Polynomial Certificates for the Vámos Riesz Obstruction

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Abstract

We determine the first ordinary polynomial-square degree that detects the negative sign of a canonical correction measure associated with the specialized four-variable Vámos quartic. If κ is the compact finite-part correction from the earlier inverse-Laplace analysis and $\nu = y\kappa$, then $\int P^2\, d\nu > 0$ for every nonzero real polynomial of degree at most 53, whereas an explicit integer polynomial of degree 54 has negative square integral. Equivalently, the moment matrix through degree 53 is positive definite and the next matrix has exactly one negative eigenvalue. The proof converts one algebraic analytic branch into rational moments and uses exact fraction-free elimination. A portable standard-library verifier reconstructs the certificate; a second implementation recovers the moments from an implicit quartic equation and independently checks the determinants and polynomial. The earlier analytic paper supplies the identification of the correction and the full inverse obstruction. The finite theorem here does not specify a mixed-derivative violation of the original quartic inverse.

1 A finite question for a signed inverse correction

The specialized Vámos quartic is

$$h_\lambda(a,b,c,d) = 4(a+b+c+d)(abc+abd+acd+bcd) + \lambda a^2 b^2. \tag{1}$$

The earlier paper [1] proves that $h_\lambda^{-1/2}$ is not completely monotone on the positive orthant for every $0 < \lambda < 64/27$. Its proof subtracts a singular positive leading term from an inverse-Laplace distribution and identifies a regular negative correction outside the leading term's support. That analytic theorem is an inherited result throughout the present paper.

Here we ask a different, finite question at $\lambda = 1$: at what degree does an ordinary polynomial square first detect the negative sign of a natural finite measure obtained from that correction? The answer is exactly 54. This includes both an explicit negative polynomial and positivity of every lower-degree polynomial square.

The correction symbol is the analytic branch

$$F(x) = -\frac{\sqrt{2+16x(1+x)+2\sqrt{1+16x(1+x)^2}}}{\sqrt{1+x}}, \qquad F(0) = -2. \tag{2}$$

Near zero, all square roots in this formula are fixed by their positive constant terms before the outside minus sign. Continuation away from zero means continuation of this branch; roots are not reset independently after substitution.

For clarity, our main objects are distinguished as follows:

Object	Role
$h_1^{-1/2}$	The original four-variable function studied in [1].
κ	A compactly supported, mass-zero finite-part distribution determined by the Cauchy transform of (2).
$\nu = y\kappa$	A finite signed measure. The degree-54 theorem concerns its ordinary polynomial-square form.

The large integers in the certificate are supplied as mathematical data, not as a substitute for the reduction. Sections 2–5 derive the exact measure, moment and matrix identities; the finite arithmetic is completely reproducible from (2).

2 The correction measure and the main theorem

We use the following precise analytic input from [1, Lemma 4.1]. Set

$$Q(z) = \frac{xF'(x)}{z}, \qquad x = -1/z. \tag{3}$$

There is $M > 0$ and a real function j supported in $[0, M]$, locally integrable away from zero, with $j(y) = O(y^{-3/2})$ as $y \downarrow 0$, such that the compact distribution

$$\langle \kappa, \phi \rangle = \int_0^M j(y)\big(\phi(y) - \phi(0)\big)\, dy \tag{4}$$

has Cauchy transform Q. The possible positive cut-node singularities of j are integrable. In particular the subtraction in (4) fixes $\langle \kappa, 1 \rangle = 0$; no unspecified multiple of δ_0 is allowed. The cited lemma derives this statement from the continued radical branch by a subtracted Cauchy contour formula. Its branch and endpoint estimates are the only analytic inversion input used by the finite theorem.

Lemma 2.1. *The distribution $\nu = y\kappa$ is the finite signed Borel measure*

$$d\nu(y) = y\, j(y)\, dy, \qquad 0 \le y \le M.$$

If $F(x) = \sum_{n\ge 0} f_n x^n$ near zero, its moments are

$$m_n := \int y^n\, d\nu(y) = (n+1)(-1)^{n+1} f_{n+1}, \qquad n \ge 0. \tag{5}$$

Proof. Multiplying the test in (4) by y kills its value at zero: $\langle y\kappa, \phi \rangle = \int_0^M y j(y)\phi(y)\, dy$. The density is $O(y^{-1/2})$ near zero, and all remaining singularities are integrable. It therefore has finite total variation and compact support.

For large z, the compact-distribution Cauchy expansion and (3) give, respectively,

$$Q(z) = \sum_{n\ge 0} \frac{\langle \kappa, y^n \rangle}{z^{n+1}}, \qquad Q(z) = \sum_{n\ge 1} \frac{n(-1)^n f_n}{z^{n+1}}.$$

These convergent expansions are compared outside the compact support. Thus $\langle \kappa, 1 \rangle = 0$ and $\langle \kappa, y^n \rangle = n(-1)^n f_n$ for $n \ge 1$. The moments of $y\kappa$ give (5). □

For a real polynomial P, define

$$\mathcal{I}(P) = \int P(y)^2 \, d\nu(y). \tag{6}$$

Compact support and finite total variation make this finite at every polynomial degree. A positive leading mass does not imply positivity of the entire form.

Theorem 2.2. *For the measure ν determined by (2)–(4):*

1. *$\mathcal{I}(P) > 0$ for every nonzero real polynomial with $\deg P \le 53$;*
2. *the integer polynomial P_{54} specified in Section 5 and in* `certificate.json` *has degree 54 and $\mathcal{I}(P_{54}) < 0$;*
3. *the matrix $(m_{i+j})_{i,j=0}^{54}$ has inertia $(54, 1, 0)$, where the entries count positive, negative and zero eigenvalues.*

Consequently 54 is the least ordinary polynomial-square degree detecting negativity of ν.

The same positive-degree statement holds for complex polynomials with P^2 replaced by $|P|^2$, because real and imaginary parts contribute additively. The minimum in the theorem concerns this fixed correction measure. It does not refer to the order of a four-variable derivative.

3 Rational moments from an algebraic branch

Put

$$B(x) = 2 + 16x + 16x^2, \qquad C(x) = 1 + 16x + 32x^2 + 16x^3.$$

Eliminating the radicals in (2) gives

$$\big((1+x)F(x)^2 - B(x)\big)^2 = 4C(x). \tag{7}$$

The derivative of the left side minus the right side with respect to F at $(x, F) = (0, -2)$ is -16. Hence this constant term determines one analytic branch and one formal power series in $\mathbb{Q}[[x]]$. It is the branch in (2).

One exact reconstruction uses three elementary operations. If $A = \sum a_n x^n$ and $S = \sum s_n x^n$ satisfy $S^2 = A$ with $s_0 \neq 0$, then

$$s_n = \frac{a_n - \sum_{j=1}^{n-1} s_j s_{n-j}}{2s_0} \quad (n \ge 1). \tag{8}$$

For $R = 1/A$ with $a_0 \neq 0$,

$$r_0 = a_0^{-1}, \qquad r_n = -a_0^{-1} \sum_{j=1}^{n} a_j r_{n-j}. \tag{9}$$

Compute $\sqrt{C}$ with constant term 1, then $\sqrt{B + 2\sqrt{C}}$ with constant term 2, and multiply its negative by $(\sqrt{1+x})^{-1}$. All operations are finite rational arithmetic at any prescribed truncation order.

The first coefficients and moments are

$$(f_0, f_1, f_2, f_3, f_4) = \left(-2, -7, \frac{93}{4}, -\frac{1283}{8}, \frac{88541}{64}\right),$$
$$(m_0, m_1, m_2, m_3) = \left(7, \frac{93}{2}, \frac{3849}{8}, \frac{88541}{16}\right).$$

Coefficients through f_{109} determine every moment used in Theorem 2.2. The verification also substitutes the reconstructed series into (7); thus the radical recurrence is checked against its source equation.

A second reconstruction uses formal Newton iteration directly on (7), starting at $F = -2$. Its denominator has nonzero constant term -16, so it is invertible in the formal series ring. This supplies an arithmetic implementation independent of the nested square-root recurrences. It does not require numerical evaluation of a branch cut.

4 Exact Hankel signs and the first negative degree

Use the scaled variable $z = 4y$ and set

$$s_n = 4^n m_n, \qquad H_d = (s_{i+j})_{i,j=0}^{d}, \qquad D_d = \det H_d. \tag{10}$$

Every s_n needed here, $0 \le n \le 108$, is an integer, as checked exactly by the coefficient reconstruction. Scaling is a congruence:

$$H_d = \operatorname{diag}(1, 4, \ldots, 4^d)(m_{i+j})_{i,j=0}^{d} \operatorname{diag}(1, 4, \ldots, 4^d).$$

It preserves definiteness and inertia.

Proposition 4.1 (Finite arithmetic certificate)**.** *The integers in* (10) *satisfy*

$$D_d > 0 \quad (0 \le d \le 53), \qquad D_{54} < 0. \tag{11}$$

Both supplied implementations reconstruct these signs from (7) *and* (5)*.*

Here is the exact computational contract for the proposition. The standard-library program reconstructs the rational series, verifies the implicit equation, forms H_{54}, and applies symmetric fraction-free elimination. With initial matrix $A = H_{54}$ and previous pivot 1, an elimination step with pivot $p = A_{kk}$ replaces

$$A_{ij} \longleftarrow \frac{pA_{ij} - A_{ik}A_{kj}}{\text{previous pivot}}, \qquad i, j > k.$$

Every integer division is checked for zero remainder. The pivot before each next step is the corresponding leading determinant. This is the classical determinant-preserving identity underlying Bareiss elimination [2]; it introduces no rounding decisions. All pivots used as divisors are nonzero, since the previously computed determinants are strictly positive. The final negative pivot is not used as a divisor.

The second program reconstructs the moments by formal Newton iteration and evaluates the 55 integer determinants separately using FLINT. It checks the proposed polynomial only after reconstructing its own moments. The first method is sufficient for reproduction without external numerical libraries.

For scale, selected results are:

Degree d	Sign of D_d	Bit length of $\lvert D_d\rvert$
0	+	3
1	+	15
2	+	34
52	+	10618
53	+	11020
54	−	11434

The verifiers reconstruct the exact determinant integers and required moments; the supplied certificate fixes the witness coefficients and exact square value. The table is not the sign-verification procedure. Only the prefixes through 54 are asserted checked.

By Sylvester's criterion, the complete list of positive leading minors makes H_{53} positive definite. This proves positivity on every polynomial space of degree at most 53, rather than positivity of a selected set of trial polynomials.

5 An explicit integer polynomial and its original integral

Write the last matrix in block form

$$H_{54} = \begin{pmatrix} A & v \\ v^T & c \end{pmatrix}, \qquad A = H_{53}.$$

Since A is positive definite, it is invertible. Define

$$p(z) = z^{54} + \sum_{i=0}^{53} a_i z^i, \qquad (a_0, \dots, a_{53})^T = -A^{-1}v. \tag{12}$$

Its square integral under the push-forward of ν by $z = 4y$ is

$$c - v^T A^{-1} v = \frac{D_{54}}{D_{53}} < 0. \tag{13}$$

This specifies the monic negative polynomial without numerical eigenvectors or a possibly ill-conditioned floating-point inverse.

For an integer version, let g be the positive greatest common divisor of the integer coefficient vector of $D_{53}p(z)$, and set

$$P_{54}(y) = \frac{D_{53}}{g}\, p(4y). \tag{14}$$

Cramer's rule shows that $D_{53}p$ has integer coefficients. Thus (14) is an integer polynomial of degree 54, with

$$\mathcal{I}(P_{54}) = \frac{D_{54}D_{53}}{g^2} < 0. \tag{15}$$

The 55 coefficients of $D_{53}p(z)/g$, in increasing degree order, are supplied in `certificate.json`. Multiplying coefficient i by 4^i gives the coefficient of y^i in P_{54}. The programs also check all 54 orthogonality relations against lower monomials in z and contract all 3025 original coefficient–moment pairs in the variable y. This independently checks the scale factor in (14) and the exact equality (15).

The block congruence

$$H_{54} \sim \begin{pmatrix} A & 0 \\ 0 & D_{54}/D_{53} \end{pmatrix}$$

has 54 positive entries in its first block and one negative scalar. It proves the claimed inertia and completes Theorem 2.2. The first-negative-degree conclusion follows from positive definiteness of the entire lower-degree space.

6 Relation to the full inverse and parameter boundary

The analytic correction studied in [1] has a regular negative density on $[3/2, 8/5]$, while the full inverse contains an additional singular positive leading term. The previous paper obtains its original-function obstruction by testing outside the support of that leading term and controlling the distributional remainder. Its result applies to every interior stable parameter $0 < \lambda < 64/27$, and implies a qualitative strict exponent gap above $1/2$ on compact interior parameter sets.

In contrast, the test in the present paper is the global polynomial $yP_{54}(y)^2$ against κ:

$$\langle \kappa, yP_{54}(y)^2 \rangle = \mathcal{I}(P_{54}) < 0.$$

The pairing is well defined by taking any smooth compact test agreeing with that polynomial near the compact support of κ. It does not annihilate the positive leading wedge of the full inverse. The new finite certificate and the older full-inverse obstruction have different objects and complementary information.

At $\lambda = 64/27$, the earlier first correction is nonnegative outside the leading support. The finite calculation concerns $\lambda = 1$.

7 Reproduction and contribution

The supplementary files contain:

- `verify.py`: an exact Python standard-library reconstruction of the branch, moments, determinant signs and integer witness;
- `verify_independent.py`: an alternative formal-Newton and FLINT determinant reconstruction;
- `certificate.json`: the integer coefficients in z, the exact square value and the leading determinant signs;
- `requirements.txt`: the optional independent checker's dependency;
- `README.md`: build and execution instructions.

The primary check is `python verify.py -certificate certificate.json`. The alternative is `python verify_independent.py -certificate certificate.json` after installing its stated dependency. Both checks use explicit failures rather than removable assertion statements and may be run with Python's `-O` option. Exact arithmetic certifies the finite claims. The interpretation as the particular signed measure relies on the cited analytic Cauchy-inverse result.

The new contribution is the exact first ordinary-square degree and portable integer witness for the canonical finite correction measure. The analytic half-power obstruction, the algebraic symbol and the classical determinant identities retain their earlier attribution. The finite result provides a reproducible constructive view of this correction.

Assistance disclosure. The manuscript and verification programs were prepared with AI assistance. The exact arithmetic claims are supplied in reproducible form.

References

[1] S. Schubr, *An Unrestricted Half-Power Obstruction for the Specialized Vámos Quartic*, preprint, October 22, 2026, edition 1. https://research.luminaryoddities.com/unrestricted-half-power-obstruction-specialized-vamos-quartic/v/1.

[2] E. H. Bareiss, *Sylvester's identity and multistep integer-preserving Gaussian elimination*, Mathematics of Computation **22** (1968), 565–578. doi:10.1090/S0025-5718-1968-0226829-0.