The root-sample threshold for interior perturbations in the Darcy inverse problem

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Abstract

We characterize root-sample estimation of a linear conductivity functional in the anisotropic Darcy problem when all perturbations are supported in a fixed interior set. If the score is nonzero at the critical point, the criterion is that some initial segment of the ordered Hessian eigenvalues, ending at a complete eigenvalue block, has sum exceeding four times its largest eigenvalue. This is an exact Sobolev transport criterion, including in the presence of resonances. It gives the same threshold in the nonlinear white-noise experiment over small fixed interior Sobolev balls at sufficiently large smoothness. The criterion allows regular interior inference at some references whose full-neighborhood exponent is strictly below one half. This resolves the root-sample part of the interior-support minimax-rate problem listed in Liu, Chang and Hu, Section 6, third bullet under 'Further open problems,' for anisotropic minima with a nonzero score there.

1 Experiment and result

The support of an unknown conductivity perturbation can change the statistical rate for a fixed linear functional. In the full-neighborhood problem one must account for transport along complete gradient trajectories. With known interior support, the adjoint equation only needs to hold near the permitted perturbations. This permits a singular adjoint built from a subset of the slow Hessian directions. The new threshold is the largest effective dimension across complete slow eigenvalue blocks.

Liu, Chang and Hu [1, §6] ask for the minimax rate with interior support at anisotropic references. We determine exactly when the root-sample rate is attainable for a score nonzero at the minimum. The example after the main theorems shows that interior support can restore that rate even when the full-neighborhood problem is slower.

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a smooth bounded connected domain, with strictly positive smooth reference conductivity f_0 and strictly positive smooth source g, both smooth up to the boundary. Write

$$L_f = \operatorname{div}(f\nabla), \qquad L_f u_f = g, \qquad u_f|_{\partial\Omega} = 0.$$

Assume $u_0 = u_{f_0}$ has a unique critical point x_0, a nondegenerate interior minimum. Let $K \Subset \Omega$ be compact with $x_0 \in \operatorname{int}(K)$. Fix $\psi \in C_c^\infty(\Omega)$ and target $\Phi(f) = \int f\psi$. The perturbations satisfy

$$\mathcal{H}_s(K, \eta) = \{h \in H^s(\Omega) : \operatorname{supp} h \subset K,\ \|h\|_{H^s(\Omega)} \leq \eta\}.$$

The nonlinear white-noise experiment has mean u_f and noise level $N^{-1/2}$, with $f = f_0 + h$ and η small enough to ensure positivity. The linearized experiment has mean $Ah = -L_{f_0}^{-1} \operatorname{div}(h\nabla u_0)$ with the same noise and target $\int h\psi$. Risk is minimax mean absolute error in each experiment.

Let $H = D^2 u_0(x_0)$ have distinct eigenvalues $0 < \lambda_1 < \cdots < \lambda_b$ with multiplicities $m_1, \dots, m_b$. At each spectral cut set

$$Q_j = \sum_{i \le j} m_i \lambda_i, \qquad \mathcal{D}(H) = \max_{1 \le j \le b} \frac{Q_j}{\lambda_j}.$$

The full effective dimension in Liu, Chang and Hu [1] is $T_0 = Q_b/\lambda_b$. The quantity $\mathcal{D}(H)$ also tests the lower spectral blocks. Only cuts after complete blocks are needed.

Theorem 1. *In the linearized experiment, an L^2 representer for the functional over L^2 perturbations supported in K exists if and only if*

$$\psi(x_0) = 0 \qquad \text{or} \qquad \mathcal{D}(H) > 4. \tag{1}$$

If a representer exists the linearized risk is $O(N^{-1/2})$. If (1) fails, the linearized risk multiplied by $\sqrt{N}$ diverges to infinity. The latter statement holds for every $s \ge 0$ and $\eta > 0$.

Theorem 2. *Assume $\psi(x_0) \neq 0$. If $\mathcal{D}(H) > 4$, the nonlinear risk on $f_0 + \mathcal{H}_s(K, \eta)$ is of order $N^{-1/2}$ for sufficiently large s and small fixed $\eta > 0$. If $\mathcal{D}(H) \le 4$, the nonlinear risk multiplied by $\sqrt{N}$ diverges to infinity on every sufficiently small fixed interior Sobolev ball, for $s > d/2$.*

Thus the case $\psi(x_0) \neq 0$ in the interior-support open problem of [1, §6] has an exact root-sample criterion. We determine the exact threshold for the parametric rate and give an upper bound governed by a retained block below that threshold. The result uses the freedom to choose an adjoint solution on a region containing K. Such a solution can be singular along the subspace of discarded fast directions.

For example, take dimension six, with five eigenvalues equal to 1 and one equal to 10. The full effective dimension is $3/2$ and $\mathcal{D}(H) = 5$. On an ellipsoid $\{x^\top H x < 1\}$ the reference $f_0 = 1$ and constant source $g = \operatorname{tr} H$ give $u_0 = (x^\top H x - 1)/2$. For $\psi(0) \neq 0$, the full-neighborhood rate is slower than root-sample by [1, Theorem 3.3(b)]. The fixed interior-support model is root-sample by Theorem 2. Locally an admissible singular part of the adjoint is a multiple of the logarithm of the radius in the five slow coordinates.

2 A local Sobolev transport criterion

Let $b = \nabla u_0$. The positive-spectrum normal form of Sternberg and Chen, in the form of [1, Lemma S2.14], provides smooth local coordinates at the minimum in which b is conjugate to a triangular polynomial field V with components

$$V_j(y) = \lambda_j y_j + P_j(y_1, \dots, y_{j-1}), \qquad y_j \in \mathbb{R}^{m_j}.$$

A resonant monomial in the jth block uses only strictly slower variables, since its weighted degree is λ_j and its ordinary degree is at least two. In particular the divergence of V restricted to its first j blocks is the constant Q_j. Denote the restriction by $V^{[j]}$.

Lemma 1. *There is a local H^2 solution of $V \cdot \nabla W = 1$ near the origin if and only if $\mathcal{D}(H) > 4$.*

Proof. For necessity suppose that $Q_j \le 4\lambda_j$ for every j. We first give the reduction for the field on its first j blocks, when

$$V^{[j]} \cdot \nabla U = -aU, \quad a > 0, \quad \text{or} \quad V^{[j]} \cdot \nabla U = 1.$$

The backward flow X_t leaves a sufficiently small ball invariant and contracts toward zero. Its volume Jacobian is $e^{-Q_j t}$. The fastest block has the exact affine formula

$$(X_t y)_j = e^{-\lambda_j t} y_j + G_t(y_1, \dots, y_{j-1}),$$

and the other blocks do not depend on y_j. For H^2 solutions the equation integrates along this flow. One way to justify this is that composing an H^1 function with the smooth flow on each finite time interval gives the weak chain rule, by smooth approximation locally, and integration in time then gives the identity in L^2 on a smaller ball. In the homogeneous equation, $U(X_t y) = e^{at} U(y)$; in the constant-source equation, $U(X_t y) = U(y) - t$. Taking two derivatives in any pair of fastest variables gives

$$(\partial_{y_j^\mu} \partial_{y_j^\nu} U)(X_t y) = e^{(2\lambda_j + a)t}\, \partial_{y_j^\mu} \partial_{y_j^\nu} U(y)$$

in the first case, and the same formula with $a = 0$ in the second. The fast dependence of X_t is affine, so the identity follows directly from the Sobolev chain rule.

For a fixed $t_0 > 0$, the annular sets $X_{\ell t_0}(B \setminus X_{t_0} B)$, $\ell \geq 0$, partition a sufficiently small starting ball B up to the point 0 and boundaries of measure zero. If one fastest second derivative had positive squared integral on the first set, its integral on subsequent sets would be multiplied by $e^{(4\lambda_j + 2a - Q_j)\ell t_0}$, with $a = 0$ for constant source. The sum diverges. Thus every fastest second derivative vanishes in B.

On a smaller product box the solution is affine in y_j,

$$U(y) = A(y_{<j}) \cdot y_j + B_0(y_{<j}),$$

with A, B_0 locally H^2. The affine assertion follows by slicing, and the coefficients inherit H^2 by integration against fixed smooth test functions of the fast variables extracting the constant and linear coefficients. Matching the linear coefficient in the equation yields

$$V^{[j-1]} \cdot \nabla A = -(a + \lambda_j) A$$

for the homogeneous equation, and the same formula with $a = 0$ for constant source. By induction on j, every H^2 solution to the homogeneous equation with $a > 0$ vanishes. Indeed the affine slope vanishes by the induction hypothesis and then B_0 satisfies the reduced homogeneous equation. In zero dimensions the equation is $0 = -aU$. In the constant-source equation the slope therefore vanishes at each stage, so one reduces to a constant-source equation on the next smaller block. In zero dimensions this is impossible. This proves necessity, including equality.

For sufficiency choose j with $Q_j > 4\lambda_j$. A local solution is the negative of the forward exit time from a small sphere for $V^{[j]}$, viewed as a function of only the first j blocks. Indeed the forward flow decreases the exit time at unit speed. Its H^2 property follows from the exit-time calculation in Lemma 2 below, or directly from the bound $O(e^{2\lambda_j t})$ on its second derivatives at backward time t and the Jacobian $e^{-Q_j t}$. Dependence on the other variables is constant on a product neighborhood. This completes the proof. □

Lemma 2. *Let B_s be a C^3 field near zero, with $B_s(0) = 0$ and symmetric positive derivative H_s at zero. Put $Q_s = \operatorname{tr} H_s$ and $\lambda_s = \lambda_{\max}(H_s)$. If $Q_s > 4\lambda_s$, the negative forward exit time ω from a sufficiently small sphere is in H^2 inside that sphere and satisfies $B_s \cdot \nabla \omega = 1$. The bounds are uniform under common C^3 and positive spectral bounds and a common positive gap $Q_s - 4\lambda_s$.*

Proof. On a small uniform ball the symmetric part of DB_s is uniformly positive. Its backward flow contracts. The time integral of the deviation of DB_s from H_s along a trajectory in the ball is

uniformly bounded, by exponential contraction and the Lipschitz bound. In exit coordinates (t, θ), with θ on the sphere, the volume element is bounded by $Ce^{-Q_s t}\, dt\, d\sigma(\theta)$ by the Jacobian formula and transversality. The forward variational equations bound derivatives of orders one and two by, respectively, $Ce^{\lambda_s t}$ and $Ce^{2\lambda_s t}$. Implicit differentiation at the uniformly transverse exit gives the same bounds for the exit time. Its value is t. The resulting squares are integrable by $Q_s > 4\lambda_s$. To identify the weak derivatives at zero, smoothly flatten the exit-time variable above level M over a transition interval of length one, with bounded first and second derivatives of the flattening. These functions are constant near zero and converge in H^2 by the integral bounds and dominated convergence. The transport equation passes to the limit in distributions. □

3 A uniform slow-coordinate factor

The preceding exact normal form suffices for a fixed smooth reference. A nonlinear upper bound also needs a uniformly bounded representer at a data-dependent pilot. Direct use of a smooth normal form could cause small denominators when nearby Hessians cross resonances. We construct a two-derivative factor using only the separation of the retained and discarded eigenvalues. The construction also gives a bounded C^k factor for any fixed $k \geq 2$ if s is sufficiently large.

Fix a spectral cut of H. At nearby Hessians retain the same number of smallest eigenvalues. If the cut is proper, there is a uniform positive gap between the retained and discarded blocks. When the retained trace exceeds four times its largest eigenvalue at the reference, that inequality also persists with a positive gap. By the uniform regularity in [1, Lemma S4.1], for sufficiently large fixed s, all nearby fields $b_p = \nabla u_p$ have a uniform C^{M+2} bound for any fixed integer M used below. We always center coordinates at the minimum and diagonalize the Hessian orthogonally. Bounds are uniform under any choice of orthonormal bases for spectral blocks.

Fix $m > 0$ smaller than one half of a uniform lower spectral bound, and $\Lambda > 0$ larger than a uniform upper spectral bound. Fix the desired regularity $k \geq 2$ and choose $M \geq k + 1$ so $(M + 1)m > (k + 1)\Lambda$. For each p, make polynomial coordinate changes of orders 2 through M in the retained output components. At order r these remove every degree-r monomial in a retained component that contains at least one discarded input. For a monomial $y^\alpha e_i$ in a coordinate change, its linear effect on that coefficient is multiplication by $\alpha \cdot \lambda - \lambda_i$, as follows by differentiating the change along the linear vector field and subtracting the linear output. The denominator is bounded below by the positive spectral gap, since one input is discarded, the output is retained, and all eigenvalues are positive. Solve at each degree and leave earlier degrees untouched. Taylor's theorem and the common finite differentiability bound give the transformed field b with retained part of the form

$$b_s(y) = B_s(y_s) + R(y), \qquad |D^\alpha R(y)| \leq C|y|^{M+1-|\alpha|} \quad (|\alpha| \leq k). \tag{2}$$

Here $y = (y_s, y_f)$ are retained and discarded coordinates, and B_s is the restriction of the retained components to $y_f = 0$. The changes and their inverses have uniform bounds on a fixed small chart. At zero the derivative is still the diagonal Hessian. If no block is discarded, one may take the identity factor and $B_s = b_p$ in centered rotated coordinates.

In the proper-cut case shrink the chart so that the symmetric part of Db is bounded below by m and the backward flow X_t stays in the chart on a common small ball. It satisfies $|X_t y| \leq e^{-mt}|y|$ and $\|D^j X_t\| \leq C_j e^{-mt}$ for $1 \leq j \leq k$ by induction in the backward variational equations. Extend B_s to all retained coordinates by cutting off only its nonlinear part in a small ball. This preserves its derivative at zero and yields a forward flow φ_t^s with $\|DB_s\| \leq \Lambda$ and bounded derivatives through order $k + 1$. Shrinking the support permits the bound by Λ because the nonlinear part

vanishes to second order and its derivative to first order. The forward variational equations give $\|D^j\varphi_t^s\| \le C_j e^{j\Lambda t}$ for $1 \le j \le k+1$.

Let $\pi_s y = y_s$ and set $P_t(y) = \varphi_t^s(\pi_s X_t y)$. The exact time derivative is

$$\partial_t P_t(y) = -D\varphi_t^s(\pi_s X_t y)\, R(X_t y).$$

Indeed the two terms involving B_s cancel by the flow identity $B_s(\varphi_t^s(z)) = D\varphi_t^s(z)B_s(z)$. By (2), the chain rule and backward contraction, derivatives of $R(X_t y)$ through order $j \le k$ are bounded by $C|y|^{M+1-j}e^{-(M+1)mt}$. Multiplying by the forward derivative bounds shows that $\partial_t P_t$ is integrable in C^k. Let $\pi = \lim_{t\to\infty} P_t$. It is uniformly C^k, with $\pi(y) = y_s + O(|y|^{M+1})$ and $D\pi(y) = D\pi_s + O(|y|^M)$. In particular $y \mapsto (\pi(y), y_f)$ is a uniformly invertible C^k change of coordinates on a smaller ball. For fixed $\tau \ge 0$ the semigroup identity in the defining limit gives

$$\pi(X_\tau y) = \varphi_{-\tau}^s(\pi(y)).$$

It follows that $D\pi\, b = B_s \circ \pi$. Localizing to a smaller neighborhood, only the unextended retained field is used. This construction gives the required factor uniformly without any nonresonance assumption among the retained eigenvalues.

4 Adjoint representers and risk

First work with the reference alone. An L^2 function z is a representer for supported perturbations if and only if $w = L_{f_0}^{-1} z \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies $b \cdot \nabla w = \psi$ almost everywhere on K. This follows by integration by parts against h, and by the Dirichlet elliptic H^2 estimate for the inverse. In particular the equation is necessary on a neighborhood of x_0. The smooth particular solution with right hand side $\psi - \psi(x_0)$ is the integral of $\psi(X_t x) - \psi(x_0)$ for $0 \le t < \infty$, with X_t the backward gradient flow in the original coordinates. After subtracting it and dividing by $\psi(x_0)$ when this value is nonzero, a representer would produce a local H^2 solution with constant source. Smooth conjugacy preserves local H^2, so Lemma 1 proves the necessity of (1).

We construct the sufficient representers simultaneously for the reference and the nonlinear pilot family. Choose a fixed regular compact sublevel of u_0 with K in its interior, contained in Ω. The proof of [1, Lemma S4.1] shows that for small enough η it is invariant under the backward flows of b_p and every point enters a common small neighborhood of the minimum in uniformly bounded time. Indeed outside that neighborhood the gradients are uniformly bounded below and u_p decreases along each backward trajectory. The strictly outward gradient on the sublevel boundary prevents backward escape. By further contracting within the local ball, choose one fixed time T so that this backward flow sends a neighborhood U of K into a ball where the factor of the preceding section is defined and where its output is in the exit-time domain of Lemma 2. Compose the factor with the local polynomial chart and with this fixed-time flow. Write the result as $\Pi_p(x)$. It satisfies

$$D\Pi_p(x) b_p(x) = B_s(\Pi_p(x)), \qquad x \in U,$$

by the factor relation and the fact that a fixed-time flow pushes its vector field to itself.

Let v_p be the preceding backward integral with $\psi(x_p)$ subtracted. On U it has bounded C^2 norm, uniformly in p. Indeed split off the fixed time until entry to the convex ball. Beyond that time the backward first and second variational derivatives decay exponentially, so the differentiated integrals converge uniformly. It satisfies $b_p \cdot \nabla v_p = \psi - \psi(x_p)$. If the spectral cut inequality holds, use ω_p from Lemma 2 and take

$$w_p = \chi\{v_p + \psi(x_p)\, \omega_p \circ \Pi_p\},$$

where χ is a fixed smooth function supported in U and equal to one near K. The second term is uniformly H^2 on U because the factor can be completed by the discarded coordinates to a uniformly controlled C^2 diffeomorphism, before composition with the fixed-time flow. In those coordinates, the time function is independent of the discarded variables. Product integration on a bounded coordinate box gives the H^2 bound, and composition with the diffeomorphisms preserves it. The weak chain rule and the transport equations show that $b_p \cdot \nabla w_p = \psi$ on K. If $\psi(x_0) = 0$ for the linearized problem, simply take $w_0 = \chi v_0$. In every sufficient case $L_p w_p$ is uniformly bounded in L^2. The exact linear estimator against this representer gives the stated linearized upper bound.

For the nonlinear parametric upper bound apply the split white-noise pilot of [1, Lemma S4.2] to the supported class. This only uses a countable prediction-dense subset of the class, so no approximation by smooth functions supported in K is needed. By that lemma, $\delta_p = \|u_f - u_p\|_2$ has all fixed positive moments of the order of $N^{-(s+1)/(2s+d+2)}$ to the corresponding powers. Take the second-sample one-step estimate using $z_p = L_p w_p$. The exact testing identity [1, Lemma S4.3] gives conditional bias

$$\int_\Omega (f - p)\, \nabla(u_f - u_p) \cdot \nabla w_p$$

because $f - p$ is supported in K. Conditional standard deviation is $\sqrt{2/N}\|z_p\|_2$. For a fixed $\sigma > d/2$ use [1, Lemma S4.4 and (S4.4)] on the larger compact sublevel, Sobolev embedding, and the uniform H^2 bound on w_p. The conditional bias in absolute value is at most

$$C\|f - p\|_{L^\infty(K)} \|\nabla(u_f - u_p)\|_{L^\infty(K)} \le C\delta_p^{[(s-1)(1-\sigma/s)+(s-\sigma)]/(s+1)}.$$

The expected value of this bound is $o(N^{-1/2})$ for sufficiently large s, since its exponent after including the pilot rate tends to one. This gives the claimed uniform parametric risk upper bound. The independent randomization of the split can be removed by conditioning on the original observation and using Jensen's inequality.

Finally suppose (1) fails. The ratio

$$\frac{|\int h\psi|}{\|Ah\|_2}, \qquad h \in C_c^\infty(\operatorname{int} K), \quad h \neq 0,$$

is unbounded. Otherwise the functional $Ah \mapsto \int h\psi$ would extend boundedly to the L^2 closure of that range and have an L^2 representer. Its weak transport equation on $\operatorname{int} K$ would contradict Lemma 1 near x_0. The denominators are nonzero: if $\operatorname{div}(hb) = 0$, then $h(X_t x) \det DX_t(x) = h(x)$, and the left-hand side tends to zero by boundedness of h and eventual exponential contraction. Hence $h(x) = 0$. For any fixed smooth direction with a prescribed large ratio, compare zero to the perturbation $t_N h$, where $t_N = c/(\sqrt{N}\|Ah\|_2)$ and $c > 0$ is a small absolute constant. It belongs to any fixed positive radius ball for all sufficiently large N. The linear Gaussian Kullback–Leibler divergence is $c^2/2$. In the nonlinear experiment it tends to the same value by differentiability of u_{f_0+th} at zero with derivative Ah. This follows directly by subtracting the elliptic equations and taking the limit in the variational inverse. The two-point testing inequality therefore lower bounds $\sqrt{N}$ times the minimax risk by a positive constant times the chosen ratio, for all sufficiently large N depending on the direction. Letting the ratio be arbitrarily large proves the divergence in both experiments. When $\psi(x_0) \neq 0$ and the spectral criterion is satisfied, the same two-point argument with any one smooth supported direction of nonzero functional gives the ordinary parametric lower bound. This proves both theorems.

5 An upper bound below the root-sample threshold

The slow factor also improves the nonparametric upper bound. Choose any fixed spectral cut retaining at least two directions, possibly the full set, and let r be the number retained. In the nonlinear interior model write

$$T_* = \inf_{p \in f_0 + \mathcal{H}_s(K,\eta)} \frac{\sum_{i=1}^{r} \lambda_i(p)}{\lambda_r(p)},$$

where here the eigenvalues at the pilot minimum are ordered with multiplicity. By taking η small, a proper reference cut stays separated from the discarded block.

Theorem 3. *For sufficiently large s and small fixed η, the nonlinear interior minimax absolute risk has the upper bounds*

$$\begin{cases} O(N^{-(2s+T_*)/(4s+8)}), & T_* < 4, \\ O(\sqrt{\log N/N}), & T_* = 4, \\ O(N^{-1/2}), & T_* > 4. \end{cases}$$

In particular one may choose a cut attaining $\mathcal{D}(H)$ at the reference. Its T_* tends to $\mathcal{D}(H)$ as $\eta \downarrow 0$.

Proof. Take an integer $k > d/2$, $k \geq 2$, and use the uniformly C^k factor obtained above. Its construction only needs the spectral gap across a proper cut, so it applies even if the retained trace is at or below four times its largest eigenvalue. Use the negative slow exit time ω_p on a common small ball. The proof of Lemma 2, with the higher variational equations through order k, gives at exit time t

$$|\omega_p| = t, \qquad |D^j \omega_p| \leq C_j e^{j\lambda_s t} \quad (1 \leq j \leq k).$$

The volume element is bounded by $Ce^{-Q_s t}\, dt\, d\sigma$. In particular ω_p lies in L^2 and $W^{1,1}$ uniformly, since $Q_s - \lambda_s$ is bounded below when at least two eigenvalues are retained. Smooth flattening at large exit time, as in that lemma, verifies the weak gradient.

Use the fixed-time extension of Π_p to U containing K. Let χ_1 and χ_2 be smooth and compactly supported in U, with $\chi_1 = 1$ near K and $\chi_2 = 1$ on a neighborhood of $\operatorname{supp} \chi_1$. Put $q = \chi_1\, \omega_p \circ \Pi_p$ and extend by zero to $\mathbb{R}^d$. Then q is uniformly in $L^2 \cap W^{1,1}$ by the controlled factor coordinates. It also satisfies the Fourier shell bound

$$\int_{R \leq |\xi| \leq 2R} |\widehat{q}(\xi)|^2\, d\xi \leq C R^{-Q_s/\lambda_s}. \tag{3}$$

Here is a finite-difference proof accounting for the change of variables and the cutoff. If E_t is the image of the small slow ball under backward flow for time t, then $\operatorname{dist}(E_{t+c_0}, \partial E_t) \gtrsim e^{-\lambda_s t}$ for a uniformly fixed c_0. This follows by comparing backward trajectories as in [1, Lemma S2.11]: the deviation of the derivative of the field from H_s is integrable along either trajectory, so distances contract by at most a constant times $e^{-\lambda_s t}$ from below. Choose the step size $|z| = c_1 e^{-\lambda_s t}$ with a sufficiently small fixed c_1. Uniform Lipschitz control of Π_p keeps the slow coordinates of all shifted points $x + \ell z$, $0 \leq \ell \leq k$, with $\Pi_p(x) \in E_t$ inside E_{t-c_0}. Only points in a fixed compact enlargement of the support need be considered.

On this inner set write $q = \chi_1(\omega_p \circ \Pi_p + t) - t\chi_1$. The integral of the squared kth difference of the first term is bounded by $Ce^{-Q_s t}$, after changing variables to the slow and complementary fast coordinates at each shifted point. Indeed the fast cross-sections are bounded, and the slow integral of $|\omega_p + t|^2$ over E_{t-c_0} is at most $Ce^{-Q_s t}$ by translation along the slow backward flow. The term $-t\chi_1$ contributes at most $Ct^2|z|^{2k}e^{-Q_s t}$. On the outer set, with slow exit time $v \leq t$, the

shifted segment stays outside E_{v+c_0}. The bound on the derivatives of ω_p, and the C^k bounds of the coordinate maps and multipliers, give an integral bounded by

$$C|z|^{2k}\left(1+\int_0^t e^{(2k\lambda_s-Q_s)v}\,dv\right)\le C|z|^{Q_s/\lambda_s},$$

since $2k>d\ge Q_s/\lambda_s$. Averaging the difference over a ball of steps and applying Plancherel proves (3).

For a smooth Fourier cutoff P_R as in [1, Lemma S2.12], use the representer potential

$$w_{p,R}=\chi_1 v_p+\psi(x_p)\chi_2 P_R q.$$

It has uniformly bounded gradient in L^1 since $q\in W^{1,1}$. The shell bound controls $\|L_p w_{p,R}\|_2$ by $CR^{2-T_*/2}$ if $T_*<4$, $C(1+\log R)^{1/2}$ if $T_*=4$, and a constant if $T_*>4$, by summation of dyadic H^2 shells. It remains to bound the tested adjoint defect. On the support of $h=f-p$ the transport derivative of the potential is $\psi-\psi(x_p)$ from its smooth part, so the defect equals $\psi(x_p)$ times

$$(I-P_R)(b_p\cdot\nabla q)+[P_R,b_p\cdot\nabla]q.$$

In this identity b_p is extended by a smooth cutoff beyond the support of χ_2, and all expressions are restricted to K. The first term uses $b_p\cdot\nabla q=\chi_1+(b_p\cdot\nabla\chi_1)(\omega_p\circ\Pi_p)$. The χ_1 contribution is rapidly decreasing. The second summand is L^1 and supported a positive distance from K, so its convolution is also rapidly decreasing on K by the Schwartz kernel estimate.

For the commutator use the pilot-dependent split of [2]. Write $b_p=b_{\mathrm{lo}}+b_{\mathrm{hi}}$ with coefficient split below and above frequencies of order $R/16$. For the low piece, the Fourier kernel acts only on input and output frequencies of order R, and its multiplier is bounded by $C|\xi-\vartheta|\,|\widehat{b}_{\mathrm{lo}}(\xi-\vartheta)|$. The weighted Fourier coefficient has bounded L^1 norm since the extension has bounded H^s norm and $s>d/2+1$. Young's inequality and (3) bound the tested defect by $CR^{-s-T_*/2}$. One can extend the supported h by zero from Ω since its support is in K. For the high piece $\|b_{\mathrm{hi}}\|_\infty\le CR^{d/2-s}$ by Cauchy–Schwarz. The commutator has L^1 norm at most that order by the uniform $W^{1,1}$ bound and convolution boundedness. Its pairing with h is therefore at most $CR^{d/2-s}\|h\|_{L^\infty(K)}$.

The preceding pilot argument bounds this last supremum norm in expectation by CN^{-A} with $A=(s-1)(1-\sigma/s)/(2s+d+2)$, and bounds the nonlinear remainder by $o(N^{-1/2})$ by the uniform gradient L^1 bound. Take s large enough that $A>d/(2s+4)$ and choose $R=N^{1/(2s+4)}$. The high commutator then contributes at most $CR^{-s-T_*/2}$. The conditional noise bound from the shell sum gives the claimed three cases. Continuity of the minimum-shifted Hessian yields the final limit for T_*. □

References

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