Typical-value anchors and weighted polynomial approximation of the Riemann xi-function

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Luminary Oddities

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Abstract

For $F(z) = \xi((1+iz)/2)/4$, we approximate growing families of localized targets by ordinary polynomial multiples of F in the original derivative-plus-absolute-potential norm. A selection of simple real zeros is equipped simultaneously with a harmonic pole bound, a global weighted graph budget, and a typical-value anchor furnished by Selberg's theorem. Transport through a root-free complex corridor and a centered Fourier tail estimate cancel the height factor in the original graph norm. The resulting sufficient log degree is $O(\log T \log\log T)$ for $\asymp T \log T$ targets with a fixed whole-span error. Consequently each parity and every sufficiently large degree N admit two rational-basis spaces of dimension at least $C\exp(c\log N/\log\log N)(\log N/\log\log N)$, with opposite correction signs and strictly positive original square form on both entire spans. This strengthens the earlier square-root-logarithmic scale. The proof retains all roots, global tails and mixed terms, and also bounds the true graph-norm Ritz residual. We also construct a fixed-width receiver for prescribed entire targets formed from F, a critical cosh factor and a bandlimited multiplier, measured in the original strip derivative energy and real-axis absolute square-form weight. A higher-derivative graph selection then gives the same rational channel spaces, of the stated dimension order, simultaneously for $0 \le y \le c(\log N/\log\log N)^{-K}$ with a fixed finite K.

1 The approximation problem and principal statements

Let

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \qquad F(z) = \tfrac{1}{4}\xi((1+iz)/2), \quad G = F'^2 - FF''.$$

The entire function F is even and real on the real line. Put $V = G/F^2$ away from its real zeros and, for real polynomials P, define

$$E(P) = \frac{1}{4\pi}\int_{\mathbb{R}} |(FP)'|^2, \qquad Q(P) = \frac{1}{8\pi}\int_{\mathbb{R}} GP^2, \qquad D = E - Q. \tag{1}$$

For raw functions use the same forms with FP replaced by u and GP^2 by $V|u|^2$. The reference Hilbert norm is

$$\|u\|_*^2 = \int_{\mathbb{R}} (|u'|^2 + (1+|V|)|u|^2). \tag{2}$$

Its domain consists of the weakly differentiable functions for which this expression is finite. Assigning values to V at isolated zeros does not change the integrals; $|V|F^2 = |G|$ almost everywhere. Both polarized forms in (1) are bounded on this domain. All polynomial coefficients below are real, and a rational basis has rational monomial coefficients.

For $y > 0$ define the original positive-width forms by

$$E_y(P) = \frac{1}{8\pi y} \int_{-y}^{y} \int_{\mathbb{R}} |(FP)'(x+iv)|^2 \, dx \, dv, \tag{3}$$

$$Q_y(P) = \frac{1}{2\pi} \int_{\mathbb{R}} \gamma_y(x) P(x)^2 \, dx, \qquad \gamma_y(x) = \frac{\partial_y |F(x+iy)|^2}{8y}. \tag{4}$$

Their endpoint limits are $E_0 = E$, $Q_0 = Q$, with $\gamma_0 = G/4$. Put $D_y = E_y - Q_y$ and define Q_y^{abs} by replacing γ_y with $|\gamma_y|$. For real entire multipliers f, these same expressions define $E_y[f]$ and $Q_y[f]$ through Ff and $f(x)$, respectively. The multiplier in Q_y is evaluated on the real axis, while E_y differentiates the entire product on the strip. Write

$$\|f\|_{\mathcal{J}_Y} = \sup_{0 \le y \le Y} \left(E_y[f] + Q_y^{\mathrm{abs}}[f]\right)^{1/2}.$$

The original strip identity (3) follows from the theta Fourier representation and Plancherel, as in [1]; all terms are finite for polynomials by the completed-zeta strip envelopes proved below.

Theorem 1.1. *For every fixed $0 < \theta < 1$ there are $c, C > 0, N_0$ such that every integer $N \ge N_0$ and each parity have two possibly different rational-basis polynomial spaces of degree at most N, each of dimension at least*

$$C \exp\left(c \frac{\log N}{\log\log N}\right) \frac{\log N}{\log\log N}. \tag{5}$$

On the entire nonzero span of the first, $Q/E \ge 2 - \theta$; on that of the second, $0 < Q/E \le \theta$. Thus the two correction indices of D are at least (5) in each parity.

Theorem 1.2 (One pair of spaces over a moving width interval)**.** *For each fixed $0 < \theta < 1$ there exist $c, C > 0$, a finite $K > 0$ and N_0 such that every integer $N \ge N_0$ and each parity have two possibly different rational-basis polynomial spaces of degree at most N and dimension at least (5), with*

$$\frac{Q_y(P)}{E_y(P)} \ge 2 - \theta \quad \text{on the first,} \qquad 0 < \frac{Q_y(P)}{E_y(P)} \le \theta \quad \text{on the second,}$$

for every nonzero P and simultaneously for

$$0 \le y \le c \left(\frac{\log N}{\log\log N}\right)^{-K}. \tag{6}$$

The bases do not depend on y in this interval. Constants and onset may depend on θ.

The mechanism is the following more precise receiving statement. The targets in it are explicitly constructed in Section 2 and below; it is not a projector bound for arbitrary vectors in an infinite-zero ideal.

Theorem 1.3 (Selected-family graph approximation)**.** *Fix either quotient margin in Theorem 1.1 and a fixed $0 < \eta < 1$. For every sufficiently large height T, each parity has $M \ge cT \log T$ disjoint-support, energy-normalized smooth targets u_r, with uniform raw graph synthesis norm, and a linear map $b \mapsto P_b$ into ordinary polynomials of a common degree n, preserving parity, such that*

$$\left\| FP_b - \sum_r b_r u_r \right\|_* \le \eta \, \|b\|_2, \qquad \log(n+2) \le C_\eta \log T \log\log T.$$

The map can be taken with rational coefficient matrix. If Π_n is the orthogonal graph projection onto the permitted span of Fx^j, then the same degree gives

$$0 \le U - B_n^{\mathsf{T}} A_n^{-1} B_n \le \eta^2 I,$$

where A_n is the original graph Gram of Fx^j, B_n its cross Gram with the targets, and U the target graph Gram.

The physical-form predecessor [1] distinguishes original positivity from correction inertia. Weighted localization [2] yields all fixed logarithmic powers; the critical-tail successor [15] yields the scale

$$C \exp(c\sqrt{\log N})\sqrt{\log N}.$$

We reuse its all-root selection, finite-order cutoff sequence and critical MP return, giving the needed details for the present proof. The additional steps are a harmonic first-pole selection, an averaged original graph budget, typical-value anchoring on the same slots, and centered full-line leakage control. Their height factors cancel, reducing the cutoff order from $O(\log T)$ to $O(\log\log T)$.

The moving-width theorem adds a constructive strip receiver for a prescribed analytic class, a complex supremum graph budget through four derivatives, and an original signed-density comparison for the same analytic target family. It does not identify the full strip zero ideal or transfer a fixed compact cap with bounded energy at a fixed positive width. These distinctions determine the choice of intermediate targets in Section 8.

Critical exponential approximation, MP expansions and strip bases have substantial precedents [12, 13, 9, 10, 11, 8]. Finite-order cutoffs are classical [14]; Selberg's limit theorem [16] and the mean-square estimates used here are classical number theory. The contribution is their quantitative combination in the unchanged original graph norm for the same selected theta targets. A typical value alone does not control global graph leakage, and an orthogonal projection identity alone does not supply this degree law.

2 All-root geometry and localized targets

The local factor and microscopic selection in this section reproduce the inputs of [15]. We add a simultaneous first-pole score, which will be needed to transport a typical-value anchor. Set $a_* = \pi/8$. The xi formula, Stirling's formula and polynomial zeta bounds on fixed vertical strips give, for every fixed b_0,

$$|F(x+iy)| \le C_{b_0}(1+|x|)^{p_{b_0}} e^{-a_*|x|}, \qquad |y| \le b_0. \tag{7}$$

For example the continuation formula $\zeta(s) = s/(s-1) - s\int_1^\infty \{u\}u^{-s-1}\,du$ first gives polynomial bounds in $\Re s > 0$; the functional equation handles the remaining fixed strip. The xi prefactors remove the apparent poles. Cauchy's estimate in fixed disks therefore gives a fixed integer $p \ge 1$ with

$$|F(x)| + |F'(x)| + \sqrt{|G(x)|} \le C(1+|x|)^p e^{-a_*|x|}. \tag{8}$$

These standard estimates are uniform through real zeros; see [6, Chapters II and IX]. We include the local product argument from [2] because the complex version is needed in our receiver.

The following lemma includes every nonreal zero in its local disk. This is necessary: the individual nonreal pole terms need not be positive on the real line.

Lemma 2.1 (Uniform local factor)**.** *There are fixed positive constants C_0, C_1, C_2 with the following property for all sufficiently large T. Set $K = \lceil C_0 \log(T+40) \rceil \geq 2$. For any real $c \in [10, T]$ choose any $b_c \in (8, 9)$ for which the circle $|z - c| = b_c$ contains no zero, and let Z_c be the multiset of all zeros inside. Then $\#Z_c \leq K$, and on the real interval $|x - c| \leq 1$, away from zeros,*

$$V(x) = \sum_{\rho \in Z_c} \frac{1}{(x-\rho)^2} + R_c(x), \qquad |R_c(x)| \leq C_1 K. \tag{9}$$

On the same interval, for any positive numbers $d_\rho \leq |x - \rho|$,

$$|F(x)| \geq e^{-\pi c/8} (T+40)^{-C_2} e^{-C_2 K} \prod_{\rho \in Z_c} \min(1, d_\rho). \tag{10}$$

All products and sums count multiplicities.

Proof. The Riemann–von Mangoldt formula with its $O(\log T)$ error [6, Chapter IX] bounds the number of all zeros, with multiplicities, in every fixed-width real window up to height $T + 20$ by $C \log(T + 40)$. All zeros have imaginary part of magnitude less than one. This bounds the zeros in every disk used in the statement, after fixing C_0.

Normalize $f_c(w) = e^{\pi c/8} F(c + w)$. Formula (7), used on the wider fixed strip needed for $|w| \leq 20$, implies $|f_c(w)| \leq C(T + 40)^p$ there. The off-axis point $w = -3i$ is a uniform anchor: its zeta argument is $2 + ic/2$, and

$$|\zeta(2 + ic/2)| \geq \frac{1}{\zeta(2)}, \qquad |\Gamma(1 + ic/4)|^2 = \frac{\pi c/4}{\sinh(\pi c/4)}.$$

The first bound follows by taking absolute values in the absolutely convergent reciprocal Euler series. The xi prefactors and the last identity give $|f_c(-3i)| \geq a_3 > 0$ for $c \geq 10$.

In the disk $|w| < b = b_c$, divide f_c by the full finite disk Blaschke product

$$B_c(w) = \prod_{\rho \in Z_c} \frac{b(w - (\rho - c))}{b^2 - \overline{(\rho - c)} w},$$

where unimodular factors may be inserted without changing any bound. The quotient $H_c = f_c / B_c$ is holomorphic and zero-free in that disk. Since $|B_c| = 1$ on its boundary, the maximum principle gives $|H_c| \leq C(T + 40)^p$ inside. Also $|B_c(-3i)| \leq 1$, so $|H_c(-3i)| \geq a_3$. For a suitably enlarged polynomial bound M, $U = \log M - \log |H_c|$ is positive harmonic. Harnack's inequality first compares $-3i$ with zero, then zero with $|w| \leq 2$. All comparison constants are uniform for $8 < b < 9$. Thus

$$0 \leq U(w) \leq C \log(T + 40), \qquad |H_c(w)| \geq e^{-C \log(T+40)} \quad (|w| \leq 2).$$

The harmonic derivative estimate on concentric disks (equivalently Borel–Caratheodory followed by Cauchy's estimate for a holomorphic logarithm) gives $|(\log H_c)'(w)| + |(\log H_c)''(w)| \leq C \log(T+40)$ for $|w| \leq 1$.

Write $\alpha = \rho - c$. The second logarithmic derivative of its Blaschke factor is

$$-\frac{1}{(w-\alpha)^2} + \frac{\overline{\alpha}^2}{(b^2 - \overline{\alpha} w)^2}.$$

The last term has magnitude at most $(b-1)^{-2} \leq 1/49$ on $|w| \leq 1$. Since $V = -(\log F)''$ locally away from zeros, summing these identities proves (9). For real $|w| \leq 1$, each factor also satisfies

$$\left| \frac{b(w-\alpha)}{b^2 - \overline{\alpha} w} \right| \geq \frac{|w - \alpha|}{b+1} > \frac{|w-\alpha|}{10}.$$

Multiply these inequalities and the lower bound for H_c, and undo the normalization $e^{\pi c/8}$. The factors $10^{-\#Z_c}$ are absorbed in e^{-C_2K}, while the quotient loss is a power of $T+40$. Clipping each assigned distance at one only weakens the bound. This proves (10) uniformly even when roots approach one another or the chosen boundary circle. □

2.1 A simultaneous microscopic selection

Let $N_F(t) = N_\zeta(t/2)$ count all zeros by real coordinate, with multiplicity. This is the correct scaling because a zeta zero $\beta+i\gamma$ corresponds to an F zero $2\gamma+i(1-2\beta)$. The classical input from Fujii [4, Lemma 1, p. 742] is only the fixed fourth increment moment. For $U \asymp T$ and $\delta = \ell/[2\log(T+40)]$, with fixed $\ell > 0$, it implies

$$\int_U^{2U} |S_\zeta(t+\delta) - S_\zeta(t)|^4\, dt \le C_\ell U.$$

Indeed $\log(3+\delta\log U)$ is bounded in that lemma, so both its fixed-moment main term and error are $O(U)$. Cauchy–Schwarz gives the second moment. The unconditional moment upper bound is also explicitly stated in [5, Theorem 4]. Riemann–von Mangoldt, a fixed finite dyadic cover and $t = 2u$ now give, for fixed $a \in (0,1)$,

$$\int_{aT/2}^{T+5} [N_F(t+\ell h) - N_F(t)]^2\, dt \le C_{a,\ell} T, \qquad h = \frac{1}{\log(T+40)}. \tag{11}$$

Its smooth main-term increment is $O_\ell(1)$; zero endpoints form a null set. No pair-correlation hypothesis or growing moment is used.

The positive-proportion simple critical-line count [3, Theorem 1.1] supplies $b_0 T\log T$ distinct simple real seeds below T. Choose $a > 0$ small enough that the all-zero count below aT is at most half that quantity eventually. This subtraction, rather than an unsupported dyadic lower count, leaves bT/h seeds in $[aT, T]$. End bins cost $O(\log T)$ and can be discarded.

Partition into bins $I_j = [jh, (j+1)h)$. Let n_j count every zero with real coordinate in I_j, including complex zeros and multiplicities, and let $s_j \le n_j$ count the simple real seeds in central bins. Applying (11) with $\ell = 3$ on the disjoint intervals $((j-1)h, jh)$ yields

$$\sum_{I_j \cap [aT-4, T+4] \ne \varnothing} n_j^2 \le CT/h. \tag{12}$$

This enlarged range covers all roots of distance less than two from a central seed. The construction is localized to the specified part of the macroscopic disk.

Define $q_j = \sum_{|l|\le 2} n_{j+l}$ and the symmetric sequence

$$w_l = \begin{cases} \log(2/h), & |l| \le 2, \\ \log_+(2/[(|l|-1)h]), & |l| \ge 3. \end{cases}$$

Its finite support and $\sum_l w_l \le C/h$ follow by comparison with $h^{-1}\int_0^2 \log(2/x)\, dx$. Truncate n_j to the enlarged range when forming convolutions and put $H_j = \sum_l w_l n_{j+l}$ and $L_j = \sum_l n_{j+l}/(1+l^2)$. Nonnegative summation and Cauchy–Schwarz in (12) give

$$\sum_{j \text{ central}} q_j^2 \le CT/h, \qquad \sum_{j \text{ central}} s_j H_j \le CT/h^2, \qquad \sum_{j \text{ central}} s_j L_j \le CT/h. \tag{13}$$

For the last inequality, each shifted inner product is at most CT/h and $\sum_l (1+l^2)^{-1} < \infty$. The first uses only the five neighboring shifts. The second uses the integrable logarithmic kernel. All shifted factors stay in the finite enlarged range; no whole-axis square-count sum is presumed.

For the additional first-pole score put

$$R_j = h^{-1}\left(q_j + \sum_{3\le|l|\le\lceil 2/h\rceil+2} \frac{n_{j+l}}{|l|-1}\right).$$

All contributing coordinates stay in the fixed enlarged range for central j and large T. The finite kernel has sum $O(\log(2/h))$; each shifted inner product $\sum_j s_j n_{j+l}$ is at most CT/h by Cauchy–Schwarz. Hence

$$\sum_{j\,\text{central}} s_j R_j \le CTh^{-2}\log(2/h).$$

Choose fixed J, B_1, B_2, B_3 large enough. The union of bins with $q_j > J$, $H_j > B_1/h$, $L_j > B_2$ or $R_j > B_3 h^{-1}\log(2/h)$ loses less than half the seeds: the respective losses are bounded by $CT/(Jh)$, $CT/(B_1 h)$, $CT/(B_2 h)$ and $CT/(B_3 h)$. Each retained bin has at most J seeds. Choose one residue class modulo four and one seed in each occupied retained bin. There are $M \asymp T/h$ selected seeds separated by at least $3h$, and every inner disk $|\rho - r| < h$ contains at most J roots.

For each retained seed apply Lemma 2.1 with $c = r$. Split its complete macroscopic root multiset into inner $|\alpha| < h$ and outer $|\alpha| \ge h$, where $\alpha = \rho - r$. For $h \le |\alpha| < 2$, a root in bin $j + l$ satisfies $|\alpha|^{-2} \le Ch^{-2}/(1 + l^2)$: use the outer condition for $|l| \le 2$, and $|\alpha| \ge (|l| - 1)h$ otherwise. Roots of distance at least two instead contribute at most $K/4$. Thus the SAME seeds obey

$$\sum_{\text{outer}} |\alpha|^{-2} \le C_* h^{-2}, \qquad \sum_{\text{outer}} \log_+(2/|\alpha|) \le B_1/h. \tag{14}$$

This treats the full macroscopic disk in the proved coordinate range. The same split at distance two gives

$$\sum_{\text{outer}} |\alpha|^{-1} \le Ch^{-1}\log(2/h). \tag{15}$$

For $h \le |\alpha| < 2$ use the near-bin bound $1/h$ or $1/[(|l|-1)h]$ in R_j; the remaining roots contribute at most $K/2$. All filters were paid as one union of exceptional sets before the residue-class extraction; unrelated positive-proportion subsets are not being intersected without an error budget.

3 Relative root budgets and localized channels

Fix an error $\epsilon > 0$ to be chosen from the desired quotient margins. Choose a fixed large $D_* > 1$ and set

$$t_0 = h/D_*, \qquad S_0 = -\log t_0.$$

For $0 < t \le t_0$, the outer poles in (9) cost at most $4C_*/D_*^2$ after multiplication by t^2. The remainder costs CKh^2/D_*^2. Both are below ϵ for fixed large D_* and large T.

For each nonzero inner root set $\lambda_\alpha = -\log|\alpha|$; for the seed set $\lambda_0 = \infty$. Take fixed large constants A_*, B_* and delete

$$[\lambda_\alpha - \log A_*, \lambda_\alpha + \log B_* + \tfrac{1}{2}\log J].$$

Outside these at most J transition intervals, each inner root has either $|\alpha| \le t/A_*$ or $|\alpha| \ge B_*\sqrt{J}\,t$, where $t = e^{-s}$. The seed is always interior. For an interior root, $|(1 - \alpha/t)^{-2} - 1| \le C/A_*$; for the remaining inner roots, the absolute total is at most $4/B_*^2$. Conjugation makes the complete sum real. Choose A_*, B_*, D_* so that on every good component

$$(1 - 2\epsilon)\nu \le t^2 V(r + t) \le (1 + 2\epsilon)\nu, \qquad 1 \le \nu \le J. \tag{16}$$

All pole contributions are included in the complete sum.

Put $\mathcal{B}(s) = \sum_{\text{inner}} \min(s, \lambda_\alpha)$. It is continuous with $1 \le \mathcal{B}' \le J$ almost everywhere; on a good component $\mathcal{B}' = \nu$. For any relative budget $L > 0$ choose the unique S satisfying $\mathcal{B}(S) - \mathcal{B}(S_0) = L$. Then

$$S_0 + L/J \le S \le S_0 + L, \qquad e^S \le D_* h^{-1} e^L. \tag{17}$$

The deleted intervals cost at most a fixed $J^2 w_J$ in weighted length. The good weighted length is therefore at least $L - J^2 w_J$, in at most $J + 1$ components.

Let $\chi \ge 0$ be any one fixed nonzero flat smooth cutoff in $(0, 1)$, and set $I_0 = \int \chi^2$, $I_1 = \int |\chi'|^2$. In a good component pack caps of length $\ell_\nu = \ell/\sqrt{\nu}$, where $\ell > 0$ is fixed. On a cap starting at a_c define

$$v_c(s) = \left(\frac{4\pi \ell_\nu}{I_1 + I_0 \ell_\nu^2/4} \right)^{1/2} \chi((s - a_c)/\ell_\nu), \qquad u(r + t) = \sqrt{t}\, v_c(-\log t), \tag{18}$$

and extend by zero. Direct substitution gives

$$4\pi E(u) = \int (|v_c'|^2 + |v_c|^2/4) = 4\pi, \tag{19}$$

$$8\pi Q(u) = \int t^2 V(r + t) |v_c|^2. \tag{20}$$

The boundary cross term vanishes. For the pure pole the quotient is

$$q_\nu = \frac{2\nu}{1 + 4(I_1/I_0)\nu/\ell^2}. \tag{21}$$

Choose ℓ large and ϵ small to obtain $Q/E \ge 2 - \theta/4$, uniformly in $1 \le \nu \le J$. Alternatively choose ℓ small and then ϵ small to obtain $q_{\min} \le Q/E \le \theta/8$ for one fixed $q_{\min} > 0$. Both choices have uniformly bounded $\|u\|_*$; the derivative part is normalized and the potential part is bounded by (16), while $\int |u|^2 \le C t_0^2 \int |v_c|^2$.

The packing count is at least $(L - J^2 w_J)/(\ell\sqrt{J}) - (J + 1)$. Hence each selected seed supplies at least cL caps once $L \ge L_0$, with L_0 independent of T. Taking exactly a fixed proportional number and using (17) gives, for either family,

$$d \asymp T \log T\, L. \tag{22}$$

Distinct caps have disjoint supports; the seed spacing exceeds $3h$ and each support is within $t_0 < h$ on the right of its seed. Reflect and divide by $\sqrt{2}$ to obtain either parity. The raw E Gram is exactly the identity, the raw Q Gram is diagonal with the stated margins, and every raw joint norm is bounded by a fixed M_0. No cross Gram between the two different families is required.

3.1 Fixed slots and root-free corridors

For the remainder of the proof fix $L \ge L_0$ once, independently of T and the cutoff order. Keep just one complete cap slot per seed for either chosen length. The complete slot, before filling it with its cutoff, has physical length between $c_L h$ and $C_L h$, and $c_L h \le t \le h/D_*$. The slots are disjoint and there are $\asymp T/h$ of them. Each may have its own $\nu \in \{1, \dots, J\}$.

For a sufficiently small fixed $c_\theta > 0$, the union of the disks $|z - (r + t)| \le c_\theta t$ along a complete good slot contains no root. Indeed an inner root has either $|\alpha| \le t/A_*$ or $|\alpha| \ge B_* \sqrt{J}\, t$, giving $|z - \rho| \ge c \max(t, |\alpha|)$, while every outer root satisfies $|z - \rho| \ge |\alpha|/2$. The constants are fixed after the margin and L are chosen. The off-axis quotient in the Blaschke factorization still contains every root outside the macroscopic disk. Its first logarithmic derivative is $O(K)$ on this entire corridor, as proved above. Thus none of these root-free assertions assumes RH, a spacing bound between individual zeros, or a lower bound on F' at a seed.

4 Finite-order cutoffs with uniform cap energies

The cutoff may depend on the requested finite approximation order. This is the classical Ehrenpreis–Hörmander cutoff-sequence principle; see, for example, the beginning of [14, Section 3]. It is different from requiring one compact function to have analytic bounds at every order. We give the construction and all uniform low-order bounds, since the two theta channels need those bounds.

Choose once a nonnegative smooth probability density φ_0 supported in $[-1/8, 1/8]$, for example the normalized function $\exp[-1/(1-64x^2)]$ there, extended by zero. Put $\delta = 1/4$. For each integer $m \geq 1$ let $w_m = \delta/m$, let H_w be the uniform probability density on $[-w/2, w/2]$, and define

$$\chi_m(x) = (\varphi_0 * H_{w_m}^{*m})(x - 1/2). \tag{23}$$

Every χ_m is nonnegative, smooth, flat at its support endpoints, has mass one, and is supported in $[1/4, 3/4]$.

Lemma 4.1. *Writing $I_{0,m} = \int \chi_m^2$ and $I_{1,m} = \int |\chi_m'|^2$, there are fixed $0 < C_{\min} \leq C_{\max} < \infty$ with*

$$2 \leq I_{0,m} \leq \|\varphi_0\|_2^2, \qquad I_{1,m} \leq \|\varphi_0'\|_2^2, \qquad C_{\min} = \pi^2 \leq I_{1,m}/I_{0,m} \leq C_{\max}. \tag{24}$$

For $0 \leq k \leq m$,

$$\left\|\chi_m^{(k)}\right\|_\infty \leq \|\varphi_0\|_\infty \, (2m/\delta)^k. \tag{25}$$

Proof. All smoothing factors are probabilities. Young's inequality proves the upper L^2 and derivative bounds, and Cauchy–Schwarz on the common support of length $1/2$ proves $I_{0,m} \geq 2$. Poincaré's inequality on $(0,1)$ gives $I_{1,m} \geq \pi^2 I_{0,m}$. Differentiate k distinct box factors as signed measures; each derivative has total variation $2/w_m$. Leave φ_0 as the bounded density and the remaining factors as probabilities. This proves (25), without any bound on derivatives above order m. □

Use χ_m in (18). By (24), one fixed long length ℓ_{long} and one fixed short length ℓ_{short} work for every m in (21). The complete long-cap quotient is at least $2 - \theta/4$. For the short caps choose $\ell_{\text{short}}^2/(2C_{\min}) \leq \theta/16$ and ϵ small; then

$$q_{\min} \leq Q/E \leq \theta/8, \qquad q_{\min} = \frac{(1-2\epsilon)2}{1 + 4C_{\max}/\ell_{\text{short}}^2} > 0.$$

The normalization factor in (18) is uniformly bounded by (24); all raw joint norms are at most one fixed M_0. Thus the fixed packing threshold L_0, cap count and both margins are independent of m.

5 Mean graph budgets and typical-value anchors

Denote the lower-height seed fraction selected in Section 2.1 by a_0. In this section write $a = \pi/8$, $g(x) = F(x)\cosh(ax)$ and introduce the actual nonnegative graph density

$$\nu(x) = |g(x)|^2 + |g'(x)|^2 + |G(x)|\cosh^2(ax). \tag{26}$$

It is even. The global estimate (8) gives $\nu(x) \leq C(1+|x|)^{2p}$ after enlarging the fixed integer p. The improvement will use an averaged estimate as well as this pointwise bound, rather than applying the global supremum on every cap.

Lemma 5.1 (Classical near-critical mean squares)**.** *For each fixed $j \le 4$ and $U \to \infty$,*

$$\int_U^{2U} |\zeta^{(j)}(1/2+it)|^2\,dt \le C_j U(\log U)^{2j+1}.$$

The estimates hold uniformly for $|\sigma - 1/2| \le C_0/\log U$ and bounded shifts of the endpoints, for every fixed C_0.

Proof. We recall an elementary upper-bound proof to specify the required uniformity. Euler–Maclaurin [7, 25.2.9] gives, for an integer X and $\sigma > -2R$,

$$\zeta(s) = \sum_{n\le X} n^{-s} + \frac{X^{1-s}}{s-1} - \frac{1}{2}X^{-s} + \sum_{l=1}^{R} \frac{B_{2l}}{(2l)!}(s)_{2l-1}X^{1-s-2l} + \mathcal{R}_R(s,X),$$

where, by the periodic Bernoulli form of the remainder,

$$|\mathcal{R}_R(s,X)| \le \frac{4|(s)_{2R+1}|}{(2\pi)^{2R+1}} \frac{X^{-\sigma-2R}}{\sigma+2R}.$$

Here $(s)_l = s(s+1)\cdots(s+l-1)$. The bound $|B_l(\{x\})|/l! \le 4/(2\pi)^l$, $l \ge 2$, follows from the Fourier formulas [7, 24.8.1–24.8.3]. For $t \in [U-1, 2U+1]$, $|\sigma - 1/2| \le C_1/\log U$, choose $X = \lceil 8U \rceil$, $R = \lceil 10 \log U \rceil$. Each entry in the rising factorial is at most $4U$ eventually. Thus the Bernoulli boundary sum is bounded by

$$CX^{-\sigma} \sum_{l\ge 1} \left(\frac{4U}{2\pi X}\right)^{2l-1} = O(X^{-\sigma}),$$

and the remainder is at most

$$CX^{1-\sigma}\left(\frac{4U}{2\pi X}\right)^{2R+1}/(\sigma+2R) = O(U^{-2}).$$

The pole-integral and endpoint terms are $O(U^{-\sigma})$. Consequently $\zeta(s) = \sum_{n\le X} n^{-s} + O(U^{-\sigma})$ uniformly on this rectangle.

For its Dirichlet polynomial $n^{-\sigma} \le C_{C_1} n^{-1/2}$. The mean-square diagonal is $O(U \log U)$. For $n > l$ the integrated oscillation is at most $2/\log(n/l) \le 2n/(n-l)$. With $k = n - l$, the absolute off-diagonal sum is bounded by

$$C \sum_{l+k\le X} \frac{\sqrt{1+k/l}}{k} \le C\left(X\log X + \sum_{l+k\le X} \frac{1}{\sqrt{lk}}\right) = O(X \log X).$$

The last double sum is $O(X)$: on each level $l + k = n$ its sum is bounded by a fixed discrete beta integral. The error term is negligible in mean square. This proves the uniform $j = 0$ estimate. Cauchy's integral formula on circles of radius $1/(2\log U)$, Cauchy–Schwarz in the circle parameter, and Fubini then give the jth derivative bound with a factor $O_j((\log U)^{2j})$. Choose $C_1 > C_0 + 1$; the enlarged rectangle covers those circles. This proves the asserted uniform derivative estimates through order four. □

On positive comparable heights write

$$g(x) = M(x)\zeta(1/2 + ix/2).$$

Stirling's formula and differentiation of the gamma prefactor give

$$|M(x)| \asymp x^{7/4}, \qquad |M^{(j)}(x)| \le C_j x^{7/4}(\log x)^j, \quad j \le 2. \tag{27}$$

The power is $2 - 1/4$: the xi quadratic prefactor contributes x^2 and $\Gamma(1/4 + ix/4)$ contributes $x^{-1/4}$; the factor $\cosh(\pi x/8)$ cancels its exponential. The coordinate is $x/2$ in the zeta ordinate throughout. Since

$$G\cosh^2(ax) = g'^2 - gg'' + a^2\operatorname{sech}^2(ax)g^2, \tag{28}$$

Lemma 5.1, the product rule and Cauchy–Schwarz imply for a fixed $a_0 \in (0,1)$ and some fixed B that

$$\int_{a_0T/2}^{2T} \nu(x)\,dx \le CT^{9/2}(\log T)^B. \tag{29}$$

A fixed finite comparable-height cover suffices; no short-interval mean-square estimate is used.

Lemma 5.2 (Full-line weighted graph budgets)**.** *Fix an integer $q > 2p + 4$ and let $w_r(x) = (1 + |x - r|)^{-q}$. After discarding at most a fixed small fraction of the selected seeds, there remain $\ge cT/h$ seeds satisfying*

$$J_r := T^{-7/2}\int_{\mathbb{R}} \nu(x)w_r(x)\,dx \le C_J(\log T)^B. \tag{30}$$

Proof. The $3h$ spacing gives $\sum_r w_r(x) \le C_q/h$ on the whole line, by comparison with the integral of $(1 + |x|)^{-q}$ and one maximal summand. On $[a_0T/2, 2T]$, summing the integrals and using (29) yields $C(T/h)(\log T)^B$. Outside this interval each seed in $[a_0T, T]$ is separated from x by at least a fixed multiple of $T + |x|$. The complete exterior sum, including the negative half-line, is therefore at most

$$CT^{-7/2}(T/h)T^{2p+1-q} = Ch^{-1}T^{2p-3/2-q}.$$

It is negligible. Thus $\sum_r J_r \le C(T/h)(\log T)^B$. Markov's inequality with a sufficiently large fixed C_J deletes an arbitrarily small fixed fraction. This proves the assertion on the same selected seeds, with both infinite tails included. □

Lemma 5.3 (Typical anchors and complex reciprocal bounds)**.** *For either fixed cap length, a further positive proportion of the budgeted seeds has, on every required complex disk of its complete slot,*

$$|1/g(z)| \le B_T := CT^{-7/4}(\log T)^C. \tag{31}$$

The slots, constants and surviving seeds can be chosen before the finite cutoff order is specified.

Proof. Selberg's unconditional limit theorem in the form of [16, Theorem 1] says that the distribution of $\log|\zeta(1/2 + it)|/\sqrt{\frac{1}{2}\log\log U}$ on $[U, 2U]$ converges to the standard Gaussian. In particular, for every fixed $\varepsilon_0 > 0$ there is a fixed C_0 such that eventually

$$\left|\left\{x \in [a_0T, T + 1] : |\zeta(1/2 + ix/2)| < e^{-C_0\sqrt{\log\log T}}\right\}\right| \le \varepsilon_0 T. \tag{32}$$

This follows from a fixed finite comparable-height cover and the Gaussian lower tail. It uses a fixed standardized threshold.

Each complete slot has length at least c_Lh, and these slots are disjoint. The budgeted seeds number at least cT/h, so their union has measure at least a fixed multiple of T. If a slot has no point outside the set in (32), its full length contributes to that exceptional measure. Choose

ε_0 small enough; at most half the slots are lost. In every remaining slot choose x_r outside the exceptional set. By (27),

$$|g(x_r)| \geq cT^{7/4}e^{-C_0\sqrt{\log\log T}}.$$

The anchor need not lie in the support of the later cutoff.

On the root-free corridor the full Blaschke quotient and its denominators have logarithmic derivatives of size $O(K)$. The bounded inner population costs $O_L(h^{-1})$ and (15), with the factor-two disk comparison, pays all outer numerators. The cosh logarithmic derivative is uniformly bounded there. Thus

$$|g'(z)/g(z)| \leq C_L h^{-1}\log(2/h).$$

Join the anchor to a real disk center along the complete good slot, then to its complex point by a radial segment. This path has length $O_L(h)$ and stays in the corridor. Integrating the logarithmic derivative loses at most $\exp(C_L\log(2/h))$. Combined with the anchor bound this proves (31), since $e^{C_0\sqrt{\log\log T}}$ is bounded by a fixed power of $\log T$. Every local root and multiplicity, and the quotient containing all exterior roots, has been retained. Cauchy differentiation of $1/g$ consequently has resolution $A = C_L/h$; the logarithmic first-pole bound is an amplitude cost, not an extra differentiation-scale factor. □

6 The critical MP receiver in the full joint norm

The critical sech-basis/strip method is closely related to [12, 8]; our return estimate additionally pays the original differential joint norm. This section separates the critical transform from the choice of compact cutoff. Write $a = a_*$, $H_M = \pi/(2a)$ and

$$\omega(x) = \frac{1}{H_M\cosh(\pi x/H_M)}, \qquad q_n(x) = P_n^{(1/2)}(x/H_M;\pi/2).$$

These real MP polynomials form an orthonormal basis in $L^2(\omega)$. Their classical generating function and recurrence [7] specialize to

$$\sum_{n\geq 0} q_n(x)z^n = (1-iz)^{-1/2+ix/H_M}(1+iz)^{-1/2-ix/H_M}, \tag{33}$$

$$xq_n = \frac{H_M}{2}[(n+1)q_{n+1} + nq_{n-1}], \qquad q_{-1} = 0. \tag{34}$$

The normalization is such that $q_0 = 1$ and $q_1 = 2x/H_M$. Completeness also follows directly: if $f \in L^2(\omega)$ is orthogonal to all polynomials, then $f\omega \in L^1$ has an analytic Fourier transform on a nonempty strip, by Cauchy–Schwarz and the exponential moments of ω. All its derivatives at zero vanish, so analytic uniqueness and Fourier injectivity give $f = 0$.

Lemma 6.1. *For the fixed p in (8), $\|Fq_n\|_* \leq C(n+1)^p$ for every $n \geq 0$.*

Proof. Repeated use of (34) gives $\|x^j q_n\|_{L^2(\omega)} \leq C_j(n+1)^j$. Differentiating (33) in x gives the finite convolution

$$q_n' = \frac{2}{H_M}\sum_{\substack{1\leq l\leq n\\ l \text{ odd}}} \frac{(-1)^{(l-1)/2}}{l} q_{n-l}.$$

After multiplication by x^j, each summand occupies only the basis indices within j of $n-l$. Each output index has at most $2j+1$ contributors. Cauchy–Schwarz with this bounded overlap, followed by $\sum l^{-2} < \infty$, yields

$$\left\|x^j q_n'\right\|_{L^2(\omega)}^2 \leq C_j(n+1)^{2j}.$$

Since $\omega \asymp e^{-2a|x|}$, the product rule and (8) prove the claim. At real zeros the potential term is paid by $|V|F^2 = |G|$ almost everywhere. No pointwise division bound is needed in this step. □

For a function e define the fixed weighted Sobolev norm

$$|e|_{p,1} = \sum_{j=0}^{p} (\left\|x^j e\right\|_2 + \left\|x^j e'\right\|_2).$$

The product rule and (8) imply

$$\|F\cosh(ax)e\|_* \le C|e|_{p,1}. \tag{35}$$

In particular $|V|F^2\cosh^2(ax) = |G|\cosh^2(ax)$ has polynomial growth globally. The derivative term includes $F'\cosh$ and $aF\sinh$.

Proposition 6.2 (Bandlimited critical receiver)**.** *Let v be real and smooth, supported in $[-T,T]$, $T \ge 1$, with $\|v\|_2 \le B$, $B \ge 1$. Fix a real even smooth frequency cutoff χ_0, equal to one on $[-1,1]$ and zero outside $[-2,2]$. For $H \ge 1$ put*

$$b_H(k) = \frac{1}{2\pi}\chi_0(k/H)\widehat{v}(k), \quad v_H(x) = \int b_H(k)e^{ikx}\,dk, \qquad \widehat{v}(k) = \int v(x)e^{-ikx}\,dx.$$

For $0 < \eta < 1$, if $|v - v_H|_{p,1} < \eta/(2C)$, there exists a real polynomial P, preserving the parity of v, with

$$\|FP - F\cosh(ax)v\|_* < \eta, \qquad \log(\deg P + 2) \le C[H + \log(B(T+2)/\eta + 2)]. \tag{36}$$

For any finite family, rational rounding preserves degree and parity with any additional positive joint-error allowance.

Proof. We give the boundary coefficient argument explicitly. Integration of (33) against $e^{ikx}\omega(x)$ gives

$$\int e^{ikx}q_n(x)\omega(x)\,dx = \operatorname{sech}(H_M k/2)[i\tanh(H_M k/2)]^n.$$

For each fixed n this identity continues to $|\Im k| < \pi/H_M$. Thus the shifts $k \mp ia$ are strictly inside the strip. Fubini for compact k support is justified by weighted L^1 integrability. The MP coefficients of $f_H = \cosh(ax)v_H \in L^2(\omega)$ are

$$c_n = \frac{1}{2}\sum_{\varepsilon=\pm1}\int b_H(k)A_\varepsilon(k)\zeta_\varepsilon(k)^n\,dk, \tag{37}$$

where

$$A_\varepsilon = \operatorname{sech}(H_M k/2 - \varepsilon i\pi/4), \qquad \zeta_\varepsilon = \varepsilon\operatorname{sech}(H_M k) + i\tanh(H_M k).$$

Each phase lies exactly on the unit circle. Set $\vartheta_+ = \arcsin(\tanh(H_M k))$ and $\vartheta_- = \pi - \vartheta_+$. These parametrize opposite half circles, with one reversed orientation, and

$$|\vartheta_\varepsilon'| = H_M\operatorname{sech}(H_M k), \qquad |A_\varepsilon|^2 = 2\operatorname{sech}(H_M k).$$

On the two arcs put $\psi(\vartheta_\varepsilon(k)) = b_H(k)A_\varepsilon(k)/(2|\vartheta_\varepsilon'(k)|)$, and extend by zero. The cutoff is flat at the finite arc endpoints. Equation (37) becomes $c_n = \int \psi(\vartheta)e^{in\vartheta}\,d\vartheta$. The order-zero change of variables cancels exactly: $|A_\varepsilon|^2/|\vartheta_\varepsilon'| = 2/H_M$. Each fixed-order circle derivative introduces at most a factor $C_s\operatorname{sech}(H_M k)^{-1}$. Logarithmic k derivatives of the remaining amplitude and Jacobian are bounded at every fixed order. Consequently

$$\|\psi\|_{H^s(\mathbb{T})} \le C_s e^{2sH_M H}\|b_H\|_{H^s(\mathbb{R})} \le C_s e^{2sH_M H}B(1+T)^s. \tag{38}$$

The last inequality is Plancherel applied to the first s spatial moments of the compact v; derivatives of $\chi_0(k/H)$ cost $H^{-l} \le 1$. Parseval for the complete integer Fourier series and Cauchy–Schwarz give, for fixed $s > p + 1/2$,

$$\sum_{n>n_0} |c_n|(n+1)^p \le C_s n_0^{p-s+1/2} e^{2sH_M H} B(1+T)^s. \tag{39}$$

The estimates retain the negative circle coefficients.

Choose the fixed integer $s = p + 2$. Lemma 6.1 shows that $F\sum_{n\le n_0} c_n q_n$ converges absolutely in the original joint norm, with tail bounded by the right side of (39), whose power is $-3/2$. MP completeness identifies the polynomial partial sums with f_H in $L^2(\omega)$. On every compact real interval multiplication by F is bounded into ordinary L^2; hence the same partial sums converge there to Ff_H. The global joint limit must therefore agree locally, and thus almost everywhere globally, with Ff_H. Closedness of the derivative and weighted L^2 norms identifies all parts of the joint limit. Also Ff_H belongs to that norm directly by (35), since v_H is Schwartz. We do not assume that multiplication by F is bounded from $L^2(\omega)$ to unweighted global L^2; its polynomial factor would invalidate that shortcut.

Taking n_0 to be the ceiling of a sufficiently large exponential in $H + \log(B(T+2)/\eta + 2)$ makes the joint tail below $\eta/2$. Equation (35) pays the other half, proving (36). A real target has real MP coefficients; the even cutoff and parity of q_n preserve either parity. Finally all $\|Fx^j\|_*$ are finite by (8). Rounding the finitely many correct-parity monomial coefficients therefore gives rational coefficients with arbitrary positive remaining error, without raising degree. □

7 Centered leakage and cancellation in the original graph norm

We now use the slots selected in Lemma 5.3. Their number $M \ge cT/h$ and their low-order margins are fixed before choosing the cutoff order. Retain the fixed integer $q > 2p + 4$ from Lemma 5.2.

Lemma 7.1 (Uniform finite jets)**.** *Fill each selected slot by the cap formula (18) with χ_m. Define $v_r = u_r/g$ on its support and extend it by zero. For every fixed integer $0 \le j \le q+1$ and the selected integer $m \ge 1$,*

$$\left\| D^m[(x-r)^j v_r] \right\|_1 \le C_j B_T m!(CA)^m, \qquad \|v_r\|_2 \le CB_T, \quad A = C_L/h. \tag{40}$$

Constants are independent of m, T, r; the profiles may depend on m.

Proof. On a fixed slot $t = x - r \asymp_L h$. If $\varphi(t) = (-\log t - a_c)/\ell_\nu$, then $|\varphi^{(j)}(t)|/j! \le K_0 t^{-j}$ for all $j \ge 1$, uniformly over $1 \le \nu \le J$. Faà di Bruno and (25) give, for $1 \le n \le m$,

$$\begin{aligned} |D^n[\chi_m(\varphi(t))]| &\le Cn! t^{-n} \sum_{k=1}^{n} \frac{(CmK_0)^k}{k!} \binom{n-1}{k-1} \\ &\le Cn!(2/t)^n e^{Cm}. \end{aligned}$$

The ordered-composition sum used here is $\sum_{\sum k_j = k, \sum j k_j = n} k!/\prod k_j! = \binom{n-1}{k-1}$. In particular the denominator $k!$ is retained. The normalization in (18) is uniformly bounded by (24). The analytic factor $\sqrt{t}/g(r+t)$ is bounded by CB_T on slightly smaller root-free disks and its lth derivative by $CB_T l!(C_L/h)^l$ by Cauchy's estimate. In Leibniz's formula the derivative factorials cancel the binomial coefficients exactly. At final order m the e^{Cm} and at most $m+1$ summands are absorbed in one fixed constant to power m. Multiplication by $(x-r)^j$ has only $j+1$ terms and bounded

centered powers on the support. Its fixed combinatorial factors are absorbed in $C_j C^m$. The support has bounded measure, giving the L^1 bound. The zeroth bound follows separately from the uniform cutoff supremum and (31); all zero extensions are smooth because the cutoffs are flat. No bound for derivatives above the selected order is used. □

Let χ_0 be the fixed even smooth Fourier cutoff of Proposition 6.2 and let $v_{r,H}$ have transform $\chi_0(k/H)\widehat{v}_r(k)$, with that proposition's Fourier convention. Write $e_r = v_r - v_{r,H}$.

Lemma 7.2 (Full-line centered error)**.** *For $H \geq 1$ and $m > q + 4$,*

$$\sup_x (1 + |x - r|)^q (|e_r(x)| + |e_r'(x)|) \leq C B_T m! (CA)^m H^{2-m}. \tag{41}$$

Consequently, with one sufficiently large fixed K_1 and $H = K_1 Am$,

$$|e_r(x)| + |e_r'(x)| \leq \varepsilon_T w_r(x), \qquad \varepsilon_T = C B_T A^2 (K_1 m)^2 e^{-cm}. \tag{42}$$

Proof. Fourier transform in the centered variable $t = x - r$. By integrating by parts m times, (40) bounds each fixed frequency derivative through order $q + 1$ by $C_j B_T m! (CA)^m |k|^{-m}$ for $k \neq 0$. Apply Fourier inversion after differentiating $(1 - \chi_0(k/H))\widehat{v}_r(k)$, or k times this expression, through each order $0 \leq j \leq q$. Cutoff derivatives have support $|k| \geq H$ and cost $H^{-l} \leq 1$; differentiating the extra k only improves its power. Thus every required inverse integral is bounded by a constant times

$$B_T m! (CA)^m \int_{|k| \geq H} |k|^{1-m} \, dk \leq C B_T m! (CA)^m H^{2-m}.$$

These estimates bound both $t^j e_r$ and $t^j e_r'$ uniformly, which implies (41). Since $m! \leq m^m$, $H = K_1 Am$ makes the right side at most $C B_T A^2 (K_1 m)^2 (C/K_1)^m$. Choose K_1 large enough and absorb its fixed constants to obtain (42). The estimate includes every off-support point, where Fourier truncation creates leakage. □

Proposition 7.3 (Weighted whole-family cancellation)**.** *For every fixed $0 < \eta < 1$, a choice*

$$m = \lceil C_\eta \log\log(T + 40) \rceil, \qquad H = K_1 Am = O_\eta(\log T \log\log T) \tag{43}$$

gives

$$\left\| g \sum_r b_r e_r \right\|_* \leq \tfrac{1}{4} \eta \, \|b\|_2$$

for all real coefficient vectors b and all sufficiently large T.

Proof. The spacing estimate $\sum_r w_r(x) \leq C/h$ and weighted Cauchy–Schwarz give

$$|\sum_r b_r e_r|^2 + |\sum_r b_r e_r'|^2 \leq C \varepsilon_T^2 h^{-1} \sum_r |b_r|^2 w_r(x).$$

The product rule and $|V| F^2 = |G|$ imply the exact domination

$$\|ge\|_*^2 \leq C \int_{\mathbb{R}} \nu(x) (|e(x)|^2 + |e'(x)|^2) \, dx. \tag{44}$$

Integrating the preceding pointwise bound and applying (30) yields

$$\left\| g \sum_r b_r e_r \right\|_*^2 \leq C \varepsilon_T^2 h^{-1} T^{7/2} (\log T)^B \, \|b\|_2^2 .$$

Its square root is at most

$$CB_T T^{7/4} A^2 m^2 h^{-1/2} (\log T)^{B/2} e^{-cm} \|b\|_2 . \tag{45}$$

The factor $T^{7/4}$ cancels the factor $T^{-7/4}$ in B_T. Every remaining factor outside e^{-cm} is a fixed power of $\log T$ or m. Choosing the fixed C_η in (43) sufficiently large pays the claimed error; increase the height threshold so $m > q + 4$. No target-count factor has been substituted for the weighted overlap. Both exterior tails are already included in (30). For varying η one may instead choose a sufficiently large fixed multiple of $1 + \log\log(T + 40) + \log(1/\eta)$, with the corresponding degree cost. □

7.1 Common MP return and true Ritz residual

For $v_b = \sum_r b_r v_r$, disjoint original supports give $\|v_b\|_2 \le CB_T \|b\|_2$; its spatial moments through the fixed order $s = p + 2$ are bounded by $CB_T(1 + T)^s \|b\|_2$. The coefficient construction in Proposition 6.2 is linear. Apply it with support parameter $T + 1$, since a slot based at $r \le T$ may extend by at most $h < 1$ beyond T; this changes no stated resource order. Its circle estimate (38) therefore applies to this entire map, giving a common polynomial degree n for which

$$\|FP_b - g v_{b,H}\|_* \le \tfrac{1}{4}\eta \|b\|_2 , \qquad \log(n + 2) \le C[H + \log(T + 2) + \log(1/\eta + 2)].$$

Here B_T may be enlarged to one in the MP bound, since it is eventually bounded. Together with Proposition 7.3 this gives a complete analytic synthesis error at most $\eta/2$ and the asserted log degree. The global limit is identified by local MP convergence and absolute graph convergence, as in the proof of Proposition 6.2; a global bounded $F : L^2(\omega) \to L^2(\mathbb{R})$ multiplier is not assumed.

For parity, reflect each positive target and divide by $\sqrt{2}$: $u_r^\varepsilon(x) = (u_r(x) + (-1)^\varepsilon u_r(-x))/\sqrt{2}$. The combined centers have the same spacing, and ν is even. Use coefficients $b_r/\sqrt{2}$ at the two centers, with the appropriate sign. All estimates and the coefficient ℓ^2 norm are preserved. The even frequency cutoff and MP polynomials preserve the requested parity.

Let $\phi_j = Fx^j$ over the permitted indices $j \le n$. Their graph Gram A_n is positive definite: F is nonzero on a real open interval, so no nonzero polynomial combination vanishes. With the target and cross Grams U, B_n, orthogonal best approximation gives

$$b^{\mathsf{T}}(U - B_n^{\mathsf{T}} A_n^{-1} B_n) b = \left\| (I - \Pi_n) \sum_r b_r u_r \right\|_*^2 .$$

The constructed map bounds this by $\eta^2 \|b\|_2^2$. These are the original graph matrices, including the full derivative and absolute potential. No numerical inverse or surrogate MP Gram is presumed evaluated. This proves the residual part of Theorem 1.3.

For rational coefficients, retain the remaining analytic allowance $\eta/2$. Every $\|Fx^j\|_*$ is finite. Round each entry of the finite polynomial coefficient matrix within

$$\frac{\eta}{2\sqrt{M} \sum_{j \le n} \|Fx^j\|_*},$$

only over the permitted parity indices. Each column changes in graph norm by at most $\eta/(2\sqrt{M})$, so the operator error is at most $\eta/2$ by the Hilbert–Schmidt bound. The degree is unchanged, and the full synthesis error is at most η. No coefficient-bit, inverse-conditioning or runtime estimate follows.

8 A constructive strip receiver and moving-width channels

The fixed-width receiver below acts on an explicitly prescribed entire class. It approximates analytic, zero-compatible intermediate targets, rather than compact caps at fixed width. A second argument then transfers the endpoint margins over a quantified shrinking interval.

8.1 Actual strip envelopes and shifted MP polynomials

For every fixed $Y < \infty$ there are an integer p_Y and C_Y such that

$$|F^{(j)}(x+iv)| \le C_Y(1+|x|)^{p_Y} e^{-a|x|}, \qquad |g^{(j)}(x+iv)| \le C_Y(1+|x|)^{p_Y}, \quad j \le 4, \tag{46}$$

for $|v| \le Y$, where $a = \pi/8$, $g = F\cosh(az)$. Here is a direct justification of the required uniformity. On the larger strip $|\Im z| \le 2$ the zeta argument has real part in $[-1/2, 3/2]$. For sufficiently large $|\Re z|$, a fixed-order Euler–Maclaurin formula, with truncation comparable to the imaginary part of that argument and enough Bernoulli terms, gives $|\zeta(s)| \le C(1+|\Im s|)^2$ uniformly. The remainder is bounded by $CX^{1-\Re s}$ times a fixed rising-factorial ratio; the pole integral is harmless at large imaginary part. Vertical-strip Stirling gives $|\Gamma(s/2)| \le C(1+|\Im s|)^{1/4} e^{-\pi|\Im s|/4}$. The quadratic xi prefactor costs at most a further square. Since $\Im s = x/2$, the exponential is $e^{-\pi|x|/8}$; a polynomial power five suffices for F there. Cauchy's formula on fixed disks gives derivatives through order four on $|\Im z| \le 1$, and $p_1 = 8$ is a conservative choice. The compact part is covered by the entire completed function, without separately bounding its apparent gamma or zeta poles. For any other fixed Y, use a larger fixed strip and an Euler–Maclaurin order exceeding its left edge. This gives (46) with constants allowed to depend on Y.

For real x, $A_x(y) = |F(x+iy)|^2$ is even in y, so $A_x'(0) = 0$ and γ_y is the average of $A_x''/8$ on $[0, y]$. Therefore

$$|\gamma_y(x)| \le C_Y(1+|x|)^{2p_Y} e^{-2a|x|}, \qquad 0 \le y \le Y. \tag{47}$$

In particular there is no singular endpoint factor $1/y$ in this bound. Direct differentiation at zero gives $\gamma_0 = G/4$, consistent with (1).

Recall that the critical MP scale is $H_M = 4$ and $\omega(x) = 1/[4\cosh(\pi x/4)]$.

Lemma 8.1 (Shifted-basis growth)**.** *For every fixed Y there are an integer b_Y and a constant K_Y such that*

$$\|q_n\|_{\mathcal{J}_Y} \le K_Y(n+1)^{b_Y}, \qquad n \ge 0. \tag{48}$$

Proof. The classical generating function gives the exact shift identity

$$q_n(x+iv) = \sum_{j=0}^{n} A_j(v) q_{n-j}(x), \qquad \sum_{j \ge 0} A_j(v) t^j = \left(\frac{1+it}{1-it}\right)^{v/4}, \quad |t| < 1, \tag{49}$$

with logarithms analytic at zero. On $|t| = r$ the modulus of the last function is at most $[(1+r)/(1-r)]^{|v|/4}$. For $n \ge 1$ choose $r = n/(n+1)$. Cauchy's inequality yields $|A_j(v)| \le e(2n+1)^{|v|/4}$ for all $j \le n$. Orthogonality gives an $L^2(\omega)$ norm at most $C_Y(n+1)^{Y/4+1/2}$ for the shifted polynomial.

Differentiation in x multiplies the generating function by

$$L(t) = \frac{i}{4}\log\frac{1-it}{1+it}.$$

On the same circle $|L(t)| \le C\log(2(n+1))$. Cauchy's coefficient bound for the product in (49) therefore gives the same norm estimate for $q_n'(x+iv)$ with this additional logarithm. For $n = 0$ the

derivative vanishes. Multiplication by x on polynomials of degree at most n costs at most $4(n+1)$ into the next degree, by the MP three-term recurrence. Iterating on the whole shifted polynomial proves, for fixed m,

$$\|(1+|x|)^m q_n(x+iv)\|_{L^2(\omega)} + \|(1+|x|)^m q_n'(x+iv)\|_{L^2(\omega)} \\ \le C_{m,Y}(n+1)^{m+Y/4+1/2}\log(n+2), \quad |v| \le Y.$$

The absolute weight is equivalent to a fixed sum of monomial weights, which justifies this notation. Product differentiation, (46) and $\omega \asymp e^{-2a|x|}$ pay the strip integral in $E_y[q_n]$; division by y is canceled by its height average. The weight (47) pays $Q_y^{\mathrm{abs}}[q_n]$ on the real line. An integer $b_Y > p_Y + Y/4 + 1$ absorbs the logarithm. The endpoint follows by its limiting identity. This proves (48) globally, without a bounded-multiplier inference from local data. □

Theorem 8.2 (Fixed-width analytic-class receiver)**.** *Fix $0 < Y < \infty$. Let v be real smooth with support in $[-T,T]$, $T \ge 1$, and let $H \ge 1$. With the fixed Fourier cutoff of Proposition 6.2, set*

$$v_H(z) = \frac{1}{2\pi}\int_{\mathbb{R}} \chi_0(k/H)\widehat{v}(k)e^{ikz}\,dk, \qquad f_H(z) = \cosh(az)v_H(z).$$

There are real MP partial sums $P_n = \sum_{j\le n} c_j q_j$ such that, for every fixed integer $s > b_Y + 1/2$,

$$\|P_n - f_H\|_{\mathcal{J}_Y} \le K_{Y,s}(n+1)^{-s+b_Y+1/2}e^{8sH}(1+T)^s\|v\|_2. \tag{50}$$

For a finite compact-input family satisfying $\|\sum_j d_j v_j\|_2 \le B\|d\|_2$, the same construction is linear and this estimate holds in whole-operator norm with B in place of $\|v\|_2$. Parity is preserved, and finite rational rounding can be included with arbitrarily small extra $\mathcal{J}_Y$ operator error and no degree increase.

Proof. The complete circle construction in Proposition 6.2 gives, for every fixed integer s,

$$\left(\sum_{j\ge1} j^{2s}|c_j|^2\right)^{1/2} \le C_s e^{8sH}(1+T)^s\|v\|_2. \tag{51}$$

The factor $8s$ is $2sH_M$ in (38). Its derivation uses only fixed spatial moments, the flat finite arc endpoints and the exact order-zero amplitude/Jacobian cancellation. It is linear in the input norm. By Lemma 8.1, the triangle inequality in each joint norm and Cauchy–Schwarz sum the tail:

$$\sum_{j>n}|c_j|(j+1)^{b_Y} \le C(n+1)^{-s+b_Y+1/2}\left(\sum_{j\ge1} j^{2s}|c_j|^2\right)^{1/2}.$$

To identify the limit, (51) is available at every fixed order, and the generating function bounds q_n polynomially in n on every compact complex set. Thus the series converges locally uniformly with derivatives to an entire function. On the real axis MP completeness identifies it locally with f_H; the identity theorem then identifies its entire limit. Absolute convergence using (48) identifies its global strip-energy and absolute-weight limits, proving (50). This is a constructive return for Ff_H, which retains every actual F-zero jet by multiplication.

For example $s = b_Y + 2$ gives the sufficient explicit choice

$$n+1 \ge \max\left\{2, \left[\frac{K_{Y,s}e^{8sH}(1+T)^s\|v\|_2}{\eta}\right]^{2/3}\right\}$$

for error η. Every step applies to a linear combination with its synthesis bound, proving the operator statement. Reality and parity follow from the even cutoff and the MP parity. For rational rounding use finiteness of $\|x^j\|_{\mathcal{J}_Y}$ and the same coefficient-matrix Hilbert–Schmidt payment used in the endpoint proof. □

8.2 Complex graph selection before the anchors

The moving-width application uses an additional filter on the same geometry-good seeds, before choosing any typical anchors or cutoff order. Enlarge the fixed envelope integer p in the preceding argument to cover (46) at width one, and choose the fixed decay order $q > 2p + 4$ accordingly. Define

$$\nu_T(x) = \sup_{|v|\le h/4} \sum_{j=0}^{4} |g^{(j)}(x+iv)|^2, \qquad w_r(x) = (1+|x-r|)^{-q}.$$

For every fixed comparable-height interval,

$$\int_{a_0T/2}^{2T} \nu_T(x)\,dx \le CT^{9/2}(\log T)^9. \tag{52}$$

Indeed Lemma 5.1, with its fixed enlarged near-critical rectangle, gives the derivative estimates through order four. Stirling's formula for the completed prefactor times cosh gives $CT^{7/4}(\log T)^j$ for its jth derivative there. Leibniz therefore gives

$$\int |g^{(j)}(x+iv)|^2\,dx \le CT^{9/2}(\log T)^{2j+1}, \quad |v| \le h/2,$$

on slightly enlarged comparable intervals.

To place the supremum inside the integral, apply the analytic submean inequality on the disk of radius $h/4$ about $x+iv$, $|v| \le h/4$. Bound that disk integral by the full rectangle $[x-h/4, x+h/4] \times [-h/2, h/2]$. This bound is independent of the chosen v. Integrating over x contributes a factor h, and integrating the uniform line estimate over the vertical interval contributes another h. They cancel the submean factor h^{-2}. The slight height enlargement absorbs the edges, and the largest logarithmic exponent is nine, proving (52).

Globally (46) bounds ν_T by $C(1+|x|)^{2p}$. The same spacing and whole-tail proof as Lemma 5.2 yields

$$\sum_r T^{-7/2} \int_{\mathbb{R}} \nu_T(x) w_r(x)\,dx \le C(T/h)(\log T)^9.$$

Markov's inequality therefore leaves $\ge cT/h$ seeds with

$$\int_{\mathbb{R}} \nu_T(x) w_r(x)\,dx \le CT^{7/2}(\log T)^9. \tag{53}$$

Its threshold may be fixed to discard an arbitrarily small fixed fraction. Make this filter before Lemma 5.3, and choose the CLT exceptional fraction relative to the remaining complete slot union. The anchor/corridor proof then gives, on the same new slots,

$$|1/g(z)| \le B_T = CT^{-7/4}(\log T)^{A_{\mathrm{rec}}} \tag{54}$$

for one fixed exponent A_{rec}. This filter is independent of m. It also pays the earlier real graph budget by (28), so all endpoint arguments remain valid on the newly selected family. This is a reselection before the anchors, rather than an unsupported intersection with a previously chosen anchor set.

8.3 Analytic targets and a second-derivative budget

Fill the newly retained slots with the long or short caps and put $v_r = u_r/g$. Use

$$m = \lceil C_m \log\log(T+40) \rceil, \qquad H = K_1(C_L/h)m = O(\log T \log\log T)$$

with constants sufficient for a small fixed endpoint graph error. The preceding centered-tail/cancellation proof gives analytic intermediates

$$U_r(z) = g(z)(v_r)_H(z) = F(z)f_r(z), \qquad f_r(z) = \cosh(az)(v_r)_H(z),$$

whose endpoint entire-span margins remain strictly stronger than the desired final margins, with bounded endpoint energy and absolute-Q operator norms. This follows by first choosing stronger raw margins and then spending a fixed small graph error. The inputs have disjoint support, $\|v_r\|_1 + \|v_r\|_2 \le CB_T$ and support within $|x - r| \le h$.

Let K_H be the inverse Fourier transform of $\chi_0(k/H)$. For $l \le 2$ and $H|v| \le 1$, scaling its compact Fourier integral and integrating by parts q times gives

$$|\partial_z^l K_H(x+iv)| \le C_q H^{l+1}(1+|x|)^{-q}.$$

The exact exponential factor is at most $e^{2H|v|}(1+H|v|)^q$, bounded in this regime. More precisely the natural scaled decay is $(1+H|x|)^{-q}$; since $H \ge 1$ it implies the displayed weaker unit-scale bound. Translation by u with $|u - r| \le h \le 1$ changes that unit-scale weight by a fixed factor. Convolution against v_r yields

$$|(v_r)_H^{(l)}(x+iv)| \le CB_T H^3 w_r(x), \qquad l \le 2. \tag{55}$$

Weighted Cauchy–Schwarz and $\sum_r w_r \le C/h$ bound the squared coefficient synthesis of each derivative by $CB_T^2 H^6 h^{-1} \sum_r |d_r|^2 w_r(x)$. Product differentiation of $U_r'' = g''(v_r)_H + 2g'(v_r)_H' + g(v_r)_H''$, followed by (53), proves

$$\sup_{|v|\le y} \| \sum_r d_r U_r''(\cdot + iv)\|_2 \le R_T \|d\|_2, \qquad R_T = CB_T H^3 h^{-1/2} T^{7/4} (\log T)^{9/2} \le CH^3 (\log T)^{A_{\mathrm{rec}}+5}, \tag{56}$$

provided $y \le h/4$ and $Hy \le 1$. Every product derivative, overlap cost and exterior tail is included. Reflection into either parity has the same bound, since the complex supremum density is even and the combined centers retain their spacing.

Fix a sufficiently small $\varepsilon > 0$ and choose

$$y_T = \min\{h/4, 1/H, \varepsilon/R_T\}. \tag{57}$$

There is a fixed finite K, for example any $K > A_{\mathrm{rec}} + 10$, and $c > 0$ such that $y_T \ge c(\log T)^{-K}$ eventually. This follows from $H = O(\log T \log\log T)$ and (56). The height onset inherits the CLT onset.

8.4 The original signed form on the same analytic states

Let $U = \sum_r d_r U_r = Ff$. Its endpoint derivative operator norm is bounded by a constant M_* from the paid endpoint construction. Integration of U'' vertically gives

$$\|U'(\cdot + iv) - U'\|_2 \le |v| R_T \|d\|_2.$$

Expanding the full squared strip derivative and averaging proves the whole energy Gram bound

$$\|\mathsf{E}_y - \mathsf{E}_0\|_{\mathrm{op}} \le C\{M_* y R_T + (yR_T)^2\}. \tag{58}$$

For Q_y, keep the multiplier $f(x)$ on the real axis. Taylor's theorem for the even function $A_x(y)$ gives

$$|\gamma_y(x) - \gamma_0(x)| \cosh^2(ax) \le C y^2 \nu_T(x), \qquad y \le h/4. \tag{59}$$

To verify the derivative payment, $A_x^{(4)}$ is a sum of products of F derivatives of total order four. For $|v| \le 1$, $|\cosh(a(x+iv))| \ge \cos(a)\cosh(ax)$, and every fixed derivative of the reciprocal is bounded by a constant times that reciprocal. Expressing $F = g/\cosh(az)$ therefore bounds $\cosh^2(ax)|A_x^{(4)}(v)|$ by $C\sum_{j\le 4}|g^{(j)}(x+iv)|^2$. The even Taylor remainder in $A_x'(y)/(8y)$ has size at most Cy^2 times this fourth derivative supremum, proving (59). In particular the fourth-jet normalization is $A_x^{(4)}(0)/48$.

Using (55) at $l = 0$, the whole-family overlap estimate and (53) in the original real-axis integral gives

$$\|\mathsf{Q}_y - \mathsf{Q}_0\|_{\mathrm{op}} \le C(yR_T)^2. \tag{60}$$

Indeed the direct bound is $Cy^2 B_T^2 H^6 h^{-1} T^{7/2} (\log T)^9$, the square of the quantity in (56). The same estimate bounds the difference of the absolute-Q forms by $||\gamma_y| - |\gamma_0|| \le |\gamma_y - \gamma_0|$. Hence the analytic target synthesis is uniformly bounded in the joint forms for $0 \le y \le y_T$. Equations (58)–(60) concern all coefficient vectors, retaining every mixed matrix entry. No sign of γ_y or complex-shift substitution for $f(x)$ was used.

8.5 One rational polynomial return for the entire interval

Proof of Theorem 1.2. Apply Theorem 8.2 with $Y = 1$ and fixed $s = b_1 + 2$ to the whole compact-input family v_r, whose synthesis bound is $CB_T\|d\|_2$ and support is contained in $[-2T, 2T]$. Choose a small fixed joint operator error η. The explicit degree choice in that theorem gives

$$\log(n+2) \le C[H + \log(T+2) + \log(1/\eta + 2)] \le C \log T \log\log T,$$

enlarging B_T to one if necessary. One common polynomial map approximates the analytic targets for every $0 \le y \le 1$. On the smaller interval $y \le y_T$, their joint operator norm is bounded by the preceding comparison. Thus the complete E_y and original Q_y Gram perturbations are at most $C(2M_*\eta + \eta^2)$ uniformly there. This follows by taking the difference of the full two bilinear products and applying Cauchy–Schwarz in the positive joint norm. Reserve half the error for rational coefficient-matrix rounding in $\mathcal{J}_1$; that gives the same simultaneous interval without increasing degree.

Start with stronger long/short raw margins, choose the endpoint error small, then ε and η small relative to the retained strict margins. Equations (58)–(60) and the final return then give the claimed quotients on the same spaces for all $0 \le y \le c(\log T)^{-K}$. Positive energy Gram gives independence and their dimension is at least $cT\log T$. The selections are performed separately for the two channel lengths and for each parity; their bases are fixed across y. Finally set $\log T = c_1 \log N/\log\log N$, with fixed small c_1, as in the endpoint degree inversion. All large integer degrees are covered, the count gives (5), and the width becomes (6). This proves the theorem with every mixed and rounding cost retained. □

The fixed-width receiver establishes a constructive sufficient class, not closure of the full holomorphic zero ideal. The moving-width theorem pays an actual rate and ordinary degree; mere

necessity of y/cap width $\to 0$ would not supply it. Its interval still shrinks to zero. Neither result supplies a fixed-positive-width all-order sign or controls the original signed complement outside the selected spaces.

9 Whole Gram transfer, all degrees and repair rank

Proof of Theorem 1.3. Fix the margin, geometric constants, a sufficiently large relative budget L, both cap lengths when needed, the graph-budget threshold, and the small exceptional-measure fraction. These choices precede T and the finite cutoff order. Lemma 5.3 leaves $M \geq cT/h$ slots for either family. The uniform cutoff energies preserve the raw quotient margins at every selected order, with energy Gram I and a joint synthesis norm $\leq M_*$ independent of M, T, m. Disjoint supports prove this latter bound quadratically. Proposition 7.3 and the common MP return provide the stated linear approximation and sufficient degree. Parity and rational rounding were proved above. Orthogonal best approximation supplies its true Ritz residual. This proves every part of the receiving statement. □

If $U_0 : b \mapsto \sum_r b_r u_r$ and $L_n : b \mapsto FP_b$ have $\|U_0\| \leq M_*$ and $\|L_n - U_0\| \leq \eta$, boundedness of the two polarized forms in (1) gives

$$\|\Delta E\|_{\mathrm{op}} \leq \frac{\eta(2M_* + \eta)}{4\pi}, \qquad \|\Delta Q\|_{\mathrm{op}} \leq \frac{\eta(2M_* + \eta)}{8\pi}. \tag{61}$$

This is a whole coefficient-vector estimate. Orthogonality of the graph residual does not make the separate E and Q errors quadratic, since their separate cross terms need not vanish. Choose a fixed sufficiently small η to preserve the raw margins $Q/E \geq 2 - \theta/4$ or $q_{\min} \leq Q/E \leq \theta/8$. Then (61) preserves $E > 0$ and respectively $Q/E \geq 2 - \theta$ or $0 < Q/E \leq \theta$ on the entire received span. Positive energy Gram also gives linear independence. Every mixed entry is paid; no cross Gram between the two different channel families is needed.

Proof of Theorem 1.1. For this fixed tolerance the sufficient degree satisfies

$$\log(n+2) \leq C \log T \log\log T.$$

Given a large integer N, choose

$$\log T = c_1 \frac{\log N}{\log\log N}$$

with one fixed $c_1 > 0$ sufficiently small. Since $\log\log T = \log\log N - \log\log\log N + O(1)$, the required log degree is at most $\frac{1}{2}\log N$ eventually. All integer roundings fit below N for every sufficiently large integer, rather than only on a subsequence. The count $M \geq cT \log T$ proves (5), separately for each channel and parity. Since $D/E = 1 - Q/E$, the two correction signs follow. □

For y in the interval of Theorem 1.2, let $E_N(y), D_N(y)$ denote the matrices of the full degree-N section in one parity. The energy matrix is positive definite. At $y = 0$, zero energy makes FP constant, and its decay forces $P = 0$. At $y > 0$, zero strip energy makes the analytic derivative vanish, giving the same conclusion. Put $K_N(y) = E_N(y)^{-1/2} D_N(y) E_N(y)^{-1/2}$.

Corollary 9.1 (A specified repair architecture)**.** *Fix $0 < c_0 < 1$, choose $0 < \theta < 1 - c_0$, and take the interval constants from Theorem 1.2 for this θ. At any y in the interval (6), if real α_N and symmetric C_N satisfy*

$$\|K_N(y) - \alpha_N I + C_N\|_{\mathrm{op}} \leq c_0,$$

then $\operatorname{rank} C_N$ *is at least* (5) *for all sufficiently large* N*, with constants depending on* c_0 *and uniform over that interval. The same holds in the full degree section even when the correction mixes the two parities.*

Proof. Choose $0 < \theta < 1 - c_0$. In energy coordinates the two constructed spaces satisfy $D_y/E_y \le -1 + \theta$ and $D_y/E_y \ge 1 - \theta$. If the rank is smaller than both dimensions, each space meets $\ker C_N$ in a nonzero vector. Testing normalized vectors gives $\alpha_N \le c_0 - 1 + \theta$ and $\alpha_N \ge 1 - \theta - c_0$, a contradiction. The argument does not require definiteness of the correction. In the full section embed either parity's spaces and use the same kernel intersection. Congruence preserves rank in the original coordinates. □

10 Scope of the receiving theorem

The index scale in (5) has logarithm of order $\log N / \log\log N$, which improves on the predecessor's $\sqrt{\log N}$ scale but remains $N^{o(1)}$. The exact true Ritz formula is an approximation certificate for the selected targets, not a sign estimate on their full complement. Both original Q channels constructed here are positive. The rank result applies to its stated scalar-centered strict-contraction architecture and is not an obstruction to every positivity argument.

The graph budget, typical anchor and root-free corridor are attached to the same retained slots. Replacing the graph integral by the earlier polynomial supremum would leave an uncanceled power of T in (45), requiring cutoff order $O(\log T)$ and losing the improved rate. Conversely dropping off-support leakage or the first-pole corridor would leave a gap. All exterior roots remain in the local quotient, and all infinite spatial tails enter the weighted budget. The proof uses no RH or zero-separation assumption.

The fixed-width strip receiver acts on a prescribed analytic, zero-compatible class; it does not identify the full zero ideal. Its application preserves the large rational channel spaces on the explicit shrinking-width interval (6). The original all-order sign and the Riemann-hypothesis question remain separate obligations.

The supplementary program `verify_strip_identities.py` uses Gaussian rational arithmetic and independently constructed full MP polynomials. It checks 203 finite polynomial shift identities and 29 derivative identities, through degree 28. Explicit exceptions retain the checks under Python optimization. These 232 bounded algebraic controls concern the exact shift and derivative formulas; they do not verify infinite analytic tails, the selected graph estimates, theta sign or the asymptotic theorems. Those conclusions are supplied by the proofs.

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