Uniform Resolvent Asymptotics for Growing Theta-Null Source Families

Scott Schubr
Luminary Oddities

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Abstract

Inverse Gram limits for a fixed collection of sources can lose their meaning when the collection grows and its leading matrix becomes ill-conditioned. We prove a relative matrix asymptotic for growing derivative-null source families in a reciprocal-theta realization of the Weil form. With $U = \log(1/\epsilon)$, the actual constrained inverse Gram approaches its explicit outer-Hardy/Cauchy leading matrix uniformly when the source count is $o(\log U)$; a fixed small multiple of $\log U$ is also admitted eventually. The estimates control every complex source combination in the full leading metric. The proof uses a positive inverse-factor kernel with a support gap, logarithmic plateau leverage, exact polar recovery and complete dual-leakage bounds. A separate displaced-wall argument gives the finite bound $2^{23}nI$ whenever $U \geq 64n^2$, and yields an explicit original-metric null-elimination estimate. A same-span perturbation construction shows how individually small source errors can produce an exponentially amplified relative Gram defect while preserving positivity, independence and the polar constraint. Together these results identify both a uniform limiting regime and the source control needed to extend it.

Keywords. constrained resolvents; inverse Gram matrices; Hardy spaces; theta functions; uniform asymptotics.
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1 From fixed families to a relative uniform limit

An inverse Gram matrix measures the joint response of an inverse operator to a collection of source vectors. A fixed-family asymptotic can describe every individual entry and still give poor control of a growing family: small errors may add coherently in a source combination whose leading energy is very small. The natural question is therefore whether the approximation remains valid relative to the entire leading matrix.

The fixed-family leading matrix of *Frequency-Source Resolvent Limits and Positive-Receiver Instability* [1] describes the positive constrained inverse at each fixed derivative order. We prove a uniform relative comparison for growing collections of its original sources, together with a separate explicit finite-parameter upper bound. Proposition 10.2 explains the distinction by constructing exponentially amplified perturbations inside the exact limiting source span. The main comparison retains the common source coefficients throughout the original-domain recovery and dual estimates.

We keep the original source and domain. For $u \geq 0$ put

$$\psi(u) = e^{u/2} \sum_{m \geq 1} (4r_m^2 - 6r_m)e^{-r_m}, \quad r_m = \pi m^2 e^{2u}, \qquad s^2 = \psi/\cosh(u/2),$$

and use its even theta continuation. Let $p = \sinh(u/2)$ and

$$X = \{h \in L^2(\mathbb{R}) : h \text{ odd}, \ \langle sp, h\rangle = 0\}.$$

Inner products are conjugate-linear first, with Fourier convention $\widehat{f}(\xi) = \int f(u)e^{-i\xi u}\,\mathrm{d}u$. The compact constrained core is completed for $\|h\|^2 + \mathfrak{a}[h]$, where

$$a(\xi) = \Re\frac{\Gamma'}{\Gamma}(1/4 + i\xi/2) - \frac{\Gamma'}{\Gamma}(1/4), \qquad \mathfrak{a}[h] = \int a(\xi)|\widehat{sh}|^2\frac{\mathrm{d}\xi}{2\pi}.$$

Write V for this minimal domain and A for its associated positive operator on X. For $i = 0, \dots, d$ set

$$\ell_i = 2i + 3, \quad z_i = (\psi^{(\ell_i)} - \psi^{(\ell_i - 2)}/4)/s, \quad b_i = \frac{(-2)^{\ell_i}}{2\pi^{1/4}}.$$

These are the complete original derivative-null rows [2, 1]; the present positive-reference theorem uses their actual formulas and exact polar moments. Put

$$\begin{aligned} n &= d + 2, & U &= \log(1/\epsilon), & R &= \tfrac{1}{2}\log(U/\pi), \\ D_U &= \operatorname{diag}(b_i U^{\ell_i + 1/4}), & G_U &= D_U^{-1} Z_d^*(A + e^{-U}I)^{-1} Z_d D_U^{-1}, & Z_d &= [z_0, \dots, z_d]. \end{aligned}$$

There are $n - 1$ sources; the extra count includes the limiting polar coordinate. For $\Re z > 0$ define the positive-normalized outer factor and leading matrix

$$O(z) = \exp\left\{\frac{1}{\pi}\int_0^\infty \frac{z\log a(t)}{z^2 + t^2}\,\mathrm{d}t\right\}, \quad B_p(z) = \frac{z - 1/2}{z + 1/2}, \quad \beta_i = 4i + 13/2,$$

$$G_\infty = 2\operatorname{diag}\left(\frac{B_p(\beta_i)}{O(\beta_i)}\right)\left[\frac{1}{\beta_i + \beta_j}\right]_{i,j=0}^{d}\operatorname{diag}\left(\frac{B_p(\beta_i)}{O(\beta_i)}\right). \tag{1}$$

Its positive definiteness and fixed-family interpretation are established in [1]; the next theorem controls changing families.

Theorem 1.1 (Uniform relative leading matrix)**.** *There are effectively computable absolute constants $C > 0$ and $U_0 \geq 128$ such that, for every $n \geq 2$, with*

$$H_n = 2\sum_{i=0}^{d}\beta_i^{-1}, \quad F_n = 2^{128n}, \quad e_n(U) = C\{\sqrt{H_n/R} + F_n U^{-1/128}\},$$

the admissions $U \geq U_0$, $F_n \leq U^{1/1024}$ and $e_n(U) \leq 1/100$ imply

$$(1 - 10e_n(U))G_\infty \preceq G_U \preceq (1 + 10e_n(U))G_\infty. \tag{2}$$

In particular $\|G_\infty^{-1/2} G_U G_\infty^{-1/2} - I\| \to 0$ uniformly along every $n = o(\log U)$. More generally, the comparison tends to zero uniformly for

$$2 \leq n \leq \frac{\log U}{131072\log 2}$$

as $U \to \infty$. The simpler schedule $n \leq 10^{-6}\log U$ is therefore sufficient eventually.

The effective constants are specified analytically; the sufficient schedule applies beyond the resulting absolute onset.

A separate finite-U result has a much larger admitted source range.

Theorem 1.2 (Complete thin-wall upper)**.** *Put* $L = 2n-1$ *and* $C_L = 2n^2 + 3n/2 + 1/4$. *For* $U \geq 64n$ *and* $T \in [U, 2U]$ *satisfying* $Te^{U-T} \leq 1$,

$$G_U \preceq 2^{21} n e^{2C_L/U} (T/U)^{2L+1/2} I. \tag{3}$$

In particular $T = U + 2\log U + 10$ *is admitted, and*

$$G_U \preceq 2^{23} nI, \qquad U \geq 64n^2. \tag{4}$$

Theorem 1.2 gives a literal finite-U upper on its larger admitted range, while Theorem 1.1 compares the full leading matrix on its stated relative range.

The scalar factorization, model-space, Fourier and variational tools used below are classical; see [3, 5, 6]. The contribution is their complete same-domain theta-source return with one paid exponential coefficient budget, and the finite-U wall improvement. The fixed-family limits and positive-receiver comparison of [1] are established inputs for the present uniform-source results.

2 The exact limiting geometry and original-domain inputs

The complete positive digamma series gives

$$\frac{16\xi^2}{1+4\xi^2} \leq a(\xi) \leq 17\xi^2, \qquad a(\xi) \leq 4 + \log(3|\xi|) \quad (|\xi| \geq 1). \tag{5}$$

The theta atoms give, for $r = \pi e^{2|u|}$,

$$2r^2 e^{-r} \leq s^2(u) \leq 16 r^2 e^{-r} \leq 169 e^{4|u|-r}. \tag{6}$$

The jump kernel of $a(D)$ is

$$k(t) = \frac{e^{-|t|/2}}{1-e^{-2|t|}}, \qquad \mathcal{E}(f) = \tfrac{1}{2} \iint k(x-y)|f(x)-f(y)|^2 \,\mathrm{d}x\,\mathrm{d}y.$$

Indeed finite positive digamma sums have this Fourier identity; monotone convergence proves it for the full kernel. Its second moment is finite, and $a \leq 17\xi^2$ bounds it by 34 in the corresponding normalization.

Let $\mathcal{H}_-$ be the energy completion of compact smooth negative-supported functions, zero-extended to $\mathbb{R}$. Their interaction with the positive half-line gives

$$\mathcal{E}(f) \geq 2 \int_{x<0} e^{x/2} |f(x)|^2 \,\mathrm{d}x. \tag{7}$$

Consequently this completion has a unique weighted representative and the Laplace moments $L_\beta(f) = \int_{-\infty}^0 e^{\beta x} f(x)\,\mathrm{d}x$ are continuous for $\beta > 1/4$. On $\ker L_{1/2}$ let v_i be the Riesz representative of L_{β_i}.

We use the exact outer-Hardy theorem of [1]: $Jf(z) = O(z) \int_{-\infty}^0 e^{zx} f(x)\,\mathrm{d}x$ maps this same completion unitarily onto half-plane H^2, with boundary measure $\mathrm{d}\xi/(2\pi)$. Its polar kernel is $B_p H^2$, and

$$Jv_i(z) = B_p(z) \frac{B_p(\beta_i)}{O(\beta_i)} \frac{1}{z+\beta_i}, \qquad [2\langle v_i, v_j\rangle_{\mathcal{E}}] = G_\infty. \tag{8}$$

The theorem uses the specified compact-core completion. Its proof establishes onto by shift invariance and outer cyclicity. The resulting literal Cauchy inverse gives, conservatively with $n = d+2$,

$$\lambda_{\min}(G_\infty) \geq \frac{36}{140777 n^2 (n+2)^2 2^{8(n+2)}}. \tag{9}$$

Here the first-pole comparison in (5) gives $O(\beta) \ge 2\beta/(\beta+1/2)$ and

$$0 < \frac{B_p(\beta_i)}{O(\beta_i)} \le \tfrac{1}{2}. \tag{10}$$

These precise predecessor inputs identify the limiting normalization.

For an original trial $h \in V$ put $f = sh$, $g_U(x) = f(R+x)$ on $x \ge -R$, zero elsewhere. Expanding the two odd arms first for finite digamma sums gives the exact identity

$$\mathcal{E}(f) = 2\mathcal{E}(g_U) + 2\sum_{j\ge 0} e^{-4q_j R}\left|\int e^{-2q_j x} g_U(x)\,\mathrm{d}x\right|^2, \quad q_j = j + 1/4. \tag{11}$$

The opposite signs of the two arms and negative convolution multipliers make the cross squares positive. Monotone convergence gives the full formula. The original polar equation is

$$L_+(g_U) = e^{-R} L_-(g_U), \qquad L_\pm(g) = \int e^{\pm x/2} g(x)\,\mathrm{d}x. \tag{12}$$

Thus the first square in (11) gives the global penalty $2e^R|L_+(g_U)|^2$.

We will also use the logarithmic-core description proved in [1]. Compactly supported $h \in X$ with sh of finite logarithmic Fourier energy belong to V. For completeness, $1+a$ is comparable to $1+\log(2+|\xi|)$. Weighted convolution bounds multiplication by smooth cutoffs uniformly, since $2+|x+y| \le (2+|x|)(2+|y|)$. Smooth cutoff and mollification approximate in that weighted norm and in original L^2; fixed compact smooth exact-moment corrections restore the polar row with coefficients tending to zero. This supplies the original minimal realization for the recovered tests below.

3 A positive inverse factor with an actual support gap

Lemma 3.1. *There are $0 < c \le 1/4$ and a positive measure μ supported on $[1/2,\infty)$ such that*

$$\frac{1}{O(z)} = \frac{c}{z} + \int_{[1/2,\infty)} \frac{\mu(\,\mathrm{d}t)}{z+t}, \qquad K(y) = c + \int e^{-ty}\mu(\,\mathrm{d}t), \quad y > 0. \tag{13}$$

Moreover $\int_0^1 K < 3$, $|K(y)-c| \le 3e^{-y/4}$ and $|K'(y)| \le 20e^{-y/4}$ for $y \ge 1$.

Proof. Set $\alpha_j = 2j + 1/2$ and

$$a_N(t) = t^2 \sum_{j=0}^{N} \frac{1}{q_j(t^2+\alpha_j^2)}, \qquad q_j = j + 1/4.$$

The rational Stieltjes function $a_N(\sqrt{z})/z$ has positive residues, negative derivative between its poles, and exactly one simple zero $-\rho_j^2$ between successive negative poles, with $\alpha_j < \rho_j < \alpha_{j+1}$. Degree counting excludes other zeros. If $A_N = \sum_{j=0}^N q_j^{-1}$, its positive outer factor is

$$O_N(z) = \sqrt{A_N}\, z \frac{\prod_{j<N}(z+\rho_j)}{\prod_{j\le N}(z+\alpha_j)}.$$

Strict interlacing gives positive residues in

$$O_N(z)^{-1} = b_N + c_N/z + \sum_{j<N} \mu_{N,j}/(z+\rho_j), \quad b_N = A_N^{-1/2}, \quad c_N = \left(\sum_{j\le N}(4q_j^3)^{-1}\right)^{-1/2}.$$

Every term is nonnegative on the positive axis. At $z = 1$ their sum is at most $1/O_0(1) = 3/4$. Thus the measures $\mu_{N,j}/(1 + \rho_j)$ have uniformly bounded mass. Compactifying at infinity and taking a weak limit retains possible escaping mass as a nonnegative constant. Poisson-log dominated convergence, between $\log a_0$ and $\log a$, gives $O_N(\beta) \to O(\beta)$ for each $\beta > 0$, and hence

$$O(z)^{-1} = b + c/z + \int_{[1/2,\infty)} (z + t)^{-1} \mu(\,\mathrm{d}t).$$

The support gap separates the zero-frequency pole from the measure term. The removable ratio $r(t) = a(t)(t^2 + 1/4)/(4t^2)$ is increasing and tends to infinity. In the actual Poisson-log factor, its correction over $t/\beta \in [1, 2]$ forces $O(\beta) \to \infty$, so $b = 0$. At zero the same ratio tends to $r_0 = 1 + \sum_{j\ge1}(4j + 1)^{-3}$; the Poisson approximate identity gives $c = 1/(4\sqrt{r_0}) \le 1/4$.

The inverse Laplace transform is the positive kernel in (13). Its Laplace value at one gives $\int_0^1 K \le e/O(1) < 3$. Split e^{-ty} into two factors and use $t \ge 1/2$, $\int \mu(\,\mathrm{d}t)/(1 + t) \le 3/4$, and $\sup_{t\ge1/2}(1 + t)e^{-t/2} < 3$ to obtain the tail bound. The same argument with one extra factor t gives the conservative derivative constant20. These estimates retain the complete symbol. General Stieltjes factors are classical [3]; theta-related interlacing factors and positive exponential mixtures appear in [4]. The finite interlacing argument also establishes the exact support gap needed here. □

Lemma 3.2 (Plateau leverage)**.** *If $v = \sum a_i v_i$ and $\mathcal{E}(v) = 1$, then its weighted representative satisfies*

$$v(-y) = K * q(y), \qquad A := \lim_{y\to\infty} v(-y) = c(Jv)(0), \qquad |A|^2 \le c^2 H_n.$$

Proof. Write $Jv = B_p k$ with k in the span of the kernels $(z+\beta_i)^{-1}$. Its norm is one. The Malmquist functions

$$e_j(z) = \frac{\sqrt{2\beta_j}}{z + \beta_j} \prod_{l<j} \frac{z - \beta_l}{z + \beta_l}$$

are orthonormal by Hardy evaluation and successive Blaschke isometries, and span that finite rational space. Since $|e_j(0)|^2 = 2/\beta_j$, Cauchy–Schwarz gives $|k(0)|^2 \le H_n$. This is evaluation on the finite kernel space just defined.

Let q be the inverse Laplace transform of Jv. Its finite exponential expansion and Lemma 3.1 show that $K * q$ has a constant limit plus an exponentially decaying remainder, and its Laplace transform is $(Jv)/O$. The weighted representative of the original v has exactly the same transforms by the outer isometry. Uniqueness after an exponential tilt identifies the two representatives. The constant part of K gives the stated limit. This identifies the original homogeneous representative. □

4 One exponential budget for every common source combination

In this and the next three sections $v = \sum a_i v_i$ is normalized by $\mathcal{E}(v) = 1$, so $a^* G_\infty a = 2$. Equation (9) yields

$$\sum |a_i| \le \frac{\sqrt{2 \cdot 140777}}{6} n^{3/2}(n + 2)2^{4(n+2)} \le 2^{16n}. \tag{14}$$

For the last bound use $n, n + 2 \le 2^n$ and $\sqrt{2 \cdot 140777}\, 2^8/6 < 2^{15}$; $6.5n + 15 \le 16n$ for $n \ge 2$. Put $d_i = B_p(\beta_i)/O(\beta_i)$, so $0 < d_i \le 1/2$. Then

$$Jv = B_p k, \quad k(z) = \sum_i \frac{a_i d_i}{z + \beta_i}, \quad k_0(y) = \sum_i a_i d_i e^{-\beta_i y}, \quad q(y) = k_0(y) - \int_0^y e^{-(y-t)/2} k_0(t)\,\mathrm{d}t.$$

The last identity is exact because $B_p(z) = 1 - (z + 1/2)^{-1}$. As $\beta_i \geq 13/2$ and $\beta_i < 4n$, direct convolution gives

$$|q(y)| \leq \tfrac{7}{12} \sum |a_i| e^{-y/2}, \qquad |q'(y)| \leq (2n+1) \sum |a_i| e^{-y/2}, \tag{15}$$
$$|(Jv)(i\xi)| \leq \frac{\sum |a_i|}{2\sqrt{1/4 + \xi^2}}.$$

The complete Riesz forcing coefficient is

$$\lambda_a = \sum a_i \lambda_i, \qquad \lambda_i = \frac{O(1/2)}{(\beta_i + 1/2) O(\beta_i)}, \qquad |\lambda_a| \leq \sum |a_i|.$$

Choose $S_n = 2^{32n}$. It pays the coefficient, q, q', Hardy boundary and polar quantities above. In particular S_n^2, $n^2 S_n$ and the products actually used below are bounded by an absolute constant times $F_n = 2^{128n}$. Constants denoted C_* below are absolute and effective; none conceals an n-dependent operator norm or compact approximation.

Convolving (15) with the gapped kernel gives, for $y \geq 2$,

$$|v(-y) - A| + |\partial_y v(-y)| \leq C_* S_n e^{-y/4}. \tag{16}$$

Indeed split the convolution with $K - c$ at its argument one; use its local integral below3 and its complete exponential tail above one. Differentiation uses the same bounds and q', together with the boundary term $(K - c)(y) q(0)$. The constant part is paid by the tail of $c \int_0^y q$.

The elementary Fourier integrals also give

$$\|Jv\|_{H^{1/4}} + \|a(D)v\|_{H^{1/4}} \leq C_* S_n. \tag{17}$$

For the second term, the squared Fourier modulus is $a(\xi)|(Jv)(-i\xi)|^2$. The quadratic zero of a removes the possible plateau pole. On $|\xi| \leq 1$ use $a \leq 17\xi^2$; outside use (5). Integrability reduces to the explicit integrals of $x^{-3/2}$ and $x^{-3/2} \log x$. Thus $\ell = a(D)v$ is a genuine full-line L^2 function, although v may have a nonzero plateau.

For an interval I of length $\eta \leq 1$, split Fourier frequencies at $1/\eta$. Cauchy–Schwarz bounds the low-frequency supremum squared by $C_* \eta^{-1/2} \|\ell\|_{H^{1/4}}^2$ and the high-frequency L^2 square by $\eta^{1/2} \|\ell\|_{H^{1/4}}^2$. Hence

$$\|1_I \ell\|_2 \leq C_* \eta^{1/4} S_n. \tag{18}$$

The estimate uses the fractional Fourier norm.

All complex conventions matter. If $e_a(x) = \sum a_i e^{\beta_i x}$, its functional is $\int \overline{e_a} g$, and on negative tests the exact identity is

$$\int \overline{e_a} g = \langle v, g \rangle_{\mathcal{E}} + \overline{\lambda_a} L_{1/2}(g). \tag{19}$$

In particular $\ell = e_a - \lambda_a e^{x/2}$ on the negative half-line, as an equality of distributions and locally L^2 representatives. This identity is kept before taking moduli.

5 Complete source profiles at the moving wall

Write $r = \pi e^{2u}$ on the positive arm. The exact derivative polynomials obey

$$P_0(r) = 4r^2 - 6r, \quad P_{k+1} = (1/2 - 2r) P_k + 2r P_k', \quad A_k = P_k / [4(-2)^k]. \tag{20}$$

Their recurrence is $A_{k+1} = (r - 1/4)A_k - rA_k'$. There is no constant coefficient, their lowest coefficient is $-(3/2)(-5/4)^k$, and for every $V \geq 1$

$$\sum_j |[r^j]A_k| V^{j-k-2} \leq e^{C_k/V}, \qquad C_k = k^2/2 + 7k/4 + 3/2. \tag{21}$$

Normalize $A_0(Vt)/V^2 = t^2 - 3t/(2V)$ and bound the next coefficient norm by the factor $1 + (k + 9/4)/V$; multiplication and $1 + x \leq e^x$ prove the formula.

The actual normalized divided profile on $x \geq -R$ is

$$w_{i,U}(x) = \frac{(z_i/s)(R + x)}{b_i U^{\ell_i + 1/4}}.$$

Its exact numerator is the complete atom sum of $P_{\ell_i} - P_{\ell_i - 2}/4$ and its denominator the complete positive atom sum of P_0, multiplied by $\cosh((R + x)/2)$. After normalization the numerator polynomial is $A_{\ell_i} - A_{\ell_i - 2}/16$. The complete atom sum and lower derivative are retained.

Let $\eta = U^{-1/8}$ and $L_0 = R/4$. The admission $F_n \leq U^{1/1024}$ implies $n \leq \log U$ and, at an absolute effective onset, all polynomial comparisons in this section. On $[-L_0, 0]$, $r \geq \pi^{1/4} U^{3/4}$. Equation (21) bounds the first-atom relative polynomial error by $6n^2/r$. The lower derivative costs $1/(16r^2)$ and the exact hyperbolic factor is $1 + \sqrt{\pi/r}$. For $r \geq 16(\ell_i + 2)$ the elementary inequality $\log m \leq (m^2 - 1)/4$ pays every higher atom by

$$\sum_{m \geq 2} m^{2\ell_i + 4} e^{-r(m^2 - 1)} \leq 2e^{-93r/32}.$$

Consecutive $m^2 - 1$ differ by at least5; the first term and a complete geometric tail give this bound. Applying it also to the denominator yields

$$|w_{i,U}(x) - e^{\beta_i x}| \leq 50 n^2 U^{-3/8} e^{\beta_i x}, \qquad -L_0 \leq x \leq 0. \tag{22}$$

On $[-R, -L_0]$ the complete global source envelope retains its individual order:

$$|w_{i,U}(x)| \leq K_i 2^{-\ell_i} e^{\beta_i x}, \qquad K_i = 1024(2\ell_i + 5)^{\ell_i}(\ell_i + 2)!.$$

For $\ell \geq 1$, the factorial bound gives

$$\log K_\ell \leq \ell\{10 \log 2 + 4 \log(2\ell + 5)\}.$$

For $n \leq \log U$, one absolute effective onset makes its brace at most $R/4$. Therefore the weighted-anchor pairing cost of each further-left source is bounded by

$$C_* K_i 2^{-\ell_i} e^{-(\beta_i - 1/4)R/4} \leq C_* e^{-\ell_i R/4 - R/16} \leq C_* (\pi/U)^{5/32}. \tag{23}$$

The coefficient sum is charged afterwards by (14). Taking the largest factorial constant first would incorrectly retain an unnecessary $e^{O(n \log n)}$ cost.

For reference, the order-normalized full-atom calculation in Lemma 8.1 below gives, whenever $r \geq \max(\pi, 2, \ell_i)$,

$$|w_{i,U}| \leq 13 e^{C_L/r} (r/U)^{\ell_i + 1/4}. \tag{24}$$

On $0 \leq x \leq \eta$, $r = Ue^{2x}$, so this is bounded by $13 \exp\{C_L/U + 2(L + 1/4)\eta\} \leq C_*$ on the admitted range. The resulting coefficient cost on that short interval is $C_* S_n$.

Put $T = Ue^{2\eta}$. The same full-atom calculation, before simplifying its exterior integral, gives

$$\int_{x\ge\eta} \frac{|w_{i,U}|^2}{V_U(x)}\,\mathrm{d}x \le C_* e^{2C_L/T}(T/U)^{2\ell_i+1/2}Te^{U-T}, \qquad V_U = e^{-U}/s(R+x)^2. \tag{25}$$

Both physical arms may be used to upper-bound this one arm. Its gamma-tail slope requirement is $T \ge 4(\ell_i + 1/4) + 2$, satisfied uniformly here. Since $T - U \ge 2U\eta$, $T \le 2U$, $n\eta \to 0$ and $n^2/U \to 0$, the square root of the right side is at most $C_*\sqrt{U}e^{-U\eta}$. The polar exponential has its separate complete potential-tail bound

$$\int_{x\ge\eta} \frac{e^x}{V_U(x)}\,\mathrm{d}x \le \frac{169}{\pi^2\sqrt{U}}T^{3/2}e^{U-T}. \tag{26}$$

Substitute $r = \pi e^{2u}$ in (6) and integrate its entire gamma tail. These estimates cover the entire unbounded exterior.

6 Uniform lower recovery in the original minimal domain

Let v have unit limiting energy as above, and retain $L_0 = R/4$ from Section5. Shift it left by $\eta = U^{-1/8}$. Equation (17) and $|e^{-i\eta\xi} - 1| \le 2\eta^{1/4}|\xi|^{1/4}$ pay its energy-norm change by $C_*S_n\eta^{1/4}$. Translation preserves its limiting polar zero.

Cut it off slowly at the far left: use a fixed smooth cutoff equal one for $y \le L_0$ and zero for $y \ge 2L_0$, with derivative bounded by C_*/L_0. The resulting profile g has support in $[-R/2, -\eta]$. A slowly changing plateau of amplitude A has derivative-square integral $O(|A|^2/L_0)$; the finite second moment of the jump kernel pays its full energy, even though the uncut plateau need not be in unweighted L^2. Equation (16) pays the cutoff remainder and its derivative. The total energy-norm error is therefore

$$C_*\sqrt{H_n/R} + C_*S_nU^{-1/32}. \tag{27}$$

This is the reason for retaining model-space plateau leverage rather than estimating its amplitude by a large residue sum.

Choose once $\phi(x) = c_\phi(x+2)^2(x+1)^2$ on $[-2,-1]$, zero elsewhere, with $L_{1/2}(\phi) = 1$. Its unnormalized moment is at least $e^{-1}/30$, so $c_\phi < 90$. Its H^1 norm and both exponential moments are fixed finite constants. Correct the original moment exactly by

$$g_U = g + t_U\phi, \qquad t_U = \frac{e^{-R}L_-(g) - L_+(g)}{1 - e^{-R}L_-(\phi)}. \tag{28}$$

The denominator is at least $1/2$ beyond one absolute onset. The cut polar error is at most $C_*S_ne^{-L_0/2}$. The convolution bound $\int_0^y K \le y/4 + 15$ and (15) give $|v(-y)| \le S_n(y+20)$, whence $|L_-(g)| \le C_*S_n(R+20)e^{L_0}$. Thus the correction costs at most $C_*S_nU^{-1/32}$ in energy norm after the same fixed-power weakening. It is exact for the full polar row.

This compact profile can have a finite-energy, non-H^1 right edge. The logarithmic-core lemma in Section2, applied after division by the smooth positive s on its compact support, puts its odd reflected original test in V. The variational argument uses this form-domain membership.

The two reflected arms are separated by at least R. Their complete interaction is at most $C_*e^{-R/2}\|g_U\|_1^2$, and $\|g_U\|_1 \le C_*S_n(R+20)^2$. On the whole recovered arm the first atom gives

$$e^{-U}/s^2 \le e^{r-U}/(2r^2).$$

Here $r \geq \pi > 2$, so this is largest at the right endpoint $r = Ue^{-2\eta}$. It is at most $C_* U^{-2} e^{-U\eta}$. Multiplication by the full $\|g_U\|_2^2 \leq C_* S_n^2 (R+20)^3$ pays the entire regularization. Both interaction and regularization are bounded by $C_* F_n U^{-1/128}$.

For the source pairing use (19). The recovered polar moment converts its last term to $\overline{\lambda_a} e^{-R} L_-(g_U)$, charged with at most S_n^2. The near source error is paid by (22) and the further-left return by (23), retaining each order's decay. Every coefficient product is bounded by S_n^2 or $n^2 S_n$ up to an absolute constant. Fixed powers $(R+20)^q$, $q \leq 4$, are absorbed by weakening $U^{-1/32}$ to $U^{-1/128}$. Enlarging one absolute constant gives the actual Hermitian two-arm pairing and trial energy

$$\Re \langle Z_d D_U^{-1} a, h_U \rangle \geq 2(1 - e_n), \qquad \mathfrak{a}[h_U] + e^{-U} \|h_U\|^2 \leq 2(1 + e_n)^2. \tag{29}$$

The same original variational quotient therefore proves, for every common coefficient combination normalized by $a^* G_\infty a = 2$,

$$a^* G_U a \geq 2 \left(\frac{1 - e_n}{1 + e_n} \right)^2. \tag{30}$$

The real part is taken only after keeping the full complex polar and source pairing. The estimate therefore applies to every common complex coefficient vector.

7 Uniform upper recovery through the complete dual leakage

Take any original trial with $\mathfrak{a}[h] + e^{-U} \|h\|^2 \leq 1$. Equations (11)–(12) give

$$\mathcal{E}(g_U) \leq 1/2, \quad \int V_U |g_U|^2 \leq 1/2, \quad |L_+(g_U)| \leq e^{-R/2}/\sqrt{2}. \tag{31}$$

The complete theta upper gives

$$V_U(x) \geq (\pi^2/169) U^{-2} e^{-4x} \exp\{U(e^{2x} - 1)\}.$$

Its logarithmic derivative is $-4 + 2Ue^{2x} > 0$ for $x \geq \eta$ at the admitted onset. In particular the trial mass on $[1,2]$ is uniformly small. The q_0 kernel $e^{-|x-y|/2}$ is below the full k. Restrict the jump energy to $x \leq 1$, $y \in [1,2]$ and use $|v - w|^2 \geq |v|^2/2 - |w|^2$. With $c_0 = \int_1^2 e^{-y/2} \, \mathrm{d}y > 0$ this gives

$$\mathcal{E}(g_U) \geq \frac{c_0}{2} \int_{x \leq 1} e^{x/2} |g_U|^2 \, \mathrm{d}x - 2e^{1/2} \int_1^2 e^{-y/2} |g_U(y)|^2 \, \mathrm{d}y.$$

Consequently

$$\int_{x \leq 1} e^{x/2} |g_U|^2 \, \mathrm{d}x \leq C_*. \tag{32}$$

This source-independent bound gives the weighted control required at the receding center.

Let $\ell = a(D)v$, the genuine full-line L^2 representative in (17). Testing the Riesz identity on negative compact functions and then using Fourier approximation gives for the actual arbitrary trial

$$\int_{x<0} \overline{e_a(x)} g_U(x) \, \mathrm{d}x = \langle v, g_U \rangle_{\mathcal{E}} + \overline{\lambda_a} L_+(g_U) - \int_{x>0} \{\overline{\ell(x)} + \overline{\lambda_a} e^{x/2}\} g_U(x) \, \mathrm{d}x. \tag{33}$$

The energy term has modulus at most $1/\sqrt{2}$. The total polar term costs at most $C_* S_n U^{-1/4}$ by (31).

On $[0, \eta]$, (18) and (32) pay the full leakage by $C_* S_n \eta^{1/4}$. The complete actual source bound (24) and the same anchor pay its common-coefficient contribution by $C_* S_n \eta^{1/2}$. The polar profile has the analogous payment.

On the whole unbounded exterior $x \geq \eta$, the increasing lower wall has inverse square root at most $C_* U e^{-U\eta}$. Thus the entire L^2 leakage norm pays its exterior pairing. The actual source and polar exponentials use their separate complete potential-tail estimates (25) and (26), with coefficient costs S_n and $|\lambda_a| \leq S_n$. All exterior source/polar mixed terms are thereby retained.

On the negative arm, (22) and (23) return the actual theta source with the same common coefficients. Combining the complete dual identity before taking a modulus gives

$$|\langle Z_d D_U^{-1} a, h\rangle| \leq \sqrt{2}(1 + e_n)$$

for every unit-energy original trial. Variational duality proves $a^* G_U a \leq 2(1 + e_n)^2$. Combine with (30) and use $e_n \leq 1/100$ to obtain Theorem 1.1.

7.1 Effective uniform admissions and their limits

The constants above depend only on the complete fixed symbol, its finite kernel second moment, the explicit inverse-kernel bounds, finite Fourier integrals, the source-independent anchor, and the single compact ϕ. Taking finite maxima and products produces one effective absolute C. Every other admission is an eventual elementary inequality after $n \leq \log U$: $n^2/U \to 0$, $nU^{-1/8} \to 0$, the source expansion range, $T \geq 4(\ell_i + 1/4) + 2$, the polar denominator and the absorption of fixed powers of R. The further-left inequality following (23) compares $O(\log\log U)$ with $R/4$. Each can be made monotone at an effective absolute threshold. Their maximum gives the absolute U_0.

If $n = o(\log U)$ then $\log F_n = o(\log U)$, $H_n = O(\log(n+1))$, and $e_n \to 0$. On the larger displayed range $n \leq \log U/(131072 \log 2)$ one has exactly $F_n \leq U^{1/1024}$ and

$$F_n U^{-1/128} \leq U^{-7/1024}, \qquad H_n/R = O(\log\log U/\log U).$$

Both error terms therefore tend to zero uniformly on that range, and all the remaining absolute-onset admissions apply. This also proves the simpler sufficient schedule stated in the theorem. The constants C, U_0 retain their analytically specified effective meaning.

The relative comparison applies under the displayed exponential coefficient admission. The separately proved finite-U upper retains its own larger source-count range.

8 Order-normalized atoms and the complete thin-wall receiver

We prove Theorem 1.2 independently of the limiting recovery argument.

Lemma 8.1 (Complete scaled atom bounds)**.** *For $r \geq \pi$, $V \geq \max(2, k)$, put*

$$S_k(r; V) = \sum_{m \geq 1} \frac{|A_k(rm^2)| e^{-r(m^2-1)}}{r^2 V^k}.$$

For $\pi \leq r \leq T = \alpha V$, $1 \leq \alpha \leq 2$, one has $S_k(r; V) \leq 3\alpha^k e^{C_k/V}$; for $r \geq \max(\pi, 2, k)$, $S_k(r; r) \leq 3e^{C_k/r}$.

Proof. For polynomial powers $j \geq 2$ put $p = j - 2 \in [0, k]$ and $d_j = |[r^j]A_k| V^{j-k-2}$. The first atom contributes $d_j (r/V)^p \leq \alpha^k d_j$. For $m \geq 2$ set $x = rm^2$. Since $m^2 - 1 \geq 3m^2/4$,

$$(r/V)^p m^{2j} e^{-r(m^2-1)} \leq m^4 (x/V)^p e^{-3x/4} \leq m^4 e^{-3\pi m^2/8}.$$

For $p \ge 1$, maximize $x^p e^{-3x/8}$ at $8p/3$ and use $p \le V$, $e > 8/3$; the case $p = 0$ is immediate. The complete remaining Gaussian series is less than $2/7$: its first term is below $1/4$ and successive ratios below $1/8$, using $\pi > 3$ and the positive Taylor bound $e^{9/2} > 64$. Equation (21) pays the sum in j.

The lowest coefficient $-(3/2)(-5/4)^k$ is handled separately. Its entire contribution is at most $(3/\pi)(5/(4V))^k < 1$, because $\sum_{m\ge1} m^2 e^{-\pi(m^2-1)} < 2$. Combining these terms gives the stated conservative constant3. This separates the negative power $j-2=-1$ instead of using an inapplicable positive-power bound. □

Let $T \in [U, 2U]$, $U \ge 64n$, $\alpha = T/U$, and $w = \frac{1}{2}\log(T/\pi)$. The exact normalized numerator $A_{\ell_i} - A_{\ell_i-2}/16$, the positive denominator $P_0(r)e^{-r} \ge 2r^2e^{-r}$, and the hyperbolic factor $1+\sqrt{\pi/r} \le 2$ give the complete core bound

$$|w_i(u)| \le 13e^{C_L/U}\alpha^{\ell_i}(r/U)^{1/4} = A_i e^{-(w-u)/2}, \quad 0 \le u \le w, \quad A_i = 13e^{C_L/U}\alpha^{\ell_i+1/4}. \tag{34}$$

The lower derivative has relative factor at most $1/(16\alpha^2U^2)$; $12(1 + 1/(16U^2)) < 13$. The full exterior estimate is (24), now for every $r \ge T$.

Put $\chi = 1_{[-w,w]}$, $I_i(u) = \int_0^u w_i$, $I_p(u) = 2(\cosh(u/2) - 1)$, and $c_i = I_i(w)/I_p(w)$. The compact forcing $f_i = \chi(w_i - c_i p)$ has integral zero on EACH half-line, so its whole primitive F_i vanishes outside $[-w, w]$ and at zero. If $H_w = (1-e^{-w/2})^{-2} \le 7$, direct exponential integration of (34) gives

$$\tfrac{1}{4}\|f_i\|^2 + \tfrac{1}{16}\|F_i\|^2 \le A_i^2\{1 + (3H_w^2 + 2H_w + 1)/2\} \le 82A_i^2. \tag{35}$$

For example $\|f_i\|^2 \le 4A_i^2(1 + H_w^2)$ and $\|F_i\|^2 \le 8A_i^2(1 + H_w)^2$, which prove the formula. The first digamma summand gives $1/a \le 1/4 + 1/(16\xi^2)$. Parseval with $\widehat{f_i} = i\xi\widehat{F_i}$ therefore pays the entire A-dual compact source by (35). The cutoff is a forcing factor in L^2; the form variational statement follows on the original minimal closure.

Let $e_i = z_i/(b_i U^{\ell_i+1/4})$. Its exact constrained decomposition is

$$e_i = P_X M_s f_i + P_X V_i, \qquad V_i = (1-\chi)(e_i - c_i sp).$$

The complete exterior polar term is present because $P_X(sp) = 0$ only for the whole polar source. Since $|c_i| \le 2H_w A_i e^{-w/2}$ and the complete theta tails give

$$\|1_{r>T} sp\|^2 \le 16T^{3/2}e^{-T}/\sqrt{\pi},$$

its squared-triangle payment is at most $128H_w^2A_i^2Te^{U-T}$. The complete source contribution follows from Lemma 8.1 and $s^2 \le 16r^2e^{-r}$:

$$e^U\|1_{r>T}e_i\|^2 \le 5408e^{2C_L/T}(T/U)^{2\ell_i+1/2}Te^{U-T}.$$

Both arms use $\mathrm{d}u_{\mathrm{total}} = \mathrm{d}r/r$; the full gamma tail is at most twice its first term because $T \ge 4(\ell_i + 1/4) + 2$. After the same squared triangle and division by A_i^2, the source term is at most $64e^{2C_L/T-2C_L/U}Te^{U-T}$. Thus $Te^{U-T} \le 1$ proves

$$e^U\|V_i\|^2 \le (64 + 128\cdot 49)A_i^2 = 6336A_i^2.$$

For a common complex coefficient vector combine the compact factors, their primitives and the exterior sources first. Cauchy–Schwarz over the finite columns and Minkowski in the inverse energy give

$$G_U \preceq (\sqrt{82} + \sqrt{6336})^2 \sum_i A_i^2 I \prec 2^{13}\sum_i A_i^2 I \preceq 2^{21} n e^{2C_L/U}\alpha^{2L+1/2} I.$$

Here $2^{13} \cdot 169 < 2^{21}$. This retains every source mixed entry and proves (3).

For $T = U + 2\log U + 10$ and $U \geq 128$, $\log U + 5 \leq \sqrt{U}$: it holds at128 since $\log 128 < 5$, $\sqrt{128} > 11$, and its difference has positive derivative thereafter. Hence $T \in [U, 2U]$ and $Te^{U-T} = Te^{-10}/U^2 < 1$. For $U \geq 64n^2$,

$$2C_L/U < 1/8, \qquad (2L + 1/2)\log(T/U) \leq 4n(2\log U + 10)/U \leq 8n/\sqrt{U} \leq 1.$$

Since $e^{9/8} < 4$, (4) follows. This completes Theorem 1.2.

9 Conditioning and exact-null elimination resources

The complete quadratic lower from [1] is retained on $U \geq 64n^2$:

$$\gamma_n I \preceq G_U, \qquad \gamma_n = \frac{2^{-67} h_n^4}{n^2(m+1)^2}, \quad h_n = \frac{2^{-26}}{(m+1)^2 75^{2m}}, \quad m = 4n + 3.$$

Together with Theorem 1.2 it gives

$$\|G_U\| \leq 2^{23} n, \quad \|G_U^{-1}\| \leq \gamma_n^{-1}, \quad \log \operatorname{cond}(G_U) = O(n), \qquad U \geq 64n^2.$$

The exponential inverse price follows directly from $\log \gamma_n^{-1} = 32(\log 75)n + O(\log n)$. On the relative range Theorem 1.1 instead transfers the exact leading-matrix condition budget with multiplicative factor $(1 + 10e_n)/(1 - 10e_n)$.

We give an original-metric application of the finite upper. Its algebra uses the complete operator-null identity for these rows [2].

Lemma 9.1 (Complete physical null Gram)**.** *For $N_d = Z_d^* Z_d$,*

$$N_d \succeq \beta_d I, \qquad \beta_d = 2^{-8d-20} \frac{3(d!)^2}{(d+1)(4d+3)^{2d+2}}.$$

Proof. The positive first theta atom gives

$$\int_{\mathbb{R}} \psi \geq \int_{\pi}^{\infty} (4r - 6)e^{-r}\, \mathrm{d}r = (4\pi - 2)e^{-\pi} > 3/8.$$

For example $\pi > 3$, $\pi < 22/7$, $e < 11/4$ and $e^{1/7} < 7/6$ imply $e^{\pi} < 25$, so the last expression exceeds $2/5$. Consecutive positive atom majorants have ratio at most $16e^{-3\pi} < 1/100$. Thus $\psi(u) \leq (400/99)r^2 e^{u/2-r}$ on the positive arm. Substitution of $r = \pi e^{2u}$, $\log^2(r/\pi) \leq r^2/\pi^2$, and $r^{13/4} \leq r^4$ for $r \geq \pi$ give

$$\int_{\mathbb{R}} u^2 \psi(u)\, \mathrm{d}u \leq \frac{2400}{99} \pi^{-9/4} < 3.$$

Consequently $\widehat{\psi}(\xi) > 1/4$ on $[0, 1/4]$ using $\cos t \geq 1 - t^2/2$. Also (6) gives $s^2 < 16$. For $z = \sum c_i z_i$,

$$\widehat{sz}(\xi) = -i\xi(\xi^2 + 1/4)\widehat{\psi}(\xi) \sum_i (-1)^i c_i \xi^{2i}.$$

Hence

$$\|z\|^2 \geq \frac{1}{4096\pi} \int_0^{1/4} \xi^2 \left| \sum_i (-1)^i c_i \xi^{2i} \right|^2 \mathrm{d}\xi.$$

For $H_d = [1/(2i + 2j + 3)]_{i,j=0}^{d}$, the exact Cauchy inverse bounds each inverse entry by $(4d + 3)^{2d+2}/[3(d!)^2]$: the denominator products are $2^{2d} i!(d-i)! j!(d-j)!$ and the remaining node sum is at least3; use the two binomial bounds $\binom{d}{i}, \binom{d}{j} \leq 2^d$. The row sum gives $\lambda_{\min}(H_d) \geq 3(d!)^2/[(d + 1)(4d + 3)^{2d+2}]$. Scale $\xi = x/4$ in the last integral and use $\pi < 4$ to obtain the stated floor. □

Let the complete signed form be $H = A - B$, B bounded selfadjoint, with $HZ_d = 0$. Set $C = A+\epsilon I$, $\mathcal{U} = C^{1/2}Z_d$, $\mathcal{R} = C^{-1/2}Z_d$, $S = Z_d^*CZ_d$ and $Q = I-\mathcal{U}S^{-1}\mathcal{U}^*$. For $L = C^{-1/2}(H+\epsilon I)C^{-1/2}$, operator nullity gives $L\mathcal{U} = \epsilon\mathcal{R}$. The leading null block is positive, and its exact Schur complement is

$$QLQ - \epsilon Q\mathcal{R}N_d^{-1}\mathcal{R}^*Q. \tag{36}$$

Its finite correction norm is

$$\delta_{\text{null}} = \epsilon\|N_d^{-1/2}(Z_d^*C^{-1}Z_d - N_dS^{-1}N_d)N_d^{-1/2}\|.$$

The matrix in parentheses is $\mathcal{R}^*Q\mathcal{R} \succeq 0$. Therefore the thin-wall upper and Lemma 9.1 prove the explicit payment

$$\delta_{\text{null}} \le \min\left\{1, e^{-U}C_{\text{wall}}\beta_d^{-1}\frac{2^{4d+4}}{\sqrt{\pi}}U^{4d+13/2}\right\}, \tag{37}$$

where C_{wall} is the scalar in (3). Every congruence uses the original physical Gram and the actual energy-normalized source map.

For $n \sim cU/\log U$ with fixed $0 < c < 1/4$, the thin wall has $\log C_{\text{wall}} = o(n)$, while Stirling gives $\log\beta_d^{-1} = (12\log 2 + 2)d + O(\log(d+2))$. Thus

$$\delta_{\text{null}} \le \exp\{-(1 - 4c + o(1))U\}.$$

At $c = 1/32$ the exponent is $7/8$. This is a paid finite correction in the full Schur expression (36); a signed application compares its complete complementary block against this matrix. The result quantifies the exact-null elimination resource supplied by the improved positive-reference upper.

10 Relative source stability and coherent perturbations

The relative metric in Theorem 1.1 controls every source combination at once. The following two statements explain the distinction between such control and a small error in each individual source. The first is classical Gram perturbation algebra; the second realizes its adverse direction inside the exact limiting source space.

Lemma 10.1 (A common relative feature bound)**.** *Let $F : \mathbb{C}^r \to \mathcal{K}$ be an injective map into a Hilbert space, $G = F^*F$, and let $\widetilde{F} : \mathbb{C}^r \to \mathcal{K}$. If*

$$\delta = \|(\widetilde{F} - F)G^{-1/2}\|,$$

then

$$\|G^{-1/2}(\widetilde{F}^*\widetilde{F} - G)G^{-1/2}\| \le 2\delta + \delta^2.$$

For $\delta < 1$ one also has the sharper lower comparison

$$(1-\delta)^2G \preceq \widetilde{F}^*\widetilde{F} \preceq (1+\delta)^2G.$$

Proof. The map $FG^{-1/2}$ is an isometry. Expand the perturbed Gram and bound the two mixed terms by δ and the quadratic term by δ^2. Alternatively, the triangle and reverse triangle inequalities bound $\|\widetilde{F}G^{-1/2}x\|$ between $(1-\delta)\|x\|$ and $(1+\delta)\|x\|$ when $\delta < 1$. Squaring proves the two Loewner comparisons. □

In the present limiting Hardy space, put

$$d_i = \frac{B_p(\beta_i)}{O(\beta_i)}, \qquad f_i(z) = \frac{d_i B_p(z)}{z+\beta_i}, \qquad Fa = \sum_{i=0}^{d} a_i f_i, \qquad G = F^*F = \tfrac{1}{2}G_\infty.$$

The columns are linearly independent and lie in the limiting polar kernel B_pH^2. Their exact norms are $\|f_i\| = d_i/\sqrt{2\beta_i}$.

Proposition 10.2 (Same-span columnwise perturbation)**.** *For every $\eta \geq 0$ there are linearly independent columns $\widetilde{f}_i \in \operatorname{Ran} F$ satisfying*

$$\|\widetilde{f}_i - f_i\| = \eta \|f_i\| \quad (0 \leq i \leq d)$$

whose positive definite Gram $\widetilde{G}$ obeys

$$\|G^{-1/2}(\widetilde{G} - G)G^{-1/2}\| \geq 2b + b^2, \qquad b \geq \sqrt{\frac{13}{8n}}\,\eta 2^d.$$

In particular, for $\eta = U^{-\alpha}$, $\alpha > 0$, and $d = c\log U + O(1)$ with $c > \alpha/\log 2$, this relative defect diverges.

Proof. The half-plane Laplace isometry and the isometry of multiplication by B_p give, on writing $t = e^{-4y}$,

$$\left\| e^{-13y/2} P(e^{-4y}) \right\|^2_{L^2(0,\infty)} = \frac{1}{4}\int_0^1 t^{9/4} |P(t)|^2 \,\mathrm{d}t.$$

Choose $P(t) = c_d(1-t)^d$, where $c_d = 2/\sqrt{\mathrm{B}(13/4, 2d+1)}$ and $\mathrm{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$. Its coefficients are $r_i = c_d(-1)^i\binom{d}{i}$. Set $a_i = r_i/d_i$ and $v = Fa$. The displayed identity gives $\|v\| = 1$. Since $\mathrm{B}(13/4, 2d+1) \leq 4/13$,

$$c_d \geq \sqrt{13}, \qquad \sum_i |r_i| = c_d 2^d.$$

Define the real numbers $s_i = (-1)^i\|f_i\|$ and put $\widetilde{f}_i = f_i + \eta s_i v$. Their individual errors have exactly the asserted norms, while

$$\widetilde{F}a = (1+b)v, \qquad b = \eta\sum_i \frac{|r_i|}{\sqrt{2\beta_i}} \geq \eta\frac{c_d 2^d}{\sqrt{8n}},$$

because $\beta_i < 4n$. The Rayleigh quotient at a is therefore $a^*(\widetilde{G} - G)a/(a^*Ga) = 2b + b^2$. Moreover $\widetilde{F} = F(I + \eta a s^*)$ and the rank-one determinant formula gives $\det(I + \eta a s^*) = 1 + b > 0$. This proves independence as well as preservation of the finite span and polar condition. Finally, the lower bound for b is a positive constant times $U^{c\log 2 - \alpha}/\sqrt{\log U}$ in the stated growing-degree regime. □

Thus a separate-column error guarantee has an exponential worst-case price even within the exact source span. The construction concerns perturbations of the limiting features. The actual finite-U comparison in Theorem 1.1 instead retains the complete theta source map, the original polar correction and both reflected arms. Its common-coefficient proof pays these structures in the leading metric before taking absolute values. A larger source-count range would require a stronger estimate for that structured map, rather than a replacement of the leading metric by individual column norms.

11 Reproducible finite controls

Three compact reader programs accompany the analytical proofs. Exact rational recurrences independently reconstruct the derivative polynomials and their normalization. Complete scalar Gaussian tails and directed Arb computations verify the atom and wall constants, including noninteger regularization parameters. Exact full Cauchy inverse calculations and common-source exponential estimates corroborate the limiting normalization and dimension budgets.

The main relative theorem gives an effective absolute recovery construction with eventual admissions. Its constants C, U_0 are specified through finite fixed-symbol bounds and elementary monotone inequalities; their numerical evaluation is a further quantitative task. The reader reports specify their concrete finite ranges and exact code identities. Build instructions and dependency versions appear in the accompanying reproduction guide.

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