Uniform permanent-derivative compensation on the conjugate-disjoint Gram locus

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Luminary Oddities

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Abstract

We prove a uniform lower bound for a compensated homogeneous permanent derivative on the complete conjugate-disjoint locus of centered complex positive semidefinite matrices of order six. The locus has rank at most two, and the theorem includes arbitrary phases, nonuniform diagonals, unequal eigenvalues and singular limits. If r is the trace, p the product of the two possible nonzero eigenvalues and δ the ordinary omission variance, the lower bound is $[1248-(1672/3)\sqrt{2}]r^4+[29568-(33920/3)\sqrt{2}]pr^2+(20160-80\sqrt{2})\delta$. All three coefficients are positive. The proof extends the earlier rank-one cancellation identity before imposing spectral relations, then uses an exterior-square Gram identity to pay the coupled remainder with the same variance reserve. Combined with an earlier conditioning-free projection and explicit Lipschitz estimate, this gives a uniform small-overlap region in the full centered positive semidefinite section.

1 Statement and relation to earlier work

All matrices have order six. Let $J=\mathbf{1}\mathbf{1}^*$ and $W=6I-J$. Positive semidefinite means Hermitian positive semidefinite, a bar means entrywise conjugation, and $\circ$ denotes entrywise multiplication. A matrix K is centered if $K\mathbf{1}=0$. Define

$$f_K(v)=\operatorname{per}(K+vJ),\qquad \mathcal{P}(K)=f_K(1)+f_K'(1),\qquad D(K)=\mathcal{P}(K)^2-\operatorname{per}|K+\sqrt{2}J|^2,$$

where the absolute square is entrywise, and set

$$E_j(K)=[x^j]\,\partial_v D(xK+vW)|_{v=0}\,,\qquad F_4(K)=E_4(K)+\frac{\operatorname{tr}K}{2}E_3(K). \tag{1}$$

The perturbation vW is additive after xK. Inserting an extra x into this direction would define a different coefficient. The functional F_4 is homogeneous of degree four over nonnegative real scalars. Put

$$\alpha_j(K)=[x^j]\operatorname{per}(J+xK),\qquad \delta(K)=\partial_v\alpha_5(K+vW)|_{v=0}\,.$$

The variance representation in Section 2 gives $\delta(K)\geq 0$ for centered positive semidefinite K.

The expression $f(1)+f'(1)$ occurs in the border identity of Li–Zhao–Luo [5, Proposition3.2(iii)]. Their complete permanent comparison is different from the signed homogeneous derivative in (1); coefficientwise positivity does not follow from positivity of a complete polynomial. The earlier work [2] classified zeros of δ and proved compensation on a flat equal-spectrum stratum. Its projection theorem reduced a uniform small-overlap conclusion to positivity on $K\overline{K}=0$. The subsequent rank-one theorem [1] proved the boundary estimate by cancelling a signed cubic contraction against ten

times the same ordinary variance. The remaining nonflat, unequal-spectrum rank-two interior was not covered there.

Here the cancellation is shown to hold on the entire exact locus. The new exterior-square payment controls both departures from rank-one geometry: $A^2 \neq A$ and $B \neq tt^T$, where $A = K+\overline{K}$, $B = |K|^2$ and $t = \operatorname{diag} K$. These terms must come from the same Gram factors. Independent worst-case choices would lose the cancellation.

Theorem 1.1 (Uniform locus compensation)**.** *Let $K \succeq 0$ be centered and satisfy $K\overline{K} = 0$. Write $r = \operatorname{tr} K$. Its rank is at most two; let p be the product of the two possible nonzero eigenvalues, with $p = 0$ in rank at most one. Then*

$$F_4(K) \geq c_0 r^4 + c_1 p r^2 + c_2 \delta(K), \tag{2}$$

where

$$c_0 = 1248 - \frac{1672}{3}\sqrt{2} > 459, \qquad c_1 = 29568 - \frac{33920}{3}\sqrt{2} > 0, \qquad c_2 = 20160 - 80\sqrt{2} > 0.$$

The rank assertion follows because the orthogonal ranges of K and $\overline{K}$ lie in the same five-dimensional complex space $\mathbf{1}^\perp$. Zero rows and arbitrary complex phases are included. The zero matrix follows directly from homogeneity. For rank one, (2), including its nonnegative variance term, is the estimate in the proof of [1, Theorem1.1]. We prove the genuine rank-two case below after normalizing $r = 1$.

This is a theorem about the auxiliary derivative F_4. The original inequality on some two-direction families was already covered by occupation arguments [3]; our new conclusion is uniform over the complete centered conjugate-disjoint locus.

2 The actual ordinary coefficient and its variance

Sections 2–5 work at trace one unless stated otherwise; the general-trace conclusion follows by homogeneity. Use the real invariants

$$t = \operatorname{diag} K, \quad d_j = \sum_i t_i^j, \quad \beta_j = \operatorname{tr}(K^j), \quad S_j = \sum_{i,k} K_{ik}^j, \quad S_{21} = \sum_{i,k} K_{ik}|K_{ik}|^2.$$

Ordinary matrix powers and entrywise powers are distinguished throughout. On the null locus set

$$b = \operatorname{diag}(K^2), \quad e = \operatorname{diag}(K^3), \quad U = t^T b, \quad X = \operatorname{Re}\sum_{i,k}(K^2)_{ik}K_{ik}^2, \quad C = \operatorname{Re}\sum_{i,k} t_i K_{ik}^3.$$

The real part in C is necessary; its unprojected sum need not be real.

Choose centered Gram linear forms g_i with Gram matrix K. In the Fischer inner product, $\|z^\gamma\|^2 = \gamma!$, and products of linear forms pair by the permanent of their Gram matrix; see [4]. Let $p_j = \sum_i g_i^j$. The omitted fourth elementary products differ by a common term from

$$\xi_i = g_i^4 - \tfrac{1}{2} g_i^2 p_2 - \tfrac{1}{3} g_i p_3.$$

Differentiating α_5 contracts their Gram matrix with W. The common term vanishes because $W\mathbf{1} = 0$, so

$$\delta = 6\sum_i \|\xi_i\|^2 - \left\|\sum_i \xi_i\right\|^2 = \sum_{i<j} \|\xi_i - \xi_j\|^2 \geq 0. \tag{3}$$

Since $\|p_2\|^2 = 2\,\mathrm{tr}(K\overline{K})$, null overlap gives $p_2 = 0$. Expanding (3) then yields

$$\delta = 144d_4 + 4S_3 + 12X - 96C - 24S_4. \tag{4}$$

Indeed $\|g_i^4\|^2 = 24t_i^4$, $\langle g_i^4, g_i p_3\rangle = 24t_i \sum_k K_{ik}^3$, $\sum_i \|g_i p_3\|^2 = 6S_3 + 18X$, and $\|p_4\|^2 = 24S_4$. The coefficients six and eighteen count the matchings with the singled-out factors paired together or separately.

Let T_4 be the actual paired derivative

$$T_4 = [x^4]\,\partial_v \,\mathrm{per}\,|\sqrt{2}J + xK + vW|^2\Big|_{v=0}.$$

The ordinary coefficient identities of [2, Section2 and AppendixA], with their normalization recalled in Appendix C, give, on the trace-one null locus,

$$\begin{aligned} F_4 &= O - T_4, \\ O &= 20160\delta + 1067904d_3 - 157824S_3 + 2880S_{21} + 16128U - 4416\beta_2 - 384\beta_3. \end{aligned} \tag{5}$$

The full ordinary variance is retained in this identity.

3 Cancellation before spectral specialization

Put $A = K + \overline{K}$ and $B = K \circ \overline{K}$. Then A is real symmetric, positive semidefinite and centered. At a null-locus basepoint,

$$B\mathbf{1} = b, \quad q := (A \circ A)\mathbf{1} = 2b, \quad \mathrm{diag}(A^3) = 2e, \quad \mathrm{tr}(A^j) = 2\beta_j. \tag{6}$$

Define

$$c = (A \circ A \circ A)\mathbf{1}, \quad h = (A \circ B)\mathbf{1}, \quad s_j = \sum_{i,k} A_{ik}^j, \quad H = \sum_{i,k} A_{ik}B_{ik}.$$

The identities used below first hold for general centered real A and symmetric B; B need not be centered. With $b_B = B\mathbf{1}$, define

$$V_4 = [x^3y]\alpha_4(xA + yB), \qquad V_3 = [xy^2]\alpha_3(xA + yB).$$

The universal mixed formulas from [1, Section3] are

$$\begin{aligned} V_4 = {} & 12\sum A_{ik}^3 B_{ik} - 8c^T b_B - 12q^T h + \tfrac{4}{3}s_3 \sum B_{ik} \\ & + 4q^T A b_B + s_2 H + 2\sum B_{ik}(A^3)_{ik}, \end{aligned} \tag{7}$$

$$V_3 = 12\sum A_{ik}B_{ik}^2 - 24h^T b_B + 6H\sum B_{ik} + 6b_B^T A b_B. \tag{8}$$

For completeness, the partition proof and real-Gram contractions needed to differentiate these identities are provided in Appendix A.

The degree-five homogeneous part of the paired permanent is $8\sqrt{2}[\alpha_5(A) + V_4 + V_3]$. Its three slot patterns are five A slots, three A and one B, or one A and two B. Under the actual perturbation $K + vW$,

$$\dot{A} = 12I - 2J, \qquad \dot{B} = 12\,\mathrm{diag}(t) - A, \qquad \frac{T_4}{8\sqrt{2}} = 2\delta(A) + \dot{V}_4 + \dot{V}_3. \tag{9}$$

Differentiation is performed *before* imposing (6). In particular, the derivative of $B\mathbf{1}$ must not be replaced by the derivative of $\mathrm{diag}(K^2)$ along a perturbation leaving the locus.

Define the additional real contractions

$$Y = \mathrm{Re}\sum_{i,k} \overline{(K^2)_{ik}} K_{ik}^2, \qquad M = \sum_{i,k} B_{ik}(A^2)_{ik}, \qquad Q = \mathrm{Re}\sum_{i,k} K_{ik}^2 B_{ik}.$$

The letter Q here denotes a scalar contraction, not a projected matrix.

Proposition 3.1 (Universal variance cancellation)**.** *On the trace-one null locus,* $T_4/(8\sqrt{2}) = 10\delta + R$*, where*

$$\begin{aligned} R = {} & 6768 d_4 - 5184 (t^{\circ 2})^T b + 696 \|b\|^2 + 576 \beta_2 d_2 + 240 t^T e \\ & - 2304 t^T h + 624 t^T A b + 24 S_3 + 432 S_{21} + 24 X + 144 Y + 216 M \\ & + 120 S_4 - 672 Q - 576 \|B\|_{\mathrm{F}}^2 - 92 \beta_2^2 - 20 \beta_4. \end{aligned} \tag{10}$$

Proof. The full derivative table in Appendix B, together with the unrestricted real-Gram contraction in Appendix A, gives

$$\begin{aligned} \frac{T_4}{8\sqrt{2}} = {} & 8208 d_4 - 5184 (t^{\circ 2})^T b + 696 \|b\|^2 + 576 \beta_2 d_2 + 240 t^T e \\ & - 480 t^T c - 864 t^T h + 624 t^T A b + 32 s_3 + 72 N - 60 s_4 \\ & - 92 \beta_2^2 - 20 \beta_4 + 120 H + 72 M - 96 G - 24 \|B\|_{\mathrm{F}}^2, \end{aligned}$$

where $N = \sum A_{ik}^2 (A^2)_{ik}$ and $G = \sum A_{ik}^2 B_{ik}$. The entrywise binomial identities are

$$\begin{gathered} t^T c = 2C + 3 t^T h, \qquad s_3 = 2 S_3 + 6 S_{21}, \qquad H = 2 S_{21}, \\ N = 2X + 2Y + 2M, \quad G = 2Q + 2\|B\|_{\mathrm{F}}^2, \quad s_4 = 2 S_4 + 8 Q + 6 \|B\|_{\mathrm{F}}^2. \end{gathered}$$

Thus $-480 t^T c - 864 t^T h = -960 C - 2304 t^T h$. Subtracting ten times (4) cancels C exactly and gives (10). No rank-one projection relation is used. □

4 Exterior-square payment of the full remainder

Let $K = \lambda u u^* + \mu v v^*$, with orthonormal eigenvectors, $\lambda \geq \mu > 0$ and $\lambda + \mu = 1$. Null overlap implies $u^T u = v^T v = u^T v = 0$. Let P be the real support projection of A and put $\rho = \mathrm{diag}(P)/2 = |u|^2 + |v|^2$. Then

$$\begin{gathered} A \preceq \lambda (I - J/6), \quad A - A^2 = pP, \quad b = t - p\rho, \quad e = (1-p)t - p\rho, \\ \beta_2 = 1 - 2p, \quad \beta_3 = 1 - 3p, \quad \beta_4 = 1 - 4p + 2p^2, \qquad p = \lambda\mu. \end{gathered} \tag{11}$$

Indeed the eigenvalues of A on its four-dimensional support are $\lambda, \lambda, \mu, \mu$. Since $P \preceq I - J/6$,

$$0 \leq t_i \leq \lambda \rho_i \leq \rho_i \leq 5/12, \quad \sum_i \rho_i = 2, \quad \|\rho - \tfrac{1}{3}\mathbf{1}\|^2 \leq 1/6. \tag{12}$$

The classical two-by-two Gram determinant identity gives the exact exterior-square correction

$$B = t t^T - pL, \qquad L_{ik} = |u_i v_k - v_i u_k|^2, \qquad L\mathbf{1} = \rho. \tag{13}$$

Here L is symmetric and entrywise nonnegative. In particular,

$$t^T L t \leq \sum_i t_i^2 \rho_i. \tag{14}$$

This follows by multiplying $2 t_i t_k \leq t_i^2 + t_k^2$ by L_{ik}.

Lemma 4.1. *On the normalized rank-two locus, with $G(s) = -2820s^2 + 5376s - 748$, the actual remainder satisfies*

$$R \le G(d_2) + \frac{4240}{3}p.$$

Proof. The Schur product theorem and positive semidefinite trace pairings give

$$0 \le X \le S_3, \qquad 0 \le Y \le S_{21}, \qquad 0 \le M \le 2S_{21}, \qquad Q \ge 0. \tag{15}$$

For example, $X = \operatorname{tr}(\overline{K}^2(K \circ K))$ and $Y = \operatorname{tr}(K^2(K \circ K))$, while $K^2 \preceq K$. Also $M = \operatorname{tr}(BA^2) \le \operatorname{tr}(BA) = 2S_{21}$ since $A^2 \preceq A$. Finally $Q = \mathbf{1}^*(K \circ K \circ B)\mathbf{1} \ge 0$. The triangle inequality gives $S_4 \le \|B\|_{\mathrm{F}}^2$. Therefore the positive X, Y, M terms in (10) leave at most $48S_3 + 1008S_{21}$, and its S_4, B terms leave at most $-456\|B\|_{\mathrm{F}}^2$.

Write $v_0 = d_2 - 1/6$. Positive semidefinite trace comparisons imply

$$S_3 \le \lambda d_2 \le d_2, \qquad S_{21} = \tfrac{1}{2}\operatorname{tr}(AB) \le \tfrac{\lambda}{2}(d_2 - \beta_2/6) = \tfrac{\lambda}{2}v_0 + \tfrac{\lambda}{6}p. \tag{16}$$

For the first, pair $\overline{K} \preceq \lambda I$ with $K \circ K$. For the second, pair $A \preceq \lambda(I - J/6)$ with B and use $\operatorname{tr} B = d_2$, $\mathbf{1}^T B\mathbf{1} = \beta_2$.

Since $A\mathbf{1} = 0$, (12) and the operator norm bound give

$$t^T At \le \lambda v_0, \quad |(t^{\circ 2})^T At| \le \tfrac{7}{12}\lambda v_0, \quad |t^T A\rho| \le \lambda\sqrt{v_0/6}. \tag{17}$$

For the middle inequality subtract $1/36$ from t_i^2 and $1/6$ from t_i, then use $|t_i^2 - 1/36| \le (7/12)|t_i - 1/6|$. The last uses (12) after subtracting $\mathbf{1}/3$ from ρ.

Set $w = (A \circ L)\mathbf{1}$. Positive semidefiniteness of A implies $A_{ik} \le 2\sqrt{t_i t_k} \le t_i + t_k$, and hence

$$t^T w \le \sum_i t_i^2\rho_i + t^T Lt \le 2\sum_i t_i^2\rho_i. \tag{18}$$

The identity (13) gives $h = t \circ (At) - pw$ and $\|B\|_{\mathrm{F}}^2 = d_2^2 - 2pt^T Lt + p^2\|L\|_{\mathrm{F}}^2$. Substitute these and (11) into (10), then apply (15)–(18). Collecting the complete p-dependent diagonal group before bounding gives

$$\begin{aligned} &10704\sum_i t_i^2\rho_i - 1632t^T\rho - 1392d_2 \le 1436d_2, \\ &(696\|\rho\|^2 - 408)p^2 \le 172p^2. \end{aligned}$$

The first uses $\rho_i \le 5/12$ in the positive term and $\rho_i \ge t_i$ in the negative term. The second uses $\|\rho\|^2 \le 5/6$. All remaining terms give

$$\begin{aligned} R \le\ & 6768d_4 - 5184d_3 + 1560d_2 - 456d_2^2 - 112 + 2472\lambda v_0 \\ &+ p(1436d_2 + 168\lambda + 448 + 624\lambda\sqrt{v_0/6}) + 172p^2. \end{aligned} \tag{19}$$

Here $2472 = 1344 + 624 + 504$ comes from the three centered terms in (17) and (16). The $168\lambda p$ term retains the β_2 correction in S_{21}.

Use $d_4 \le 5d_3/12$, $d_3 \ge d_2^2$, and $2472\lambda \le 3816$. These turn the first line of (19) into at most $G(d_2)$. Finally $d_2 \le 5/12$, $v_0 \le 1/4$, $p \le 1/4$ and $\lambda \le 1$ give

$$1436d_2 + 168\lambda + 448 + 624\lambda\sqrt{v_0/6} + 172p \le 1436\tfrac{5}{12} + 168 + 448 + 156 + 43 = \tfrac{4240}{3}.$$

This proves the lemma without separating the coupled Gram variables. □

5 Completion of the uniform bound

From (11) and (12), $U = t^T b = d_2 - pt^T\rho \geq d_2 - 5p/12$. In (5), discard $S_{21} \geq 0$, use $S_3 \leq \lambda d_2$, and use $d_3 \geq d_2^2$. With $f(s) = 1067904s^2 - 141696s - 4800$, this yields

$$\begin{aligned} O &\geq 20160\delta + f(d_2) + 157824\mu d_2 + 3264p \\ &\geq 20160\delta + f(d_2) + 29568p, \end{aligned}$$

where the last step uses $d_2 \geq 1/6$ and $\mu \geq p$. Combining Proposition 3.1 and Lemma 4.1 now gives

$$F_4 \geq c_2\delta + [f(d_2) - 8\sqrt{2}G(d_2)] + c_1 p.$$

The derivative of $f(s) - 8\sqrt{2}G(s)$ is increasing and at $s = 1/6$ equals $214272 - 35488\sqrt{2} > 0$. Its minimum for $s \geq 1/6$ therefore occurs at $1/6$, where $f = 1248$ and $G = 209/3$. This proves (2) at trace one. Homogeneity gives the general trace statement: δ has degree four and the spectral product p has degree two. The rational inequality $\sqrt{2} < 99/70$ proves $c_0 > 459$; the other coefficients are plainly positive. All estimates allow zero coordinates and arbitrarily small positive μ. The rank-one endpoint is already included as noted following Theorem 1.1.

6 A uniform small-overlap consequence

Let $\mathcal{D} = \{K \succeq 0 : K\mathbf{1} = 0,\ \operatorname{tr} K = 1\}$ and $\mathcal{Z} = \{K \in \mathcal{D} : K\overline{K} = 0\}$. This is a nonlinear locus, not a convex face of the positive semidefinite cone. The inherited estimates [2, Lemma5.1 and Theorem6.1] are

$$|F_4(K) - F_4(H)| \leq C_0\|K - H\|_{\mathrm{F}}, \qquad C_0 = 3190000000, \quad K, H \in \mathcal{D}, \tag{20}$$

$$Q_0 = \frac{(K - \overline{K})_+}{\operatorname{tr}((K - \overline{K})_+)} \in \mathcal{Z}, \qquad \|K - Q_0\|_{\mathrm{F}} \leq \sqrt{6\eta}, \qquad \eta = \operatorname{tr}(K\overline{K}) < 1/5. \tag{21}$$

The strict threshold in (21) ensures a nonzero denominator. Neither estimate assumes an inverse, a spectral gap or a condition-number bound.

Corollary 6.1. *For every $K \in \mathcal{D}$ with $\eta < 1/5$,*

$$F_4(K) \geq c_0 - C_0\sqrt{6\eta}.$$

In particular $F_4(K) > 0$ whenever

$$\eta < \frac{459^2}{6C_0^2}.$$

The latter endpoint is strict.

Proof. Apply Theorem 1.1 to Q_0 in (21), then use (20). The displayed conservative threshold is less than $1/5$ and $c_0 > 459$. □

The corollary imposes no circulant, repeated-spectrum or phase restriction on K. It includes singular matrices and centered rank-five matrices. Positive anchors $K + sJ$, $s > 0$, have rank six when K has centered rank five; the asserted derivative is still the functional of the centered residual K. Moderate overlap, the remaining homogeneous blocks and the full Chollet inequality require additional arguments.

7 Why coefficientwise spectral positivity is unnecessary

The proof pays a coupled polynomial; it does not require every mixed eigenweight coefficient to be nonnegative. The following example, implicit in the two-direction formula of [1], makes this distinction explicit.

Proposition 7.1. *Let $\omega = (-1+\sqrt{-3})/2$, and let U, V be the trace-one orthogonal projectors onto $(1, \omega, \omega^2, 0, 0, 0)$ and $(0, 0, 0, 1, \omega, \omega^2)$ respectively. Then*

$$\begin{aligned} F_4(\lambda U + \mu V) = (66624 - 448\sqrt{2}/3)(\lambda^4 + \mu^4) \\ + (89088 - 512\sqrt{2}/3)(\lambda^3\mu + \lambda\mu^3) \\ - (8832 + 1600\sqrt{2})\lambda^2\mu^2. \end{aligned}$$

Thus the middle mixed coefficient is negative even on actual centered positive semidefinite null-locus matrices.

Proof. In the Gram forms $(u, \omega u, \omega^2 u, v, \omega v, \omega^2 v)$, take orthogonal u, v of squared norms $\lambda/3, \mu/3$ and substitute into the exact trine polynomial in [1, Section5]. Equivalently, with $r = \lambda + \mu$ and $p = \lambda\mu$, the result is

$$(66624 - 448\sqrt{2}/3)r^4 + (-177408 + 1280\sqrt{2}/3)pr^2 + (-53760 - 4672\sqrt{2}/3)p^2.$$

Expanding r and p gives the displayed coefficients. The two projector ranges and their conjugates are mutually orthogonal, so the matrices satisfy every stated constraint for $\lambda, \mu \geq 0$. □

A Partition formulas and the real-Gram contraction

For ordered residual slots, α_j has factor $(6-j)!/j!$ before the distinct-row and distinct-column sum. Partition-lattice inclusion–exclusion has block weight $(-1)^{m-1}(m-1)!$ for a block of size m. In A^3B, a singleton in an A slot vanishes by centering. For the distinguished B slot the surviving row partitions are U (one four-block, weight -6), V (three A slots together and B singleton, weight 2), and the three pairings P of B with one A (weight 1). The columns have the same types. Their grouped contractions, before multiplication by $4 \cdot 2!/4! = 1/3$, are

(U, U)	$36 \sum A_{ik}^3 B_{ik}$
$(U, V) + (V, U)$	$-24 c^T b_B$
$(U, P) + (P, U)$	$-36 q^T h$
(V, V)	$4 s_3 \sum B_{ik}$
$(V, P) + (P, V)$	$12 q^T A b_B$
(P, P), same partner	$3 s_2 H$
(P, P), different partners	$6 \sum B_{ik}(A^3)_{ik}$.

For AB^2, the surviving partitions are the triple block of weight 2 and the two pairings of A with a B, each of weight -1. The grouped contractions are $4 \sum A_{ik} B_{ik}^2$, $-8 h^T b_B$, $2H \sum B_{ik}$ and $2 b_B^T A b_B$. Multiplication by $3 \cdot 3!/3! = 3$ proves (8); the preceding table proves (7).

For a general real centered Gram matrix A write $a_i = A_{ii}$. The omitted Fischer states obey

$$\begin{aligned}\|\xi_i\|^2 &= 24a_i^4 + q_i^2 + 4a_i(A^3)_{ii} + a_i^2 s_2 + \tfrac{2}{3}a_i s_3 + 2(A(A\circ A)A)_{ii} \\ &\quad - 24a_i^2 q_i - 16a_i c_i + 4\sum_{j,k} A_{ik}^2 A_{ji}A_{jk} + 4a_i(Aq)_i, \\ \left\|\sum_i \xi_i\right\|^2 &= 24s_4 - 24\sum_i q_i^2 + 2s_2^2 + 4\,\mathrm{tr}(A^4).\end{aligned}$$

These identities follow by matching the four factors in each product. For example the square of $g_i^2 p_2/2$ contributes $q_i^2 + 4a_i(A^3)_{ii} + a_i^2 s_2$, and the square of $g_i p_3/3$ contributes $2a_i s_3/3 + 2(A(A\circ A)A)_{ii}$. The cross terms with g_i^4 give $-24a_i^2 q_i - 16a_i c_i$; the remaining cross term gives the final two terms. Also $\sum_i \xi_i = p_4 - p_2^2/2$, which gives the total-state norm.

At the null-locus basepoint (6), substitution into these unrestricted contractions yields

$$\begin{aligned}\sum_i \|\xi_i\|^2 &= 384d_4 + 4\|b\|^2 + 16t^T e + 8\beta_2 d_2 + \tfrac{4}{3}s_3 + 6N \\ &\quad - 192(t^{\circ 2})^T b - 32t^T c + 16t^T Ab, \\ \left\|\sum_i \xi_i\right\|^2 &= 24s_4 - 96\|b\|^2 + 8\beta_2^2 + 8\beta_4.\end{aligned}$$

The coefficient $6N$ is the sum of two and four times the same contraction after summing over i. Hence

$$\begin{aligned}2\delta(A) &= 4608d_4 + 240\|b\|^2 + 192t^T e + 96\beta_2 d_2 + 16s_3 + 72N \\ &\quad - 2304(t^{\circ 2})^T b - 384t^T c + 192t^T Ab - 48s_4 - 16\beta_2^2 - 16\beta_4.\end{aligned}$$

B The complete mixed derivative table

From (9), differentiating before restriction gives

$$\begin{gathered}\dot b_B = 12t, \quad \dot q = 48t, \quad \dot c = 144t^{\circ 2} - 12b, \quad \dot h = 36t^{\circ 2} - 4b, \\ \dot s_2 = 48, \quad \dot s_3 = 144d_2 - 12\beta_2, \quad \dot H = 36d_2 - 4\beta_2, \quad (\textstyle\sum B_{ik})^{\cdot} = 12, \quad (A^3)^{\cdot} = 36A^2.\end{gathered}$$

Writing $G = \sum A_{ik}^2 B_{ik}$ and evaluating $b_B = b$ only after differentiation, the eleven terms are

term	derivative
$12\sum A_{ik}^3 B_{ik}$	$2880d_4 - 72G - 12s_4$
$-8c^T b_B$	$-1152(t^{\circ 2})^T b + 96\|b\|^2 - 96t^T c$
$-12q^T h$	$-576t^T h - 864(t^{\circ 2})^T b + 96\|b\|^2$
$\frac{4}{3}s_3\sum B_{ik}$	$192\beta_2 d_2 - 16\beta_2^2 + 16s_3$
$4q^T Ab_B$	$288t^T Ab + 96\|b\|^2 - 16\beta_2^2$
$s_2 H$	$48H + 72\beta_2 d_2 - 8\beta_2^2$
$2\sum B_{ik}(A^3)_{ik}$	$48t^T e - 4\beta_4 + 72M$
$12\sum A_{ik}B_{ik}^2$	$720d_4 - 24\|B\|_{\mathrm{F}}^2 - 24G$
$-24h^T b_B$	$-864(t^{\circ 2})^T b + 96\|b\|^2 - 288t^T h$
$6H\sum B_{ik}$	$216\beta_2 d_2 - 24\beta_2^2 + 72H$
$6b_B^T Ab_B$	$144t^T Ab + 72\|b\|^2 - 12\beta_2^2$.

For example the first row is $12\sum(3A_{ik}^2\dot{A}_{ik}B_{ik} + A_{ik}^3\dot{B}_{ik})$. Its diagonal contributions give $2880d_4$; the others give $-72G - 12s_4$. Summing the table with the preceding real-Gram variance proves the unrestricted expression used in Proposition 3.1.

C Ordinary normalization and portable exact extraction

The centered residual-slot identities give $\alpha_0 = 720$, $\alpha_1 = 0$, $\alpha_2 = 12S_2$, $\alpha_3 = 4S_3$, where $S_2 = \operatorname{tr}(K\overline{K})$. The four-slot surviving partition types are 4 and $2+2$, giving

$$\alpha_4 = 3\sum K_{ik}^4 - 3\,\mathrm{Re}\sum_i(\sum_k K_{ik}^2)^2 + \tfrac{1}{4}S_2^2 + \tfrac{1}{2}\operatorname{tr}((K\overline{K})^2).$$

On the null locus its derivative is $\partial_W\alpha_4 = 72d_3 - 12S_3$. Since $\mathcal{P}(K) = \sum_j(7-j)\alpha_j(K)$, $\partial_W\alpha_2 = 144r$, and $\partial_W\alpha_3 = 12(6d_2 - S_2)$, ordinary coefficient extraction gives

$$[x^4]\partial_v\mathcal{P}(xK+vW)^2 = 20160\delta + 23040rS_3, \qquad [x^3]\partial_v\mathcal{P}(xK+vW)^2 = 30240(72d_3 - 12S_3).$$

For the paired cubic derivative, the degree-four polynomial before differentiation is

$$\begin{aligned}16\alpha_4(A) + 192\sum A_{ik}^2B_{ik} - 192q^Tb_B + 48s_2\sum B_{ik}\\ + 192[(\sum B_{ik})^2 - 2\|b_B\|^2 + \|B\|_{\mathrm{F}}^2].\end{aligned}$$

Differentiating along (9) before imposing null overlap gives $41472d_3 - 1152S_3 - 5760S_{21} - 32256U + 8832r\beta_2 + 768\beta_3$. Subtracting this from the ordinary cubic derivative and combining with the quartic expression gives (5) at $r = 1$.

An exact extraction independent of the contraction table uses the product rule over the 720 permutations. For $N_{ik}(x) = 2 + \sqrt{2}xA_{ik} + x^2B_{ik}$,

$$T_4 = [x^4]\sum_{\pi\in S_6}\sum_i(2\sqrt{2}W_{i,\pi(i)} + xW_{i,\pi(i)}A_{i,\pi(i)})\prod_{j\neq i}N_{j,\pi(j)}(x).$$

Alternatively, a subset recurrence builds each permanent: for a column set S of size i, extend by an unused column j with $P_{S\cup\{j\}} \mathrel{+}= P_S n_{ij}$ and $\dot{P}_{S\cup\{j\}} \mathrel{+}= \dot{P}_S n_{ij} + P_S \dot{n}_{ij}$, starting with $P_\varnothing = 1$, $\dot{P}_\varnothing = 0$. Truncating at degree four is exact for these derivatives. The ordinary slots are $n_{ij} = 1 + xK_{ij}$, $\dot{n}_{ij} = W_{ij}$; paired slots are N_{ij} and the displayed paired direction.

These finite algebraic procedures can check coefficient arithmetic, boundary fixtures and Proposition 7.1 in rational quadratic fields. The universal proof is the partition calculation and the inequalities above.

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