An Unrestricted Half-Power Obstruction for the Specialized Vámos Quartic

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Abstract

We prove that the inverse square root of the specialized four-variable Vámos quartic is not completely monotone. More generally, this fails for every interior stable member of $h_\lambda = 4e_1e_3 + \lambda a^2b^2$, namely $0 < \lambda < 64/27$. The proof computes a regular correction to a singular positive leading inverse, establishes convergence after distributional Laplace inversion by uniform complex estimates, and finds a negative density outside the leading term's support. A smooth-section argument transfers that sign to the full unrestricted inverse. For the original quartic an explicit interval and uniform negative bound are obtained. The result implies a qualitative strict exponent gap above $1/2$, uniform on compact interior parameter sets. At the sharp stable parameter $64/27$, the same first correction is nonnegative outside the leading support.

1 Introduction and statements

Let e_j denote the elementary symmetric polynomial of degree j in a, b, c, d, and put

$$h_\lambda = 4(a+b+c+d)(abc+abd+acd+bcd) + \lambda a^2b^2, \qquad L = \frac{64}{27}. \tag{1}$$

The polynomial h_1 is the specialized Vámos quartic. It appears in the study of hyperbolicity, spectrahedral cones and determinantal representations [2, 5, 3, 4]. Its cone is spectrahedral, although no positive integer power of the polynomial has a definite symmetric determinantal representation [1, 2]. Neither fact determines complete monotonicity of its inverse square root: the determinantal construction is sufficient, and its failure is not a converse positivity theorem.

A smooth function f on the positive orthant is *completely monotone* if

$$(-1)^{|\alpha|}\partial^\alpha f(x) \geq 0 \quad (x \in \mathbb{R}^4_{>0},\ \alpha \in \mathbb{Z}^4_{\geq 0}).$$

By the multivariate Bernstein–Hausdorff–Widder theorem, this is equivalent to a representation

$$f(x) = \int_{\mathbb{R}^4_{\geq 0}} e^{-x\cdot Y}\, d\mu(Y) \tag{2}$$

with a positive measure, the integral being finite on the orthant; see [8, Section 2] and [3, Section 2]. The corresponding definition on an open convex cone uses derivatives in directions from that cone. For inverse powers of a hyperbolic polynomial, complete monotonicity on its full hyperbolicity cone is equivalent to complete monotonicity on any open convex subcone [3, Proposition 4.15].

The real-stability interval of (1) is $[0, L]$ [7, Theorem 2.2]. Thus, for $0 < \lambda \leq L$, h_λ is positive and hyperbolic in every positive direction. For $\lambda = 1$ these facts also follow directly from the classical Vámos construction recorded in [2]. We use this stability input; we do not repeat the separate stability and geometric classification of [7].

Theorem 1.1. *For every $0 < \lambda < L$, the function $h_\lambda^{-1/2}$ is not completely monotone on $\mathbb{R}^4_{>0}$, or on its full hyperbolicity cone. In particular this holds for the specialized Vámos quartic h_1.*

The obstruction concerns the unrestricted four-variable inverse. It is not a failure of a particular latent-variable construction. Moreover, failure for the full eight-variable Vámos basis polynomial was already classical: its $U_{2,4}$ minor and the minor-closure and quadratic criteria of [8, Propositions 7.6 and 7.9] exclude the half power. Non-complete-monotonicity need not survive identification of variables, so that argument does not decide the four-variable specialization. Likewise, real stability alone does not imply half-power positivity, as the classical Lorentz examples already show [8, Theorem 1.9]. The content of Theorem 1.1 is the sign of the unrestricted inverse of this specialization and its coefficient deformation.

Our method gives the following consequence.

Corollary 1.2. *For each compact set $J \subset (0, L)$, there exists $\epsilon_J > 0$ such that $h_\lambda^{-\alpha}$ is not completely monotone whenever $\lambda \in J$ and $0 < \alpha < 1/2 + \epsilon_J$.*

Recent large-exponent results are in a different range; the Vámos application in [9, Section 5] gives a sufficient bound $\alpha \geq 65536$. Our argument neither uses that bound nor contradicts it.

The proof has three analytic ingredients. First, an oscillatory Fourier inverse is reduced to a K_0 integral. Subtracting its leading model leaves a symbol of degree zero. Second, estimates on complex Bromwich planes justify convergence after inverse Laplace transformation. Third, a Cauchy jump computes a negative part of the correction outside the support of the leading term. The last step would be invalid if based only on formal moment asymptotics. We supply both the complex domination and the regularity needed to test a section of the full inverse.

2 Coordinates and the partial Fourier inverse

Use the primal coordinates

$$s = a + b, \quad \delta = a - b, \quad U = c + d, \quad V = c - d, \qquad P = ab = \frac{s^2 - \delta^2}{4}.$$

The dual coordinates are

$$w = \frac{Y_1 + Y_2}{2}, \quad \rho = \frac{Y_1 - Y_2}{2}, \quad p = \frac{Y_3 + Y_4}{2}, \quad q = \frac{Y_3 - Y_4}{2}, \tag{3}$$

so that $x \cdot Y = sw + \delta\rho + Up + Vq$. An invertible real coordinate change preserves positivity of a measure under push-forward. We use inverse Fourier factors $1/(2\pi)$ and inverse Laplace transforms on fixed vertical lines.

A direct expansion gives

$$\begin{aligned} h_\lambda &= A_\lambda - BV^2, \\ A_\lambda &= \lambda P^2 + 4sPU + (4P + s^2)U^2 + sU^3, \qquad B = s(s + U). \end{aligned} \tag{4}$$

On $\delta = i\xi$, define

$$L_\lambda = 2s^2 + mU(s + U), \qquad C_\lambda = s^4 + msU(s + U)^2, \qquad m = 16/\lambda. \tag{5}$$

Then

$$16A_\lambda = \lambda(\xi^4 + L_\lambda \xi^2 + C_\lambda). \tag{6}$$

We always fix $\operatorname{Re} s = \sigma > 0$ and $\operatorname{Re} U = \upsilon > 0$. Branches are continued from the positive real quadrant; in particular the square roots of B and A_λ/B have positive real part. These assertions, including quantitative bounds, are proved below.

For $q > 0$, Fourier inversion in $V = i\eta$ gives

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{iq\eta}(A_\lambda + B\eta^2)^{-1/2}\,d\eta = \frac{K_0(q\sqrt{A_\lambda/B})}{\pi\sqrt{B}}. \tag{7}$$

The integral is oscillatory. For positive real parameters the identity follows from the gamma integral and a Gaussian Fourier integral. For complex parameters, subtract $B^{-1/2}(1+\eta^2)^{-1/2}$ from the integrand. The difference is $O(|\eta|^{-3})$, locally uniformly in the parameters, and the subtracted model has the known inverse $K_0(q)/(\pi\sqrt{B})$. This proves holomorphic distributional continuation of the identity; nonvanishing on the tube fixes its branch. This interpretation avoids claiming absolute convergence of the unsubtracted integral. Classical Bessel formulas used here and below can be found in [6, Sections 10.29–10.32 and 10.43].

At $\rho = 0$ the double partial inverse has Laplace symbol

$$T_{\lambda,q}(s,U) = \frac{1}{2\pi^2\sqrt{B}}\int_{\mathbb{R}} K_0(qf_\lambda(\xi))\,d\xi, \qquad f_\lambda = \sqrt{A_\lambda/B}. \tag{8}$$

For a general ρ, insert $e^{i\rho\xi}$ in this integral.

3 Uniform complex subtraction

Lemma 3.1 (Tube bounds)**.** *For λ in a compact subset of $(0, L]$, let $R = 1 + |\operatorname{Im} s| + |\operatorname{Im} U|$ on the fixed Bromwich planes. There are positive constants, uniform in λ, such that*

$$c \le |B| \le CR^2, \qquad \operatorname{Re}(1/\sqrt{B}) \ge c/R^2, \tag{9}$$

$$|f_\lambda| \le C(R^2 + \xi^2), \qquad \operatorname{Re} f_\lambda \ge \frac{c}{R^2(R^2+\xi^2)}. \tag{10}$$

Put $g_\lambda = \sqrt{\lambda}\,\xi^2/(4\sqrt{B})$. For a sufficiently large fixed C_0 and $X_0 = C_0R^3$,

$$|f_\lambda - g_\lambda| \le CR^4, \qquad \operatorname{Re} f_\lambda, \operatorname{Re} g_\lambda \ge c\xi^2/R^2 \qquad (|\xi| \ge X_0), \tag{11}$$

$$|f_\lambda - g_\lambda| \le CR^6, \qquad \operatorname{Re} f_\lambda \ge cR^{-8} \qquad (|\xi| \le X_0). \tag{12}$$

Proof. For a positive real base point x, hyperbolicity and homogeneity give $h_\lambda(x+ty) = h_\lambda(x)\prod_j(1+t\gamma_j)$ with real γ_j, allowing zero factors for a reduced degree. Consequently

$$|h_\lambda(x+iy)| \ge h_\lambda(x) > 0.$$

Apply this to $x = (\sigma/2, \sigma/2, \upsilon/2, \upsilon/2)$. Varying the imaginary coordinates gives

$$|A_\lambda + B\eta^2| \ge c > 0 \quad (\eta \in \mathbb{R}), \qquad \operatorname{dist}(A_\lambda/B, (-\infty, 0]) \ge c/|B|.$$

The constant is uniform on compact positive parameter sets. Also, for $\operatorname{Re} z, \operatorname{Re} t > 0$,

$$\operatorname{Re}\sqrt{zt} \ge \sqrt{\operatorname{Re} z\,\operatorname{Re} t},$$

because $|z||t| \ge \operatorname{Re} z \operatorname{Re} t + \operatorname{Im} z \operatorname{Im} t$. Thus $\operatorname{Re}\sqrt{B} \ge \sqrt{\sigma(\sigma+\upsilon)}$, which proves (9). Polynomial upper bounds on A_λ, together with

$$\operatorname{Re}\sqrt{z} \ge \frac{\operatorname{dist}(z, (-\infty, 0])}{2\sqrt{|z|}},$$

prove (10).

The exact identity

$$(f_\lambda - g_\lambda)(f_\lambda + g_\lambda) = \frac{\lambda(L_\lambda \xi^2 + C_\lambda)}{16B}$$

and $|f_\lambda + g_\lambda| \geq \operatorname{Re} g_\lambda \geq c\xi^2/R^2$ give $|f_\lambda - g_\lambda| \leq C(R^4 + R^6/\xi^2)$. At $|\xi| \geq C_0 R^3$, choosing C_0 large makes the error less than half the lower bound for $\operatorname{Re} g_\lambda$. This gives (11). The remaining estimates follow from (10) on $|\xi| \leq C_0 R^3$. □

Lemma 3.2 (Subtraction bound and limit)**.** *Uniformly for* $0 < q \leq 1$*,*

$$\int_{\mathbb{R}} |K_0(qf_\lambda) - K_0(qg_\lambda)|\, d\xi \leq CR^{18}. \tag{13}$$

On each fixed pair (s, U)*, the difference integral converges to* $-\pi(r_+ + r_-)$*, where*

$$r_+^2 + r_-^2 = L_\lambda, \qquad r_+^2 r_-^2 = C_\lambda, \tag{14}$$

and $r_\pm > 0$ *initially for* $s, U > 0$*.*

Proof. The integral representation of K_1 yields $|K_1(z)| \leq K_1(\operatorname{Re} z)$ in the right half-plane. The identity $(xK_1(x))' = -xK_0(x)$ and its limit at zero give $K_1(x) \leq 1/x$. Integrating $K_0' = -K_1$ along the straight segment from g_λ to f_λ therefore gives

$$|K_0(qf_\lambda) - K_0(qg_\lambda)| \leq |f_\lambda - g_\lambda| \int_0^1 \frac{dt}{(1-t)\operatorname{Re} g_\lambda + t \operatorname{Re} f_\lambda}. \tag{15}$$

The factor q cancels. For positive a, b, this last integral is $(\log a - \log b)/(a - b)$, with its continuous value at equality, and is at most $2(1 + |\log(a/b)|)/\max(a, b)$. On the tail, (11) bounds (15) by CR^6/ξ^2. On the inner region it is at most

$$CR^{14}(1 + \log R + |\log|\xi||).$$

Here the lower bound cR^{-8} for $\operatorname{Re} f_\lambda$ controls the maximum in the denominator, and the quadratic zero of $\operatorname{Re} g_\lambda$ contributes only the integrable logarithm. Integrating over the inner interval of length $O(R^3)$, and then the tail, proves (13).

For fixed nonzero ξ, the small-argument expansion of K_0 gives the limit $-\log f_\lambda + \log g_\lambda$, with logarithms on the right half-plane. The preceding majorants permit dominated convergence. For real positive s, U,

$$L_\lambda^2 - 4C_\lambda = mU^2(s + U)((m-4)s + mU) > 0,$$

and $L_\lambda, C_\lambda > 0$, so $r_\pm$ exist as positive roots. Using

$$\int_{\mathbb{R}} \log \frac{\xi^2 + r^2}{\xi^2}\, d\xi = 2\pi r \quad (r > 0)$$

gives the stated limit. The identity extends holomorphically to the fixed complex tube. Outside the positive quadrant, the root sum means this continuation, not a new selection of individual principal roots. □

Define the degree-zero symbol

$$F_\lambda(x) = -\frac{\sqrt{2 + mx(1+x) + 2\sqrt{1 + mx(1+x)^2}}}{\sqrt{1+x}}, \qquad m = 16/\lambda, \tag{16}$$

continued from $x > 0$, so $F_\lambda(0) = -2$. Equations (14) give $F_\lambda(U/s) = -(r_+ + r_-)/\sqrt{B}$. The degree-zero property follows either from this expression or by scaling (s, U, ξ) in the logarithmic difference integral. It holds holomorphically, so the symbol depends only on U/s. Each ratio off the nonpositive real axis has a representative with $\operatorname{Re} s, \operatorname{Re} U > 0$; the set of such representatives for a fixed ratio is connected. Thus F_λ is holomorphic on this slit plane.

Proposition 3.3 (Inverse-distribution limit)**.** *Let $K_\lambda = \mathcal{L}^{-1}_{s,U}[F_\lambda(U/s)]$. Then, in distributions in (w, p), as $q \downarrow 0$,*

$$\mathcal{L}^{-1}_{s,U} T_{\lambda,q} = c_\lambda q^{-1/2} \mathcal{B} + \frac{K_\lambda}{2\pi} + o(1), \qquad c_\lambda = \frac{\Gamma(1/4)^2 \lambda^{-1/4}}{2\sqrt{2}\pi^2}, \tag{17}$$

where

$$\mathcal{B}(w, p) = \frac{\mathbf{1}_{\{0<p<w\}} (w-p)^{-3/4} p^{-3/4}}{\Gamma(1/4)^2}. \tag{18}$$

In particular $\operatorname{supp} \mathcal{B} \subset \{0 \le p \le w\}$.

Proof. The Mellin integral of K_0 gives exactly

$$\int_{\mathbb{R}} K_0(q g_\lambda)\, d\xi = \frac{\Gamma(1/4)^2 \lambda^{-1/4} B^{1/4}}{\sqrt{2}\sqrt{q}}.$$

Initially this is a real positive identity; holomorphy gives its complex continuation. Lemma 3.2, with the prefactor in (8), therefore gives the symbol expansion with regular term $F_\lambda(U/s)/(2\pi)$.

For a smooth compactly supported test $\phi(w, p)$, Fourier inversion on the fixed Bromwich planes pairs the symbol with the Fourier transform of $e^{\sigma w + \upsilon p}\phi(w, p)$, up to the fixed Fourier normalization and sign. This transform is Schwartz. The uniform polynomial majorant (13) and $|B|^{-1/2} \le C$ justify dominated convergence in the two remaining Fourier variables. This proves (17), rather than only pointwise convergence of real Laplace transforms. Formula (18) follows by convolving the one-dimensional inverse gamma densities of $s^{-1/4}$ and $(s + U)^{-1/4}$. □

Lemma 3.4 (Positive sections)**.** *For fixed $q > 0$, the full inverse is smooth in (ρ, q), with values in distributions in (w, p). If $h_\lambda^{-1/2}$ were completely monotone, every such section would be a positive distribution.*

Proof. Insert $e^{i\rho\xi}$ in (8). On the inner region of Lemma 3.1,

$$|K_0(q f_\lambda)| \le K_0(q \operatorname{Re} f_\lambda) \le C_q(1 + \log R).$$

On the tail $\operatorname{Re} f_\lambda \ge c\xi^2/R^2$ gives Gaussian decay. Inserting any power of ξ leaves an integrable majorant with polynomial growth in R. Derivatives in q, on a compact positive q-interval, introduce powers of f_λ and integral-order modified Bessel functions, which obey the same kind of polynomial inner bounds and Gaussian tail bounds. Pairing with a compact smooth (w, p) test as above proves the asserted smoothness.

Under (2), the inverse is the push-forward of a positive measure in the coordinates (3). Pair it with a nonnegative compact test in (w, p) and nonnegative approximate identities about $(\rho, q) = (\rho_0, q_0)$, where $q_0 > 0$. The limit is exactly the continuous section evaluated on that test, and is nonnegative. This argument uses an actual continuous section, not an arbitrarily chosen conditional value on a null event. □

4 An elementary finite-part Cauchy inverse

We give a direct construction that also specifies endpoint terms. For each parameter define

$$Q_\lambda(z) = \frac{xF'_\lambda(x)}{z}, \qquad x = -1/z, \qquad j_\lambda(y) = -\frac{1}{\pi}\operatorname{Im} Q_\lambda(y + i0). \tag{19}$$

Lemma 4.1 (Subtracted Cauchy formula)**.** *There is an $M > 0$ such that j_λ is locally integrable on $(0, M]$, vanishes for $y > M$, and is $O(y^{-3/2})$ at zero. The compactly supported distribution*

$$\langle \kappa_\lambda, \phi\rangle = \int_0^M j_\lambda(y)\big(\phi(y) - \phi(0)\big)\, dy \tag{20}$$

has Cauchy transform Q_λ. On every regular open interval away from zero it has the ordinary density

$$j_\lambda(y) = \frac{\operatorname{Im} F'_\lambda(-1/y + i0)}{\pi y^2}. \tag{21}$$

Proof. The inner cubic $1 + mx(1 + x)^2$ has three negative simple roots when $0 < \lambda < L$. Indeed, after $u = 1 + x$, it has one root below zero, one in $(0, 2/3)$ and one in $(2/3, 1)$; its derivative has turning points at $0, 2/3$, and its value at the latter is $1 - 4m/27 < 0$. At $\lambda = L$, the latter two roots merge, with the exact factorization $1 + (27/4)x(1+x)^2 = (1+3x)^2(1+3x/4)$. Possible outer zeros, besides the removable point at zero, are given by

$$\left(1 + \frac{m}{2}x(1 + x)\right)^2 - \left(1 + mx(1 + x)^2\right) = \frac{m}{4}x^2(1 + x)(m(1 + x) - 4). \tag{22}$$

Thus all possible finite singularities lie on the negative real axis, at the cubic roots, -1, and $\lambda/4 - 1$. The branch is analytic at $x = 0$, so Q_λ is analytic off a compact positive cut and is $O(z^{-2})$ at infinity.

At a simple inner root with nonzero outer radicand, F'_λ has at worst order $-1/2$. The same is true at a simple outer zero away from an inner root. A simultaneous zero occurs only at $\lambda = 2, x = -1/2$; the resulting fourth-root behavior gives order $-3/4$ for the derivative. At $\lambda = L$ the doubled inner root is removable in the selected square root and the outer radicand there is nonzero. At $x = -1$, continuation past the two inner roots gives $\sqrt{1 + mx(1 + x)^2} = -1$. Writing $u = 1 + x$, the numerator of F_λ^2, divided by u, is $-m + 2mu + O(u^2)$; hence this apparent singularity is removable. These facts prove local integrability at every positive cut node. As $x \to \infty$, the radical formula gives $F_\lambda(x) = O(|x|^{1/2})$ and $F'_\lambda(x) = O(|x|^{-1/2})$. Consequently $Q_\lambda(z) = O(|z|^{-3/2})$ at zero. These bounds hold on the selected branches on either side of the cut.

Formula (20) now defines a distribution: subtraction of $\phi(0)$ makes the integrand $O(y^{-1/2})$ at zero. To obtain its Cauchy transform, apply the keyhole Cauchy formula to $Q_\lambda(\zeta)$ with kernel

$$\frac{1}{\zeta - z} + \frac{1}{z} = \frac{\zeta}{z(\zeta - z)}.$$

The kernel vanishes at zero. The small endpoint circle is $O(\varepsilon^{1/2})$, the outer circle is $O(R^{-1})$, and the other node circles vanish by their integrable singular orders. The upper and lower jumps, using Schwarz symmetry, give

$$Q_\lambda(z) = \int_0^M j_\lambda(y)\left(\frac{1}{z - y} - \frac{1}{z}\right) dy = \langle \kappa_\lambda(y), (z - y)^{-1}\rangle.$$

This fixes the mass-zero counterterm rather than leaving a hidden multiple of δ_0. Finally, $x = -1/z$ maps the upper half-plane to itself, and (21) follows from (19). □

Proposition 4.2 (Local homogeneous inverse)**.** *On $w > 0$,*

$$K_\lambda(w,p) = w^{-2}\kappa_\lambda(p/w). \tag{23}$$

In a regular region this is an identity of ordinary densities.

Proof. Differentiate the degree-zero symbol. Lemma 4.1 gives

$$\partial_s F_\lambda(U/s) = \frac{1}{U} Q_\lambda(-s/U) = -\langle \kappa_\lambda(y), (s+Uy)^{-1}\rangle.$$

The inverse of the last resolvent is $\int_0^\infty e^{-t(s+Uy)}\,dt$. Since differentiation in s corresponds to multiplication of its inverse by $-w$, inversion gives

$$-wK_\lambda(w,p) = -\langle \kappa_\lambda(y), \delta(p-wy)\rangle.$$

For $w > 0$, the change $p = wy$ proves (23). Terms supported at $w = 0$ do not affect this assertion or any test used below. General distributional Laplace inversion is discussed in [10]; the identities here follow directly by pairing the displayed ray kernels with compact tests. □

5 A negative correction for the original quartic

Proposition 5.1. *For $\lambda = 1$, $j_1(3/2) = -2/(\pi\sqrt{5})$. More explicitly,*

$$j_1(y) \le -\frac{275}{3456\pi} \qquad \left(\frac{3}{2} \le y \le \frac{8}{5}\right). \tag{24}$$

Consequently, on $1 \le w \le 2$, $3/2 \le p/w \le 8/5$,

$$\frac{K_1(w,p)}{2\pi} \le -\frac{275}{27648\pi^2}. \tag{25}$$

Here K_1 denotes the correction inverse from Proposition 4.2, not the modified Bessel function.

Proof. At $x = -2/3 + i0$, continuation from zero through the upper half-plane gives

$$\sqrt{1+16x+32x^2+16x^3} = \frac{i\sqrt{15}}{9}, \qquad \sqrt{2+16x+16x^2+2\sqrt{1+16x+32x^2+16x^3}} = \frac{1+i\sqrt{15}}{3}.$$

There is exactly one inner root between $-2/3$ and zero; the continued inner root has positive imaginary part after crossing it. The outer radicand then lies in the upper half-plane. In particular no root is independently reset to a principal value after substitution. Differentiation gives

$$F_1'(-2/3) = -\frac{21 + 27i/\sqrt{15}}{2\sqrt{3}}.$$

Formula (21) at $y = 3/2$ yields the exact value.

For the interval bound, let $u = 1 - 1/y$. On $u \in [1/3, 3/8]$, the inner cubic $C(u) = 1 + 16(u-1)u^2$ is negative: it equals $-5/27$ at the left endpoint and has negative derivative there and throughout the interval. The selected branch has $\operatorname{Im} F_1(u-1) = -H(u)$, where

$$H(u)^2 = \frac{1-u}{2}(16 + R(u)) - \frac{1}{u}, \qquad R(u) = \sqrt{16(16 - 4/u)}. \tag{26}$$

This follows by taking the imaginary part of a square root with positive imaginary part; the same computation is given for general parameters in the next section. Here $8 \le R(u) < 10$. Differentiating,

$$E(u) := (H^2)' = -\frac{16+R}{2} + \frac{16(1-u)}{Ru^2} + \frac{1}{u^2} \ge -13 + \frac{64}{9} + \frac{64}{9} = \frac{11}{9}.$$

Also $0 < H^2 < 26/3 < 9$, so $H' > 11/54$. Since $\operatorname{Im} F_1' = -H'$, (21) and $y \le 8/5$ give (24). Equation (25) follows from (23) and $w^{-2} \ge 1/4$. □

Choose a nonzero nonnegative smooth compact test supported in the interior of the region in (25). It annihilates $\mathcal{B}$, because $p > w$ there. By Proposition 3.3, its pairing with the actual section $\mathcal{L}^{-1}T_{1,q}$ is strictly negative for every sufficiently small $q > 0$. Lemma 3.4 contradicts a positive representation (2). This proves Theorem 1.1 at $\lambda = 1$. The compact test is explicit in its allowed support and sign; the present estimates do not give a numerical small-q threshold.

6 Every interior deformation parameter

Proof of Theorem 1.1. Set $m = 16/\lambda > 27/4$ and use $u = 1 + x$. The cubic

$$C(u) = 1 + m(u-1)u^2$$

is negative between its middle root $u_m \in (0, 2/3)$ and upper root $u_o \in (2/3, 1)$. Continuation from $x > 0$ through the upper half-plane gives $\sqrt{C} = +i\sqrt{-C}$ on this interval. The outer radicand divided by $1 + x$ is

$$G(u) = 2/u + m(u-1) + 2i\sqrt{-C(u)}/u.$$

It has positive imaginary part. At u_o its limiting value is $(2u_o - 1)/u_o^2 > 0$, so the continued square root is in the first quadrant. Write $H(u) = \operatorname{Im}\sqrt{G(u)} > 0$, giving $\operatorname{Im} F_\lambda(u-1) = -H(u)$.

The inequality $C < 0$ implies $m > 1/((1-u)u^2) \ge 4/u$. Put $R(u) = \sqrt{m(m-4/u)}$. Since

$$|G(u)|^2 = (1-u)^2 m(m - 4/u),$$

the elementary square-root formula yields

$$H(u)^2 = \frac{1-u}{2}(m + R(u)) - \frac{1}{u}. \tag{27}$$

At the middle root, $\lambda = 16(1-u_m)u_m^2$, a strictly increasing function of $u_m \in (0, 2/3)$. Thus u_m is below, equal to, or above $1/2$ as λ is below, equal to, or above 2. Differentiating (27) at fixed m, then taking the one-sided limit $u \downarrow u_m$, gives

$$\lim (H^2)' = \begin{cases} \dfrac{u_m - 1}{u_m^2(2u_m - 1)} > 0, & 0 < u_m < 1/2, \\ \dfrac{3u_m - 2}{u_m(u_m - 1)(2u_m - 1)} > 0, & 1/2 < u_m < 2/3. \end{cases} \tag{28}$$

For verification, before substitution the derivative is

$$-(m+R)/2 + \frac{m(1-u)}{u^2 R} + \frac{1}{u^2},$$

and at the root $R = |1 - 2u_m|/((1 - u_m)u_m^2)$, which gives the two displayed rational expressions. In either case $H' > 0$ on a nonempty open subinterval just above u_m.

For $\lambda = 2$, the middle root is $1/2$, and G tends to zero there, so H extends continuously by zero. It is positive inside (u_m, u_o). The mean value theorem on a closed interval from u_m to an interior point supplies a point with $H' > 0$. Analyticity in the interior makes this strict inequality persist on an open subinterval. No derivative at the singular endpoint is required.

On the subinterval found in every case, $\operatorname{Im} F'_\lambda = -H' < 0$. Equation (21) gives a strictly negative regular density at $y = 1/(1 - u) > 1$. Proposition 4.2 supplies a negative correction on an open region with $p > w > 0$. Choose a nonnegative smooth compact test there. Its leading contribution in (17) is identically zero, while its regular limit is negative. For sufficiently small positive q, the actual section is therefore negative on that test, contradicting Lemma 3.4 under complete monotonicity. This proves failure on the orthant. Failure on the full cone follows by restriction, or by the cone/subcone equivalence cited above. □

Proof of Corollary 1.2. For each $\lambda_0 \in J$, Theorem 1.1 implies a strict violation of a signed mixed derivative at some orthant point when $\alpha = 1/2$. That derivative depends continuously on (λ, α). A product neighborhood of $(\lambda_0, 1/2)$ retains the same violation. A finite cover of the compact set J provides a common positive exponent radius.

For smaller positive exponents, fix $a = 1$ and scale (b, c, d) by ε. The first nonzero term of h_λ/ε^2 is

$$Q_\lambda = \lambda b^2 + 4b(c + d) + 4cd = (U + 2b)^2 - (4 - \lambda)b^2 - V^2.$$

This is a nondegenerate Lorentz quadratic in three effective variables. Complete monotonicity of the original inverse power would pass to the scaled limit on an open positive subcone. The classical quadratic criterion [8, Theorem 1.9] excludes $0 < \alpha < 1/2$. Combining this with the finite-cover neighborhood of $1/2$ proves the corollary. □

7 The sharp parameter endpoint

The same first-correction calculation has the opposite sign behavior at $\lambda = L$. This identifies a boundary of the present method.

Proposition 7.1. *On the open region $y > 1$, the distribution κ_L is nonnegative. Its density is positive for $1 < y < 27/11$, vanishes for $y > 27/11$, and has a locally integrable inverse-square-root singularity at $27/11$. This does not determine positivity of the full inverse of $h_L^{-1/2}$.*

Proof. Let $r = \sqrt{1 + 3x/4}$. The selected inner square root is $(1 + 3x)r$, not its absolute value. The identities

$$\begin{aligned} 1 + \tfrac{27}{4}x(1 + x)^2 &= (1 + 3x)^2(1 + 3x/4), \\ 12r^4 + 8r^3 - 15r^2 - 6r + 5 &= (r + 1)^2(2r - 1)(6r - 5), \\ 1 + x &= (4r^2 - 1)/3 \end{aligned}$$

give the exact simplification

$$F_L(x) = -\sqrt{3}(r + 1)\sqrt{\frac{6r - 5}{2r + 1}}.$$

For $y > 1$, $x = -1/y \in (-1, 0)$ and $r \in (1/2, 1)$. The expression is real for $r > 5/6$. On $1/2 < r < 5/6$ write $F_L = -iH$; then

$$H^2 = \frac{3(r+1)^2(5-6r)}{2r+1}, \qquad \frac{dH^2}{dr} = -\frac{6(r+1)(12r^2+4r+3)}{(2r+1)^2} < 0.$$

Since $dr/dx = 3/(8r) > 0$, $\operatorname{Im} F_L' > 0$. Equation (21) proves the density signs. The point $r = 5/6$ corresponds to $x = -11/27$ and $y = 27/11$. It is a simple outer zero, so the derivative has an integrable inverse-square-root singularity. Continuity of F_L there creates no point mass. The doubled inner root is removable. These observations also exclude a hidden negative term on the open region $y > 1$. □

Interior failure cannot be extended to L by continuity: the completely monotone parameter set is closed, so an excluded open range may have an admissible boundary point. A higher correction or a different construction is needed to decide it.

8 Scope and reproducibility

The conclusion is an analytic counterexample for a specific four-variable polynomial and a one-parameter interior family. It does not settle a general large-exponent conjecture, classify all admissible exponents, or give a new obstruction to the generalized Lax conjecture. The essential sign is outside the full support of the leading positive inverse; a negative formal moment correction alone would not have sufficed.

The supplementary `verify.py` uses only the Python standard library and exact rational arithmetic. It checks polynomial factorizations, the continued local radical arithmetic, the explicit interval bounds, and the deformation and endpoint identities. The analytic convergence and positivity arguments are supplied above.

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