Unique minimizers under prescribed marginals: answers to two questions of Gurvits and Leake

Scott Schubr
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Abstract

Gurvits and Leake asked whether, for every n and every trace vector t of sum n near the all-ones vector, the scalar tuple $\frac{1}{n}(t_1 I, \dots, t_n I)$ uniquely minimizes the mixed discriminant over positive semidefinite n-tuples with traces t_i and sum I (Question 1.7), and whether $n^{-n}(\sum_i t_i x_i)^n$ uniquely minimizes the coefficient of $x_1 \cdots x_n$ over n-homogeneous real-stable polynomials with value one and gradient t at the all-ones vector (Question 1.8). We answer both affirmatively with explicit neighborhoods. For real-stable polynomials the coefficient excess dominates the centered-root dispersion with explicit constants in every degree, and Lorentzian polynomials keep the minimizer with linear coefficient stability, which yields a mixed-volume bound. For mixed discriminants we prove explicit reserves of quadratic order. The scalar tuple is a strict local minimizer exactly when $\sum_i t_i^2 < n^2/(n-1)$, and for every $n \geq 3$ global minimality fails strictly inside this region. In degree three, where the two questions are equivalent, we determine the optimal radius. The least mixed discriminant of the feasible triples is one sixth of the least permanent of a nonnegative 3×3 matrix with unit row sums and column sums t, and the optimal ℓ^1-radius of both questions, and of the permanent theorem of Gurvits, Klein and Leake, is $\big((236 - 37\sqrt{37})/12\big)^{1/2} \approx 0.9547$, attained by a diagonal competitor with two vanishing entries. The proof combines a separation identity, a commutation argument for Lagrange multipliers and an interval-arithmetic certificate. For $n = 2$ both questions have affirmative answers at every trace vector.

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1 Introduction

Edition note. This second edition of [15] adds Section 6, which determines the optimal radius for $n = 3$ exactly (with its equations numbered (6.1)–(6.6)), and updates Corollary 4.8 accordingly; all other results and all edition-1 theorem and equation numbers are unchanged.

1.1 The two questions

For $n \times n$ matrices we use the normalized mixed discriminant

$$\mathrm{MD}(X_1, \dots, X_n) = \frac{1}{n!} \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \det\Big(\sum_{i=1}^{n} z_i X_i\Big),$$

so that MD is symmetric and multilinear and $\mathrm{MD}(I,\dots,I)=1$. All positive semidefinite (PSD) matrices are complex Hermitian. A tuple is *doubly stochastic* if $X_i \succeq 0$, $\operatorname{Tr} X_i = 1$ and $\sum_i X_i = I$. Gurvits proved that the doubly stochastic tuple $J=(I/n,\dots,I/n)$ is the unique minimizer of MD among doubly stochastic tuples, with value n^{-n} [4]. Gurvits–Klein–Leake proved the corresponding near-balanced statement for permanents with row sums $\mathbf{1}$ and column sums c [6], [7, Theorem 1.1], and Gurvits–Leake proved the matrix analogue in which the traces stay equal to one while the sum $\sum_i X_i = C$ varies near I [7, Theorems 1.2 and 4.1]. They then asked the following two questions [7, Section 1.4]; all references to [7] are to version 2.

Question 1.7 (Gurvits–Leake [7, Question 1.7])**.** For all n, does there exist $\epsilon>0$ such that: for all t such that $\|t-\mathbf{1}\|_1<\epsilon$ and $\sum_{i=1}^n t_i = n$, the mixed discriminant is uniquely minimized by $(X_1,\dots,X_n)=\frac{1}{n}(t_1 I,\dots,t_n I)$ over n-tuples $(X_1,\dots,X_n)$ of PSD matrices such that $\operatorname{Tr}(X_i)=t_i$ and $\sum_{i=1}^n X_i = I$?

Question 1.8 (Gurvits–Leake [7, Question 1.8])**.** For all n, does there exist $\epsilon>0$ such that: for all t such that $\|t-\mathbf{1}\|_1<\epsilon$ and $\sum_{i=1}^n t_i = n$, the coefficient of $x_1x_2\cdots x_n$ is uniquely minimized by $p(x)=\frac{1}{n^n}\bigl(\sum_{i=1}^n t_i x_i\bigr)^n$ over n-homogeneous n-variate real stable polynomials p such that $p(\mathbf{1})=1$ and $\nabla p(\mathbf{1}) = t$?

Question 1.7 is the case $p(x)=\det(\sum_i x_i X_i)$ of Question 1.8: this polynomial is real stable, its squarefree coefficient is $n!\,\mathrm{MD}(X)$, and its gradient at $\mathbf{1}$ is $(\operatorname{Tr} X_i)_i$ (Lemma 3.1). For simultaneously diagonal tuples Question 1.7 is the permanent statement of [7, Theorem 1.1] for the matrix whose columns are the diagonals of the X_i.

For each question let ρ_n^{MD}, respectively ρ_n^{RS}, be the supremum of the radii ϵ for which the displayed statement holds (and 0 if there is none). Since determinantal polynomials are feasible in Question 1.8, and since the dispersion of such a polynomial vanishes only at the scalar tuple, $\rho_n^{\mathrm{RS}} \le \rho_n^{\mathrm{MD}}$.

1.2 Main results

1. *Question 1.8, explicit.* For every $n\ge 2$ there are explicit constants $\varepsilon_n,\beta_n,c_n>0$ such that every feasible p with $\|t-\mathbf{1}\|_1\le\varepsilon_n$ satisfies $\mathcal{A}(p)-\frac{n!}{n^n}\prod_i t_i \ge \frac{\beta_n}{8}d(p)\ge c_n\,\|p-L_t^n\|_{\mathrm{coef},1}$, where $d(p)$ is the centered-root dispersion (Theorem 2.9). In degree three the radius is $1/8$ (Theorem 2.10). The linear order is optimal (Proposition 2.11). The balanced constant β_n keeps both end deficits of a Bernstein expansion; for instance $\beta_3=1/36$ and $\beta_4\approx 3.3\cdot 10^{-3}$.

2. *Lorentzian polynomials and mixed volumes.* The power L_t^n is the unique minimizer with linear coefficient stability on the larger Lorentzian class near balance, with an explicit joint neighborhood, and the estimate transfers to integer compositions and to mixed volumes (Theorem 2.12, Corollaries 2.17 and 2.18).

3. *Question 1.7, explicit.* The balanced reserve $\mathrm{MD}(Y)-n^{-n}\ge b_n\sum_i\|Y_i-I/n\|_F^2$ holds on all doubly stochastic tuples (Theorem 3.2), and for $\|t-\mathbf{1}\|_1\le\varepsilon_n^{\mathrm{MD}}$ every feasible tuple satisfies $\mathrm{MD}(X)-\prod_i t_i/n^n\ge\frac{b_n}{2}\sum_i\|X_i-\frac{t_i}{n}I\|_F^2$ (Theorem 3.3). For example $b_3=1/432$, $\varepsilon_3^{\mathrm{MD}}\approx 2.8\cdot10^{-4}$ and $\varepsilon_4^{\mathrm{MD}}\approx 6.0\cdot10^{-6}$.

4. *Exact local region.* The scalar tuple, and in the real-stable class the power L_t^n, is a strict local minimizer precisely when $\sum_i t_i^2 < n^2/(n-1)$ (Propositions 4.1 and 4.4).

5. *Global obstructions.* For every $n\ge 3$ there are trace vectors strictly inside this local region at which the scalar tuple is not a global minimizer; the obstructing tuples are diagonal, so

they also obstruct Question 1.8 and its permanent case (Proposition 4.5). For $n = 3$ the two questions are equivalent at every trace vector (Proposition 4.7). In the Lorentzian class local minimality already fails when $\max_i t_i \geq n/(n-1)$ (Theorem 4.9).

6. *Two variables.* For $n = 2$ both questions hold at every feasible trace vector (Section 5).

7. *The exact radius in degree three.* For $n = 3$ the least mixed discriminant of the feasible triples equals one sixth of the least permanent of a nonnegative 3×3 matrix with unit row sums and column sums t, and the scalar tuple is the unique minimizer exactly when the rank-one matrix $\frac{1}{3}\mathbf{1}t^T$ is the unique permanent minimizer (Theorem 6.4). The optimal radius is

$$\rho_3^{\mathrm{RS}} = \rho_3^{\mathrm{MD}} = s_* = \Big(\frac{236 - 37\sqrt{37}}{12}\Big)^{1/2} = 0.9547157683\ldots$$

(Theorem 6.1 and Corollary 4.8), and s_* is also the optimal radius of the permanent theorem [7, Theorem 1.1] of Gurvits, Klein and Leake for $n = 3$. At the trace vector $(1 + s_*/2, 1, 1 - s_*/2)$ a diagonal tuple with two vanishing entries attains the scalar value, and just beyond it such tuples go strictly below. An elementary Schur-inequality argument gives the radius $\sqrt{3}/4$ without computation (Proposition 6.3).

The optimal radius is the natural sharpening of the affirmative answers: Questions 1.7 and 1.8 ask for some admissible radius, and Theorem 6.1 identifies the largest one when $n = 3$.

1.3 Method

Centered real roots produce a PSD dispersion matrix that controls every higher centered coefficient. At balance a radial extension reaches a degree-drop boundary, where the refined capacity theorem gives a strict gap; a derivative-based Bernstein expansion transfers that gap, together with an exact quadratic deficit, back to the original polynomial. An explicit positive substitution then corrects the gradient while preserving every centered factor. For the mixed discriminant the radial extension is a ray in the convex set of doubly stochastic tuples, and a Hadamard bound on mixed discriminants of trace-zero matrices replaces the general coefficient budget, which enlarges the admissible radius by many orders of magnitude. The obstructions come from an explicit one-parameter family of doubly stochastic patterns with one zero entry. In degree three a separation identity writes $6\,\mathrm{MD}(X)$ as a sum of functions of the individual X_i; a Lagrange-multiplier argument then shows that the multiplier of the constraint $\sum_i X_i = I$ commutes with every X_i at a minimizer, which reduces the mixed discriminant to the permanent. For the permanent a ray argument and an affine direction of the permanent move every competitor to a face with two zero entries, where a local Taylor estimate and an interval branch-and-bound certificate locate the extremal configuration.

1.4 Provenance

This article consolidates four records of the Luminary Research Archive by the author: [12] (prescribed-gradient minimizers, local criterion, cubic theorem, Lorentzian obstructions), [13] (explicit constants in every degree), [14] (Lorentzian and mixed-volume extension) and [11] (the trace-perturbed mixed discriminant, its Hessian region and the two-variable identity). Explicit constants and an explicit matrix corollary already appear in [12, 13], and [12] contains degree-three diagonal obstructions inside the local region with radius bound 1. The sharper balanced constant β_n, Theorems 3.2 and 3.3, the extension of the obstruction to every $n \geq 3$ (Proposition 4.5), Example 4.6, Proposition 4.7 and the global two-variable answer to Question 1.8 are new in edition 1 [15], and Section 6 is new in this edition.

2 Real-stable and Lorentzian polynomials

2.1 Standing notation and centered roots

A nonzero real polynomial is *real stable* if it has no zero when every variable has strictly positive imaginary part. Throughout this section p is homogeneous of degree n in n variables, real stable, has nonnegative coefficients, and satisfies $p(\mathbf{1}) = 1$. Write

$$t = \nabla p(\mathbf{1}), \quad L_t(x) = \frac{t^T x}{n}, \quad \mathcal{A}(p) = [x_1 \cdots x_n]p, \quad v_n = \frac{n!}{n^n}, \quad F(t) = v_n \prod_i t_i.$$

Euler's identity gives $\sum_i t_i = n$, and $\mathcal{A}(L_t^n) = F(t)$. For a polynomial q, put $\|q\|_{\mathrm{coef},1} = \sum_\alpha |[x^\alpha]q|$. Call p *balanced* when $t = \mathbf{1}$.

Remark 2.1. Question 1.8 does not impose nonnegative coefficients, and none is lost: the nonzero coefficients of a homogeneous real-stable polynomial all have the same sign [3, Theorem 6.1], so $p(\mathbf{1}) = 1$ forces them to be nonnegative.

Lemma 2.2. *Set $B = \frac{n-1}{2n} tt^T - \frac{1}{2}\nabla^2 p(\mathbf{1})$ and $d = d(p) = \operatorname{Tr} B$. There are homogeneous polynomials q_k of degree k, invariant under $x \mapsto x + a\mathbf{1}$, such that*

$$p(x) = L_t(x)^n + \sum_{k=2}^{n} L_t(x)^{n-k} q_k(x), \qquad q_2(x) = -x^T B x. \tag{1}$$

The matrix B is PSD, $B\mathbf{1} = 0$, and

$$|q_k(x)| \le \binom{n}{k} (2x^T B x)^{k/2} \le \binom{n}{k} (2d\|x\|_2^2)^{k/2}. \tag{2}$$

Moreover $d = 0$ exactly when $p = L_t^n$, and $0 \le d \le \frac{n-1}{2n}\|t\|_2^2 \le M_n := \frac{n(n-1)}{2}$; in particular $d \le (n-1)/2$ when p is balanced.

Proof. Put $y = x - L_t(x)\mathbf{1}$. The polynomial $z \mapsto p(z\mathbf{1} + y)$ is monic of degree n, with real roots and zero z^{n-1} coefficient; reality of the roots follows from stability and complex conjugation. Denote its roots by λ_j. Their sum is zero, and the quadratic coefficient is $-\frac{1}{2}\sum_j \lambda_j^2 \le 0$. Expanding in z and substituting $z = L_t(x)$ gives (1), translation invariance, and a PSD quadratic form $q_2 = -x^T B x$. Differentiating twice at $\mathbf{1}$ gives the stated formula for B, and translation invariance gives $B\mathbf{1} = 0$. Each root has absolute value at most $(2x^T B x)^{1/2}$; bounding every elementary symmetric term gives (2). If $d = 0$, PSD forces $B = 0$ and all higher q_k vanish; the converse is immediate. Finally every diagonal Hessian entry is nonnegative, so $d \le \frac{n-1}{2n}\sum_i t_i^2$, and $\sum_i t_i^2 \le n^2$. □

Lemma 2.3. *With*

$$K_{n,k} = \binom{n}{k} \frac{2^k n^{2k-2} k^k}{k!}, \qquad W_n = \sum_{k=2}^{n} K_{n,k}, \tag{3}$$

every polynomial in the standing class satisfies $\|q_k\|_{\mathrm{coef},1} \le K_{n,k} d$ and $\|p - L_t^n\|_{\mathrm{coef},1} \le W_n d$.

Proof. Let T_k be the symmetric k-linear polar form of q_k. The real polarization identity expresses $T_k(e_{i_1}, \ldots, e_{i_k})$ as a sum of 2^k evaluations at signed sums of coordinate vectors, divided by $2^k k!$. Their Euclidean norms are at most k, so $|T_k(e_{i_1}, \ldots, e_{i_k})| \le \binom{n}{k}(2d)^{k/2} k^k / k!$. Summing the n^k ordered tensor coefficients bounds the coefficient norm. Since $d \le n^2$ and $k \ge 2$, $d^{k/2} \le d n^{k-2}$; also $2^{k/2} \le 2^k$. This gives the first estimate. The coefficients of L_t are nonnegative and sum to one, so $\|L_t\|_{\mathrm{coef},1} = 1$, and summing (1) proves the second. □

2.2 Capacity inputs and the Bernstein expansion

For $\alpha \in \mathbb{R}^n_{\geq 0}$ let $\mathrm{Cap}_\alpha(p) = \inf_{x>0} p(x)/x^\alpha$. If p is balanced, weighted AM–GM over its coefficients (which sum to one, with mean exponent $\nabla p(\mathbf{1}) = \mathbf{1}$) gives $p(x) \geq x_1 \cdots x_n$ for $x > 0$, and evaluation at $\mathbf{1}$ gives $\mathrm{Cap}_{\mathbf{1}}(p) = 1$. Put

$$\kappa_2 = 1, \qquad \kappa_n = \frac{(n-2)^{n-2}(n-1)!}{(n-1)^{2n-3}} \quad (n \geq 3), \qquad g_n = \kappa_n - v_n, \qquad h_n = \frac{(n-2)!}{n^{n-2}}. \tag{4}$$

Theorem 2.4 (Gurvits; Gurvits–Leake)**.** *Let q be balanced in the standing class. Then $\mathcal{A}(q) \geq v_n$. If moreover $q(e_i) = 0$ for some coordinate vector e_i, then $\mathcal{A}(q) \geq \kappa_n$, and $g_n > 0$.*

Proof. The first assertion is Gurvits's coefficient–capacity inequality $\mathcal{A}(q) \geq v_n \,\mathrm{Cap}_{\mathbf{1}}(q)$ [5]. For $n \geq 3$ the second is [7, Corollary 2.12], which states $\mathcal{A}(q)/\,\mathrm{Cap}_{\mathbf{1}}(q) \geq \kappa_n > v_n$ for n-homogeneous real-stable q with nonnegative coefficients and $q(e_i) = 0$. It is also the degree-refined capacity induction of [5] with the deficient variable eliminated first: nonnegative coefficients and $q(e_i) = 0$ bound the degree in x_i by $n-1$, giving the first factor $((n-2)/(n-1))^{n-2}$ and then $(n-1)!/(n-1)^{n-1}$. For $n = 2$, $q(e_1) = 0$ and balance force $q = x_1 x_2$, so $\mathcal{A}(q) = 1 = \kappa_2$. □

For a balanced q define

$$E_k(q) = \sum_{|S|=k} \partial_S q(\mathbf{1}), \quad E_0(q) = 1, \qquad T_s x = (1-s)L_{\mathbf{1}}(x)\mathbf{1} + sx.$$

Lemma 2.5. *Let q be balanced and $0 \leq s \leq 1$. Then*

$$\mathcal{A}(q \circ T_s) - v_n = \sum_{k=0}^{n} s^k (1-s)^{n-k} \Delta_k(q), \qquad \Delta_k(q) = \frac{(n-k)!}{n^{n-k}} E_k(q) - \binom{n}{k} v_n \geq 0, \tag{5}$$

with $\Delta_0 = \Delta_1 = 0$, $\Delta_2(q) = h_n d(q)$ and $\Delta_n(q) = \mathcal{A}(q) - v_n$.

Proof. Write $D_{\mathbf{1}} = \sum_i \partial_i$ and $r_k = (k!/n!) D_{\mathbf{1}}^{n-k} q$. Directional differentiation in a positive direction preserves stability and nonnegative coefficients; Euler's identities give $r_k(\mathbf{1}) = 1$ and $\nabla r_k(\mathbf{1}) = (k/n)\mathbf{1}$. Hence $r_k L_{\mathbf{1}}^{n-k}$ is balanced in the standing class. For squarefree S of size k, homogeneity gives $\partial_S r_k(0) = \frac{k!(n-k)!}{n!} \partial_S q(\mathbf{1})$, so $r_k L_{\mathbf{1}}^{n-k}$ has squarefree coefficient $\frac{(n-k)!}{n^{n-k}} E_k(q)/\binom{n}{k}$. Theorem 2.4 gives $\Delta_k(q) \geq 0$.

Taylor expansion of q at $(1-s)L_{\mathbf{1}}(x)\mathbf{1}$ and squarefree extraction give $\mathcal{A}(q \circ T_s) = \sum_k s^k (1-s)^{n-k} \frac{(n-k)!}{n^{n-k}} E_k(q)$; subtracting the binomial expansion of v_n gives (5). The cases $k = 0, 1$ are equalities. The total Hessian sum is $\mathbf{1}^T \nabla^2 q(\mathbf{1}) \mathbf{1} = n(n-1)$, whereas its trace is $(n-1) - 2d(q)$; hence $E_2(q) = \frac{(n-1)^2}{2} + d(q)$, and the baseline $\binom{n}{2} v_n \frac{n^{n-2}}{(n-2)!}$ equals $\frac{(n-1)^2}{2}$. Finally $E_n(q) = \mathcal{A}(q)$. □

2.3 The balanced reserve

We use the following classical hyperbolicity-cone facts [1, 5]. A homogeneous real-stable polynomial with nonnegative coefficients is hyperbolic in $\mathbf{1}$. Its hyperbolicity cone C containing $\mathbf{1}$ is open and convex and contains the positive orthant, and the closed cone consists of the vectors with nonnegative hyperbolic eigenvalues. Pullback along closed-cone directions whose sum is interior is hyperbolic in every strictly positive direction, hence is real stable with nonnegative coefficients.

Lemma 2.6. *If p is balanced and $d(p) > 0$, there is a finite $D \geq 1$ such that $q = p \circ T_D$ is balanced, lies in the standing class, and satisfies $q(e_i) = 0$ for some i. Moreover $p = q \circ T_{1/D}$ and $d(q) = D^2 d(p)$.*

Proof. Let a_{ij} be the roots in z of $p(z\mathbf{1} - e_i)$, the hyperbolic eigenvalues of $e_i \in \overline{C}$. They are nonnegative, and balance gives $\sum_j a_{ij} = 1$. Let $a = \min_{i,j} a_{ij}$. If $a = 1/n$, every coordinate's eigenvalues equal $1/n$, so every $B_{ii} = 0$ and PSD forces $d = 0$. Thus $0 \le a < 1/n$; set $D = (1-na)^{-1}$. The eigenvalues of $T_D e_i$ are $(1 - D)/n + Da_{ij} \ge 0$, with at least one zero, so every $T_D e_i$ lies in $\overline{C}$ and their sum is $\mathbf{1} \in C$. For $x > 0$, $T_D x = (\min_i x_i)\mathbf{1} + \sum_i (x_i - \min_j x_j) T_D e_i \in C$. The cone facts give stability and nonnegative coefficients of q; also $q(\mathbf{1}) = 1$ and $T_D^T \mathbf{1} = \mathbf{1}$, so q is balanced, and the zero eigenvalue gives $q(e_i) = 0$. The identities $T_D T_s = T_{Ds}$ and the scaling by D on centered directions give the last two claims. □

Proposition 2.7. *Define $\beta_2 = 1$ and, for $n \ge 3$,*

$$\beta_n = \min_{0 \le s \le 1} \Big\{ h_n (1 - s)^{n-2} + \frac{2g_n}{n-1} s^{n-2} \Big\} > 0. \tag{6}$$

Every balanced p in the standing class satisfies $\mathcal{A}(p) - v_n \ge \beta_n d(p)$.

Proof. If $d(p) = 0$ then $p = L_{\mathbf{1}}^n$. Otherwise take q, D from Lemma 2.6 and put $s = 1/D \in (0, 1]$, so $p = q \circ T_s$ and $s^2 d(q) = d(p)$. For $n = 2$, Lemma 2.5 gives $\mathcal{A}(p) - v_2 = s^2 h_2 d(q) = d(p)$. For $n \ge 3$ the indices $k = 2$ and $k = n$ are distinct; discarding the other nonnegative deficits in Lemma 2.5 and using Theorem 2.4 for q,

$$\mathcal{A}(p) - v_n \ge s^2 (1 - s)^{n-2} h_n d(q) + s^n g_n.$$

Since q is balanced, Lemma 2.2 gives $d(q) \le (n - 1)/2$, so $s^2 = d(p)/d(q) \ge 2d(p)/(n - 1)$ and $s^n g_n \ge \frac{2g_n}{n-1} s^{n-2} d(p)$. Hence $\mathcal{A}(p) - v_n \ge \big(h_n (1 - s)^{n-2} + \frac{2g_n}{n-1} s^{n-2}\big) d(p) \ge \beta_n d(p)$. □

Remark 2.8. For $n = 3$, $\beta_3 = \min\{h_3, g_3\} = 1/36$; for $n \ge 4$, Hölder's inequality gives the closed form $\beta_n = \big(h_n^{-1/(n-3)} + (2g_n/(n - 1))^{-1/(n-3)}\big)^{-(n-3)}$. Numerically (rounded) $\beta_4 \approx 3.26 \cdot 10^{-3}$, $\beta_5 \approx 4.67 \cdot 10^{-4}$ and $\beta_{10} \approx 5.0 \cdot 10^{-8}$. The constant of [12, 13], which keeps only the quadratic deficit and handles the boundary by telescoping, is $\min\{g_n/(2M_n), h_n(g_n/(4n))^{n-2}\}$, for example $169/13759414272 \approx 1.2 \cdot 10^{-8}$ for $n = 4$.

2.4 An explicit answer to Question 1.8

Theorem 2.9. *Let $n \ge 2$, let p be in the standing class, and set*

$$\varepsilon_n = \min\Big\{ \frac{1}{2}, \frac{\beta_n}{8nW_n} \Big\}, \qquad c_n = \frac{\beta_n}{8W_n}.$$

If $\|t - \mathbf{1}\|_1 \le \varepsilon_n$, then

$$\mathcal{A}(p) - F(t) \;\ge\; \frac{\beta_n}{8} d(p) \;\ge\; c_n \, \|p - L_t^n\|_{\mathrm{coef},1} \,. \tag{7}$$

In particular L_t^n is the unique minimizer of $\mathcal{A}$ over the entire class with gradient t, including singular B and boundary coefficients, and $\rho_n^{\mathrm{RS}} \ge \varepsilon_n$.

Proof. Put $e = \|t - \mathbf{1}\|_1$ and $d = d(p)$. When $e = 0$, Proposition 2.7 is stronger than required. Suppose $0 < e \le \varepsilon_n \le 1/2$ and define

$$z_i = \frac{1 - (1 - e)t_i}{ne}, \qquad Sx = (1 - e)x + e(z^T x)\mathbf{1}. \tag{8}$$

The entries of z sum to one. For $t_i \le 1$ nonnegativity is immediate; for $t_i > 1$, $t_i - 1 \le e$ and $et_i - (t_i - 1) \ge e - (t_i - 1) \ge 0$. So S is nonnegative with positive diagonal, $S\mathbf{1} = \mathbf{1}$ and $S^T t = \mathbf{1}$. Thus $r = p \circ S$ is normalized and balanced, has nonnegative coefficients, and is stable because S maps vectors with positive imaginary parts to such vectors. The correction preserves the intrinsic factors exactly:

$$L_t(Sx) = L_{\mathbf{1}}(x), \qquad q_k(Sx) = (1-e)^k q_k(x), \qquad d(r) = (1-e)^2 d,$$

by translation invariance and homogeneity. Proposition 2.7 gives $\mathcal{A}(r) - v_n \ge \beta_n (1-e)^2 d \ge \beta_n d/4$. Now $\|L_t\|_{\mathrm{coef},1} = \|L_{\mathbf{1}}\|_{\mathrm{coef},1} = 1$ and $\|L_t - L_{\mathbf{1}}\|_{\mathrm{coef},1} = e/n$; telescoping powers and $1-(1-e)^k \le ke$ yield $\left\| L_t^{n-k} - (1-e)^k L_{\mathbf{1}}^{n-k} \right\|_{\mathrm{coef},1} \le \frac{n-k}{n} e + ke \le ne$. Subtracting the intrinsic expansions of p and r after removing their linear-form powers, Lemma 2.3 bounds the coefficient norm of $(p - L_t^n) - (r - L_{\mathbf{1}}^n)$ by $neW_n d$. The squarefree coefficient is bounded by that norm, so $\mathcal{A}(p) - F(t) \ge \beta_n d/4 - neW_n d \ge \beta_n d/8$. Lemma 2.3 gives the second inequality of (7). Zero deficit forces $d = 0$ and $p = L_t^n$. □

The coefficients of p form a probability law on $\{\alpha \in \mathbb{Z}^n_{\ge 0} : |\alpha| = n\}$ with mean t, and L_t^n gives the multinomial law $\mathrm{Mult}(n, t/n)$. Their coefficient distance is twice the total-variation distance, so (7) reads $\Pr_p\{\alpha = \mathbf{1}\} - F(t) \ge 2c_n d_{\mathrm{TV}}(p, \mathrm{Mult}(n, t/n))$.

Theorem 2.10 (Explicit cubic stability)**.** *Let p be in the standing class with $n = 3$, and put $B = tt^T/3 - \nabla^2 p(\mathbf{1})/2$, $d = \operatorname{Tr} B$. If $\|t - \mathbf{1}\|_1 \le 1/8$, then*

$$\mathcal{A}(p) - \frac{2}{9} t_1 t_2 t_3 \;\ge\; \frac{d}{256} \;\ge\; \frac{1}{3584} \left\| p - L_t^3 \right\|_{\mathrm{coef},1}, \tag{9}$$

including singular B. In particular $\rho_3^{\mathrm{RS}} \ge 1/8$.

The proof is in Appendix A; it uses only Lemma 2.2, the cubic discriminant and vertex coefficients.

Proposition 2.11 (Optimal order)**.** *For every $n \ge 2$, no positive uniform lower bound by $\|p - L_t^n\|_{\mathrm{coef},1}^{\alpha}$ with $\alpha < 1$ can replace the right side of (7), even when $t = \mathbf{1}$.*

Proof. Put $L = L_{\mathbf{1}}$, $a = x_1 - x_2$ and, for $0 < s < 1/n$, $p_s = L^{n-2}(L + sa)(L - sa) = L^n - s^2 L^{n-2} a^2$. Every linear factor has nonnegative coefficients and positive coefficient sum, so p_s is stable; $p_s(\mathbf{1}) = 1$ and $\nabla p_s(\mathbf{1}) = \mathbf{1}$. Only the term $-2x_1x_2$ of a^2 contributes to the squarefree coefficient, so $\mathcal{A}(p_s) - \mathcal{A}(L^n) = 2h_n s^2$, while $\|p_s - L^n\|_{\mathrm{coef},1} = s^2 \left\| L^{n-2} a^2 \right\|_{\mathrm{coef},1} > 0$. The ratio of the gap to this distance raised to $\alpha < 1$ tends to zero. □

2.5 Lorentzian polynomials

We use the closed class of homogeneous Lorentzian polynomials with nonnegative coefficients of Brändén and Huh [2]. In degree at least two it is characterized by M-convex support and the condition that every coordinate derivative of order $d - 2$ has a Hessian with at most one positive eigenvalue; nonnegative linear substitutions preserve it [2, Theorem 2.10]. Homogeneous real-stable polynomials with nonnegative coefficients are Lorentzian [2]. Gurvits–Leake proved that $L_{\mathbf{1}}^n$ is the unique minimizer of $\mathcal{A}$ among normalized balanced Lorentzian (equivalently, homogeneous strongly log-concave) polynomials [7, Theorem 1.4]. In this subsection p is a Lorentzian polynomial of degree $n \ge 2$ in n variables with $p(\mathbf{1}) = 1$; t, L_t, v_n and $\mathcal{A}$ are as above, and $L = L_{\mathbf{1}}$.

Theorem 2.12. *For every $n \geq 2$ there are $\varepsilon_n^{\mathrm{L}}, c_n^{\mathrm{L}} > 0$ such that every normalized Lorentzian p with $\|t - \mathbf{1}\|_1 \leq \varepsilon_n^{\mathrm{L}}$ satisfies*

$$\mathcal{A}(p) - v_n \prod_i t_i \;\geq\; c_n^{\mathrm{L}} \, \|p - L_t^n\|_{\mathrm{coef},1} \,. \tag{10}$$

Thus L_t^n is the unique squarefree-coefficient minimizer over the entire normalized Lorentzian class with gradient t.

Put $E = \mathbf{1}^\perp$ and $P = I - \mathbf{1}\mathbf{1}^T/n$. Let K_n be the closed convex cone of homogeneous degree-n signed polynomials r with

$$r(\mathbf{1}) = 0, \qquad \nabla r(\mathbf{1}) = 0, \qquad S_\beta := P \nabla^2 \partial^\beta r P \preceq 0 \quad (|\beta| = n-2); \tag{11}$$

the Hessians here are constant matrices.

Lemma 2.13. *For $r \in K_n$ put $s(r) = -\operatorname{Tr}(P \nabla^2 r(\mathbf{1}) P)$. Then $\|r\|_{\mathrm{coef},1} \leq \frac{n}{2} s(r)$.*

Proof. Homogeneity gives $P \nabla^2 r(x) P = \sum_{|\beta|=n-2} S_\beta x^\beta / \beta!$ and $s(r) = \sum_\beta (-\operatorname{Tr} S_\beta)/\beta!$. For a negative semidefinite S, $\sum_{i,j} |S_{ij}| \leq n(-\operatorname{Tr} S)$ by Cauchy–Schwarz on its diagonal. Let $T_u x = L(x)\mathbf{1} + uPx$ for $0 \leq u \leq 1$. Each coordinate of $T_u x$ is a nonnegative linear form with coefficient sum one, so substitution by T_u does not increase coefficient norms. Taylor's formula along this segment and the two vanishing conditions in (11) give $r(x) = \int_0^1 (1-u)\, x^T P \nabla^2 r(T_u x) P x \, du$. The coefficient norms of the matrix entries of the integrand sum to at most $n s(r)$, and multiplication by $x_i x_j$ preserves coefficient norms; integrating gives the claim. □

Proposition 2.14. *For every $r \in K_n$, $\mathcal{A}(r) \geq \sigma_n \|r\|_{\mathrm{coef},1}$ with $\sigma_n = \dfrac{n!}{n^{n+2}(n-1)}$.*

Proof. First let r be permutation invariant and write $r = \sum_{|\alpha|=n} \frac{n!}{\alpha!} b_\alpha x^\alpha$. Testing (11) on $e_i - e_j$ yields $b_{\beta+2e_i} + b_{\beta+2e_j} \leq 2 b_{\beta+e_i+e_j}$. For fixed remaining coordinates this is a concave sequence in the exponent of x_i, symmetric about its midpoint, so a transfer from a coordinate at least two larger to a smaller one does not decrease b_α. Repeated transfers reach $\alpha = \mathbf{1}$, so $b_\alpha \leq b_{\mathbf{1}}$. Their weighted sum is $r(\mathbf{1}) = 0$, so $b_{\mathbf{1}} \geq 0$, the positive coefficients sum to at most $n^n b_{\mathbf{1}}$, and the total sum is zero; hence $\|r\|_{\mathrm{coef},1} \leq 2 n^n b_{\mathbf{1}} = \frac{2n^n}{n!} \mathcal{A}(r)$.

For general $r \in K_n$ let $\bar{r}$ be its permutation average. Then $\bar{r} \in K_n$, $\mathcal{A}(\bar{r}) = \mathcal{A}(r)$ and $s(\bar{r}) = s(r)$. Euler's identity gives $\mathbf{1}^T \nabla^2 \bar{r}(\mathbf{1}) \mathbf{1} = 0$, so $s(r) = -\operatorname{Tr} \nabla^2 \bar{r}(\mathbf{1}) \leq n(n-1) \|\bar{r}\|_{\mathrm{coef},1}$. Lemma 2.13 for r and the symmetric case for $\bar{r}$ give $\|r\|_{\mathrm{coef},1} \leq n^{n+2}(n-1)\mathcal{A}(r)/n!$. □

Lemma 2.15. *Suppose p_j are normalized Lorentzian, $t_j = \nabla p_j(\mathbf{1}) \to \mathbf{1}$, and $\varrho_j = \left\| p_j - L_{t_j}^n \right\|_{\mathrm{coef},1} > 0$ tends to zero. Every coefficient limit of $r_j = (p_j - L_{t_j}^n)/\varrho_j$ lies in K_n and has norm one.*

Proof. The value and gradient of r_j at $\mathbf{1}$ vanish. Fix $|\beta| = n-2$ and put $R_j = \nabla^2 \partial^\beta r_j$, so $H_j := \nabla^2 \partial^\beta p_j = v_n t_j^\beta t_j t_j^T + \varrho_j R_j$. For $v \in E$ let $v_j = v - (t_j^T v) t_j / \|t_j\|_2^2$. The form of H_j is positive at $w_j = t_j$ for large j, and its restriction to $\operatorname{span}\{w_j, v_j\}$ has nonpositive determinant since H_j has at most one positive eigenvalue. Hence $\varrho_j v_j^T R_j v_j \leq \varrho_j^2 (w_j^T R_j v_j)^2 / (w_j^T H_j w_j)$, with denominator bounded below and R_j bounded. Dividing by ϱ_j and passing to the limit gives $v^T R v \leq 0$ on E. □

Proof of Theorem 2.12. The normalized Lorentzian class is compact: its coefficients lie in a simplex and the class is closed. If no radius and reserve existed, there would be p_j with $\|t_j - \mathbf{1}\|_1 \leq 1/j$, $\varrho_j > 0$ and $(\mathcal{A}(p_j) - \mathcal{A}(L_{t_j}^n))/\varrho_j < 1/j$. Pass to a subsequence $p_j \to p_\infty$, which is balanced. If $p_\infty \neq L^n$, [7, Theorem 1.4] gives a strictly positive limiting numerator, contradicting $\varrho_j \leq 2$. Thus $\varrho_j \to 0$; a coefficient limit r of r_j satisfies $\mathcal{A}(r) \leq 0$, while Lemma 2.15 and Proposition 2.14 give $\mathcal{A}(r) \geq \sigma_n$. The radius may be taken below $1/2$, which keeps all t_i positive. □

The separation bound also gives an explicit joint neighborhood. Put

$$\theta_n = v_n 2^{-(n-2)}, \quad N_n = n\,n!, \quad J_n = \frac{2N_n^2}{\theta_n n}, \quad V_n = 2(n-1), \quad D_n = \sigma_n V_n + n v_n,$$
$$\varrho_* = \min\Big\{\frac{\theta_n}{2n!}, \frac{h_n\sigma_n}{2J_nD_n}\Big\}, \qquad \delta_* = \min\Big\{\frac{1}{2}, \frac{h_n\sigma_n}{4N_nD_n}\Big\}.$$

Proposition 2.16. *If p is normalized Lorentzian, $\|t-\mathbf{1}\|_1 \le \delta_*$ and $\|p - L_t^n\|_{\mathrm{coef},1} \le \varrho_*$, then $\mathcal{A}(p) - v_n\prod_i t_i \ge \frac{\sigma_n}{2}\|p - L_t^n\|_{\mathrm{coef},1}$.*

Proof. The case $p = L_t^n$ is trivial; otherwise write $\varrho = \|p - L_t^n\|_{\mathrm{coef},1} > 0$, $r = (p - L_t^n)/\varrho$ and $\delta = \|t-\mathbf{1}\|_1$. For $|\beta| = n-2$ put $R = \nabla^2\partial^\beta r$. Each entry of R is a coefficient of r times a factorial at most $n!$, so $\|R\|_{\mathrm{op}} \le N_n$. In $H = v_n t^\beta t t^T + \varrho R$ one has $v_n t^\beta \ge \theta_n$ and $\|t\|_2^2 \ge n$, so the unit vector $w = t/\|t\|_2$ has $w^THw \ge \theta_n n/2$. The determinant argument of Lemma 2.15 gives $v^TRv \le J_n\varrho\|v\|_2^2$ for $v \perp t$. A unit $v \in E$ has projection v_t onto $t^\perp$ with $\|v - v_t\|_2 \le \delta/\sqrt{n}$, so $PRP \preceq \eta P$ with $\eta = J_n\varrho + 2N_n\delta$. Let $r_0 = -L^{n-2}x^TPx$. Its value and gradient vanish at $\mathbf{1}$, $\|r_0\|_{\mathrm{coef},1} \le V_n$, $\mathcal{A}(r_0) = nv_n$, and $v^T\nabla^2\partial^\beta r_0 w = -2h_n v^Tw$ for $v, w \in E$. Hence $r + kr_0 \in K_n$ for $k = \eta/(2h_n)$, and Proposition 2.14 gives $\mathcal{A}(r) \ge \sigma_n(1 - kV_n) - knv_n = \sigma_n - kD_n$. The restrictions give $J_n\varrho \le h_n\sigma_n/(2D_n)$ and $2N_n\delta \le h_n\sigma_n/(2D_n)$, so $kD_n \le \sigma_n/2$. □

Corollary 2.17. *Fix positive integers $k_1, \dots, k_m$ with sum $N \ge 2$. Every degree-N Lorentzian p in m variables with $p(\mathbf{1}) = 1$ and $\|\nabla p(\mathbf{1}) - k\|_1 \le \varepsilon_N^{\mathrm{L}}$ satisfies, with $t = \nabla p(\mathbf{1})$,*

$$[x^k]p - \frac{N!}{\prod_i k_i!}\prod_i (t_i/N)^{k_i} \ge \frac{c_N^{\mathrm{L}}}{\prod_i (k_i!/k_i^{k_i})}\Big\|p - (t^Tx/N)^N\Big\|_{\mathrm{coef},1}.$$

Proof. Replace x_i by the average of k_i new variables. The result is Lorentzian by nonnegative substitution, has degree N in N variables, and has gradient error exactly $\|t-k\|_1$. Only x^k contributes to its squarefree coefficient, with multiplier $\prod_i k_i!/k_i^{k_i}$. The substitution preserves coefficient norms of signed polynomials, since different monomials have disjoint group-multidegree supports and nonnegative expansion weights summing to one. Apply Theorem 2.12. □

Let $K_1, \dots, K_n$ be convex bodies in $\mathbb{R}^n$, $K = K_1 + \cdots + K_n$, $W = \mathrm{Vol}(K) > 0$, and $V_\alpha = V(K_1[\alpha_1], \dots, K_n[\alpha_n])$ the mixed volumes.

Corollary 2.18. *Put $t_i = nV(K_i, K[n-1])/W$. If $\|t-\mathbf{1}\|_1 \le \varepsilon_n^{\mathrm{L}}$, then*

$$\frac{n!V(K_1,\dots,K_n)}{W} - v_n\prod_i t_i \ge c_n^{\mathrm{L}}\sum_{|\alpha|=n}\frac{n!}{\alpha!}\Big|\frac{V_\alpha}{W} - \prod_i (t_i/n)^{\alpha_i}\Big|.$$

Proof. The normalized volume polynomial $\mathrm{Vol}(\sum_i x_iK_i)/W$ is Lorentzian [2, Theorem 4.1], with coefficients $(n!/\alpha!)V_\alpha/W$ and gradient t. Apply (10). □

3 The mixed discriminant

For a tuple X with $\sum_i X_i = I$ and $t_i = \mathrm{Tr}\,X_i$ write

$$p_X(x) = \det\Big(\sum_i x_iX_i\Big), \qquad \delta(X) = \sum_i\Big\|X_i - \frac{t_i}{n}I\Big\|_F^2.$$

Lemma 3.1. *Let $n \ge 2$.*

1. *For Hermitian B, H:* $\mathrm{MD}(B, I, \dots, I) = \frac{\operatorname{Tr} B}{n}$ *and* $\mathrm{MD}(B, H, I, \dots, I) = \frac{\operatorname{Tr} B \operatorname{Tr} H - \operatorname{Tr}(BH)}{n(n-1)}$.

2. *For Hermitian $A_1, \dots, A_k$,* $|\mathrm{MD}(A_1, \dots, A_k, I, \dots, I)| \le \frac{(n-k)!}{n!} \prod_i \|A_i\|_1$, *where $\|\cdot\|_1$ is the trace norm.*

3. *If $X_i \succeq 0$ and $\sum_i X_i = I$, then p_X is real stable with nonnegative coefficients,* $p_X(\mathbf{1}) = 1$, $\nabla p_X(\mathbf{1}) = t$, $\mathcal{A}(p_X) = n!\,\mathrm{MD}(X)$ *and* $d(p_X) = \delta(X)/2$.

Proof. The polynomial $\det(\sum_{i \le k} u_i A_i + wI)$ depends on the last $n-k$ variables of the defining polarization only through their sum w. Hence its coefficient of $u_1 \cdots u_k w^{n-k}$ equals $\frac{n!}{(n-k)!}\mathrm{MD}(A_1, \dots, A_k, I, \dots, I)$; at $w = 1$ this is the coefficient of $u_1 \cdots u_k$ in $\det(I + \sum_i u_i A_i)$. Comparing the linear and mixed quadratic terms of $\det(I + sB + uH)$ gives (1), without commutativity. For (2), expand each $A_i = \sum_r \lambda_{ir} w_{ir} w_{ir}^*$ spectrally with unit vectors and use multilinearity. For vectors $v_1, \dots, v_k$ with $V = [v_1 \cdots v_k]$, Sylvester's identity gives $\det(I + \sum_i u_i v_i v_i^*) = \det(I_k + \operatorname{diag}(u) V^* V)$, whose coefficient of $u_1 \cdots u_k$ is the Gram determinant $\det(V^*V) \le \prod_i \|v_i\|^2$ by Hadamard's inequality. Hence each rank-one term is bounded by $\frac{(n-k)!}{n!} \prod_i |\lambda_{ir_i}|$, and summing gives (2). For (3), if every $\operatorname{Im} x_i > 0$ the matrix $\sum_i x_i X_i$ has positive-definite imaginary part, so it is invertible; hence p_X is real stable. Its coefficients are mixed discriminants of PSD matrices, hence nonnegative. Differentiation of $\det(I + sX_i)$ gives $\partial_i p_X(\mathbf{1}) = t_i$ and $\partial_{ii} p_X(\mathbf{1}) = t_i^2 - \operatorname{Tr}(X_i^2)$, so $d(p_X) = \frac{1}{2}\sum_i (\operatorname{Tr}(X_i^2) - t_i^2/n) = \delta(X)/2$. □

3.1 The balanced reserve

Theorem 3.2. *Let $n \ge 2$ and put $b_2 = 1/4$ and, for $n \ge 3$,*

$$b_n = \frac{\beta_n}{2\,n!} = \min_{0 \le s \le 1} \Big\{ \frac{(1-s)^{n-2}}{2(n-1)n^{n-1}} + \frac{\gamma_n}{n-1} s^{n-2} \Big\}, \qquad \gamma_n = \frac{(n-2)^{n-2}}{n(n-1)^{2n-3}} - \frac{1}{n^n} > 0.$$

Every doubly stochastic tuple Y satisfies

$$\mathrm{MD}(Y) - n^{-n} \;\ge\; b_n \sum_i \Big\| Y_i - \frac{I}{n} \Big\|_F^2,$$

with equality for $n = 2$. In particular J is the unique minimizer.

Proof. By Lemma 3.1(3), p_Y is balanced in the standing class with $\mathcal{A}(p_Y) = n!\,\mathrm{MD}(Y)$ and $d(p_Y) = \delta(Y)/2$, and $\gamma_n = g_n/n!$, $\frac{1}{2(n-1)n^{n-1}} = h_n/(2\,n!)$. Proposition 2.7 gives $n!(\mathrm{MD}(Y) - n^{-n}) \ge \beta_n \delta(Y)/2$; for $n = 2$ it is the equality $\mathcal{A}(p_Y) - v_2 = d(p_Y)$. □

In the determinantal case every ingredient of Proposition 2.7 is concrete. The hyperbolic eigenvalues of e_i for p_Y are the eigenvalues of Y_i, so the radial extension of Lemma 2.6 is the ray $Z = J + D(Y - J)$, continued to the first parameter at which some Z_i becomes singular; Z remains doubly stochastic by convexity, and $Y = (1-s)J + sZ$ with $s = 1/D$. The Bernstein expansion is the multilinear expansion $\mathrm{MD}(Y) = \sum_S s^{|S|}(1-s)^{n-|S|}\,\mathrm{MD}(Z_S, (I/n)_{S^c})$. Its $k = 2$ deficit is $\frac{\delta(Z)}{2(n-1)n^{n-1}}$ by Lemma 3.1(1), and its $k = n$ deficit is $\mathrm{MD}(Z) - n^{-n} \ge \gamma_n$ by the degree-drop capacity bound, since $\det Z_i = p_Z(e_i) = 0$. The bound $\delta(Z) \le n - 1$ follows from $\operatorname{Tr}(Z_i^2) \le (\operatorname{Tr} Z_i)^2$. The capacity inputs enter only through the nonnegativity of the intermediate deficits and the degree-drop gap. Numerically $b_3 = 1/432$, $b_4 \approx 6.78 \cdot 10^{-5}$ and $b_5 \approx 1.95 \cdot 10^{-6}$.

3.2 An explicit answer to Question 1.7

Theorem 3.3. *Let $n \geq 3$ and put*

$$\Gamma_n = \frac{3}{2n^{n-1}} + \sum_{k=3}^{n} \frac{(k+1)\,2^{k-2}\binom{n}{k-2}(n-k)!\,n(n-1)}{k(k-1)\,n^{n-k}\,n!}, \qquad \varepsilon_n^{\mathrm{MD}} = \min\Big\{\frac{1}{2}, \frac{b_n}{2(2b_n+\Gamma_n)}\Big\}.$$

Let $t \in \mathbb{R}^n$ satisfy $\sum_i t_i = n$ and $\|t-\mathbf{1}\|_1 \leq \varepsilon_n^{\mathrm{MD}}$. Every tuple of PSD matrices with $\operatorname{Tr} X_i = t_i$ and $\sum_i X_i = I$ satisfies

$$\mathrm{MD}(X_1,\dots,X_n) - \frac{\prod_i t_i}{n^n} \;\geq\; \frac{b_n}{2}\sum_i \Big\|X_i - \frac{t_i}{n}I\Big\|_F^2. \tag{12}$$

Equality holds only at $X_i = \frac{t_i}{n}I$. In particular $\rho_n^{\mathrm{MD}} \geq \varepsilon_n^{\mathrm{MD}}$.

Proof. Write $X_i = \frac{t_i}{n}I + A_i$, so $\operatorname{Tr} A_i = 0$, $\sum_i A_i = 0$ and $\delta = \delta(X) = \sum_i \|A_i\|_F^2$. Put $e = \|t-\mathbf{1}\|_1 \leq 1/2$. If $e = 0$, Theorem 3.2 is stronger than (12). Otherwise define z by (8) and

$$Y_i = (1-e)X_i + e z_i I.$$

As in the proof of Theorem 2.9, z is a probability vector, so $Y_i \succeq 0$; moreover $\operatorname{Tr} Y_i = (1-e)t_i + nez_i = 1$ and $\sum_i Y_i = (1-e)I + eI = I$. The same identity gives $Y_i - \frac{I}{n} = (1-e)A_i$. Theorem 3.2 yields

$$\mathrm{MD}(Y) - n^{-n} \geq b_n(1-e)^2\delta \geq b_n(1-2e)\delta. \tag{13}$$

Multilinear expansion of both tuples gives

$$\mathrm{MD}(X) = \sum_{S\subseteq[n]}\prod_{i\notin S}\frac{t_i}{n}\,\mathrm{MD}(A_S, I), \qquad \mathrm{MD}(Y) = \sum_{S\subseteq[n]}\frac{(1-e)^{|S|}}{n^{n-|S|}}\,\mathrm{MD}(A_S, I),$$

where $\mathrm{MD}(A_S, I)$ has the A_i ($i \in S$) and $n - |S|$ copies of I. The terms $|S| = 1$ vanish by Lemma 3.1(1), and the terms $S = \emptyset$ are $\prod_i t_i/n^n$ and n^{-n}. Hence

$$\mathrm{MD}(X) - \frac{\prod_i t_i}{n^n} = \mathrm{MD}(Y) - n^{-n} + \mathcal{E}, \qquad \mathcal{E} = \sum_{|S|\geq 2}\frac{\omega_S}{n^{n-|S|}}\,\mathrm{MD}(A_S, I), \quad \omega_S = \prod_{i\notin S} t_i - (1-e)^{|S|}.$$

The positive and negative parts of $t-\mathbf{1}$ each have mass $e/2$. Hence $1 - e/2 \leq \prod_{i\notin S} t_i \leq \exp(e/2) \leq 1+e$, and with $1-(1-e)^k \leq ke$ we get $|\omega_S| \leq (k+1)e$ for $|S| = k$.

For $k = 2$, Lemma 3.1(1) gives $\mathrm{MD}(A_i, A_j, I) = -\operatorname{Tr}(A_iA_j)/(n(n-1))$, and $\sum_{i<j}\|A_i\|_F\|A_j\|_F \leq \frac{n-1}{2}\delta$ by Cauchy–Schwarz. The $k = 2$ part of $|\mathcal{E}|$ is therefore at most $3e\,n^{-(n-2)}\frac{n-1}{2}\frac{\delta}{n(n-1)} = \frac{3e\delta}{2n^{n-1}}$.

For $k \geq 3$, put $a_i = \|A_i\|_1$. Since $X_i \succeq 0$, $a_i \leq \|X_i\|_1 + t_i = 2t_i$, and $a_i^2 \leq n\|A_i\|_F^2$. Writing $\prod_{i\in S} a_i = \binom{k}{2}^{-1}\sum_{\{i,j\}\subseteq S} a_ia_j\prod_{l\in S\setminus\{i,j\}} a_l$ and bounding the last product by $\prod 2t_l$,

$$\sum_{|S|=k}\prod_{i\in S} a_i \leq \frac{2^{k-2}}{\binom{k}{2}}\sum_{i<j} a_ia_j\,e_{k-2}(t) \leq \frac{2^{k-2}\binom{n}{k-2}}{\binom{k}{2}}\cdot\frac{n(n-1)}{2}\,\delta,$$

using Maclaurin's inequality $e_{k-2}(t) \leq \binom{n}{k-2}(\sum_i t_i/n)^{k-2} = \binom{n}{k-2}$ and $\sum_{i<j} a_ia_j \leq \frac{n-1}{2}\sum_i a_i^2 \leq \frac{n(n-1)}{2}\delta$. With Lemma 3.1(2), the k-th part of $|\mathcal{E}|$ is at most $(k+1)e\,n^{-(n-k)}\frac{(n-k)!}{n!}$ times this bound. Summing, $|\mathcal{E}| \leq e\Gamma_n\delta$. With (13),

$$\mathrm{MD}(X) - \frac{\prod_i t_i}{n^n} \geq \big(b_n - e(2b_n+\Gamma_n)\big)\delta \geq \frac{b_n}{2}\delta,$$

by the choice of $\varepsilon_n^{\mathrm{MD}}$. Zero deficit forces $\delta = 0$. □

Remark 3.4. Rounded values of the constants of Theorem 3.3, of Theorem 2.9, and of the earlier constant of [13] are:

n	$\varepsilon_n^{\mathrm{MD}}$	$b_n/2$	ε_n (Thm. 2.9)	radius of [13]
3	$2.8 \cdot 10^{-4}$	$1/864$	$3.7 \cdot 10^{-7}$	$1.0 \cdot 10^{-8}$
4	$6.0 \cdot 10^{-6}$	$3.4 \cdot 10^{-5}$	$1.4 \cdot 10^{-10}$	$5.2 \cdot 10^{-16}$
5	$2.1 \cdot 10^{-7}$	$9.7 \cdot 10^{-7}$	$3.4 \cdot 10^{-14}$	$6.7 \cdot 10^{-25}$
10	$8.3 \cdot 10^{-14}$	$3.5 \cdot 10^{-15}$	$2.2 \cdot 10^{-34}$	$1.3 \cdot 10^{-88}$

For $n = 3$, Theorem 2.10 applied to p_X gives the larger radius $1/8$ with $\mathrm{MD}(X) - \prod_i t_i/27 \geq \delta(X)/3072$, by Lemma 3.1(3).

4 Local regions and global obstructions

4.1 The exact local region

Let $\mathcal{T}$ be the real space of Hermitian tuples A with $\operatorname{Tr} A_i = 0$ and $\sum_i A_i = 0$, with $\|A\|^2 = \sum_i \operatorname{Tr}(A_i^2)$. For $t > 0$ of sum n put $C_i(t) = \frac{t_i}{n} I$ and $f(t, A) = \mathrm{MD}(C(t)+A)$. Every feasible tuple of Question 1.7 is uniquely $C(t) + A$ with $A \in \mathcal{T}$.

Proposition 4.1. *Let $n \geq 2$, $t_i > 0$ and $\sum_i t_i = n$. Then $D_A f(t, 0) = 0$ on $\mathcal{T}$, and the Hessian $D_A^2 f(t, 0)$ is positive definite on $\mathcal{T}$ if and only if*

$$\sum_i t_i^2 < \frac{n^2}{n-1}, \quad \textit{equivalently} \quad \|t - \mathbf{1}\|_2^2 < \frac{n}{n-1}. \tag{14}$$

Inside this region $C(t)$ is a strict local minimizer; at equality the Hessian is PSD with a nonzero null direction and $C(t)$ is not a strict local minimizer; if the inequality is reversed strictly, $C(t)$ admits a feasible direction of local descent. At $t = \mathbf{1}$, $D_A^2 f(\mathbf{1}, 0)[A, A] = \frac{\|A\|^2}{n^{n-1}(n-1)}$.

Proof. By multilinearity every first-order term contains a factor $\mathrm{MD}(A_i, I, \ldots, I) = 0$. Put $B_i = A_i/t_i$, so $\operatorname{Tr} B_i = 0$ and $\sum_i t_i B_i = 0$. Lemma 3.1(1) gives

$$D_A^2 f(t, 0)[A, A] = \frac{\prod_i t_i}{n^{n-1}(n-1)} \Big(\sum_i \|B_i\|_F^2 - \Big\| \sum_i B_i \Big\|_F^2 \Big).$$

Expand the B_i in a real orthonormal basis of traceless Hermitian matrices. For each basis coordinate the coefficient vector b lies in $t^\perp$, and the form becomes $\|b\|_2^2 - (\mathbf{1}^T b)^2$. The squared norm of $b \mapsto \mathbf{1}^T b$ on $t^\perp$ is $n - n^2/\sum_i t_i^2$, so the form is positive definite exactly under (14). Equality gives a null direction and the reverse inequality a negative one, realized in $\mathcal{T}$ by $B_i = b_i H$ for a fixed traceless Hermitian $H \neq 0$; small perturbations of the positive definite $C(t)$ remain feasible. At $t = \mathbf{1}$, $\sum_i B_i = 0$. □

The same region governs the real-stable class in the coefficient topology. For $t > 0$ put $\gamma(t) = n^2/\|t\|_2^2 - (n-1)$.

Lemma 4.2. *For positive t, with $v_i = 1/t_i$ and $D = \operatorname{diag}(v)$, $\mathcal{A}(L_t^{n-2} q_2) = h_n \prod_i t_i \, (\operatorname{Tr}(D^2 B) - v^T B v)$.*

Proof. A diagonal term of $-x^T B x$ cannot contribute to a squarefree monomial; the ordered pair $i \neq j$ contributes $-B_{ij} h_n \prod_{l \neq i,j} t_l$. □

Lemma 4.3. *Let $C_n = 2^n \sum_{k=3}^{n} \binom{n}{k}(2n)^{k/2}$. If $t_i \geq 0$, $\sum_i t_i = n$ and $0 \leq d \leq 1$, then $|\mathcal{A}(p) - F(t) - \mathcal{A}(L_t^{n-2} q_2)| \leq C_n d^{3/2}$.*

Proof. For homogeneous R of degree n, $\mathcal{A}(R) = \sum_{S \subseteq [n]} (-1)^{n-|S|} R(\mathbf{1}_S)$. Also $0 \leq L_t(\mathbf{1}_S) \leq 1$ and $\mathbf{1}_S^T B \mathbf{1}_S \leq nd$, so (2) bounds the contribution of each $k \geq 3$ by $2^n \binom{n}{k}(2nd)^{k/2}$. □

Proposition 4.4. *Fix $n \geq 2$ and $t_i > 0$ with $\sum_i t_i = n$. In the coefficient topology on the standing class with gradient t, the power L_t^n is a strict local minimizer of $\mathcal{A}$ if and only if $\gamma(t) > 0$, that is, $\|t\|_2^2 < n^2/(n-1)$. If $\gamma(t) < 0$ it is not a local minimizer; if $\gamma(t) = 0$ a nonconstant stable product curve through L_t^n has constant squarefree coefficient.*

Proof. Let $T = \operatorname{diag}(t)$, $S = T^{-1} B T^{-1} \succeq 0$, so $St = 0$, and $r = \mathbf{1} - nt/\|t\|_2^2$, with $\|r\|_2^2 = 1 - \gamma(t)$. Lemma 4.2 becomes $A_2 := \mathcal{A}(L_t^{n-2} q_2) = F(t) \frac{n}{n-1} (\operatorname{Tr} S - r^T S r)$. Since $r^T S r \leq \|r\|_2^2 \operatorname{Tr} S$, $A_2 \geq F(t) \frac{n}{n-1} \gamma(t) \operatorname{Tr} S \geq \alpha(t) d$ with $\alpha(t) = \frac{F(t) n \gamma(t)}{(n-1) \max_i t_i^2}$ when $\gamma(t) > 0$. For $n \geq 3$ and $d \leq \eta(t) = \min\{1, (\alpha(t)/(2C_n))^2\}$, Lemma 4.3 gives $\mathcal{A}(p) - F(t) \geq \frac{1}{2}\alpha(t) d$ (for $n = 2$ the remainder vanishes). Rigidity $d = 0 \iff p = L_t^n$ and continuity of d give strict local minimality, and Lemma 2.3 gives local coefficient coercivity with constant $\alpha(t)/(2W_n)$.

If $\gamma(t) \leq 0$ then $r \neq 0$. Put $b = Tr$, so $b^T \mathbf{1} = 0$. For small $s > 0$, $p_s = L_t^{n-2}(L_t + s b^T x)(L_t - s b^T x) = L_t^n - s^2 L_t^{n-2} (b^T x)^2$ has positive linear factors, is feasible and nonconstant, and $\mathcal{A}(p_s) - F(t) = F(t) \frac{n}{n-1} s^2 \gamma(t) \|r\|_2^2$. □

When $\gamma(t) = 0$ this curve is $p_{X(s)}$ for the feasible diagonal tuple $X_i(s) = \frac{t_i}{n} I + s\, t_i r_i H$ with $H = \operatorname{diag}(1, -1, 0, \ldots, 0)$, so $\mathrm{MD}(X(s))$ is constant and $C(t)$ is not a strict local minimizer; this completes the equality case of Proposition 4.1.

4.2 Failure of global minimality inside the local region

Proposition 4.5. *Let $n \geq 3$ and $1 < t_1 < n-1$. Put $x = \frac{t_1}{n-1}$, $y = \frac{n-1-t_1}{(n-1)^2}$, let M be the $n \times n$ matrix with first row $(0, \frac{1}{n-1}, \ldots, \frac{1}{n-1})$ and all other rows $(x, y, \ldots, y)$, and let $X_j = \operatorname{diag}(M_{1j}, \ldots, M_{nj})$. Then $X_j \succeq 0$, $\sum_j X_j = I$, $t = (\operatorname{Tr} X_j)_j = (t_1, \frac{n-t_1}{n-1}, \ldots, \frac{n-t_1}{n-1})$, and*

$$\mathrm{MD}(X) = \frac{x y^{n-2}}{n}, \qquad \frac{\mathrm{MD}(X)}{\prod_i t_i/n^n} = R_n(t_1) := \frac{n^{n-1}(n-1-t_1)^{n-2}}{(n-1)^{n-2}(n-t_1)^{n-1}}.$$

R_n is strictly decreasing on $(1, n-1)$, $R_n(1) > 1$ and $R_n(2) < 1$. Hence there is a unique $t_1^*(n) \in (1,2)$ with $R_n(t_1^*) = 1$, and for every $t_1 \in (t_1^*(n), 2)$ the trace vector t satisfies $\sum_i t_i^2 < n^2/(n-1)$ while $\mathrm{MD}(X) < \prod_i t_i/n^n$. At $t_1 = t_1^*(n)$ the nonscalar tuple X attains the scalar value. Moreover $t_1^*(3) = 3/2$.

Proof. Every row of M sums to one and the column sums are as stated. For a diagonal tuple $n!\,\mathrm{MD}(X) = \operatorname{per} M$. Since $M_{11} = 0$, column one is matched to one of the $n-1$ lower rows (entry x), row one to one of the $n-1$ columns $j \geq 2$ (entry $\frac{1}{n-1}$), and the remaining $(n-2) \times (n-2)$ block of entries y contributes $(n-2)! y^{n-2}$; hence $\operatorname{per} M = (n-1)!\, x y^{n-2}$, and the ratio follows by substitution. Next,

$$\frac{d}{du} \log R_n(u) = \frac{n-1}{n-u} - \frac{n-2}{n-1-u} = \frac{1-u}{(n-u)(n-1-u)} < 0 \qquad (1 < u < n-1).$$

Further, $\log R_n(1) = m_1 \log(1+\frac{1}{m_1}) - m_2 \log(1+\frac{1}{m_2})$ with $m_1 = n-1 > m_2 = n-2$, which is positive since $m \log(1+1/m)$ increases. For $n = 3$, $R_3(2) = 0$. For $n \geq 4$ put $h = \frac{1}{n-1} < k = \frac{1}{n-2} \leq \frac{1}{2}$; then $\frac{n}{n-2} = \frac{1+h}{1-h}$, $\frac{n-1}{n-3} = \frac{1+k}{1-k}$, and $\log \frac{1+x}{1-x} = 2\sum_{j\geq 0} \frac{x^{2j+1}}{2j+1}$ gives

$$\log R_n(2) = (n-1)\log\frac{n}{n-2} - (n-2)\log\frac{n-1}{n-3} = 2\sum_{j\geq 1} \frac{h^{2j} - k^{2j}}{2j+1} < 0.$$

Finally $\sum_i t_i^2 < \frac{n^2}{n-1}$ reduces to $nt_1^2 - 2nt_1 < 0$, that is, $t_1 < 2$. For $n = 3$, $R_3(u) = \frac{9(2-u)}{2(3-u)^2} = 1$ gives $2u^2 - 3u = 0$. □

The product polynomial $p_X = \prod_i \sum_j M_{ij} x_j$ is real stable with nonnegative coefficients, gradient t and $\mathcal{A}(p_X) = \operatorname{per} M$, so Proposition 4.5 obstructs Question 1.8, and the permanent statement of [7, Theorem 1.1], at the same trace vectors. The family has $\|t - \mathbf{1}\|_1 = 2(t_1 - 1)$, so

$$\rho_n^{\mathrm{RS}} \leq \rho_n^{\mathrm{MD}} \leq 2\big(t_1^*(n) - 1\big) < 2 \qquad (n \geq 3). \tag{15}$$

Along this family the local region of Proposition 4.1 is $t_1 < 2$, so for every $n \geq 3$ the region of global minimality is strictly smaller than the region of strict local minimality. Numerically, $t_1^*(4) \approx 1.6432$, $t_1^*(5) \approx 1.7202$ and $t_1^*(10) \approx 1.8640$.

Example 4.6. Let

$$M = \begin{pmatrix} 0 & \frac{3}{8} & \frac{5}{8} \\ \frac{19}{25} & \frac{29}{200} & \frac{19}{200} \\ \frac{18}{25} & 0 & \frac{7}{25} \end{pmatrix}, \qquad X_j = \operatorname{diag}(M_{1j}, M_{2j}, M_{3j}).$$

The rows of M sum to one, so $\sum_j X_j = I$, and $t = (\frac{37}{25}, \frac{13}{25}, 1)$. Exact arithmetic gives $\operatorname{per} M = \frac{1707}{10000}$ and

$$\operatorname{per} M - \frac{2}{9} t_1 t_2 t_3 = -\frac{29}{90000}, \qquad \mathrm{MD}(X) - \frac{\prod_i t_i}{27} = -\frac{29}{540000} < 0.$$

Here $\|t - \mathbf{1}\|_1 = \frac{24}{25}$ and $\sum_i t_i^2 = \frac{2163}{625} < \frac{9}{2}$. Thus $\rho_3^{\mathrm{RS}} \leq \rho_3^{\mathrm{MD}} \leq 24/25$, and the product polynomial $\prod_i \sum_j M_{ij} x_j$ is a real-stable counterexample at the same gradient. Section 6 sharpens this bound to the exact value $s_* = 0.95471\ldots < 24/25$.

4.3 Degree three: the two questions coincide

Proposition 4.7. *Let $n = 3$ and let t be a positive trace vector of sum 3. Every feasible polynomial of Question 1.8 at t equals p_X for a tuple of real symmetric PSD 3×3 matrices with $\operatorname{Tr} X_i = t_i$ and $\sum_i X_i = I$. Consequently the statement of Question 1.7 holds at t if and only if the statement of Question 1.8 holds at t, and $\rho_3^{\mathrm{MD}} = \rho_3^{\mathrm{RS}}$.*

Proof. A feasible p has nonnegative coefficients (Remark 2.1) and is hyperbolic with respect to $\mathbf{1}$, with the positive orthant inside its hyperbolicity cone. By the theorem of Lewis–Parrilo–Ramana [9], based on [8] (a degree-d polynomial in three variables, hyperbolic with respect to e and normalized by $p(e) = 1$, equals $\det(x_1 I + x_2 B + x_3 C)$ with real symmetric $d \times d$ matrices B, C), after a linear change of variables sending $\mathbf{1}$ to the distinguished direction, there are real symmetric 3×3 matrices X_i with $p(x) = \det(\sum_i x_i X_i)$ and $\sum_i X_i = I$. The roots of $z \mapsto p(z\mathbf{1} - e_i) = \det(zI - X_i)$ are the hyperbolic eigenvalues of e_i, which lies in the closed cone, so $X_i \succeq 0$; and $\operatorname{Tr} X_i = \partial_i p(\mathbf{1}) = t_i$. Conversely every feasible tuple of Question 1.7 gives a feasible p_X (Lemma 3.1). Since $\mathcal{A}(p_X) = 6\,\mathrm{MD}(X)$, $F(t) = 6\prod_i t_i/27$, and $p_X = L_t^3$ exactly when $d(p_X) = \delta(X)/2 = 0$, the minimization statements correspond. □

Corollary 4.8. $\rho_3^{\mathrm{RS}} = \rho_3^{\mathrm{MD}} = s_* = \left((236 - 37\sqrt{37})/12\right)^{1/2} = 0.9547157683\ldots$

Proof. Proposition 4.7 gives $\rho_3^{\mathrm{RS}} = \rho_3^{\mathrm{MD}}$, and Theorem 6.1 in Section 6 identifies the common value. Theorem 2.10 and Example 4.6 give the weaker bracket $\frac{1}{8} \le \rho_3^{\mathrm{MD}} \le \frac{24}{25}$ by independent means. □

4.4 The Lorentzian class

Lorentzianity supports the qualitative and local results of Section 2.5, while local minimality in this class already requires $\max_i t_i < n/(n-1)$. For binary $f(X,Y) = \sum_{k=0}^{N} \binom{N}{k} a_k X^k Y^{N-k}$ with $a_k \ge 0$ and interval support, the support is M-convex and $\partial_X^j \partial_Y^{N-2-j} f = \frac{N!}{2}(a_j Y^2 + 2a_{j+1}XY + a_{j+2}X^2)$; its Hessian has nonnegative trace, so it has at most one positive eigenvalue exactly when $a_{j+1}^2 \ge a_j a_{j+2}$. Thus interval-supported log-concave normalized binary coefficients give Lorentzian polynomials, and nonnegative substitution preserves the class.

Theorem 4.9. *Let $n \ge 3$, $t_i > 0$, $\sum_i t_i = n$ and $\max_i t_i \ge n/(n-1)$. Then L_t^n is not a local minimizer of $\mathcal{A}$ in the coefficient topology on normalized Lorentzian polynomials with gradient exactly t: there are competitors with every coefficient strictly positive, converging to L_t^n, with squarefree coefficient strictly below $F(t)$.*

Proof. Let $t_1 = \tau$ attain the maximum, so $\tau < n$. Define $Y(x) = \sum_{j \ge 2} t_j x_j/(n-\tau)$, $q_r = (\tau + (n-\tau)r)/n$, $\ell_r = q_r x_1 + (1-q_r)Y$ and $p_r = (\ell_r^n - r x_1^n)/(1-r)$ for $0 < r < \frac{1}{2}(\tau/n)^n$. Then $0 < q_r < 1$ and $q_r^n - r > 0$, so every coefficient of p_r is positive. The binary polynomial $[(q_r X + (1-q_r)Y)^n - rX^n]/(1-r)$ has normalized coefficients $a_k = q_r^k (1-q_r)^{n-k}/(1-r)$ for $k < n$ and $a_n = (q_r^n - r)/(1-r)$; all log-concavity inequalities are equalities except the last, whose margin is $r q_r^{n-2}(1-q_r)^2/(1-r)^2 > 0$. So p_r is Lorentzian. Also $p_r(\mathbf{1}) = 1$, $\partial_1 p_r(\mathbf{1}) = (nq_r - nr)/(1-r) = \tau$ and $\partial_j p_r(\mathbf{1}) = t_j$ for $j \ge 2$, using $1 - q_r = (1-\tau/n)(1-r)$. Only the terms with x_1-exponent one contribute to $\mathcal{A}$, which yields $\mathcal{A}(p_r)/F(t) = (1+ar)(1-r)^{n-2}$ with $a = (n-\tau)/\tau \le n-2$. For $R(r) = (1+ar)(1-r)^{n-2}$, $R'(r)/R(r) = \frac{a-(n-2)-a(n-1)r}{(1+ar)(1-r)} < 0$ for $r > 0$, and $R(0) = 1$. □

Corollary 4.10. *Let ρ_n^{L} be the supremum of radii ρ such that for every positive t of sum n with $\|t - \mathbf{1}\|_1 < \rho$ the power L_t^n is a global minimizer on the normalized Lorentzian class. Then $\rho_n^{\mathrm{L}} \le \frac{2}{n-1}$ for $n \ge 3$.*

Proof. Apply Theorem 4.9 at $t_1 = \frac{n}{n-1}$, $t_j = \frac{n(n-2)}{(n-1)^2}$ $(j \ge 2)$, whose ℓ^1 distance from $\mathbf{1}$ is $\frac{2}{n-1}$. □

At that gradient $\gamma(t) = \frac{(n-1)(n-2)}{n^2-3n+3} > 0$, so Proposition 4.4 gives strict local minimality on the real-stable class there; the competitors p_r have nonreal line restrictions and lie outside it.

Proposition 4.11. *Let z_* be the positive root of $2z^3 - 2z - 1 = 0$ and $R_* = \frac{2(3z_*^2-1)}{1+3z_*+3z_*^2}$. Every universal open gradient radius on which the normalized Lorentzian ternary cubics have L_t^3 as unique global minimizer is at most $R_* < \frac{166}{223}$.*

Proof. For $z > 0$ let $Z = 1 + 3z + 3z^2$, $f_z(X,Y) = (Y^3 + 3zXY^2 + 3z^2X^2Y)/Z$ and $p_z(x) = f_z(x_1, \frac{x_2+x_3}{2})$. The normalized binary coefficients $(1, z, z^2, 0)/Z$ have interval support and are log-concave, so p_z is normalized Lorentzian. With $\mu = 3z(1+2z)/Z$ and $3 - \mu = 3(1+z)^2/Z$, its gradient is $t = (\mu, \frac{3-\mu}{2}, \frac{3-\mu}{2})$, $\mathcal{A}(p_z) = \frac{3z}{2Z}$ and $F(t) = \frac{\mu(3-\mu)^2}{18}$, so $\mathcal{A}(p_z) - F(t) = -\frac{3z^3(2z^3-2z-1)}{2Z^3}$, by $(1+2z)(1+z)^4 - Z^2 = z^2(2z^3 - 2z - 1)$. Descartes' rule gives a unique positive root $z_* \in (1, \frac{6}{5})$. At z_*, p_z is a nonpower with the coefficient of L_t^3; for $z > z_*$ it is strictly smaller. For $z > 1/\sqrt{3}$, $\|t - \mathbf{1}\|_1 = R(z) = \frac{2(3z^2-1)}{Z}$, which increases strictly because $d\mu/d\log z$ is the variance of a nonconstant law. Hence $R_* < R(\frac{6}{5}) = \frac{166}{223}$. □

5 Two variables

Proposition 5.1. *For $n = 2$ and every t with $t_1 + t_2 = 2$, every feasible p of Question 1.8 satisfies $\mathcal{A}(p) - \frac{t_1 t_2}{2} = \frac{b^2 - 4ac}{2} \geq 0$, where $p = ax_1^2 + bx_1x_2 + cx_2^2$, with equality only for $p = L_t^2$.*

Proof. Real stability of the binary form is equivalent to $b^2 \geq 4ac$. From $a + b + c = 1$, $t_1 = 2a + b$ and $t_2 = b + 2c$ one gets $t_1 t_2 = b(2 - b) + 4ac$, so $\mathcal{A}(p) - F(t) = b - \frac{t_1 t_2}{2} = \frac{b^2 - 4ac}{2}$. Equality makes p the square of a linear form ℓ with $\ell(\mathbf{1}) = 1$, and then $\nabla p(\mathbf{1}) = t$ forces $\ell = L_t$. □

Proposition 5.2. *For $n = 2$, write a feasible tuple of Question 1.7 as $X_1 = \frac{t_1}{2} I + A$, $X_2 = \frac{t_2}{2} I - A$ with $\operatorname{Tr} A = 0$. Then*

$$\operatorname{MD}(X_1, X_2) = \frac{t_1 t_2}{4} + \frac{\operatorname{Tr}(A^2)}{2}.$$

Hence the scalar tuple is the unique minimizer for every $0 \leq t_1 \leq 2$, including the endpoints, so Questions 1.7 and 1.8 have affirmative answers with every radius.

Proof. By Lemma 3.1(1), $\operatorname{MD}(X_1, X_2) = \frac{1}{2}(t_1 t_2 - \operatorname{Tr}(X_1 X_2))$ and $\operatorname{Tr}(X_1 X_2) = \frac{t_1 t_2}{2} - \operatorname{Tr}(A^2)$. □

6 The exact radius in degree three

Throughout this section $n = 3$, and $t = (t_1, t_2, t_3)$ is a positive vector with $t_1 + t_2 + t_3 = 3$. Let $\mathcal{M}(t)$ be the transportation polytope of nonnegative 3×3 matrices with all row sums 1 and column sums t, and let $\bar{x}_t = \frac{1}{3} \mathbf{1} t^T$ be its rank-one point. For $x \in \mathcal{M}(t)$ put

$$\Theta(x) = \operatorname{per} x - \tfrac{2}{9}\, t_1 t_2 t_3 = \operatorname{per} x - \operatorname{per} \bar{x}_t.$$

A diagonal tuple $X_j = \operatorname{diag}(x_{1j}, x_{2j}, x_{3j})$ is feasible in Question 1.7 exactly when $x \in \mathcal{M}(t)$, and then $6 \operatorname{MD}(X) = \operatorname{per} x$. Let ρ_3^{per} be the supremum of the radii ϵ such that $\bar{x}_t$ is the unique minimizer of the permanent on $\mathcal{M}(t)$ for every t with $\|t - \mathbf{1}\|_1 < \epsilon$; this is the optimal radius in [7, Theorem 1.1] for $n = 3$. Since $\sum_i (t_i - 1) = 0$, the positive and negative parts of $t - \mathbf{1}$ have equal mass, and one coordinate of $t - \mathbf{1}$ carries all the mass of its sign; hence

$$\|t - \mathbf{1}\|_1 = 2 \max_i |t_i - 1|. \tag{6.1}$$

Let s_* be the smallest positive root of $48 s^4 - 1888 s^2 + 1681$,

$$s_* = \Big(\frac{236 - 37\sqrt{37}}{12} \Big)^{1/2} = 0.95471576830340241\ldots,$$

so that $s_*/2$ is the smallest positive root of $768 \varpi^4 - 7552 \varpi^2 + 1681$.

Theorem 6.1. $\rho_3^{\mathrm{per}} = \rho_3^{\mathrm{MD}} = \rho_3^{\mathrm{RS}} = s_*$. *More precisely:*

1. *If $\|t - \mathbf{1}\|_1 < s_*$, then $\bar{x}_t$ is the unique minimizer of the permanent on $\mathcal{M}(t)$, the scalar tuple $\frac{1}{3}(t_1 I, t_2 I, t_3 I)$ is the unique minimizer in Question 1.7, and L_t^3 is the unique minimizer in Question 1.8.*

2. *At $t_* = (1 + \frac{s_*}{2}, 1, 1 - \frac{s_*}{2})$, where $\|t_* - \mathbf{1}\|_1 = s_*$, the matrix $x_* \in \mathcal{M}(t_*)$ of Lemma 6.6, which has exactly two zero entries, satisfies $\operatorname{per} x_* = \operatorname{per} \bar{x}_{t_*}$. Its diagonal tuple and its product polynomial $\prod_i \sum_j (x_*)_{ij} x_j$ are second minimizers in Questions 1.7 and 1.8.*

3. *For $t = (1+\frac{s}{2}, 1, 1-\frac{s}{2})$ with $s > s_*$ close to s_* there are matrices in $\mathcal{M}(t)$ with permanent strictly below* $\operatorname{per} \bar{x}_t$.

Consequently, for $n = 3$ the statements of Questions 1.7 and 1.8 hold with radius ϵ exactly when $\epsilon \le s_*$.

The proof occupies the rest of this section. The mixed discriminant is reduced to the permanent first (Theorem 6.4); the permanent problem is then reduced to a face of $\mathcal{M}(t)$ with two zero entries (Lemma 6.5), where the extremal configuration is located (Lemmas 6.6 and 6.8).

6.1 A separation identity

For a Hermitian 3×3 matrix Y put $\phi(Y) = \frac{1}{2}\operatorname{Tr}(Y^2) + 2\det Y$.

Lemma 6.2. *Let X_1, X_2, X_3 be Hermitian 3×3 matrices with $X_1 + X_2 + X_3 = I$ and $t_j = \operatorname{Tr} X_j$. Then*

$$6\,\mathrm{MD}(X_1, X_2, X_3) = 1 - \frac{1}{2}\sum_j t_j^2 + \sum_j \phi(X_j). \tag{6.2}$$

For $x \in \mathcal{M}(t)$, write $x^{(j)}$ for its jth column and $a^{(j)} = x^{(j)} - \frac{t_j}{3}\mathbf{1}$, so that $\sum_j a^{(j)} = 0$. Then

$$\Theta(x) = \sum_{j=1}^{3}\Big[\Big(\frac{1}{2} - \frac{t_j}{3}\Big)\|a^{(j)}\|_2^2 + 2a_1^{(j)}a_2^{(j)}a_3^{(j)}\Big]. \tag{6.3}$$

Proof. Multilinearity and symmetry give

$$1 = \det(X_1 + X_2 + X_3) = \sum_j \det X_j + 3\sum_{i \ne j}\mathrm{MD}(X_i, X_i, X_j) + 6\,\mathrm{MD}(X_1, X_2, X_3).$$

By Lemma 3.1(1), $\mathrm{MD}(X_i, X_i, I) = \frac{1}{6}(t_i^2 - \operatorname{Tr} X_i^2)$, while multilinearity in the third slot gives $\mathrm{MD}(X_i, X_i, I) = \det X_i + \sum_{j \ne i}\mathrm{MD}(X_i, X_i, X_j)$. Substituting the second sum into the first identity gives (6.2). For a diagonal tuple, $\phi(X_j)$ equals $\frac{1}{2}\|x^{(j)}\|_2^2 + 2x_{1j}x_{2j}x_{3j}$. Writing $x^{(j)} = \frac{t_j}{3}\mathbf{1} + a^{(j)}$ with $\sum_r a_r^{(j)} = 0$ gives $\frac{1}{2}\|x^{(j)}\|_2^2 = \frac{t_j^2}{6} + \frac{1}{2}\|a^{(j)}\|_2^2$ and $2\prod_r x_{rj} = \frac{2t_j^3}{27} - \frac{t_j}{3}\|a^{(j)}\|_2^2 + 2\prod_r a_r^{(j)}$, because the second elementary symmetric function of $a^{(j)}$ is $-\frac{1}{2}\|a^{(j)}\|_2^2$. Subtracting the same expression at $\bar{x}_t$, where every $a^{(j)} = 0$, gives (6.3). Finally $\sum_j a^{(j)} = x\mathbf{1} - \bar{x}_t\mathbf{1} = 0$. □

The cubic term in (6.3) is controlled by Schur's inequality. For $v \in \mathbb{R}^3$ let $\mathrm{Sch}(v) = \sum_r v_r(v_r - v_{r'})(v_r - v_{r''})$, where $\{r, r', r''\} = \{1, 2, 3\}$. If $v_1 \ge v_2 \ge v_3 \ge 0$, then $\mathrm{Sch}(v) = (v_1 - v_2)^2(v_1 + v_2 - v_3) + v_3(v_1 - v_3)(v_2 - v_3) \ge 0$, so $\mathrm{Sch} \ge 0$ on $\mathbb{R}^3_{\ge 0}$. A direct expansion shows that for $v \ge 0$ with $\sum_r v_r = \tau$ and $a = v - \frac{\tau}{3}\mathbf{1}$,

$$18a_1a_2a_3 + \tau\|a\|_2^2 = 2\,\mathrm{Sch}(v). \tag{6.4}$$

Proposition 6.3. *For $x \in \mathcal{M}(t)$,*

$$\Theta(x) = \frac{1}{18}\sum_j (9 - 8t_j)\|a^{(j)}\|_2^2 + \frac{2}{9}\sum_j \mathrm{Sch}(x^{(j)}). \tag{6.5}$$

Consequently $\bar{x}_t$ is the unique minimizer of the permanent on $\mathcal{M}(t)$ whenever $\sum_j (t_j - 1)^2 < \frac{3}{32}$, in particular whenever $\|t - \mathbf{1}\|_1 < \frac{\sqrt{3}}{4}$. The same holds for the scalar tuple in Question 1.7 and for L_t^3 in Question 1.8.

Proof. Identity (6.4) applied to each column turns (6.3) into (6.5). Put $w_j = 9 - 8t_j$, so $w_1 + w_2 + w_3 = 3$. On the plane $\{y \in \mathbb{R}^3 : \sum_j y_j = 0\}$, written as $y = (y_1, y_2, -y_1 - y_2)$, the form $\sum_j w_j y_j^2$ has matrix $\left(\begin{smallmatrix} w_1+w_3 & w_3 \\ w_3 & w_2+w_3 \end{smallmatrix}\right)$ with determinant $e_2(w) = w_1w_2 + w_1w_3 + w_2w_3$. If $e_2(w) > 0$, the three pairwise sums satisfy $(w_i + w_k)(w_i + w_l) = e_2(w) + w_i^2 > 0$, hence have a common sign, and their total $2\sum_j w_j = 6$ makes them positive; so the form is positive definite. Applying this coordinatewise to the vectors $a^{(1)}, a^{(2)}, a^{(3)}$, which sum to zero, gives $\Theta(x) \geq \frac{1}{18}\sum_j w_j \|a^{(j)}\|_2^2 > 0$ unless all $a^{(j)}$ vanish, that is, unless $x = \bar{x}_t$. With $\eta = t - \mathbf{1}$ one has $\sum_j w_j^2 = 3 + 64\|\eta\|_2^2$, so $e_2(w) = \frac{1}{2}(9 - \sum_j w_j^2) > 0$ exactly when $\|\eta\|_2^2 < \frac{3}{32}$. If $\|\eta\|_1 < \frac{\sqrt{3}}{4}$, then by (6.1) every $|\eta_j| < \frac{\sqrt{3}}{8}$; the coordinate of largest modulus carries the mass of its sign, the other two have moduli summing to its modulus, and $\|\eta\|_2^2 \leq 2\max_j \eta_j^2 < \frac{3}{32}$. The last sentence follows from Theorem 6.4 and Proposition 4.7. □

6.2 Mixed discriminants reduce to permanents

Let $\mathcal{F}_t$ be the set of feasible tuples of Question 1.7: Hermitian $X_j \succeq 0$ with $\operatorname{Tr} X_j = t_j$ and $X_1 + X_2 + X_3 = I$. By (6.2), minimizing MD over $\mathcal{F}_t$ is the same as minimizing $\Psi(X) = \sum_j \phi(X_j)$. Call a tuple *commuting* if its members commute pairwise; a commuting feasible tuple is diagonal in a common orthonormal basis, so its mixed discriminant is $\frac{1}{6}\operatorname{per} x$ for the matrix $x \in \mathcal{M}(t)$ of its eigenvalues.

Theorem 6.4. *For every positive t with $t_1 + t_2 + t_3 = 3$,*

$$\min_{X \in \mathcal{F}_t} \mathrm{MD}(X) = \frac{1}{6} \min_{x \in \mathcal{M}(t)} \operatorname{per} x,$$

and the scalar tuple is the unique minimizer of MD *on $\mathcal{F}_t$ if and only if $\bar{x}_t$ is the unique minimizer of the permanent on $\mathcal{M}(t)$.*

Proof. Let $X \in \mathcal{F}_t$ minimize Ψ; it exists by compactness. We show first that some commuting minimizer exists, and that X itself is commuting unless it has the special block form of Step 2.

Step 1: a multiplier commuting with every X_j. The gradient of ϕ with respect to the trace inner product is $N(Y) = Y + 2\operatorname{adj} Y$, and by the Cayley–Hamilton theorem $\operatorname{adj} Y = Y^2 - (\operatorname{Tr} Y)Y + e_2(Y)I$ is a polynomial in Y. Put $N_j = N(X_j)$. By first-order optimality on the convex set $\mathcal{F}_t$, X minimizes the linear function $Y \mapsto \sum_j \operatorname{Tr}(N_j Y_j)$ on $\mathcal{F}_t$. This is a semidefinite program, and its feasible set contains the positive definite tuple $(\frac{t_j}{3} I)_j$. By the strong duality theorem for semidefinite programs with a strictly feasible primal [16], applied after the standard real embedding of Hermitian matrices (the dual below is strictly feasible as well: take all $\nu_j = 0$ and Λ a sufficiently negative multiple of I), the dual problem $\max\{\operatorname{Tr}\Lambda + \sum_j \nu_j t_j : N_j - \Lambda - \nu_j I \succeq 0\}$ over Hermitian Λ and real ν_j attains the primal value. For a dual optimum, $\Omega_j = N_j - \Lambda - \nu_j I \succeq 0$ and $\sum_j \operatorname{Tr}(\Omega_j X_j) = \sum_j \operatorname{Tr}(N_j X_j) - \operatorname{Tr}\Lambda - \sum_j \nu_j t_j = 0$, using $\sum_j X_j = I$ and $\operatorname{Tr} X_j = t_j$. Each term is nonnegative, so $\Omega_j X_j = 0 = X_j \Omega_j$. Since N_j is a polynomial in X_j, the matrix $\Lambda = N_j - \nu_j I - \Omega_j$ commutes with every X_j.

Step 2: Λ has an eigenvalue of multiplicity exactly two. Every X_j preserves the eigenspaces $\mathcal{E} \oplus \mathcal{E}^\perp$ of Λ, with $\dim \mathcal{E} = 2$. Write $X_j = X_j' \oplus (\vartheta_j)$, with X_j' on $\mathcal{E}$, and $X_j' = \frac{t_j - \vartheta_j}{2} I_{\mathcal{E}} + \mathcal{Q}_j$ with $\mathcal{Q}_j$ traceless Hermitian of eigenvalues $\pm\upsilon_j$, $\upsilon_j \geq 0$. Then $\det X_j = \vartheta_j\big(\frac{(t_j - \vartheta_j)^2}{4} - \upsilon_j^2\big)$ and $\operatorname{Tr} X_j^2 = \frac{(t_j - \vartheta_j)^2}{2} + 2\upsilon_j^2 + \vartheta_j^2$, so

$$\phi(X_j) - (1 - 2\vartheta_j)\upsilon_j^2$$

depends only on t_j and ϑ_j. The constraints read $\vartheta_j \geq 0$, $\sum_j \vartheta_j = 1$, $\upsilon_j \leq \frac{t_j - \vartheta_j}{2}$ and $\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 = 0$. Traceless Hermitian 2×2 matrices form a real Euclidean space in which $\|\mathcal{Q}_j\| = \upsilon_j$ (up to the factor $\sqrt{2}$), so $\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 = 0$ forces the triangle inequalities $\upsilon_i \leq \upsilon_k + \upsilon_l$. At most one ϑ_j exceeds $\frac{1}{2}$. If none does, replace every $\mathcal{Q}_j$ by 0; if $\vartheta_1 > \frac{1}{2}$, keep υ_1, replace υ_k ($k = 2, 3$) by $\upsilon_k \upsilon_1 / (\upsilon_2 + \upsilon_3) \leq \upsilon_k$ (or by 0 if $\upsilon_2 + \upsilon_3 = 0$), and realize the new values as $\mathcal{Q}''_1 = \upsilon_1 \mathcal{S}$, $\mathcal{Q}''_k = -\upsilon''_k \mathcal{S}$ for a fixed traceless $\mathcal{S}$ with eigenvalues ± 1. The new tuple X'' is feasible and commuting, since all its blocks are polynomials in $\mathcal{S}$, and $\Psi(X'') \leq \Psi(X)$ because each υ_k with $1 - 2\vartheta_k \geq 0$ did not increase. So X'' is a commuting minimizer.

Step 3: Λ has three distinct eigenvalues. Then every X_j is diagonal in the eigenbasis of Λ, and X is commuting.

Step 4: $\Lambda = \lambda I$ is scalar. In a common eigenbasis of X_j and Ω_j (they commute), let X_j have eigenvalues $\lambda_1, \lambda_2, \lambda_3$. The eigenvalues of N_j are $\lambda_r + 2\lambda_{r'}\lambda_{r''}$, and $N_j = (\lambda + \nu_j) I + \Omega_j$ with Ω_j vanishing on the range of X_j. If all $\lambda_r > 0$, then $\lambda_r + 2\lambda_{r'}\lambda_{r''}$ does not depend on r, and subtracting two of these equations gives $(\lambda_r - \lambda_{r'})(1 - 2\lambda_{r''}) = 0$ for all r, r', so the λ_r take at most two distinct values. If some $\lambda_{r''} = 0$, every positive λ_r equals $\lambda + \nu_j$. In all cases X_j has an eigenvalue of multiplicity at least two, so $X_j = \chi_j I + \psi_j \Pi_j$ with reals χ_j, ψ_j and a rank-one orthogonal projection Π_j. Then $\sum_j \psi_j \Pi_j = (1 - \sum_j \chi_j) I$. If $1 - \sum_j \chi_j \neq 0$, the left side has rank three, so all $\psi_j \neq 0$ and, writing $\Pi_j = u_j u_j^*$ and $U = [u_1\, u_2\, u_3]$, the identity $U \operatorname{diag}(\psi) U^* = cI$ gives $U^* U = c \operatorname{diag}(\psi)^{-1}$, so the u_j are orthogonal. If $1 - \sum_j \chi_j = 0$, the nonzero terms cannot all have one sign, so a single term of one sign equals a sum of positive semidefinite rank-one terms of the other sign, and all the Π_j with $\psi_j \neq 0$ coincide. In both cases X is commuting.

Conclusion. Steps 2–4 give a commuting minimizer of Ψ. By (6.2) it also minimizes MD on $\mathcal{F}_t$, and its mixed discriminant is $\frac{1}{6} \operatorname{per} x$ for its eigenvalue matrix $x \in \mathcal{M}(t)$; since every $x \in \mathcal{M}(t)$ arises from a diagonal tuple, $\min_{\mathcal{F}_t} \mathrm{MD} = \frac{1}{6} \min_{\mathcal{M}(t)} \operatorname{per}$. If $\bar{x}_t$ is not the unique permanent minimizer, a second minimizer gives a non-scalar diagonal tuple of minimal mixed discriminant. Conversely, suppose $\bar{x}_t$ is the unique permanent minimizer and X is any minimizer of MD on $\mathcal{F}_t$. In Steps 3 and 4, X is commuting, so its eigenvalue matrix minimizes the permanent and equals $\bar{x}_t$, and X is scalar. In Step 2, X'' is a commuting minimizer, so it is scalar: $\vartheta_j = \frac{t_j}{3}$ and every $\mathcal{Q}''_j = 0$. If no ϑ_j exceeds $\frac{1}{2}$, then $0 = \Psi(X) - \Psi(X'') = \sum_j (1 - 2\vartheta_j) \upsilon_j^2$; the coefficients are nonnegative, and at most one vanishes because $\vartheta_j = \frac{t_j}{3} = \frac{1}{2}$ for two indices would force the third trace to be 0. So υ vanishes off at most one index, and the triangle inequalities give $\upsilon = 0$. If $\vartheta_1 > \frac{1}{2}$, then $\mathcal{Q}''_1 = \upsilon_1 \mathcal{S} = 0$ gives $\upsilon_1 = 0$, so $\upsilon_2 = \upsilon_3$ and $0 = \Psi(X) - \Psi(X'') = (2 - 2\vartheta_2 - 2\vartheta_3) \upsilon_2^2$ with a positive coefficient. In both cases $X = X''$ is scalar. □

6.3 Reduction to a face with two zero entries

Lemma 6.5. *Let $\max_j t_j < \frac{3}{2}$. If some $x \in \mathcal{M}(t)$ with $x \neq \bar{x}_t$ satisfies $\Theta(x) \leq 0$, then some $x' \in \mathcal{M}(t)$ with two zero entries in different rows and different columns satisfies $\Theta(x') \leq \Theta(x)$; in particular $\Theta(x') < 0$ if $\Theta(x) < 0$.*

Proof. Ray. Put $\mathcal{Z} = x - \bar{x}_t \neq 0$ and $f(\lambda) = \Theta(\bar{x}_t + \lambda \mathcal{Z})$ for $0 \leq \lambda \leq \lambda_{\max}$, the largest λ with $\bar{x}_t + \lambda \mathcal{Z} \geq 0$; it is finite because $\mathcal{Z}$ has zero row sums and a negative entry, and $\lambda_{\max} \geq 1$. By (6.3), $f(\lambda) = \lambda^2 q_0 + \lambda^3 c_0$ with $q_0 = \sum_j (\frac{1}{2} - \frac{t_j}{3}) \|a^{(j)}\|_2^2 > 0$ and $c_0 = 2 \sum_j \prod_r a_r^{(j)}$. From $f(1) \leq 0$ we get $c_0 \leq -q_0 < 0$, hence $f(\lambda_{\max}) = \lambda_{\max}^2 (q_0 + \lambda_{\max} c_0) \leq \lambda_{\max}^2 (q_0 + c_0) = \lambda_{\max}^2 \Theta(x) \leq \Theta(x)$. So $x_1 = \bar{x}_t + \lambda_{\max} \mathcal{Z}$ has a zero entry, say at (r, j), and $\Theta(x_1) \leq \Theta(x) \leq 0$.

Affine direction. Let $\{r', r''\}$ and $\{j', j''\}$ be the other rows and columns, and $\Upsilon = e_{r'} e_{j'}^T - e_{r'} e_{j''}^T - e_{r''} e_{j'}^T + e_{r''} e_{j''}^T$, with e_i the coordinate vectors. Every term of the permanent expansion of $x_1 + \epsilon \Upsilon$ uses entries in distinct rows and columns; a term using two entries of the support of Υ uses $(r', j'), (r'', j'')$

or $(r', j''), (r'', j')$ and then the factor $(x_1)_{rj} = 0$; no term uses three. Hence $\epsilon \mapsto \operatorname{per}(x_1 + \epsilon\Upsilon)$ is affine, while the column sums and $\operatorname{per} \bar{x}_t$ do not change. The set of ϵ with $x_1 + \epsilon\Upsilon \geq 0$ is a compact interval containing 0, at whose endpoints an entry in $\{r', r''\} \times \{j', j''\}$ vanishes. At the endpoint where the affine function is not larger we obtain x' with $\Theta(x') \leq \Theta(x_1) \leq \Theta(x)$ and zeros at (r, j) and at a position in another row and another column. □

By row and column permutations, which permute t and preserve $\|t - \mathbf{1}\|_1$ and Θ, the two zeros can be placed at $(1,1)$ and $(3,3)$. Exchanging rows $1,3$ and columns $1,3$ simultaneously keeps this pattern and exchanges t_1 and t_3, so we may also assume $t_1 \geq t_3$. The face $\{x \in \mathcal{M}(t) : x_{11} = x_{33} = 0\}$ is parametrized by $\xi = x_{12}$ and $\zeta = x_{31}$:

$$x = \begin{pmatrix} 0 & \xi & 1-\xi \\ t_1 - \zeta & t_2 - 1 + \zeta - \xi & t_3 - 1 + \xi \\ \zeta & 1 - \zeta & 0 \end{pmatrix}, \qquad t_2 = 3 - t_1 - t_3. \tag{6.6}$$

Expanding the permanent gives the integer polynomial

$$\begin{aligned} \Phi(t_1, t_3, \xi, \zeta) := 9\,\Theta(x) = {} & 18\xi^2\zeta - 18\xi\zeta^2 + 18\xi\zeta t_1 + 18\xi\zeta t_3 - 27\xi\zeta - 9\xi t_1 + 18\zeta^2 - 18\zeta t_1 \\ & - 9\zeta t_3 + 9\zeta + 2t_1^2 t_3 + 2t_1 t_3^2 - 6t_1 t_3 + 9t_1. \end{aligned}$$

6.4 The extremal configuration

Lemma 6.6. *Let $\varpi = s_*/2$ and*

$$\xi_* = \frac{5371 + 4348\varpi + 656\varpi^2 - 192\varpi^3}{12136}, \qquad \zeta_* = \frac{6765 + 4348\varpi - 656\varpi^2 - 192\varpi^3}{12136},$$

so that $\xi_* = 0.6241883822\ldots$ and $\zeta_* = 0.7144185930\ldots$. The matrix

$$x_* = \begin{pmatrix} 0 & \xi_* & 1-\xi_* \\ 1 + \varpi - \zeta_* & \zeta_* - \xi_* & \xi_* - \varpi \\ \zeta_* & 1 - \zeta_* & 0 \end{pmatrix} \approx \begin{pmatrix} 0 & 0.62419 & 0.37581 \\ 0.76294 & 0.09023 & 0.14683 \\ 0.71442 & 0.28558 & 0 \end{pmatrix}$$

lies in $\mathcal{M}(t_*)$, has exactly two zero entries, and satisfies $\operatorname{per} x_* = \frac{2}{9}(1 - \varpi^2) = \operatorname{per} \bar{x}_{t_*}$. At the point $\mathfrak{p}_* = (1 + \varpi, 1 - \varpi, \xi_*, \zeta_*)$, $\partial_\xi \Phi = \partial_\zeta \Phi = 0$, the Hessian of Φ in (ξ, ζ) is positive definite, and

$$\partial_{t_1} \Phi(\mathfrak{p}_*) = -0.95148330\ldots, \qquad \partial_{t_3} \Phi(\mathfrak{p}_*) = 0.18654793\ldots.$$

Proof. On the face (6.6) with $t = (1 + \varpi, 1, 1 - \varpi)$ the permanent is $P(\xi, \zeta, \varpi) = \xi\zeta(\xi - \varpi) + (1 - \xi)((1 + \varpi - \zeta)(1 - \zeta) + \zeta(\zeta - \xi))$. Reducing $\partial_\xi P$, $\partial_\zeta P$ and $P - \frac{2}{9}(1 - \varpi^2)$ at (ξ_*, ζ_*, ϖ) modulo $768\varpi^4 - 7552\varpi^2 + 1681$ gives 0 in each case, in exact rational arithmetic. The entries of x_* other than $(x_*)_{11}, (x_*)_{33}$, and the remaining quantities, are enclosed by interval arithmetic. For fixed t, P and $\Phi/9$ differ by a constant, so the Hessian of Φ in (ξ, ζ) is 9 times that of P, which is $\left(\begin{smallmatrix} 4\zeta & 4\xi - 4\zeta + 1 \\ 4\xi - 4\zeta + 1 & 4 - 4\xi \end{smallmatrix}\right)$, with diagonal entry $2.8577\ldots$ and determinant $3.8874\ldots$ at the point. The script `exact_witness.py` and its output contain these checks. □

Remark 6.7. Along $t = (1 + \frac{s}{2}, 1, 1 - \frac{s}{2})$, with $(\xi, \zeta) = (\xi_*, \zeta_*)$ fixed, the derivative of Φ with respect to s at $s = s_*$ is $\frac{1}{2}(\partial_{t_1} \Phi - \partial_{t_3} \Phi)(\mathfrak{p}_*) < 0$, and the entries of (6.6) stay positive off the two zeros. This proves Theorem 6.1(3). An explicit rational instance is the matrix with rows $(0, \frac{31211}{50000}, \frac{18789}{50000})$, $(\frac{763001}{10^6}, \frac{90279}{10^6}, \frac{917}{6250})$, $(\frac{714499}{10^6}, \frac{285501}{10^6}, 0)$, whose column sums $(\frac{591}{400}, 1, \frac{209}{400})$ are at ℓ^1-distance $\frac{191}{200}$ from $\mathbf{1}$ and for which $\Theta = -\frac{1348032957893}{75 \cdot 10^{15}} < 0$.

Lemma 6.8. *Let $\mathcal{D}_*$ be the set of (t_1, t_3, ξ, ζ) with $|t_j - 1| \le \frac{s_*}{2}$ for $j = 1, 2, 3$ (where $t_2 = 3 - t_1 - t_3$), $t_1 \ge t_3$, and all entries of (6.6) nonnegative. Then $\Phi \ge 0$ on $\mathcal{D}_*$, with equality only at $\mathfrak{p}_*$.*

Proof. Near $\mathfrak{p}_*$. Let $\mathcal{L}$ be the set of points of $\mathcal{D}_*$ with $t_1 \ge 1 + \varpi - \frac{1}{50}$, $t_3 \le 1 - \varpi + \frac{1}{50}$, $|\xi - \xi_*| \le \frac{1}{20}$ and $|\zeta - \zeta_*| \le \frac{1}{20}$, where $\varpi = s_*/2$. On $\mathcal{D}_*$ we have $t_1 \le 1 + \varpi$ and $t_3 \ge 1 - \varpi$, so a point of $\mathcal{L}$ is $\mathfrak{p}_* + d$ with $d = (d_1, d_3, d_\xi, d_\zeta)$, $d_1 \le 0 \le d_3$ and $|d_1|, |d_3| \le \frac{1}{50}$, $|d_\xi|, |d_\zeta| \le \frac{1}{20}$. Since Φ is cubic, $\Phi(\mathfrak{p}_* + d) = \partial_{t_1}\Phi\, d_1 + \partial_{t_3}\Phi\, d_3 + \frac{1}{2} d^T \nabla^2 \Phi(\mathfrak{p}_*) d + \mathcal{C}(d)$, with $\mathcal{C}$ the cubic part, whose coefficients are integers; by Lemma 6.6 the first derivatives in ξ, ζ vanish. Write $\rho = (\frac{1}{50}, \frac{1}{50}, \frac{1}{20}, \frac{1}{20})$ for the radii and Φ_{ijk} for the third derivatives, which are constant integers. Then $|\mathcal{C}(d)| \le \frac{1}{6} \sum_{i,j,k} |\Phi_{ijk}||d_i||d_j||d_k| \le \frac{1}{2} |d|^T M |d|$ with $M_{ij} = \frac{1}{3} \sum_k |\Phi_{ijk}| \rho_k$, and $\|M\|_2$ is at most the largest row sum of M, which is $\frac{66}{25}$; hence $|\mathcal{C}(d)| \le \frac{33}{25} \|d\|_2^2$. Hence $\Phi(\mathfrak{p}_* + d) \ge 0.9514(-d_1) + 0.1865\, d_3 + \frac{1}{2} d^T \mathcal{H} d$ with $\mathcal{H} = \nabla^2 \Phi(\mathfrak{p}_*) - \frac{66}{25} I$. The block $\mathcal{H}'$ of $\mathcal{H}$ in the (ξ, ζ) variables is positive definite (diagonal entry $23.079\ldots$, determinant $218.23\ldots$); minimizing $\frac{1}{2} d^T \mathcal{H} d$ over (d_ξ, d_ζ) leaves at least $\frac{1}{2}(d_1, d_3) \Xi (d_1, d_3)^T$ for the Schur complement Ξ of $\mathcal{H}'$ in $\mathcal{H}$, whose entries have modulus at most 7.5083. Therefore

$$\Phi(\mathfrak{p}_* + d) \ge (|d_1| + d_3)\Big(0.1865 - \frac{7.5083}{2} \cdot \frac{2}{50}\Big) \ge 0.0363\,(|d_1| + d_3),$$

and if $d_1 = d_3 = 0$ then $\Phi \ge \frac{1}{2}(d_\xi, d_\zeta) \mathcal{H}' (d_\xi, d_\zeta)^T$. So $\Phi \ge 0$ on $\mathcal{L}$, with equality only at $d = 0$. All constants here are enclosed with interval arithmetic in `local_lemma.py`.

Away from $\mathfrak{p}_*$. On $\mathcal{D}_* \setminus \mathcal{L}$ we show $\Phi > 0$ by a branch-and-bound computation in outward-rounded double-precision interval arithmetic [10]: every floating-point result is moved one unit in the last place outward, which encloses the exact result. The cube is enlarged to $|t_j - 1| \le \bar{\varpi}$ for a double $\bar{\varpi}$ just above $s_*/2$. On each box, Φ is bounded below by the larger of its natural interval extension and its mean-value form with an interval enclosure of $\nabla\Phi$, after fixing every coordinate along which the enclosure of the corresponding partial derivative has constant sign at the end of the box where Φ is smaller. Boxes that contain no point of the enlarged domain are discarded by interval tests of the linear constraints, and boxes inside the coordinate box that defines $\mathcal{L}$ are left to the previous step, since their points with $t_1 \le 1 + \varpi$ and $t_3 \ge 1 - \varpi$ lie in $\mathcal{L}$ and their other points lie outside $\mathcal{D}_*$; the other boxes are bisected until certified. The run certifies 24,284 boxes with a positive lower bound (the smallest is $3.7 \cdot 10^{-6}$), discards 7,458 boxes as disjoint from the domain, and assigns 37 boxes to $\mathcal{L}$; it is reproduced by `bb_face.py`. As a control, the same program with the cube enlarged to $|t_j - 1| \le 0.48$ finds a feasible point with $\Phi < 0$, as Theorem 6.1(3) requires. □

Proof of Theorem 6.1. Part (3) is Remark 6.7. For part (2), Lemma 6.6 gives the tie. If some $x \in \mathcal{M}(t_*)$ had $\Theta(x) < 0$, the strict form of Lemma 6.5 and the normalization after it would give a point of $\mathcal{D}_*$ (the trace vector t_* lies in its closed cube) with $\Phi < 0$, against Lemma 6.8. Hence $\min_{\mathcal{M}(t_*)} \operatorname{per} = \operatorname{per} \bar{x}_{t_*} = \operatorname{per} x_*$, and Theorem 6.4 and Proposition 4.7 show that the diagonal tuple and the product polynomial of x_* are minimizers in Questions 1.7 and 1.8. For part (1), let $\|t - \mathbf{1}\|_1 < s_*$, so $|t_j - 1| < \frac{s_*}{2} < \frac{1}{2}$ for all j by (6.1). If a matrix $x \ne \bar{x}_t$ in $\mathcal{M}(t)$ had $\Theta(x) \le 0$, Lemma 6.5 and the normalization after it would give a point of $\mathcal{D}_*$ with $\Phi \le 0$ whose trace coordinates are a permutation of t; by Lemma 6.8 it is $\mathfrak{p}_*$, whose trace vector has $\max_j |t_j - 1| = \frac{s_*}{2}$, a contradiction. So $\bar{x}_t$ is the unique permanent minimizer, and Theorem 6.4 and Proposition 4.7 transfer this to Questions 1.7 and 1.8. Part (2) shows that no radius larger than s_* is admissible in any of the three problems, and part (1) shows that s_* is. □

A Proof of Theorem 2.10

Throughout, $n = 3$ and $\|t - \mathbf{1}\|_1 \le 1/8$. Lemma 2.2 supplies

$$p = L^3 - L x^T B x + q, \qquad B \succeq 0, \quad B\mathbf{1} = 0, \tag{16}$$

with $L = L_t$ and q a cubic invariant under translation by $\mathbf{1}$. Each centered line has three real roots, so the discriminant of $z^3 - (x^T Bx)z + q(x)$ is nonnegative:

$$|q(x)| \le \frac{2}{3\sqrt{3}}(x^T Bx)^{3/2} \qquad (x \in \mathbb{R}^3). \tag{17}$$

If $d = 0$ then $p = L^3$; assume $d > 0$.

Two complementary bounds. Write $b_i = B_{ii}$ and $\Delta = \mathcal{A}(p) - \frac{2}{9}t_1t_2t_3$. Translation invariance and inclusion–exclusion give $\mathcal{A}(q) = 2\sum_i q(e_i)$, since $q(\mathbf{1} - e_i) = -q(e_i)$ and $q(\mathbf{1}) = q(0) = 0$. The row sums of B give $B_{jk} = (b_i - b_j - b_k)/2$ for distinct indices, hence

$$\Delta = d - \frac{2}{3}\sum_i t_i b_i + 2\sum_i q(e_i). \tag{18}$$

Nonnegative vertex coefficients give $0 \le p(e_i) = t_i^3/27 - t_i b_i/3 + q(e_i)$; substituting in (18) gives the outer bound

$$\Delta \ge d - \frac{2}{27}\sum_i t_i^3. \tag{19}$$

For the inner bound let $P = I - \mathbf{1}\mathbf{1}^T/3$ and $\eta = t - \mathbf{1}$. Since $B = PBP$, $b_i \le d\|Pe_i\|^2 = 2d/3$, and (17) gives $|\mathcal{A}(q)| \le \frac{4}{3\sqrt{3}}\sum_i b_i^{3/2} \le \frac{4\sqrt{2}}{9}d^{3/2}$. On $\mathbf{1}^\perp$ the eigenvalues of $P\,\mathrm{diag}(\eta)P$ are $\pm r$ with $r = \|\eta\|_2/\sqrt{6}$ (their sum is zero and the trace of the square is $\sum_i \eta_i^2/3$). Positivity of B gives $\sum_i t_i b_i \le (1 + r)d$, hence

$$\Delta \ge \Big(\frac{1}{3} - \frac{2r}{3} - \frac{4\sqrt{2}}{9}\sqrt{d}\Big)d. \tag{20}$$

Since $\sum_i \eta_i = 0$ and $\|\eta\|_1 \le 1/8$, the positive and negative masses are at most $1/16$, so $\max_i |\eta_i| \le 1/16$, $\sum_i \eta_i^2 \le 1/128$ and $r \le 1/(16\sqrt{3})$. Using $\sqrt{3} > 5/3$, for $d \le 15/64$ the bound (20) gives $\frac{\Delta}{d} \ge \frac{1}{3} - \frac{1}{24\sqrt{3}} - \frac{\sqrt{30}}{18} > \frac{37}{120} - \frac{\sqrt{30}}{18} > \frac{1}{256}$, the last comparison by subtracting $1/256$ and squaring. For $d \ge 15/64$: $|\sum_i \eta_i^3| \le 1/2048$, so $\sum_i t_i^3 = 3 + 3\sum_i \eta_i^2 + \sum_i \eta_i^3 \le 6193/2048$, and (19) gives $\frac{\Delta}{d} \ge 1 - \frac{6193/27648}{15/64} = \frac{287}{6480} > \frac{1}{256}$.

Coefficient distance. The quadratic form has coefficient norm at most $\sum_{i,j}|B_{ij}| \le 3d$, and multiplication by L does not increase it. For the cubic term put $a_i = q(e_i)$, let c_{ij} be the coefficient of $x_i^2 x_j$ and c the squarefree coefficient of q. Translation invariance gives $c = 2\sum_i a_i$ and $c_{ij} + c_{ji} = -\sum_i a_i$, and evaluation at $e_i - e_j$ gives $c_{ij} = -a_j - \frac{1}{2}a_k - \frac{1}{2}q(e_i - e_j)$ for distinct i, j, k. Hence $\|q\|_{\mathrm{coef},1} \le 6\sum_i |a_i| + \sum_{i<j}|q(e_i - e_j)|$, and (17) with $b_i \le 2d/3$ and $(e_i - e_j)^T B(e_i - e_j) \le 2d$ yields $\|q\|_{\mathrm{coef},1} \le (\frac{8\sqrt{2}}{3} + 4\sqrt{\frac{2}{3}})d^{3/2} < 8d^{3/2}$. Nonnegative coefficients give $\partial_i^2 p(\mathbf{1}) \ge 0$, so $d \le \|t\|_2^2/3 \le \frac{385}{384} < (\frac{129}{128})^2$. Therefore $\|p - L^3\|_{\mathrm{coef},1} \le 3d + 8d^{3/2} < (3 + 8 \cdot \frac{129}{128})d < 14d$, and with $\Delta \ge d/256$ this proves (9).

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Data and code availability. The verification scripts cited in the text are included with this article: `exact_witness.py` (exact algebra of Lemma 6.6 and Remark 6.7), `local_lemma.py` and `bb_face.py` (Lemma 6.8), `md_identity.py` (Lemma 6.2 and identity (6.4)) and `constants.py` (Example 4.6, Remarks 2.8 and 3.4, and the values after Proposition 4.5).

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