertion. □

Lemma 3.5 (Positive sections)**.** *For fixed $q > 0$, the full inverse is smooth in (ρ, q), with values in distributions in (w, p). If $h_\lambda^{-1/2}$ were completely monotone, every such section would be a positive distribution.*

Proof. Insert $e^{i\rho\xi}$ in (8). On the inner region of Lemma 3.1,

$$|K_0(qf_\lambda)| \le K_0(q \operatorname{Re} f_\lambda) \le C_q(1 + \log R).$$

On the tail $\operatorname{Re} f_\lambda \ge c\xi^2/R^2$ gives Gaussian decay. Inserting any power of ξ leaves an integrable majorant with polynomial growth in R. Derivatives in q, on a compact positive q-interval, introduce powers of f_λ and integral-order modified Bessel functions, which obey the same kind of polynomial inner bounds and Gaussian tail bounds. Pairing with a compact smooth (w, p) test as above proves the asserted smoothness.

Under (2), the inverse is the push-forward of a positive measure in the coordinates (3). Pair it with a nonnegative compact test in (w, p) and nonnegative approximate identities about $(\rho, q) = (\rho_0, q_0)$, where $q_0 > 0$. The limit is exactly the continuous section evaluated on that test, and is nonnegative. This argument uses an actual continuous section, not an arbitrarily chosen conditional value on a null event. □

4 An elementary finite-part Cauchy inverse

We give a direct construction that also specifies endpoint terms. For each parameter define

$$Q_\lambda(z) = \frac{xF_\lambda'(x)}{z}, \qquad x = -1/z, \qquad j_\lambda(y) = -\frac{1}{\pi}\operatorname{Im} Q_\lambda(y + i0). \tag{22}$$

Lemma 4.1 (Subtracted Cauchy formula)**.** *There is an $M > 0$ such that j_λ is locally integrable on $(0, M]$, vanishes for $y > M$, and is $O(y^{-3/2})$ at zero. The compactly supported distribution*

$$\langle \kappa_\lambda, \phi \rangle = \int_0^M j_\lambda(y)(\phi(y) - \phi(0))\, dy \tag{23}$$

has Cauchy transform Q_λ. On every regular open interval away from zero it has the ordinary density

$$j_\lambda(y) = \frac{\operatorname{Im} F_\lambda'(-1/y + i0)}{\pi y^2}. \tag{24}$$

Proof. The inner cubic $1 + mx(1+x)^2$ has three negative simple roots when $0 < \lambda < L$. Indeed, after $u = 1 + x$, it has one root below zero, one in $(0, 2/3)$ and one in $(2/3, 1)$; its derivative has turning points at $0, 2/3$, and its value at the latter is $1 - 4m/27 < 0$. At $\lambda = L$, the latter two roots merge, with the exact factorization $1 + (27/4)x(1+x)^2 = (1+3x)^2(1+3x/4)$. Possible outer zeros, besides the removable point at zero, are given by

$$\left(1 + \frac{m}{2}x(1+x)\right)^2 - \left(1 + mx(1+x)^2\right) = \frac{m}{4}x^2(1+x)(m(1+x) - 4). \tag{25}$$

Thus all possible finite singularities lie on the negative real axis, at the cubic roots, -1, and $\lambda/4 - 1$. The branch is analytic at $x = 0$, so Q_λ is analytic off a compact positive cut and is $O(z^{-2})$ at infinity.

At a simple inner root with nonzero outer radicand, F_λ' has at worst order $-1/2$. The same is true at a simple outer zero away from an inner root. A simultaneous zero occurs only at $\lambda = 2, x = -1/2$; the resulting fourth-root behavior gives order $-3/4$ for the derivative. At $\lambda = L$ the doubled inner root is removable in the selected square root and the outer radicand there is nonzero. At $x = -1$, continuation past the two inner roots gives $\sqrt{1 + mx(1+x)^2} = -1$. Writing $u = 1 + x$, the numerator of F_λ^2, divided by u, is $-m + 2mu + O(u^2)$; hence this apparent singularity is removable. These facts prove local integrability at every positive cut node. As $x \to \infty$, the radical formula gives $F_\lambda(x) = O(|x|^{1/2})$ and $F_\lambda'(x) = O(|x|^{-1/2})$. Consequently $Q_\lambda(z) = O(|z|^{-3/2})$ at zero. These bounds hold on the selected branches on either side of the cut.

Formula (23) now defines a distribution: subtraction of $\phi(0)$ makes the integrand $O(y^{-1/2})$ at zero. To obtain its Cauchy transform, apply the keyhole Cauchy formula to $Q_\lambda(\zeta)$ with kernel

$$\frac{1}{\zeta - z} + \frac{1}{z} = \frac{\zeta}{z(\zeta - z)}.$$

The kernel vanishes at zero. The small endpoint circle is $O(\varepsilon^{1/2})$, the outer circle is $O(R^{-1})$, and the other node circles vanish by their integrable singular orders. The upper and lower jumps, using Schwarz symmetry, give

$$Q_\lambda(z) = \int_0^M j_\lambda(y) \left(\frac{1}{z - y} - \frac{1}{z}\right) dy = \langle \kappa_\lambda(y), (z - y)^{-1} \rangle.$$

This fixes the mass-zero counterterm rather than leaving a hidden multiple of δ_0. Finally, $x = -1/z$ maps the upper half-plane to itself, and (24) follows from (22). □

Proposition 4.2 (Local homogeneous inverse)**.** *On $w > 0$,*

$$K_\lambda(w, p) = w^{-2} \kappa_\lambda(p/w). \tag{26}$$

In a regular region this is an identity of ordinary densities.

Proof. Differentiate the degree-zero symbol. Lemma 4.1 gives

$$\partial_s F_\lambda(U/s) = \frac{1}{U} Q_\lambda(-s/U) = -\langle \kappa_\lambda(y), (s+Uy)^{-1} \rangle.$$

The inverse of the last resolvent is $\int_0^\infty e^{-t(s+Uy)}\,dt$. Since differentiation in s corresponds to multiplication of its inverse by $-w$, inversion gives

$$-wK_\lambda(w,p) = -\langle \kappa_\lambda(y), \delta(p-wy) \rangle.$$

For $w > 0$, the change $p = wy$ proves (26). Terms supported at $w = 0$ do not affect this assertion or any test used below. General distributional Laplace inversion is discussed in [12]; the identities here follow directly by pairing the displayed ray kernels with compact tests. □

5 A negative correction for the original quartic

Proposition 5.1. *For $\lambda = 1$, $j_1(3/2) = -2/(\pi\sqrt{5})$. More explicitly,*

$$j_1(y) \le -\frac{275}{3456\pi} \qquad \left(\frac{3}{2} \le y \le \frac{8}{5}\right). \tag{27}$$

Consequently, on $1 \le w \le 2$, $3/2 \le p/w \le 8/5$,

$$\frac{K_1(w,p)}{2\pi} \le -\frac{275}{27648\pi^2}. \tag{28}$$

Here K_1 denotes the correction inverse from Proposition 4.2, not the modified Bessel function.

Proof. At $x = -2/3 + i0$, continuation from zero through the upper half-plane gives

$$\sqrt{1+16x+32x^2+16x^3} = \frac{i\sqrt{15}}{9}, \qquad \sqrt{2+16x+16x^2+2\sqrt{1+16x+32x^2+16x^3}} = \frac{1+i\sqrt{15}}{3}.$$

There is exactly one inner root between $-2/3$ and zero; the continued inner root has positive imaginary part after crossing it. The outer radicand then lies in the upper half-plane. In particular no root is independently reset to a principal value after substitution. Differentiation gives

$$F_1'(-2/3) = -\frac{21 + 27i/\sqrt{15}}{2\sqrt{3}}.$$

Formula (24) at $y = 3/2$ yields the exact value.

For the interval bound, let $u = 1 - 1/y$. On $u \in [1/3, 3/8]$, the inner cubic $C(u) = 1 + 16(u-1)u^2$ is negative: it equals $-5/27$ at the left endpoint and has negative derivative there and throughout the interval. The selected branch has $\operatorname{Im} F_1(u-1) = -H(u)$, where

$$H(u)^2 = \frac{1-u}{2}(16 + R(u)) - \frac{1}{u}, \qquad R(u) = \sqrt{16(16 - 4/u)}. \tag{29}$$

This follows by taking the imaginary part of a square root with positive imaginary part; the same computation is given for general parameters in the next section. Here $8 \le R(u) < 10$. Differentiating,

$$E(u) := (H^2)' = -\frac{16+R}{2} + \frac{16(1-u)}{Ru^2} + \frac{1}{u^2} \ge -13 + \frac{64}{9} + \frac{64}{9} = \frac{11}{9}.$$

Also $0 < H^2 < 26/3 < 9$, so $H' > 11/54$. Since $\operatorname{Im} F_1' = -H'$, (24) and $y \le 8/5$ give (27). Equation (28) follows from (26) and $w^{-2} \ge 1/4$. □

Choose a nonzero nonnegative smooth compact test supported in the interior of the region in (28). It annihilates $\mathcal{B}$, because $p > w$ there. By Proposition 3.3, its pairing with the actual section $\mathcal{L}^{-1}T_{1,q}$ is strictly negative for every sufficiently small $q > 0$. Lemma 3.5 contradicts a positive representation (2). This proves Theorem 1.1 at $\lambda = 1$. More quantitatively, set $\delta = -\langle K_1/(2\pi), \phi\rangle > 0$ for this test. Proposition 3.4 makes its pairing with the original section negative whenever

$$C_\phi \sqrt{q}(1 + |\log q|) < \delta.$$

For example, $q < \min\{1, (\delta/(5C_\phi))^4\}$ suffices, because $\sqrt{q}(1 + |\log q|) \le 5q^{1/4}$ on $(0, 1]$. The constant C_ϕ is specified by the contour estimate and test seminorm in that proposition.

6 Every interior deformation parameter

Proof of Theorem 1.1. Set $m = 16/\lambda > 27/4$ and use $u = 1 + x$. The cubic

$$C(u) = 1 + m(u-1)u^2$$

is negative between its middle root $u_m \in (0, 2/3)$ and upper root $u_o \in (2/3, 1)$. Continuation from $x > 0$ through the upper half-plane gives $\sqrt{C} = +i\sqrt{-C}$ on this interval. The outer radicand divided by $1 + x$ is

$$G(u) = 2/u + m(u-1) + 2i\sqrt{-C(u)}/u.$$

It has positive imaginary part. At u_o its limiting value is $(2u_o - 1)/u_o^2 > 0$, so the continued square root is in the first quadrant. Write $H(u) = \operatorname{Im}\sqrt{G(u)} > 0$, giving $\operatorname{Im} F_\lambda(u-1) = -H(u)$.

The inequality $C < 0$ implies $m > 1/((1-u)u^2) \ge 4/u$. Put $R(u) = \sqrt{m(m - 4/u)}$. Since

$$|G(u)|^2 = (1-u)^2 m(m - 4/u),$$

the elementary square-root formula yields

$$H(u)^2 = \frac{1-u}{2}(m + R(u)) - \frac{1}{u}. \tag{30}$$

At the middle root, $\lambda = 16(1 - u_m)u_m^2$, a strictly increasing function of $u_m \in (0, 2/3)$. Thus u_m is below, equal to, or above $1/2$ as λ is below, equal to, or above 2. Differentiating (30) at fixed m, then taking the one-sided limit $u \downarrow u_m$, gives

$$\lim (H^2)' = \begin{cases} \dfrac{u_m - 1}{u_m^2(2u_m - 1)} > 0, & 0 < u_m < 1/2, \\ \dfrac{3u_m - 2}{u_m(u_m - 1)(2u_m - 1)} > 0, & 1/2 < u_m < 2/3. \end{cases} \tag{31}$$

For verification, before substitution the derivative is

$$-(m + R)/2 + \frac{m(1-u)}{u^2 R} + \frac{1}{u^2},$$

and at the root $R = |1 - 2u_m|/((1 - u_m)u_m^2)$, which gives the two displayed rational expressions. In either case $H' > 0$ on a nonempty open subinterval just above u_m.

For $\lambda = 2$, the middle root is $1/2$, and G tends to zero there, so H extends continuously by zero. It is positive inside (u_m, u_o). The mean value theorem on a closed interval from u_m to an interior

point supplies a point with $H' > 0$. Analyticity in the interior makes this strict inequality persist on an open subinterval. No derivative at the singular endpoint is required.

On the subinterval found in every case, $\operatorname{Im} F'_\lambda = -H' < 0$. Equation (24) gives a strictly negative regular density at $y = 1/(1-u) > 1$. Proposition 4.2 supplies a negative correction on an open region with $p > w > 0$. Choose a nonnegative smooth compact test there. Its leading contribution in (17) is identically zero, while its regular limit is negative. For sufficiently small positive q, the actual section is therefore negative on that test, contradicting Lemma 3.5 under complete monotonicity. This proves failure on the orthant. Failure on the full cone follows by restriction, or by the cone/subcone equivalence cited above. □

Proof of Corollary 1.2. For each $\lambda_0 \in J$, Theorem 1.1 implies a strict violation of a signed mixed derivative at some orthant point when $\alpha = 1/2$. That derivative depends continuously on (λ, α). A product neighborhood of $(\lambda_0, 1/2)$ retains the same violation. A finite cover of the compact set J provides a common positive exponent radius.

For smaller positive exponents, fix $a = 1$ and scale (b, c, d) by ε. The first nonzero term of h_λ/ε^2 is

$$Q_\lambda = \lambda b^2 + 4b(c+d) + 4cd = (U+2b)^2 - (4-\lambda)b^2 - V^2.$$

This is a nondegenerate Lorentz quadratic in three effective variables. Complete monotonicity of the original inverse power would pass to the scaled limit on an open positive subcone. The classical quadratic criterion [6, Theorem 1.9] excludes $0 < \alpha < 1/2$. Combining this with the finite-cover neighborhood of $1/2$ proves the corollary. The finite-cover argument gives a qualitative positive radius ϵ_J. □

7 The finite correction measure and its exact degree threshold

We now specialize the shared analytic construction to $\lambda = 1$, writing $F = F_1$, $Q = Q_1$, $\kappa = \kappa_1$ and $j = j_1$. Equations (16), (22) and (23) fix the branch, its Cauchy transform and the mass-zero subtraction. In particular, Lemma 4.1 gives compact support, integrable singularities at positive cut nodes, and $j(y) = O(y^{-3/2})$ at zero. These facts identify the measure whose moments are reconstructed below.

Lemma 7.1. *The distribution $\nu = y\kappa$ is the finite signed Borel measure*

$$d\nu(y) = y\, j(y)\, dy, \qquad 0 \le y \le M.$$

If $F(x) = \sum_{n\ge 0} f_n x^n$ near zero, its moments are

$$m_n := \int y^n \, d\nu(y) = (n+1)(-1)^{n+1} f_{n+1}, \qquad n \ge 0. \tag{32}$$

Proof. Multiplying the test in (23) by y kills its value at zero: $\langle y\kappa, \phi\rangle = \int_0^M y j(y)\phi(y)\, dy$. The density is $O(y^{-1/2})$ near zero, and all remaining singularities are integrable. It therefore has finite total variation and compact support.

For large ζ, the compact-distribution Cauchy expansion and (22) give, respectively,

$$Q(\zeta) = \sum_{n\ge 0} \frac{\langle \kappa, y^n\rangle}{\zeta^{n+1}}, \qquad Q(\zeta) = \sum_{n\ge 1} \frac{n(-1)^n f_n}{\zeta^{n+1}}.$$

These convergent expansions are compared outside the compact support. Thus $\langle \kappa, 1\rangle = 0$ and $\langle \kappa, y^n\rangle = n(-1)^n f_n$ for $n \ge 1$. The moments of $y\kappa$ give (32). □

For a real polynomial P, define

$$\mathcal{I}(P) = \int P(y)^2 \, d\nu(y). \tag{33}$$

Compact support and finite total variation make this finite at every polynomial degree. A positive leading mass does not imply positivity of the entire form.

Theorem 7.2. *For the measure ν determined by (16)–(23):*

1. *$\mathcal{I}(P) > 0$ for every nonzero real polynomial with $\deg P \le 53$;*
2. *the integer polynomial P_{54} specified in Section 10 and in* `certificate.json` *has degree 54 and $\mathcal{I}(P_{54}) < 0$;*
3. *the matrix $(m_{i+j})_{i,j=0}^{54}$ has inertia $(54, 1, 0)$, where the entries count positive, negative and zero eigenvalues.*

Consequently 54 is the least ordinary polynomial-square degree detecting negativity of ν.

The same positivity statement through degree 53 holds for complex polynomials with P^2 replaced by $|P|^2$, because real and imaginary parts contribute additively. The minimum in the theorem concerns this fixed correction measure. Its degree is measured in the one-variable square form $\mathcal{I}$.

8 Rational moments from an algebraic branch

Put

$$B(x) = 2 + 16x + 16x^2, \qquad C(x) = 1 + 16x + 32x^2 + 16x^3.$$

Eliminating the radicals in (16) gives

$$\big((1+x)F(x)^2 - B(x)\big)^2 = 4C(x). \tag{34}$$

The derivative of the left side minus the right side with respect to F at $(x, F) = (0, -2)$ is -16. Hence this constant term determines one analytic branch and one formal power series in $\mathbb{Q}[[x]]$. It is the branch in (16).

One exact reconstruction uses three elementary operations. If $A = \sum a_n x^n$ and $S = \sum s_n x^n$ satisfy $S^2 = A$ with $s_0 \neq 0$, then

$$s_n = \frac{a_n - \sum_{j=1}^{n-1} s_j s_{n-j}}{2s_0} \quad (n \ge 1). \tag{35}$$

For $R = 1/A$ with $a_0 \neq 0$,

$$r_0 = a_0^{-1}, \qquad r_n = -a_0^{-1} \sum_{j=1}^{n} a_j r_{n-j}. \tag{36}$$

Compute $\sqrt{C}$ with constant term 1, then $\sqrt{B + 2\sqrt{C}}$ with constant term 2, and multiply its negative by $(\sqrt{1+x})^{-1}$. All operations are finite rational arithmetic at any prescribed truncation order.

The first coefficients and moments are

$$(f_0, f_1, f_2, f_3, f_4) = \left(-2, -7, \frac{93}{4}, -\frac{1283}{8}, \frac{88541}{64}\right),$$
$$(m_0, m_1, m_2, m_3) = \left(7, \frac{93}{2}, \frac{3849}{8}, \frac{88541}{16}\right).$$

Coefficients through f_{109} determine every moment used in Theorem 7.2. The verification also substitutes the reconstructed series into (34); thus the radical recurrence is checked against its source equation.

A second reconstruction uses formal Newton iteration directly on (34), starting at $F = -2$. Its denominator has nonzero constant term -16, so it is invertible in the formal series ring. This supplies an arithmetic implementation independent of the nested square-root recurrences. It uses only the uniquely determined formal branch at zero.

9 Exact Hankel signs and the first negative degree

Use the scaled variable $z = 4y$ and set

$$s_n = 4^n m_n, \qquad H_d = (s_{i+j})_{i,j=0}^{d}, \qquad D_d = \det H_d. \tag{37}$$

Every s_n needed here, $0 \le n \le 108$, is an integer, as checked exactly by the coefficient reconstruction. Scaling is a congruence:

$$H_d = \operatorname{diag}(1, 4, \dots, 4^d)(m_{i+j})_{i,j=0}^{d} \operatorname{diag}(1, 4, \dots, 4^d).$$

It preserves definiteness and inertia.

Proposition 9.1 (Finite arithmetic certificate)**.** *The integers in* (37) *satisfy*

$$D_d > 0 \quad (0 \le d \le 53), \qquad D_{54} < 0. \tag{38}$$

Both supplied implementations reconstruct these signs from (34) *and* (32)*.*

Here is the exact computational contract for the proposition. The standard-library program reconstructs the rational series, verifies the implicit equation, forms H_{54}, and applies symmetric fraction-free elimination. With initial matrix $A = H_{54}$ and previous pivot 1, an elimination step with pivot $p = A_{kk}$ replaces

$$A_{ij} \longleftarrow \frac{pA_{ij} - A_{ik}A_{kj}}{\text{previous pivot}}, \qquad i, j > k.$$

Every integer division is checked for zero remainder. The pivot before each next step is the corresponding leading determinant. This is the classical determinant-preserving identity underlying Bareiss elimination [13]; it introduces no rounding decisions. All pivots used as divisors are nonzero, since the previously computed determinants are strictly positive. The final negative pivot is not used as a divisor.

The second program reconstructs the moments by formal Newton iteration and evaluates the 55 integer determinants separately using FLINT. It checks the proposed polynomial only after reconstructing its own moments. The first method is sufficient for reproduction without external numerical libraries.

For scale, selected results are:

Degree d	Sign of D_d	Bit length of $\lvert D_d\rvert$
0	+	3
1	+	15
2	+	34
52	+	10618
53	+	11020
54	−	11434

The verifiers reconstruct the exact determinant integers and required moments; the supplied certificate fixes the witness coefficients and exact square value. The exact reconstruction establishes all prefixes through 54.

By Sylvester's criterion, the complete list of positive leading minors makes H_{53} positive definite. This proves positivity on every polynomial space of degree at most 53, rather than positivity of a selected set of trial polynomials.

10 An explicit integer polynomial and its original integral

Write the last matrix in block form

$$H_{54} = \begin{pmatrix} A & v \\ v^T & c \end{pmatrix}, \qquad A = H_{53}.$$

Since A is positive definite, it is invertible. Define

$$p(z) = z^{54} + \sum_{i=0}^{53} a_i z^i, \qquad (a_0, \dots, a_{53})^T = -A^{-1}v. \tag{39}$$

Its square integral under the push-forward of ν by $z = 4y$ is

$$c - v^T A^{-1} v = \frac{D_{54}}{D_{53}} < 0. \tag{40}$$

This specifies the monic negative polynomial without numerical eigenvectors or a possibly ill-conditioned floating-point inverse.

For an integer version, let g be the positive greatest common divisor of the integer coefficient vector of $D_{53}p(z)$, and set

$$P_{54}(y) = \frac{D_{53}}{g}\, p(4y). \tag{41}$$

Cramer's rule shows that $D_{53}p$ has integer coefficients. Thus (41) is an integer polynomial of degree 54, with

$$\mathcal{I}(P_{54}) = \frac{D_{54}D_{53}}{g^2} < 0. \tag{42}$$

The 55 coefficients of $D_{53}p(z)/g$, in increasing degree order, are supplied in `certificate.json`. Multiplying coefficient i by 4^i gives the coefficient of y^i in P_{54}. The programs also check all 54 orthogonality relations against lower monomials in z and contract all 3025 original coefficient–moment pairs in the variable y. This independently checks the scale factor in (41) and the exact equality (42).

The block congruence

$$H_{54} \sim \begin{pmatrix} A & 0 \\ 0 & D_{54}/D_{53} \end{pmatrix}$$

has 54 positive entries in its first block and one negative scalar. It proves the claimed inertia and completes Theorem 7.2. The first-negative-degree conclusion follows from positive definiteness of the entire lower-degree space.

10.1 Finite detection throughout the interior family

The same moment form is defined at every interior parameter. This gives a qualitative bridge between the open negative-density region and the exact degree calculation at $\lambda = 1$.

Corollary 10.1. *For $0 < \lambda < L$, let $\nu_\lambda = y\kappa_\lambda$, and define*

$$d(\lambda) = \min\left\{ d \in \mathbb{Z}_{\geq 0} : \int P(y)^2 \, d\nu_\lambda(y) < 0 \text{ for some } P \in \mathbb{R}[y],\ \deg P \leq d \right\}.$$

Then $d(\lambda)$ is finite and upper semicontinuous on $(0, L)$. In particular, it is bounded on every compact subset of $(0, L)$. There is an open interval $I \subset (0, L)$ containing 1 such that $d(\lambda) = 54$ for every $\lambda \in I$. The same integer polynomial P_{54} is a negative-square witness throughout I.

Proof. Lemma 4.1 shows that multiplication by y removes the counterterm at zero and gives a compactly supported finite signed measure with density $yj_\lambda(y)$. Near zero its absolute density is $O(y^{-1/2})$; the other cut-node singularities are integrable. The proof of Theorem 1.1 gives a regular open interval in $y > 1$ on which this density is strictly negative. Choose a nonzero continuous function g, supported in a compact subinterval there. Then $\int g^2 \, d\nu_\lambda < 0$. On a compact interval containing the measure's support, polynomials approximate g uniformly. For such an approximating polynomial P,

$$\left| \int (P^2 - g^2) \, d\nu_\lambda \right| \leq \|\nu_\lambda\|_{\mathrm{TV}} \|P - g\|_\infty (\|P\|_\infty + \|g\|_\infty).$$

The right side tends to zero, so a polynomial square has negative integral. This proves finiteness at each fixed interior parameter.

For each fixed n, the Taylor coefficient $f_n(\lambda)$ of F_λ at zero is real analytic in $\lambda > 0$. Indeed, the two radicands at zero have the fixed nonzero values 1 and 4, and their selected roots are 1 and 2; analytic implicit inversion therefore applies locally in (x, λ). The proof of Lemma 7.1 applies without change to ν_λ, giving

$$\int y^n \, d\nu_\lambda(y) = (n+1)(-1)^{n+1} f_{n+1}(\lambda).$$

Thus the square integral of every fixed polynomial is continuous in λ, without requiring a common pointwise density bound at moving cut nodes. A negative witness at λ_0 remains negative in a neighborhood, so $d(\lambda) \leq d(\lambda_0)$ there. This is upper semicontinuity; a finite subcover gives the asserted bound on each compact set.

At $\lambda = 1$, the exact leading minors in Proposition 9.1 make the moment form positive definite on the entire degree-53 polynomial space. Its matrix entries are continuous in λ, so positive definiteness persists on a neighborhood of 1. The strict negative value of the fixed integer polynomial P_{54} persists on another neighborhood. On their intersection no polynomial of degree at most 53 is negative and a degree-54 polynomial is negative. Hence the least negative degree is exactly 54 there. □

11 The sharp parameter endpoint

The same first-correction calculation has the opposite sign behavior at $\lambda = L$. The following exact sign description complements the interior negative-density result.

Proposition 11.1. *On the open region $y > 1$, the distribution κ_L is nonnegative. Its density is positive for $1 < y < 27/11$, vanishes for $y > 27/11$, and has a locally integrable inverse-square-root singularity at $27/11$. The assertion concerns the first correction κ_L.*

Proof. Let $r = \sqrt{1 + 3x/4}$. The selected inner square root is $(1 + 3x)r$, not its absolute value. The identities

$$\begin{aligned} 1 + \tfrac{27}{4}x(1+x)^2 &= (1+3x)^2(1+3x/4), \\ 12r^4 + 8r^3 - 15r^2 - 6r + 5 &= (r+1)^2(2r-1)(6r-5), \\ 1 + x &= (4r^2 - 1)/3 \end{aligned}$$

give the exact simplification

$$F_L(x) = -\sqrt{3}(r+1)\sqrt{\frac{6r-5}{2r+1}}.$$

For $y > 1$, $x = -1/y \in (-1, 0)$ and $r \in (1/2, 1)$. The expression is real for $r > 5/6$. On $1/2 < r < 5/6$ write $F_L = -iH$; then

$$H^2 = \frac{3(r+1)^2(5-6r)}{2r+1}, \qquad \frac{dH^2}{dr} = -\frac{6(r+1)(12r^2+4r+3)}{(2r+1)^2} < 0.$$

Since $dr/dx = 3/(8r) > 0$, $\operatorname{Im} F_L' > 0$. Equation (24) proves the density signs. The point $r = 5/6$ corresponds to $x = -11/27$ and $y = 27/11$. It is a simple outer zero, so the derivative has an integrable inverse-square-root singularity. Continuity of F_L there creates no point mass. The doubled inner root is removable. These observations also exclude a hidden negative term on the open region $y > 1$. □

12 Reproducible arithmetic and the two test mechanisms

The full-inverse proof uses a nonnegative compact test in the region $p > w > 0$, where the leading density $\mathcal{B}$ vanishes. The finite certificate uses the global correction-measure pairing

$$\langle \kappa_1, yP_{54}(y)^2 \rangle = \int P_{54}(y)^2 \, d\nu(y) < 0.$$

Compact support defines this pairing by any smooth compact extension agreeing with the polynomial near the support of κ_1. Thus the two arguments supply complementary tests of the shared inverse correction: a support-separated analytic sign and an exact polynomial threshold. The anisotropic positive corner profile studied in [14] describes a further local scaling regime.

Three programs reproduce the finite arithmetic. The standard-library verifier `verify_analytic.py` checks the factorizations, continued local radical arithmetic, explicit interval bounds, and deformation and endpoint identities used in the analytic proof. The program `verify.py` reconstructs the rational moments, all 55 determinant signs, the 54 orthogonality equations, and all 3025 coefficient–moment pairs for the integer square witness. The alternative `verify_independent.py` obtains the moments by formal Newton iteration and computes determinants separately with FLINT. The integer witness and its exact integral are in `certificate.json`. All executable checks use explicit

failures and support Python's `-O` option. The analytic convergence and section arguments are proved in the text; the programs reproduce their finite algebraic identities and the complete finite certificate. Build and execution commands are given in `README.md`.

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