Three-point positivity for weighted Riemann theta products

Scott Schubr

Abstract

We give an explicit curvature argument for three-point positivity of weighted products of the Riemann theta density. If Φ is the usual positive even density, then for every $\alpha \geq 5/8$ and every nonnegative measure ν with $0 < \int \Phi(s)^{2\alpha}\, d\nu(s) < \infty$, the function $\int \Phi(s+t/2)^{\alpha}\Phi(s-t/2)^{\alpha}\, d\nu(s)$ has positive definite three-point translation matrices at distinct real positions. Repeated positions give positive semidefinite matrices. The proof combines a general curvature criterion with positive-Laplace estimates for the complete theta-series tail. All infinite-tail bounds and rational interval estimates are provided explicitly. The result concerns three-point matrices.

1 Statement and context

Use the normalization

$$\Phi(s) = \sum_{n\geq 1} (2\pi^2 n^4 e^{9s} - 3\pi n^2 e^{5s}) e^{-\pi n^2 e^{4s}}, \qquad s \in \mathbb{R}. \tag{1}$$

The series defines the classical positive even Riemann theta density; we verify the properties needed here below. A real even function H has three-point positivity if $[H(t_i - t_j)]_{i,j=1}^{3}$ is positive semidefinite for all real t_1, t_2, t_3.

Theorem 1. *Let $\alpha \geq 5/8$, and let ν be a nonnegative Borel measure on $\mathbb{R}$ such that $0 < Z = \int_{\mathbb{R}} \Phi(s)^{2\alpha}\, d\nu(s) < \infty$. Then*

$$H_{\alpha,\nu}(t) = \int_{\mathbb{R}} \Phi(s + t/2)^{\alpha}\Phi(s - t/2)^{\alpha}\, d\nu(s) \tag{2}$$

is finite and even, and its three-point translation matrix is positive definite when the three positions are distinct. For arbitrary positions it is positive semidefinite. The same assertions hold for $H_{\alpha,\nu}/Z$.

In particular, for $y > 0$ the kernel

$$K_y(t) = 2\int_0^{\infty} s \sinh(2ys)\Phi(s + t/2)\Phi(s - t/2)\, ds \tag{3}$$

has this property. At $y = 0$ the kernel K_0 is identically zero; the meaningful nonzero limit is

$$Q_1(t) = \lim_{y\downarrow 0} \frac{K_y(t)}{2y} = 2\int_0^{\infty} s^2\Phi(s + t/2)\Phi(s - t/2)\, ds. \tag{4}$$

It too is strictly positive on distinct three-point configurations. Arbitrary nonnegative polynomial weights are also covered. The fractional exponent in Theorem 1 is applied to the density *before* integration.

Weighted associated kernels and their relationship to Laguerre inequalities are classical; see Csordas [1, Sections 3–4]. Positivity for matrices of all orders is a substantially stronger question

than the theorem here. Umihara [3] proves the qualitative inequality $(\log\Phi)'''' < 0$ in this same normalization, by separating three initial theta terms and controlling the remaining tail. Our proof below uses a different positive-Laplace representation and also obtains the quantitative curvature estimate needed for the fractional density range. The cosine characterization of a 3×3 correlation matrix is classical elementary Gram geometry; see also Tomás, Alsina and Rubio-Martinez [4]. Its derivation is included. The general entrywise-power theorem of FitzGerald and Horn [2] gives the critical exponent 1 for arbitrary nonnegative 3×3 positive semidefinite matrices. Our exponent $5/8$ pertains to the specific theta-product family, not to that general class.

2 A curvature criterion

Theorem 2. *Let $\varphi > 0$ be C^4 on $\mathbb{R}$, and suppose $k = -(\log\varphi)''$ satisfies*

$$k > 0, \qquad 0 \le k'' \le 2k^2 \quad \text{on } \mathbb{R}. \tag{5}$$

For each real s put $p_s(t) = \varphi(s+t/2)\varphi(s-t/2)/\varphi(s)^2$. Every p_s has strictly positive definite three-point translation matrices at distinct positions. Any probability mixture of these functions has the same property. Neither evenness of φ nor a reflection assumption is required in this criterion.

Proof. Fix s and set $q(t) = -\log p_s(t)$ for $t \ge 0$. The exchange of the two arguments makes q even, with $q(0) = q'(0) = q'''(0) = 0$. Write $k_\pm = k(s \pm t/2)$. Differentiation gives

$$q'' = (k_+ + k_-)/4 > 0, \qquad q''' = [k'(s+t/2) - k'(s-t/2)]/8 \ge 0, \tag{6}$$

$$q'''' = (k_+'' + k_-'')/16, \qquad 2(q'')^2 - q'''' \ge k_+k_-/4 > 0. \tag{7}$$

The third-derivative sign uses convexity of k across the entire ordered interval, including negative arguments. In particular $q, q' > 0$ for $t > 0$. Define

$$F = 2q'q'' - q''', \qquad J = (q')^2 - q''(1 - e^{-2q}).$$

Both vanish at zero, and

$$F' = 2(q'')^2 + 2q'q''' - q'''' > 0, \qquad J' = (1 - e^{-2q})F > 0.$$

Consequently $\theta(t) = \arccos(e^{-q(t)})$ is increasing and strictly concave:

$$\theta' = \frac{q'}{\sqrt{e^{2q} - 1}} > 0, \qquad \theta'' = -\frac{e^{2q}J}{(e^{2q}-1)^{3/2}} < 0 \quad (t > 0).$$

At zero, $q(t) = k(s)t^2/4 + O(t^4)$, so $\theta(0) = 0$ and $\theta'(0+) = \sqrt{k(s)/2}$. Thus concavity extends to the closed half-line.

We now use the classical three-positive correlation geometry [4]. Sort and translate three distinct positions, with successive gaps $u, v > 0$. Strict concavity with zero origin implies strict subadditivity: apply concavity at u and v as fractions of $u + v$ and add. With $a = \theta(u), c = \theta(v), b = \theta(u+v)$, monotonicity therefore gives

$$|a - c| < b < a + c, \qquad 0 < a, b, c < \pi/2, \qquad a + c < \pi.$$

Let $A = p_s(u), C = p_s(v), B = p_s(u+v)$. Taking cosines on $[0, \pi]$ yields

$$AC - \sqrt{(1-A^2)(1-C^2)} < B < AC + \sqrt{(1-A^2)(1-C^2)}.$$

The matrix with successive rows $(1, A, B), (A, 1, C), (B, C, 1)$ has determinant

$$(1 - A^2)(1 - C^2) - (B - AC)^2 > 0,$$

and all its one- and two-point principal minors are positive. It is therefore positive definite, also for complex coefficient vectors. Repeated positions follow by continuity. Finally a common probability measure averages the same quadratic forms in every entry. Each nonzero form is strictly positive at every s when the positions are distinct, so its integral is strictly positive. □

3 The complete theta tail

Put $\ell = \log \Phi$ and $\kappa = -\ell''$. We will prove

$$\ell'' \le -42, \qquad \ell'''' < 0, \qquad \ell'''' + 2(\ell'')^2 \ge 3456, \tag{8}$$

and then strengthen the curvature ratio.

Let $\Theta(X) = 1+2\sum_{n\ge1} e^{-\pi n^2 X}$. Its Poisson identity implies $f(s) = e^s\Theta(e^{4s}) = f(-s)$. Termwise differentiation shows $\Phi = (f'' - f)/16$. The convergence of all derivatives on compact intervals follows from the exponential theta tail. Thus Φ is smooth and even. For $s \ge 0$ every summand of (1) is positive, and reflection proves positivity on $\mathbb{R}$.

For $s \ge 0$ write

$$x = \pi e^{4s} > 3, \quad a = \tfrac{3}{2}, \quad \tau_n = n^2 - 1, \quad D = 4x\frac{d}{dx}.$$

The first theta term and the exact remainder are

$$\Phi_1 = 2\pi e^{5s}(x - a)e^{-x}, \qquad \Phi = \Phi_1(1 + R), \qquad R = \sum_{n\ge2}\left[n^4 + \frac{an^2\tau_n}{x - a}\right] e^{-\tau_n x}. \tag{9}$$

In particular R is a Laplace transform of a positive measure:

$$R = \int_3^\infty e^{-vx}\, d\sigma(v), \quad d\sigma = \sum_{n\ge2}\left[n^4\delta_{\tau_n} + an^2\tau_n e^{a(v-\tau_n)}\mathbf{1}_{v\ge\tau_n}\, dv\right]. \tag{10}$$

Although σ need not have finite total mass, all exponentially weighted polynomial moments used below converge. Expanding the integrated moments gives explicit polynomial-in-n times $e^{-3(n^2-1)}$ majorants. This also justifies every differentiation and interchange of sums and integrals below.

Since $a/(x - a) \le 1$, the nth coefficient in (9) is at most $2n^4$. The first term of the resulting majorant is $32e^{-9}$ and the ratio of successive terms is at most $(81/16)e^{-15}$. The positive Taylor polynomial of e^3 through degree five exceeds 16. Hence

$$0 < R < \frac{32/16^3}{1 - 81/16^6} = \frac{131072}{16777135} < \frac{1}{64}. \tag{11}$$

3.1 A positive logarithmic fourth derivative of the remainder

Set $A_j = \int (vx)^j e^{-vx}\, d\sigma(v)$, with $A_0 = R$, and let $R_j = D^j R$. Because $vx \ge 9$,

$$R_1 = -4A_1, \qquad R_2 = 16B_2, \qquad R_3 = -64B_3, \qquad R_4 = 256B_4,$$
$$B_2 = A_2 - A_1, \qquad B_3 = A_3 - 3A_2 + A_1, \qquad B_4 = A_4 - 6A_3 + 7A_2 - A_1.$$

We have $B_2 \geq 8A_2/9$, $0 < B_3 \leq A_3$, $B_2 \leq A_2$, and $B_4 \geq c_0 A_4$, where $c_0 = 305/729$. Indeed $P_4(z) = z^4 - 6z^3 + 7z^2 - z$ satisfies $P_4(9)/9^4 = c_0$, and $(P_4(z)/z^4)' = (6z^2 - 14z + 3)/z^4 > 0$ for $z \geq 9$. Positive-measure moment inequalities give

$$A_1^2 \leq RA_2, \qquad A_2^2 \leq RA_4, \qquad A_1 A_3 \leq RA_4. \tag{12}$$

These follow from Cauchy–Schwarz or log-convexity of moments.

Write $U = 1 + R$ and $h = \log U$. The exact chain rule gives

$$h'''' = \frac{R_4}{U} - \frac{4R_1R_3 + 3R_2^2}{U^2} + \frac{12R_1^2R_2}{U^3} - \frac{6R_1^4}{U^4}. \tag{13}$$

The final two terms equal $1536A_1^2(2UB_2 - A_1^2)/U^4 \geq 0$, since $2UB_2 - A_1^2 \geq (16+7R)A_2/9$. By (12), $4R_1R_3 + 3R_2^2 \leq 256 \cdot 7RA_4 \leq (7R/c_0)R_4$. Using $R/U < 1/65$ in (13) proves

$$h'''' \geq \left(1 - \frac{7R}{c_0 U}\right)\frac{R_4}{U} > \frac{14722}{19825}\frac{R_4}{U} > 0. \tag{14}$$

3.2 Uniform first and second derivative envelopes

For $k = 1, 2, 4$, the absolute physical derivative of e^{-vx} is bounded by $4^k(vx)^k e^{-vx}$ on $vx \geq 9$. The integrated contribution of the nth term is therefore bounded by

$$M_{n,k}(x) = 4^k x^k e^{-\tau_n x}\left[n^4\tau_n^k + an^2\tau_n \sum_{j=0}^{k}\binom{k}{j}\frac{\tau_n^{k-j}\,j!}{(x-a)^{j+1}}\right]. \tag{15}$$

Each positive summand has logarithmic derivative at most $k/x - \tau_n < 0$. For $k = 2$, multiplication by $x + 2$ still leaves a negative derivative, bounded above by $1/(x+2) + 2/x - \tau_n \leq -2$. At $x = 3$ direct simplification gives

$$\begin{aligned} M_{n,1}(3) &= 4n^2\tau_n(6n^2 - 1)e^{-3\tau_n} \leq 24n^6 e^{-3\tau_n}, \\ M_{n,2}(3) &= 16n^2\tau_n(18n^4 - 15n^2 + 5)e^{-3\tau_n} \leq 288n^8 e^{-3\tau_n}. \end{aligned}$$

For $n = 2$ the exact coefficients are respectively 1104 and 44736. For $n \geq 3$, successive majorant terms have ratio less than $(4/3)^{10}2^{-21} < 1/100$, and $e^{-24} < 2^{-24}$. The Taylor polynomial for e^3 through degree eight exceeds 20, so $e^9 > 8000$. Thus

$$|R_1| \leq \frac{1104}{8000} + \frac{100}{99}\frac{24 \cdot 3^6}{2^{24}} < \frac{1}{4}, \tag{16}$$

$$0 < R_2 \leq U_2 := \frac{44736}{8000} + \frac{100}{99}\frac{288 \cdot 3^8}{2^{24}} = \frac{1028315733}{180224000} < 6, \tag{17}$$

$$(x+2)R_2 \leq 5U_2 < 30. \tag{18}$$

4 The global curvature band

Put $\ell_1 = \log \Phi_1$. Direct physical differentiation gives

$$\ell_1'' = -16A(x), \qquad A(x) = x + \frac{6x}{(2x-3)^2} \leq x + 2. \tag{19}$$

Also

$$\frac{\ell_1'''' + 2(\ell_1'')^2}{256} = \frac{xP_2(x)}{(2x-3)^4},$$

where $P_2 = 32x^5 - 208x^4 + 624x^3 - 960x^2 + 522x - 135$. The identity

$$\begin{aligned}(3+u)P_2(3+u) - 21(3+2u)^4 \\ = 2349u + 3762u^2 + 3072u^3 + 1488u^4 + 368u^5 + 32u^6\end{aligned}$$

proves $\ell_1'''' + 2(\ell_1'')^2 \geq 5376$ for $x \geq 3$. Since $h'' = R_2/U - R_1^2/U^2 \leq R_2$, the negativity of ℓ_1'' and (14) imply

$$\begin{aligned}\ell'''' + 2(\ell'')^2 &= \ell_1'''' + 2(\ell_1'')^2 + h'''' + 4\ell_1''h'' + 2(h'')^2 \\ &\geq 5376 - 64AR_2 \geq 5376 - 1920 = 3456.\end{aligned}$$

There is no assumption that h'' is positive. Furthermore, $\ell'' \leq -16x + R_2 \leq -42$.

4.1 Strict negativity of the fourth derivative

The polynomial P_4 above obeys

$$(P_4 - P_4')(9+v) = 1162 + 921v + 241v^2 + 26v^3 + v^4 > 0.$$

Hence $256P_4(vx)e^{-vx}$ is positive and decreasing in $x \geq 3$. Integration against the fixed measure σ proves that $R_4(x)$ decreases. Its exact $n = 2$ contribution is $e^{-3x}C(x)$, with

$$C(x) = 256\left[16P_4(3x) + \frac{18}{x-a}\sum_{j=0}^{4} P_4^{(j)}(3x)\left(\frac{x}{x-a}\right)^j\right]. \tag{20}$$

Here the derivatives are ordinary derivatives. The factorials from the Taylor expansion cancel those from the integrated exponential moments. The four exact endpoint values are

$$\begin{aligned}C(3) &= 43140096, \qquad C(16/5) = \frac{42636828818522112}{887410625}, \\ C(7/2) &= 59421432, \qquad C(4) = \frac{278436163584}{3125}.\end{aligned} \tag{21}$$

For all $n \geq 3$, (15) gives

$$M_{n,4}(3) \leq C_4 n^{12} e^{-3\tau_n}, \qquad C_4 = 12^4\left[1 + \sum_{j=0}^{4}\binom{4}{j} j!(2/27)^j\right] = \frac{107597312}{2187} < 62208.$$

The ratio of the majorant terms is less than $(4/3)^{12}/20^7 < 1/100$. Therefore the entire $n \geq 3$ fourth tail is less than

$$\frac{100}{99}\frac{62208 \cdot 3^{12}}{20^8} = \frac{14348907}{11000000} < 2. \tag{22}$$

Dropping the negative terms of (13) and using (16)–(17) gives $h'''' \leq R_4 + 9/2$.

The leading fourth derivative is $\ell_1'''' = -256B(x)$, where

$$B(x) = x + T(a/x), \qquad T(r) = \frac{r(1+4r+r^2)}{(1-r)^4}.$$

The derivative of T is positive, with numerator $1 + 11r + 11r^2 + r^3$. On $[L, H]$ we may use $B(x) \geq L + T(a/H)$; on $x \geq 4$ use $B(x) \geq 4$. The following rational bounds complete the sign proof. The middle column is a lower bound for e^{3L}; the next is a strict upper bound for $C(L)e^{-3L}$.

Interval	exponential bound	$n=2$ fourth term	lower bound for $256B$
$[3, 16/5]$	8000	5393	5430
$[16/5, 7/2]$	14400	3337	3801
$[7/2, 4]$	34000	1748	2557
$[4, \infty)$	160000	557	1024

The finite exponential certificates, beyond $e^3 > 20$, are $e^{3/5} > 1 + 3/5 + 9/50 + 27/750 > 9/5$ and $e^{3/2} > \sum_{j=0}^{4} (3/2)^j/j! > 17/4$. They imply the four displayed exponential bounds. All other table entries follow by direct substitution in (21) and T. In every row the last entry exceeds the preceding entry plus $2 + 9/2$. Together with (22), this proves $\ell'''' < 0$ on the half-line. Evenness extends all estimates to $\mathbb{R}$, proving (8). In particular $\kappa \geq 42$, $\kappa'' > 0$, and $\kappa'' \leq 2\kappa^2$. Since $\ell'(0) = 0$, twice integrating the second-derivative bound gives

$$\Phi(s) \leq \Phi(0) e^{-21 s^2} \qquad (s \in \mathbb{R}). \tag{23}$$

5 A stronger ratio and fractional densities

Proposition 3. *For every real s, the theta density satisfies*

$$0 < \frac{\kappa''(s)}{2\kappa(s)^2} < \frac{5}{8}. \tag{24}$$

Proof. It suffices to consider $s \geq 0$. The inequalities $h'''' > 0$ and $h'' < 6$ give $\kappa'' < 256B$ and $\kappa \geq 16A - 6$. First $A \geq 24/5$ for $x \geq 3$. Multiplication by $5(2x-3)^2$ reduces this to $P(x) = 20x^3 - 156x^2 + 363x - 216 \geq 0$. On $x = 3 + u/2$, $0 \leq u \leq 1$,

$$P(3 + u/2) = 9(1-u)^3 + \tfrac{21}{2} u(1-u)^2 + u^3 > 0.$$

On $x = 7/2 + v$, $v \geq 0$, it equals $20v^3 + 54v^2 + 6v + 1$. Thus $\kappa \geq 354/5$.

The classical Archimedean lower bound $\pi > 223/71 > 157/50$ allows a compact split. If $157/50 \leq x \leq 4$, monotonicity of T gives

$$\frac{\kappa''}{2\kappa^2} < \frac{128(4 + T(75/157))}{(354/5)^2} = \frac{54596377700}{88528266369} < \frac{5}{8}.$$

The difference from $5/8$ is $5870310245/708226130952 > 0$. For $x \geq 4$, we have $T(3/(2x)) \leq T(3/8) = 4056/625 < 13/2$ and $\kappa \geq 16x - 6$. The ratio $128(x + 13/2)/(16x-6)^2$ decreases there: the derivative has the sign of $-16x - 214$. Its value at 4 is $336/841 < 5/8$. This proves (24). □

Proof of Theorem 1. For $\varphi = \Phi^\alpha$ its logarithmic curvature is $k = \alpha\kappa$. If $\alpha \geq 5/8$, Proposition 3 gives $k > 0$ and $0 < k'' < 2k^2$. Apply Theorem 2. The mixture measure is $d\mu(s) = Z^{-1}\Phi(s)^{2\alpha}\, d\nu(s)$, which is a probability measure. Log-concavity of Φ implies $0 < p_s(t) \leq 1$, so all integrals in the theorem are finite. The normalized weighted kernel is precisely the common mixture of $p_s(t)^\alpha$, proving all assertions. □

For (3) and its fractional-density counterparts, Gaussian domination from (23) makes Z finite for every fixed $y > 0$ and $\alpha \geq 5/8$. For (4), the bound $\sinh(2ys)/(2y) \leq s\cosh(2y_0 s)$ for $0 < y \leq y_0$ supplies integrable domination, uniformly for t in compact sets. The normalization at $y = 0$ is therefore exactly the limit displayed in (4); it does not use the vanishing value $K_0(0)$.

6 Finite-order controls

The power threshold above is sufficient. The distinction from a necessary local threshold can be seen directly. At a fixed s,

$$p_s(t)^\alpha = 1 - at^2 + bt^4 + O(t^6), \quad a = \frac{\alpha\kappa(s)}{4}, \quad b = \frac{\alpha^2\kappa(s)^2}{32} - \frac{\alpha\kappa''(s)}{192}.$$

The determinant on $0, t, 2t$ has leading term $8a(6b - a^2)t^6$. Thus a necessary local condition is $\alpha \geq \kappa''(s)/(4\kappa(s)^2)$. It is distinct from the sufficient ratio condition used in Theorem 2.

Even the complete original curvature band does not imply four-point positivity. Indeed $\varphi(u) = \exp(-u^2 - u^4/3)$ has $k = 2 + 4u^2$ and $k'' = 8 \leq 2k^2$. At $s = 0$ its normalized product is $p(t) = \exp(-t^2/2 - t^4/24)$. The odd-reversal block for the four equally spaced positions $0, t, 2t, 3t$ has determinant

$$(1 - p(3t))(1 - p(t)) - (p(t) - p(2t))^2 = -t^8 + \frac{35}{8}t^{10} + O(t^{12}).$$

It is negative for sufficiently small nonzero t. This is a generic pointwise obstruction to extending the curvature method. It is not a counterexample for the actual theta density or its averaged kernels, and it does not settle their higher-order positivity.

The proofs consist of analytic domination and moment inequalities plus finite polynomial and rational identities. Those identities can be verified directly from the formulas displayed here; numerical sampling is not a premise.

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