The nonlinear minimax exponent for Darcy inference at anisotropic minima, with power-law and saddle critical points

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Abstract

We estimate linear functionals of a conductivity from white-noise observations of its Darcy potential. For a reference potential with one nondegenerate minimum, Liu, Chang and Hu showed that the linearized rate is governed by the effective dimension of the minimum's Hessian, trace over largest eigenvalue. We prove their Conjecture 6.1 that, when this dimension is at most four and their transport compatibility condition fails, the linearized exponent is the local exponent of the nonlinear experiment on shrinking full Sobolev neighborhoods, for sufficiently large smoothness s. On each sufficiently small fixed Sobolev ball the exponent is $\min\{(2s + T)/(4s+8), 1/2\}$, where T, the ball's infimal effective dimension, lies strictly below the reference value. The adjoint's singular time function answers three further problems of theirs in specific models. Under interior support constraints with nonzero score at the minimum, root-sample estimation is possible exactly when some complete lower block of Hessian eigenvalues has sum exceeding four times its largest member; if the Hessian trace is below four times the smallest eigenvalue, the rate is isotropic, $N^{-(2s+d)/(4s+8)}$. The radial power-law minimum of even order p has critical dimension $2p$. At a two-block linear saddle with block rates λ, μ, dimensions m, n below four and local perturbations, the linearized rate is $N^{-(2s+M)/(4s+8)}$ for $s > 0$, with the same exponent at $s = 0$; the effective dimension $M \geq \max(m, n)$ is strictly larger for equal dimensions and unequal rates. These hold nonlinearly at large smoothness: radially at the power-law minimum, as lower bounds at the saddle.

Keywords. Darcy inverse problem, minimax rates, linear functionals, gradient flow, transport equation, degenerate critical point, saddle point.
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1 Introduction

1.1 The question

How accurately can a weighted average of a conductivity be estimated from a noisy observation of the potential that it induces? Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be smooth, bounded and connected. Let g, f_0 be smooth up to the boundary and bounded below by positive constants. For a scalar conductivity f write u_f for the solution of

$$L_f u_f = g, \qquad L_f = \operatorname{div}(f\nabla), \qquad u_f|_{\partial\Omega} = 0.$$

The data are the Gaussian white-noise observation with mean u_f and noise level $N^{-1/2}$ in $L^2(\Omega)$. Fix nonzero $\psi \in C_c^\infty(\Omega)$ and set $\Phi(f) = \int_\Omega f\psi$. Liu, Chang and Hu [5], henceforth LCH, treat this problem when the reference $u_0 = u_{f_0}$ has one critical point, a nondegenerate minimum x_0 in the interior (their Assumption 2.1), over the full Sobolev ball

$$\mathcal{F}_\eta = \{f = f_0 + h : \|h\|_{H^s(\Omega)} \leq \eta,\ \inf_\Omega f > 0\},$$

which they denote $\mathcal{F}_s(f_0, \eta)$. For small fixed $\eta > 0$ and the smoothness used below, each u_f has a unique minimum x_f and positive Hessian H_f there. Write $\lambda_f = \lambda_{\max}(H_f)$, $T_f = \operatorname{tr}(H_f)/\lambda_f$ and $T_-(\eta) = \inf_{f\in\mathcal{F}_\eta} T_f$. The number T_f is the effective dimension of LCH, and anisotropy refers to the eigenvalues of H_f. Use absolute risk

$$\mathcal{R}_N(\mathcal{F}_\eta) = \inf_{\widehat{\Phi}} \sup_{f\in\mathcal{F}_\eta} \mathbb{E}_f|\widehat{\Phi} - \Phi(f)|, \qquad e(\mathcal{F}_\eta) = \liminf_{N\to\infty} \frac{\log(1/\mathcal{R}_N(\mathcal{F}_\eta))}{\log N}.$$

The LCH condition (C) [5, Definition 2.2] holds at a smooth reference if $\psi(x_f) = 0$ and the integral of $-\psi$ is independent of the complete forward trajectory. Here each trajectory is parametrized by $\dot{x} = \nabla u_f(x)$, approaches x_f as $t \to -\infty$, and ends on $\partial\Omega$. LCH relate this condition to the efficient information analysis of Nickl and Paternain [6]. In the linearized experiment at f_0, LCH Theorem 3.1 gives the minimax rate $N^{-1/2}$ if (C) holds, and otherwise

$$a_N(s, T_{f_0}) = \begin{cases} N^{-(2s+T_{f_0})/(4s+8)}, & T_{f_0} < 4, \\ \sqrt{\log N/N}, & T_{f_0} = 4, \\ N^{-1/2}, & T_{f_0} > 4. \end{cases}$$

In the nonlinear experiment, at a reference failing (C) with $T_{f_0} \leq 4$, their Theorem 3.3(b) gives the lower bound $\mathcal{R}_N(\mathcal{F}_\eta) \gtrsim a_N(s, T_{f_0})$, and Theorem 3.3(c) gives $\mathcal{R}_N(\mathcal{F}_\eta) \gtrsim N^{-1/2}$ in every case. The rate is $N^{-1/2}$ when $T_{f_0} > 4$ (their Theorem 3.7), and at the canonical isotropic model in dimensions two to four their Corollary 3.8 identifies the limit of the local exponent as $\eta \downarrow 0$. For anisotropic references below the elbow $T_{f_0} = 4$, LCH name the minimax rate on full neighborhoods as the main question of their Section 6 and conjecture the following. In it, $N \to \infty$ is taken first, at fixed η, in the definition of e, and then $\eta \downarrow 0$.

Conjecture 6.1 (Liu, Chang and Hu [5, §6])**.** *Under Assumption 2.1 with $T_{f_0} \leq 4$, if* (C) *fails at f_0, then for all sufficiently large s,*

$$\lim_{\eta\downarrow 0} e(\mathcal{F}_\eta) = \min\left\{\frac{2s + T_{f_0}}{4s + 8}, \frac{1}{2}\right\}.$$

1.2 Main results

Theorem 1 proves Conjecture 6.1; any $s > 2d + 4$ suffices. It also gives the exponent on each fixed ball: if (C) fails at f_0, then for every sufficiently small fixed $\eta > 0$,

$$e(\mathcal{F}_\eta) = \min\left\{\frac{2s + T_-(\eta)}{4s + 8}, \frac{1}{2}\right\},$$

and (1) gives the corresponding upper bound on the risk. Corollary 2 shows that $T_-(\eta) < T_{f_0}$. Hence, when (C) fails and $T_{f_0} \le 4$, every fixed full neighborhood has a strictly smaller exponent than the conjectured limit, which is reached only as the radius shrinks. This extends the strict inequality of LCH Corollary 3.8 from the canonical model to every such reference. LCH remark after the conjecture that the truncation of the transport representer used in the linearized analysis would have to be made uniform over the data-dependent pilot. Sections 2 and 3 make it uniform, using only a fixed Sobolev bound on the pilot.

LCH then list further problems for their nondegenerate theory [5, §6]. The second bullet concerns several, degenerate or saddle-type critical points, which arise for instance for sources of variable sign. The third concerns the minimax rate under interior support constraints or known boundary values, for anisotropic references with $\psi(x_0) \neq 0$; their Section 2.1 singles out the interior-support case. Parts II–V answer these problems in the following cases.

Interior support (Parts II and V). Let the perturbations be supported in a fixed compact $K \subset \Omega$ with $x_0 \in \operatorname{int} K$, and let $\psi(x_0) \neq 0$. Theorems 3 and 4 show that the rate is $N^{-1/2}$ exactly when $\mathcal{D}(H) > 4$, where $\mathcal{D}(H)$ is the largest effective dimension of a complete lower block of eigenvalues of $H = H_{f_0}$ (Section 5). This holds in the linearized experiment for every $s \ge 0$ and in the nonlinear experiment for sufficiently large s, and Theorem 7 gives an upper bound below the threshold. When $d \le 3$ and $\operatorname{tr} H \le 4\lambda_{\min}(H)$ (in the plane, $\lambda_{\max} \le 3\lambda_{\min}$), Theorem 26 in Part V gives the rate below the threshold. If $\operatorname{tr} H < 4\lambda_{\min}(H)$ it is $N^{-(2s+d)/(4s+8)}$, the rate of an isotropic minimum, in the linearized experiment for every $s \ge 0$ and in the nonlinear experiment for $s > 2d+4$. If $\operatorname{tr} H = 4\lambda_{\min}(H)$ the linearized risk lies between $N^{-(2s+d)/(4s+8)}$ and $(N/\log N)^{-(2s+d)/(4s+8)}$. For anisotropic H this is polynomially faster than the spectral-block bound (Corollary 27). The interior-support case of the third bullet is thus answered, for $\psi(x_0) \neq 0$, at the level of the root-sample threshold at every nondegenerate minimum, with the exact rate when $\operatorname{tr} H < 4\lambda_{\min}(H)$, and with the exact linearized exponent when $d \le 3$ and $\operatorname{tr} H = 4\lambda_{\min}(H)$. In the remaining anisotropic cases below the threshold, $\operatorname{tr} H > 4\lambda_{\min}(H)$ and $\mathcal{D}(H) \le 4$, the risk lies between the lower bound $N^{-(2s+d)/(4s+8)}$ of LCH Proposition S3.14(ii) and, for sufficiently large s, the upper bound of Theorem 7. The case $\psi(x_0) = 0$ is LCH Remark 3.9. For example, in dimension six with Hessian eigenvalues $1, 1, 1, 1, 1, 10$ one has $T_{f_0} = 3/2$ and $\mathcal{D}(H) = 5$, so the rate on full neighborhoods is slower than root-sample while the interior-support rate is root-sample (Section 5).

A power-law degenerate minimum (Part III). For the radial reference $u_0 = (|x|^p - 1)/p$ on the unit ball, with $p \ge 4$ even, the Hessian vanishes at the minimum. Theorems 8 and 9 give the rate $N^{-(2s+d)/(4s+4p)}$ for $d < 2p$, $\sqrt{\log N/N}$ for $d = 2p$ and $N^{-1/2}$ for $d > 2p$, in the linearized experiment for every $s \ge 0$ and in the nonlinear experiment over radial conductivities for sufficiently large s. The critical dimension is therefore $2p$. This answers the degenerate case of the second bullet for this radial family.

A two-block saddle (Part IV). Let $f_0 = 1$ and let the reference gradient equal $(\lambda x, -\mu y)$ on a product neighborhood of an interior critical point in $\mathbb{R}^m \times \mathbb{R}^n$, with ψ nonzero at that point and perturbations supported near it. Theorem 11 shows that an L^2 representer exists exactly when

$\max(m,n) > 4$, for all $\lambda, \mu > 0$. Theorem 12 gives the linearized rate for all block rates and dimensions, with matching nonlinear lower bounds for $s > (m+n)/2+1$: $\sqrt{\log N/N}$ when $\max(m,n) = 4$, and $N^{-(2s+M)/(4s+8)}$ when $m, n \le 3$ (including $s = 0$ when $m\lambda \ne n\mu$), with the effective dimension M of (19). When $s = 0$ and $m\lambda = n\mu$ the risk lies between $N^{-M/8}(\log N)^{-(4-M)/8}$ and $N^{-M/8}$, so the exponent $(2s + M)/(4s + 8)$ holds for every $s \ge 0$. By Corollary 14, $M = \max(m,n)$ exactly when one block has both the larger product of dimension and rate and the smaller product of rate and four minus dimension; otherwise, and in particular for equal dimensions with $\lambda \ne \mu$, the rate is strictly faster than the bound obtained from a single block. For $\lambda = \mu$ and $m = n$ there is no logarithm at $s > 0$, and Theorem 15 gives the factor $(\log N)^{-(4-m)/8}$ at $s = 0$. The reference may have other critical points, and Section 18 gives an example with positive source and 3^n nondegenerate critical points, which for $m = n$ and $\lambda > \mu$ has $M > \max(m,n)$. This answers the saddle-type case of the second bullet for this two-block model in the supported linearized experiment, at every block rate and block dimension for $s > 0$ and, at $s = 0$, whenever $\max(m,n) \ge 4$, $m\lambda \ne n\mu$ or $(\lambda, m) = (\mu, n)$, and at the level of the exponent in the remaining case.

1.3 Mechanism and organization

In each geometry the representer potential solves $b \cdot \nabla w = \psi$ with $b = \nabla u_0$, and its singular part is a multiple of the time spent near the critical point. The square integrability of the second derivatives of that time function fixes a critical dimension, and below it the exponent takes the form $(2s+D)/(4s+4\kappa)$, where D is an effective dimension (the volume growth of the time function in Parts I–III and the effective dimension M of (19) in Part IV) and κ the derivative growth of the time function. This form is sharp in Parts I, III and IV, and in Part V, with $D = d$ and $\kappa = 2$, when $\operatorname{tr} H < 4\lambda_{\min}(H)$; in Part II it is an upper bound. In Part I the sharp quantity on a fixed ball is $T_-(\eta)$, the infimum of the effective dimension over the ball, and T_{f_0} enters as its limit:

geometry	critical condition	reference
nondegenerate minimum, full support	$T_{f_0} > 4$	Part I, [5]
nondegenerate minimum, interior support	$\mathcal{D}(H) > 4$	Part II, Part V
power-law minimum of order p	$d > 2p$	Part III
two-block saddle	$\max(m,n) > 4$	Part IV

Part I (Sections 2–4) proves Theorem 1 and Corollary 2. Parts II (Sections 5–9), III (Sections 10–13), IV (Sections 14–19) and V (Section 20) each open with their setting and results, and reuse the shell and commutator estimates of Part I where they apply.

Edition note. This third edition extends Part IV from equal block rates to arbitrary block rates and equal block dimensions, keeping the numbers of Theorems 11 and 12 and Remark 13, with the second-edition statement as the case $\lambda = \mu$, $m \ne n$ of Theorem 12, and adds Part V, the interior-support rate when the Hessian trace is at most four times its smallest eigenvalue, after the second edition [10] added Parts II–IV, which consolidate the first editions of three Luminary Research Archive articles [11, 7, 8], to the proof of Conjecture 6.1 in the first edition [9].

Part I. The anisotropic minimum: Conjecture 6.1

This part determines the polynomial exponent on each sufficiently small fixed full Sobolev ball. The controlling effective dimension is the infimum over the ball, and the conjectured local exponent follows as the radius tends to zero. A perturbation in a top Hessian eigendirection shows that this infimum is strictly smaller than the effective dimension of the reference (Corollary 2), so every fixed ball has a strictly smaller exponent than the limit. The setting, the ball $\mathcal{F}_\eta$, the quantities T_f and $T_-(\eta)$, the exponent e and condition (C) are those of Section 1.1.

Theorem 1. *For all sufficiently large s and sufficiently small fixed $\eta > 0$,*

$$\mathcal{R}_N(\mathcal{F}_\eta) \lesssim \begin{cases} N^{-(2s+T_-(\eta))/(4s+8)}, & T_-(\eta) < 4, \\ \sqrt{\log N/N}, & T_-(\eta) = 4, \\ N^{-1/2}, & T_-(\eta) > 4. \end{cases} \tag{1}$$

If (C) fails at f_0, the exact exponent on each such ball is

$$e(\mathcal{F}_\eta) = \min\left\{\frac{2s + T_-(\eta)}{4s+8}, \frac{1}{2}\right\}.$$

In particular, if $T_{f_0} \le 4$ and (C) fails there, then

$$\lim_{\eta \downarrow 0} e(\mathcal{F}_\eta) = \min\left\{\frac{2s + T_{f_0}}{4s+8}, \frac{1}{2}\right\},$$

which is LCH Conjecture 6.1.

For $T_-(\eta) > 4$, the upper bound and LCH Theorem 3.3(c) give the root-N rate. At $T_-(\eta) = 4$, the exponent is $1/2$ and the displayed upper bound carries a logarithmic factor. The constants in the risk bounds depend on the fixed ball, and the exact exponent describes the power of N on that ball. The proof applies, for example, whenever $s > 2d + 4$.

The proof uses the LCH pilot and exact conditional bias identity. The new issue is to regularize the transport solution uniformly over pilot fields with only a fixed Sobolev bound. We first obtain a uniform Fourier-shell estimate and an integrable gradient for the singular solution. Then we split the commutator with the Fourier cutoff. Low coefficient frequencies use the shell estimate. High coefficient frequencies are paired directly with the small pilot error in supremum norm. The integrable gradient also controls the nonlinear remainder.

2 Uniform bounds for characteristic solutions

Fix an integer $k > d/2$ with $k \ge 2$, and take $s > d/2 + k$. Additional strict inequalities on s appear below. We use LCH Lemma S4.1. On a fixed compact regular sublevel domain $K \Subset \Omega$, the characteristic solution w_p of

$$b_p \cdot \nabla w_p = \psi, \qquad b_p = \nabla u_p, \qquad w_p|_{\partial\Omega} = 0$$

is supported in a fixed compact subset of the interior of K. The backward flow stays in K and reaches a uniformly convex ball about the minimum in uniformly bounded time. We may apply the following bounds to smooth pilots, since smooth conductivities in the ball are dense in it in H^s: approximate the perturbation by smooth ones after a contraction toward the center. The finite derivative estimates below hold uniformly. In particular $\|u_p\|_{H^{s+1}(\Omega)}$ and $\|b_p\|_{C^k}$ are bounded, whereas $\|u_p - u_0\|_{C^2} = O(\eta)$.

Use the following notation with the pilot index suppressed. Write x_p for the minimum, H for the Hessian, $Q = \operatorname{tr} H$, $\lambda = \lambda_{\max}(H)$, and $T = Q/\lambda$. In a common-radius ball B about x_p, $Db \ge mI$ with $m > 0$ uniform. Denote the flow of $-b$ by X_t. For $t \ge 0$ and $x, y \in B$,

$$\begin{aligned} |X_t x - x_p| &\le e^{-mt}|x - x_p|, \\ \det DX_t(x) &\asymp e^{-Qt}, \\ |X_t x - X_t y| &\ge c\, e^{-\lambda t}|x - y|. \end{aligned} \tag{2}$$

The first bound is the differential inequality from convexity. Liouville's identity gives the second because, along each orbit, the time integral of $|\operatorname{div} b - Q|$ is uniformly bounded. For the last, $Db(z) - H = O(|z - x_p|)$ and the differential inequality for the separation has an integrable error, uniformly in initial points. Every point in $B \setminus \{x_p\}$ is $X_r\theta$ for a unique $\theta \in \partial B$ and exit time $r \geq 0$ from the ball under the forward flow. Its volume element is comparable to $e^{-Qr}\, dr\, d\sigma(\theta)$. The comparison follows from (2) and the uniform transversality of b on ∂B.

Set $c_p = \psi(x_p)$ and, in B, define the particular solution

$$v_p(x) = \int_0^\infty [\psi(X_t x) - c_p]\, dt.$$

It satisfies $b \cdot \nabla v_p = \psi - c_p$. Indeed, $v_p(X_\tau x) - v_p(x) = -\int_0^\tau [\psi(X_t x) - c_p]\, dt$. Its C^k norm is uniformly bounded: the zeroth-order integrand decays by (2). The first derivative of X_t decays as e^{-mt}, and the higher variational equations have the same contracting propagator and forcing terms that are products of at least two lower derivatives. Induction gives the same exponential bound up to order k. On compact sets separated from the minimum, w_p has uniformly bounded derivatives through order k. Choose a fixed regular level surface of u_0 between the support of ψ and ∂K, high enough to contain the convex balls on its inner side. The characteristic integral ends there because the forward flow points out of the sublevel set and cannot return. In the region between a smaller convex ball and this surface, $\nabla u_0 \cdot b_p$ is uniformly positive by C^1 closeness and compactness. The forward flow cannot enter the smaller ball from outside, so the travel time is uniformly bounded. The same positivity gives transversality at the surface. Ordinary variational and implicit function bounds give the required derivatives. The same bounds apply to the values of w_p and v_p on ∂B.

Set $F = w_p - v_p$ on $B \setminus \{x_p\}$. Then $b \cdot \nabla F = c_p$. In particular, with the preceding flow coordinates,

$$F(X_r\theta) = F(\theta) - c_p r.$$

For $1 \leq j \leq k$, the forward variational and exit-time equations give

$$|D^j F(X_r\theta)| \leq C_j e^{j\lambda r}, \qquad |F(X_r\theta)| \leq C(1 + r). \tag{3}$$

Indeed, the derivative of the forward flow along a segment of length t in B is at most $Ce^{\lambda t}$ by the integrable deviation of Db from H. In a higher variational equation the forcing products of total order j are bounded by $Ce^{j\lambda t}$. Integrating them against the first-order propagator gives the same bound at the endpoint for $j \geq 2$. Implicit differentiation of the exit condition, whose time derivative has a uniform positive lower bound, gives $C_j e^{j\lambda r}$ for each derivative of the exit time and exit position of order j. The chain rule for the bounded sphere data gives (3).

Choose a real smooth cutoff ζ inside B, equal to one near x_p, by translating a fixed function. Put $q_p = \zeta F$ on $\mathbb{R}^d$ by zero extension and $w_p^{\mathrm{reg}} = w_p - q_p$. The latter has uniformly bounded C^k norm and compact support inside K. We claim

$$\|q_p\|_{W^{1,1}(\mathbb{R}^d)} \leq C, \tag{4}$$

$$\int_{R \leq |\xi| \leq 2R} |\widehat{q}_p(\xi)|^2\, d\xi \leq CR^{-T_p}, \qquad R \geq 1. \tag{5}$$

For (4), integrate (3) in exit coordinates. The integral for the gradient converges uniformly because $Q - \lambda$ is the sum of the other positive eigenvalues and is bounded below when $d \geq 2$. This classical gradient is also the weak gradient across the point x_p. Indeed the first bound of (2) gives $r \lesssim 1 + |\log|x - x_p||$, and the integration-by-parts cutoff error tends to zero at rate $O(\epsilon^{d-1}(1 + |\log \epsilon|))$ in a ball of radius ϵ. We also have $\|q_p\|_2 \leq C$ from the logarithmic growth bound.

For (5) use finite differences, as in LCH Lemma S2.11. Here we give the uniform estimate with the finite number k of derivatives. Let $E_t = X_t(B)$. Choose fixed c_0 such that $X_{c_0}(B)$ lies in the half-radius ball. Apply X_t to $X_{c_0}(B)$ and ∂B. The separation bound in (2) gives $\operatorname{dist}(E_{t+c_0}, \partial E_t) \gtrsim e^{-\lambda t}$. For a difference step z, choose large t by $|z| = c_1 e^{-\lambda t}$ with c_1 uniform and sufficiently small. Then all shifts by jz, $0 \le j \le k$, of E_t lie in E_{t-c_0}, where $\zeta = 1$ for sufficiently large t. Adding $c_p t$ to each shifted value leaves the kth difference unchanged. Bounding its square by a fixed multiple of the sum of the squared shifted values and translating each integral bounds the contribution on E_t by a constant times

$$\int_{E_{t-c_0}} |F(x) + c_p t|^2 \, dx \le C e^{-Qt} \int_B |F(y) + c_p c_0|^2 \, dy \le C e^{-Qt}.$$

For an outside point of exit time $r \le t$, the segment of length at most $k|z|$ stays outside E_{r+c_0} after reducing c_1 if needed. The same separation estimate gives this because the point lies on ∂E_r, and the segment is shorter than the distance to E_{r+c_0}. For bounded r and away from B, the regular derivatives alone suffice. Thus the outer squared difference is bounded by

$$C|z|^{2k} \left(1 + \int_0^t e^{(2k\lambda - Q)r} \, dr\right) \le C|z|^T.$$

The final constant is uniform since $T \le d < 2k$ and λ is bounded below. Also $e^{-Qt} \asymp |z|^T$ uniformly. Steps outside this small-$|z|$ range are bounded using the uniform L^2 norm of q_p. Plancherel and averaging the difference bounds over z in a ball of radius R^{-1} prove (5): the average of $|e^{i\xi \cdot z} - 1|^{2k}$ is bounded below on $R \le |\xi| \le 2R$ by scaling and compactness of the unit annulus.

3 A uniformly regularized pilot correction

Take two real cutoffs centered at the pilot minimum, with $\zeta_2 = 1$ on a neighborhood of $\operatorname{supp} \zeta$ and ζ_2 compactly supported in B. Let P_R have a real even symbol $\mu(\xi/R)$, where $\mu \in C_c^\infty$ equals one on the unit ball and vanishes outside the ball of radius two. Put

$$w_{p,R} = w_p^{\mathrm{reg}} + \zeta_2 P_R q_p, \qquad z_{p,R} = L_p w_{p,R}.$$

All supports lie in K. The convolution kernel of P_R has uniformly bounded L^1 norm. By (4),

$$\|\nabla w_{p,R}\|_1 \le C. \tag{6}$$

The second-order product estimate and the upper bounds on $p, \nabla p$ give $\|z_{p,R}\|_2 \le C\|w_{p,R}\|_{H^2}$. Write $T_* = T_-(\eta)$. The shell sum in (5) yields

$$\|z_{p,R}\|_2 \lesssim \begin{cases} R^{2-T_*/2}, & T_* < 4, \\ (1 + \log R)^{1/2}, & T_* = 4, \\ 1, & T_* > 4, \end{cases} \tag{7}$$

uniformly in p. In each dyadic annulus the contribution to the squared H^2 norm is at most $C R_j^{4-T_*}$, with R_j its radius, and only $R_j \lesssim R$ contribute. The low frequencies use $\|q_p\|_2 \le C$.

Write $b = b_p$ and use $\widetilde{b}$ for its compact extension obtained by a fixed smooth cutoff equal to one near the supports of ζ_2 for all p. Then $\|\widetilde{b}\|_{H^s(\mathbb{R}^d)} \le C$. For the adjoint defect $r_{p,R} = \psi - b \cdot \nabla w_{p,R}$, the exact decomposition on Ω is

$$r_{p,R} = \zeta_2 (I - P_R)(\widetilde{b} \cdot \nabla q_p) + \zeta_2 [P_R, \widetilde{b} \cdot \nabla] q_p - (b \cdot \nabla \zeta_2) P_R q_p. \tag{8}$$

Here $[P_R, \widetilde{b}\cdot\nabla] = P_R(\widetilde{b}\cdot\nabla) - (\widetilde{b}\cdot\nabla)P_R$. The weak gradient in (4) makes these identities valid across the minimum. The decomposition follows by expanding $b\cdot\nabla(\zeta_2 P_R q_p)$, and using $b\cdot\nabla q_p = \psi - b\cdot\nabla w_p^{\mathrm{reg}}$, supported where $\zeta_2 = 1$. The derivative of the outer cutoff is supported at positive uniform distance from q_p, so the last term is $O(R^{-L})$ in supremum norm for every fixed L by the Schwartz kernel bound and $\|q_p\|_1 \le C$. The exact right hand side

$$\widetilde{b}\cdot\nabla q_p = c_p\zeta + F\, b\cdot\nabla\zeta$$

has bounded H^k norm, because the derivative of ζ is supported away from the minimum. Thus the first term in (8), tested against $h = f - p$, is bounded by CR^{-s-k} using $\|\zeta_2 h\|_{H^s} \le C$ and the frequency support of $I - P_R$ away from zero.

To treat the commutator, split $\widetilde{b} = b_{\mathrm{lo}} + b_{\mathrm{hi}}$ by a smooth frequency cutoff equal to one on $|\omega| \le R/32$ and supported on $|\omega| \le R/16$. The low coefficient piece uses the argument of LCH Lemma S2.12. Its Fourier kernel, aside from transform constants, is

$$i\widehat{b}_{\mathrm{lo}}(\xi-\vartheta)\cdot\vartheta\,[\mu(\xi/R) - \mu(\vartheta/R)].$$

The factor $|\vartheta|\,|\mu(\xi/R) - \mu(\vartheta/R)|$ is at most $C|\xi - \vartheta|$. For a nonzero low-piece kernel, both input and output frequencies lie between fixed multiples of R. This follows from $|\xi - \vartheta| \le R/16$ and the constant regions of μ. Young's convolution inequality, $\||\omega|\widehat{b}_{\mathrm{lo}}(\omega)\|_1 \le C$ by $s > d/2+1$, and (5) give an L^2 bound of $CR^{-T_p/2}$. Its H^{-s} bound is therefore $CR^{-s-T_p/2}$. Pairing against $\zeta_2 h$, extended by zero from the interior, preserves this order.

For the high coefficient piece, Cauchy–Schwarz in Fourier space gives $\|b_{\mathrm{hi}}\|_\infty \le CR^{d/2-s}$. The two terms are bounded individually in L^1 by $C\|b_{\mathrm{hi}}\|_\infty\|\nabla q_p\|_1$. For the first term apply the L^1 convolution bound after multiplication. For the second use $\nabla P_R q_p = P_R\nabla q_p$ and apply the convolution bound before multiplication. Thus each term is $O(R^{d/2-s})$ in L^1. We pair this part against the small supremum norm of h on K. Since $k > T_p/2$, the result is

$$|\langle h, r_{p,R}\rangle| \le C\big(R^{-s-T_*/2} + R^{d/2-s}\|h\|_{L^\infty(K)}\big). \tag{9}$$

This estimate requires only the fixed H^s budget on the pilot field. The high coefficient frequencies need no uniform higher Sobolev bounds.

4 Risk and the fixed-neighborhood exponent

Use the split white-noise prediction pilot of LCH Lemma S4.2 with $\mathcal{G} = \mathcal{F}_\eta$ and the countable smooth dense subset above. The resulting p takes countably many values and

$$\Big(\mathbb{E}_f \delta_p^q\Big)^{1/q} \le C_q N^{-(s+1)/(2s+d+2)}, \qquad \delta_p = \|u_f - u_p\|_2, \quad q \ge 1.$$

With the independent second observation Y_2 at noise level $\sqrt{2/N}$, use

$$\widehat{\Phi} = \langle p, \psi\rangle + Y_2(z_{p,R}) - \langle u_p, z_{p,R}\rangle.$$

Countable selection ensures measurability of this use of the second observation. If a nonrandomized estimator is desired, condition on the original observation to remove the independent randomization used to split the noise. Jensen's inequality preserves the risk.

For $v = u_f - u_p$ and $h = f - p$, LCH Lemma S4.3 gives the exact conditional bias

$$-\langle h, r_{p,R}\rangle + \int_\Omega h\,\nabla v\cdot\nabla w_{p,R}.$$

This follows by testing $L_p v = -\operatorname{div}(h\nabla u_f)$ against $w_{p,R}$. By (6), the last term is bounded by $C\|h\|_{L^\infty(K)}\|\nabla v\|_{L^\infty(K)}$ uniformly in R.

Fix $\sigma > d/2$ independent of s. The local transport stability estimate and interpolation in LCH Lemma S4.4 and (S4.4) give

$$\|h\|_{L^\infty(K)} \le C\,\delta_p^{(s-1)(1-\sigma/s)/(s+1)}, \qquad \|\nabla v\|_{L^\infty(K)} \le C\,\delta_p^{(s-\sigma)/(s+1)}.$$

We used $H^\sigma(K) \hookrightarrow L^\infty(K)$ and $H^{1+\sigma}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$. The pilot moments then yield

$$\mathbb{E}_f\|h\|_{L^\infty(K)} \lesssim N^{-A},$$
$$\mathbb{E}_f\big[\|h\|_{L^\infty(K)}\|\nabla v\|_{L^\infty(K)}\big] \lesssim N^{-A-B}, \qquad A = \frac{(s-1)(1-\sigma/s)}{2s+d+2}, \quad B = \frac{s-\sigma}{2s+d+2}.$$

The moment bounds also apply to exponents below one by Jensen. Besides $s > d/2+k$ and $s > \sigma$, we require $A+B > 1/2$ and $A > d/(2s+4)$. These hold for all sufficiently large s since $A, B \to 1/2$ as $s \to \infty$. For the sufficient condition $s > 2d+4$, choose $\sigma = d/2+1$ and $k = \max(2, \lfloor d/2 \rfloor + 1)$. Then $s-1 \ge 7s/8$, $\sigma < s/4$ and $2s+d+2 < 5s/2$, giving $A > 21/80$, $B > 3/10$ and $d/(2s+4) < 1/4$. Use $R = N^{1/(2s+4)}$. Since $T_* \le d$, $A > d/(2s+4)$ implies

$$R^{d/2-s}N^{-A} \le R^{-s-T_*/2}.$$

The conditional Gaussian fluctuation has expected absolute value bounded by $\sqrt{2/N}\|z_{p,R}\|_2$. Combining (7), (9) and the nonlinear remainder bound now gives (1) in all three cases.

It remains to match the polynomial exponent. First, failure of (C) persists under small perturbation of the smooth reference in this class. If $\psi(x_0) \neq 0$, this follows from continuity of the critical point. Otherwise use the values of $F = w_p - v_p$ on the translated sphere ∂B. At the reference the profile is nonconstant, and when $\psi(x_p) = 0$ these sphere values are exactly the limits of w_p along the respective backward trajectories. The values depend continuously on p for the above uniform local geometry. For w_p, use its finite forward exit integral up to the fixed transverse surface. For v_p, use continuous dependence of the backward flow on each fixed time interval and the uniform exponential tail. Hence two distinct sphere values at the reference persist for sufficiently small η. This proves the persistence, including at nearby pilots where $\psi(x_p) \neq 0$.

The map $p \mapsto T_p$ is continuous in H^s on this ball, by the regularity and continuity of u_p in H^{s+1} and the nondegenerate minimum. If $T_* < 4$, approximate its infimum arbitrarily closely by smooth strict interior points p of the ball. Around each such p there is a smaller full H^s ball contained in $\mathcal{F}_\eta$. Its smooth reference satisfies LCH Assumption 2.1 and fails (C). For the single critical point required by Assumption 2.1, C^1-closeness of u_p (LCH Lemma S4.1(a)) to u_0, whose gradient is bounded below on $\overline{\Omega} \setminus B$ (by compactness in the interior and by the Hopf lemma at $\partial\Omega$), gives a single critical point of u_p on $\overline{\Omega}$, the minimum $x_p \in B$. Theorem 3.3(b) of LCH applies on sufficiently small balls about the new center p, and lower bounds over a small ball hold over every larger ball (LCH, Section 2.1). For $T_p < 4$ this bounds $e(\mathcal{F}_\eta)$ above by $(2s+T_p)/(4s+8)$. Letting these T_p approach T_* gives the stated exponent upper bound. If $T_* \ge 4$, the parametric lower bound of Theorem 3.3(c) suffices for the exponent. This proves equality of the exponents for fixed η. Finally $T_-(\eta) \to T_{f_0}$ as $\eta \downarrow 0$ by the same continuity, proving the conjectured limit.

Corollary 2. *For each sufficiently small $\eta > 0$, one has $T_-(\eta) < T_{f_0}$. If (C) fails at f_0 and $T_{f_0} \le 4$, the exponent on every such fixed ball is strictly smaller than the local limit in Theorem 1.*

Proof. Let a be a unit eigenvector for the largest eigenvalue of H_{f_0}. Choose $V \in C_c^\infty(\Omega)$ equal to $(a \cdot (x - x_0))^2/2$ near x_0. By the transport inverse in LCH Lemma S2.2 there is a smooth perturbation h in $H^s(\Omega)$ such that

$$\operatorname{div}(h\nabla u_0) = -L_{f_0}V.$$

Indeed, apply the inverse to $-L_{f_0}V$, which belongs to every Sobolev space. For small $t > 0$ put $f_t = f_0 + th$. Substitution into the forward equation gives

$$L_{f_t}(u_{f_t} - u_0 - tV) = -t^2 \operatorname{div}(h\nabla V).$$

The uniform H^{s-1} to H^{s+1} Dirichlet inverse bound from LCH Lemma S4.1 and Sobolev embedding give an $O(t^2)$ error in C^2. Since $\nabla V(x_0) = 0$ and the Hessian stays uniformly positive near x_0, the minimum moves by $O(t^2)$. Consequently $H_{f_t} = H_{f_0} + taa^{\mathsf{T}} + O(t^2)$. Write $Q_0 = \operatorname{tr} H_{f_0}$ and $\lambda_0 = \lambda_{\max}(H_{f_0})$. The trace increases by $t + O(t^2)$. The largest eigenvalue is $\lambda_0 + t + O(t^2)$ by the variational formula: the Rayleigh quotient of $H_{f_0} + taa^{\mathsf{T}}$ is at most $\lambda_0 + t$ and equals it at a. This holds for any unit top eigenvector a, also when λ_0 has multiplicity greater than one. Hence

$$T_{f_t} = T_{f_0} - \frac{Q_0 - \lambda_0}{\lambda_0^2} t + O(t^2).$$

Since $d \geq 2$ and H_{f_0} is positive definite, $Q_0 > \lambda_0$. Choose $t > 0$ small enough that $f_t \in \mathcal{F}_\eta$ and $T_{f_t} < T_{f_0}$. The exponent comparison now follows from Theorem 1. □

For s as in Theorem 1, Corollary 2 extends the strict inequality that LCH Corollary 3.8 proves at the canonical model for $d \in \{2, 3\}$ to every reference satisfying LCH Assumption 2.1 with $T_{f_0} \leq 4$ at which (C) fails, including $T_{f_0} = 4$. Since $\mathcal{R}_N(\mathcal{F}_\eta)$ is nondecreasing in η, it holds for every larger fixed ball as well.

Part II. Interior support: the root-sample threshold

LCH, Section 6, third bullet, asks for the minimax rate under interior support constraints or known boundary values, for anisotropic references with $\psi(x_0) \neq 0$. Their Remark 3.9 treats interior perturbations in the linearized model when $\psi(x_0) = 0$, where the parametric rate holds, and isotropic Hessians, where $a_N(s, d)$ persists. This part answers the question for interior support constraints at the level of the root-sample threshold, at every nondegenerate minimum with nonzero score, and so complements Remark 3.9: with known interior support the adjoint equation needs to hold only near the permitted perturbations, so a singular adjoint may be built from a subset of the slow Hessian directions, and the threshold becomes the largest effective dimension across complete slow eigenvalue blocks. The criterion contains Remark 3.9 as its $\psi(x_0) = 0$ branch and recovers its isotropic threshold through $\mathcal{D}(H) = d$.

5 Experiment and result

Keep the domain, source and reference of Section 1.1. Let $K \Subset \Omega$ be compact with $x_0 \in \operatorname{int}(K)$. Fix $\psi \in C_c^\infty(\Omega)$ and target $\Phi(f) = \int f\psi$. The perturbations satisfy

$$\mathcal{H}_s(K, \eta) = \{h \in H^s(\Omega) : \operatorname{supp} h \subset K,\ \|h\|_{H^s(\Omega)} \leq \eta\}.$$

The nonlinear white-noise experiment has mean u_f and noise level $N^{-1/2}$, with $f = f_0 + h$ and η small enough to ensure positivity. The linearized experiment has mean $Ah = -L_{f_0}^{-1} \operatorname{div}(h\nabla u_0)$ with the same noise and target $\int h\psi$. Risk is minimax mean absolute error in each experiment.

Let $H = D^2u_0(x_0)$ have distinct eigenvalues $0 < \lambda_1 < \cdots < \lambda_b$ with multiplicities $m_1, \ldots, m_b$. At each spectral cut set

$$Q_j = \sum_{i \leq j} m_i \lambda_i, \qquad \mathcal{D}(H) = \max_{1 \leq j \leq b} \frac{Q_j}{\lambda_j}.$$

The full effective dimension of LCH is $T_{f_0} = Q_b/\lambda_b$. The quantity $\mathcal{D}(H)$ also tests the lower spectral blocks; only cuts after complete blocks are needed.

Theorem 3. *In the linearized experiment, an L^2 representer for the functional over L^2 perturbations supported in K exists if and only if*

$$\psi(x_0) = 0 \qquad or \qquad \mathcal{D}(H) > 4. \tag{10}$$

If a representer exists the linearized risk is $O(N^{-1/2})$. If (10) fails, the linearized risk multiplied by $\sqrt{N}$ diverges to infinity. The latter statement holds for every $s \geq 0$ and $\eta > 0$.

The branch $\psi(x_0) = 0$ is LCH Remark 3.9, and for H a multiple of the identity $\mathcal{D}(H) = d$ recovers their isotropic threshold.

Theorem 4. *Assume $\psi(x_0) \neq 0$. If $\mathcal{D}(H) > 4$, the nonlinear risk on $f_0 + \mathcal{H}_s(K, \eta)$ is of order $N^{-1/2}$ for sufficiently large s and small fixed $\eta > 0$. If $\mathcal{D}(H) \leq 4$, the nonlinear risk multiplied by $\sqrt{N}$ diverges to infinity on every sufficiently small fixed interior Sobolev ball, for $s > d/2$.*

Thus the case $\psi(x_0) \neq 0$ of the interior-support question of LCH, Section 6, has an exact root-sample criterion, and Theorem 7 below gives an upper bound governed by a retained block below that threshold. The result uses the freedom to choose an adjoint solution on a region containing K. Such a solution can be singular along the subspace of discarded fast directions.

For example, take dimension six, with five eigenvalues equal to 1 and one equal to 10. The full effective dimension is $3/2$ and $\mathcal{D}(H) = 5$. On an ellipsoid $\{x^\top H x < 1\}$ the reference $f_0 = 1$ and constant source $g = \operatorname{tr} H$ give $u_0 = (x^\top H x - 1)/2$. For $\psi(0) \neq 0$, the full-neighborhood rate is slower than root-sample by LCH Theorem 3.3(b). The fixed interior-support model is root-sample by Theorem 4. Locally an admissible singular part of the adjoint is a multiple of the logarithm of the radius in the five slow coordinates.

6 A local Sobolev transport criterion

Let $b = \nabla u_0$. The positive-spectrum normal form of Sternberg and Chen [12, 1], in the form of LCH Lemma S2.14, provides smooth local coordinates at the minimum in which b is conjugate to a triangular polynomial field V with components

$$V_j(y) = \lambda_j y_j + P_j(y_1, \ldots, y_{j-1}), \qquad y_j \in \mathbb{R}^{m_j}.$$

A resonant monomial in the jth block uses only strictly slower variables, since its weighted degree is λ_j and its ordinary degree is at least two. In particular the divergence of V restricted to its first j blocks is the constant Q_j. Denote the restriction by $V^{[j]}$.

Lemma 5. *There is a local H^2 solution of $V \cdot \nabla W = 1$ near the origin if and only if $\mathcal{D}(H) > 4$.*

Proof. For necessity suppose that $Q_j \leq 4\lambda_j$ for every j. We first give the reduction for the field on its first j blocks, when

$$V^{[j]} \cdot \nabla U = -aU, \quad a > 0, \quad \text{or} \quad V^{[j]} \cdot \nabla U = 1.$$

The backward flow X_t leaves a sufficiently small ball invariant and contracts toward zero. Its volume Jacobian is $e^{-Q_j t}$. The fastest block has the exact affine formula

$$(X_t y)_j = e^{-\lambda_j t} y_j + G_t(y_1, \ldots, y_{j-1}),$$

and the other blocks do not depend on y_j. For H^2 solutions the equation integrates along this flow: composing an H^1 function with the smooth flow on each finite time interval gives the weak chain rule, by smooth approximation locally, and integration in time then gives the identity in L^2 on a smaller ball. In the homogeneous equation, $U(X_t y) = e^{at} U(y)$; in the constant-source equation, $U(X_t y) = U(y) - t$. Taking two derivatives in any pair of fastest variables gives

$$(\partial_{y_j^\mu} \partial_{y_j^\nu} U)(X_t y) = e^{(2\lambda_j + a)t} \, \partial_{y_j^\mu} \partial_{y_j^\nu} U(y)$$

in the first case, and the same formula with $a = 0$ in the second. The fast dependence of X_t is affine, so the identity follows directly from the Sobolev chain rule.

For a fixed $t_0 > 0$, the annular sets $X_{\ell t_0}(B \setminus X_{t_0} B)$, $\ell \geq 0$, partition a sufficiently small starting ball B up to the point 0 and boundaries of measure zero. If one fastest second derivative had positive squared integral on the first set, its integral on subsequent sets would be multiplied by $e^{(4\lambda_j + 2a - Q_j)\ell t_0}$, with $a = 0$ for constant source. The sum diverges. Thus every fastest second derivative vanishes in B.

On a smaller product box the solution is affine in y_j,

$$U(y) = A(y_{<j}) \cdot y_j + B_0(y_{<j}),$$

with A, B_0 locally H^2. The affine assertion follows by slicing, and the coefficients inherit H^2 by integration against fixed smooth test functions of the fast variables extracting the constant and linear coefficients. Matching the linear coefficient in the equation yields

$$V^{[j-1]} \cdot \nabla A = -(a + \lambda_j) A$$

for the homogeneous equation, and the same formula with $a = 0$ for constant source. By induction on j, every H^2 solution to the homogeneous equation with $a > 0$ vanishes. Indeed the affine slope vanishes by the induction hypothesis and then B_0 satisfies the reduced homogeneous equation. In zero dimensions the equation is $0 = -aU$. In the constant-source equation the slope therefore vanishes at each stage, so one reduces to a constant-source equation on the next smaller block. In zero dimensions this is impossible. This proves necessity, including equality.

For sufficiency choose j with $Q_j > 4\lambda_j$. A local solution is the negative of the forward exit time from a small sphere for $V^{[j]}$, viewed as a function of only the first j blocks. Indeed the forward flow decreases the exit time at unit speed. Its H^2 property follows from Lemma 6 below. Dependence on the other variables is constant on a product neighborhood. □

Lemma 6. *Let B_s be a C^3 field near zero, with $B_s(0) = 0$ and symmetric positive derivative H_s at zero. Put $Q_s = \operatorname{tr} H_s$ and $\lambda_s = \lambda_{\max}(H_s)$. If $Q_s > 4\lambda_s$, the negative forward exit time ω from a sufficiently small sphere is in H^2 inside that sphere and satisfies $B_s \cdot \nabla \omega = 1$. The bounds are uniform under common C^3 and positive spectral bounds and a common positive gap $Q_s - 4\lambda_s$.*

Proof. On a small uniform ball the symmetric part of DB_s is uniformly positive. Its backward flow contracts. The time integral of the deviation of DB_s from H_s along a trajectory in the ball is uniformly bounded, by exponential contraction and the Lipschitz bound. In exit coordinates (t, θ), with θ on the sphere, the volume element is bounded by $Ce^{-Q_s t} \, dt \, d\sigma(\theta)$ by the Jacobian formula and transversality. The forward variational equations bound derivatives of orders one and two by, respectively, $Ce^{\lambda_s t}$ and $Ce^{2\lambda_s t}$, as in (3). Implicit differentiation at the uniformly transverse exit gives the same bounds for the exit time. Its value is t. The resulting squares are integrable by $Q_s > 4\lambda_s$. To identify the weak derivatives at zero, smoothly flatten the exit-time variable above level L_0 over a transition interval of length one, with bounded first and second derivatives of the flattening. These functions are constant near zero and converge in H^2 by the integral bounds and dominated convergence. The transport equation passes to the limit in distributions. □

7 A uniform slow-coordinate factor

The preceding exact normal form suffices for a fixed smooth reference. A nonlinear upper bound also needs a uniformly bounded representer at a data-dependent pilot. Direct use of a smooth normal form could cause small denominators when nearby Hessians cross resonances. We construct a factor using only the separation of the retained and discarded eigenvalues. It is uniformly C^k for any fixed $k \geq 2$ when s is sufficiently large.

Fix a spectral cut of H. At nearby Hessians retain the same number of smallest eigenvalues. If the cut is proper, there is a uniform positive gap between the retained and discarded blocks. By the uniform regularity in LCH Lemma S4.1, for sufficiently large fixed s, all nearby fields $b_p = \nabla u_p$ have a uniform C^{M+2} bound for any fixed integer M used below, and $\|u_p - u_0\|_{C^2} = O(\eta)$. Hence the retained eigenvalues at x_p are within $O(\eta)$ of those at x_0, by continuity of eigenvalues. In particular, if the cut satisfies $Q_j > 4\lambda_j$ at the reference, then for η small every pilot's retained block satisfies

$$Q_s(p) - 4\lambda_s(p) \;\geq\; \tfrac{1}{2}\,(Q_j - 4\lambda_j) > 0, \tag{11}$$

which is the common gap required in Lemma 6. We always center coordinates at the minimum and diagonalize the Hessian orthogonally. Bounds are uniform under any choice of orthonormal bases for spectral blocks.

Fix $m > 0$ smaller than one half of a uniform lower spectral bound, and $\Lambda > 0$ larger than a uniform upper spectral bound. Fix the desired regularity $k \geq 2$ and choose $M \geq k+1$ so $(M+1)m > (k+1)\Lambda$. For each p, make polynomial coordinate changes of orders 2 through M in the retained output components. At order r these remove every degree-r monomial in a retained component that contains at least one discarded input. For a monomial $y^\alpha e_i$ in a coordinate change, its linear effect on that coefficient is multiplication by $\alpha \cdot \lambda - \lambda_i$, as follows by differentiating the change along the linear vector field and subtracting the linear output. The denominator is bounded below by the positive spectral gap, since one input is discarded, the output is retained, and all eigenvalues are positive. Solve at each degree and leave earlier degrees untouched. Taylor's theorem and the common finite differentiability bound give the transformed field b with retained part of the form

$$b_s(y) = B_s(y_s) + R(y), \qquad |D^\alpha R(y)| \leq C|y|^{M+1-|\alpha|} \quad (|\alpha| \leq k). \tag{12}$$

Here $y = (y_s, y_f)$ are retained and discarded coordinates, and B_s is the restriction of the retained components to $y_f = 0$. The changes and their inverses have uniform bounds on a fixed small chart. At zero the derivative is still the diagonal Hessian. If no block is discarded, one may take the identity factor and $B_s = b_p$ in centered rotated coordinates.

In the proper-cut case shrink the chart so that the symmetric part of Db is bounded below by m and the backward flow X_t stays in the chart on a common small ball. It satisfies $|X_t y| \leq e^{-mt}|y|$ and $\|D^j X_t\| \leq C_j e^{-mt}$ for $1 \leq j \leq k$ by induction in the backward variational equations. Extend B_s to all retained coordinates by cutting off only its nonlinear part in a small ball. This preserves its derivative at zero and yields a forward flow φ_t^s with $\|DB_s\| \leq \Lambda$ and bounded derivatives through order $k+1$. Shrinking the support permits the bound by Λ because the nonlinear part vanishes to second order and its derivative to first order. The forward variational equations give $\|D^j \varphi_t^s\| \leq C_j e^{j\Lambda t}$ for $1 \leq j \leq k+1$.

Let $\pi_s y = y_s$ and set $P_t(y) = \varphi_t^s(\pi_s X_t y)$. The exact time derivative is

$$\partial_t P_t(y) = -D\varphi_t^s(\pi_s X_t y)\, R(X_t y).$$

Indeed the two terms involving B_s cancel by the flow identity $B_s(\varphi_t^s(z)) = D\varphi_t^s(z) B_s(z)$. By (12), the chain rule and backward contraction, derivatives of $R(X_t y)$ through order $j \leq k$ are bounded by

$C|y|^{M+1-j}e^{-(M+1)mt}$. Multiplying by the forward derivative bounds shows that $\partial_t P_t$ is integrable in C^k. Let $\pi = \lim_{t\to\infty} P_t$. It is uniformly C^k, with $\pi(y) = y_s + O(|y|^{M+1})$ and $D\pi(y) = D\pi_s + O(|y|^M)$. In particular $y \mapsto (\pi(y), y_f)$ is a uniformly invertible C^k change of coordinates on a smaller ball. For fixed $\tau \geq 0$ the semigroup identity in the defining limit gives

$$\pi(X_\tau y) = \varphi^s_{-\tau}(\pi(y)).$$

It follows that $D\pi\, b = B_s \circ \pi$. Localizing to a smaller neighborhood, only the unextended retained field is used. This construction gives the required factor uniformly with no nonresonance hypothesis among the retained eigenvalues.

8 Adjoint representers and risk

First work with the reference alone. An L^2 function z is a representer for supported perturbations if and only if $w = L_{f_0}^{-1} z \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies $b \cdot \nabla w = \psi$ almost everywhere on K. This follows by integration by parts against h, and by the Dirichlet elliptic H^2 estimate for the inverse. In particular the equation is necessary on a neighborhood of x_0. The smooth particular solution with right hand side $\psi - \psi(x_0)$ is the integral of $\psi(X_t x) - \psi(x_0)$ for $0 \leq t < \infty$, with X_t the backward gradient flow in the original coordinates. After subtracting it and dividing by $\psi(x_0)$ when this value is nonzero, a representer would produce a local H^2 solution with constant source. Smooth conjugacy preserves local H^2, so Lemma 5 proves the necessity of (10).

We construct the sufficient representers simultaneously for the reference and the nonlinear pilot family. Choose a fixed regular compact sublevel of u_0 with K in its interior, contained in Ω. The proof of LCH Lemma S4.1 shows that for small enough η it is invariant under the backward flows of b_p and every point enters a common small neighborhood of the minimum in uniformly bounded time. Indeed outside that neighborhood the gradients are uniformly bounded below and u_p decreases along each backward trajectory. The strictly outward gradient on the sublevel boundary prevents backward escape. By further contracting within the local ball, choose one fixed time t_1 so that this backward flow sends a neighborhood U of K into a ball where the factor of Section 7 is defined and where its output is in the exit-time domain of Lemma 6. Compose the factor with the local polynomial chart and with this fixed-time flow. Write the result as $\Pi_p(x)$. It satisfies

$$D\Pi_p(x) b_p(x) = B_s(\Pi_p(x)), \qquad x \in U,$$

by the factor relation and the fact that a fixed-time flow pushes its vector field to itself.

Let v_p be the preceding backward integral with $\psi(x_p)$ subtracted. On U it has bounded C^2 norm, uniformly in p. Indeed split off the fixed time until entry to the convex ball. Beyond that time the backward first and second variational derivatives decay exponentially, so the differentiated integrals converge uniformly. It satisfies $b_p \cdot \nabla v_p = \psi - \psi(x_p)$. If the spectral cut inequality holds, use ω_p from Lemma 6 and take

$$w_p = \chi\{v_p + \psi(x_p)\, \omega_p \circ \Pi_p\},$$

where χ is a fixed smooth function supported in U and equal to one near K. By the uniform gap (11), the bounds of Lemma 6 hold uniformly over the pilots. The second term is therefore uniformly H^2 on U, because the factor can be completed by the discarded coordinates to a uniformly controlled C^2 diffeomorphism, before composition with the fixed-time flow. In those coordinates, the time function is independent of the discarded variables. Product integration on a bounded coordinate box gives the H^2 bound, and composition with the diffeomorphisms preserves it. The weak chain rule and the transport equations show that $b_p \cdot \nabla w_p = \psi$ on K. If $\psi(x_0) = 0$ for the linearized

problem, simply take $w_0 = \chi v_0$. In every sufficient case $\|w_p\|_{H^2}$ and hence $\|\nabla w_p\|_{L^1(K)}$ and $\|L_p w_p\|_2$ are bounded uniformly in p. The exact linear estimator against this representer gives the stated linearized upper bound.

For the nonlinear parametric upper bound apply the split white-noise pilot of LCH Lemma S4.2 to the supported class. This only uses a countable prediction-dense subset of the class, so no approximation by smooth functions supported in K is needed. By that lemma, $\delta_p = \|u_f - u_p\|_2$ has all fixed positive moments of the order of $N^{-(s+1)/(2s+d+2)}$ to the corresponding powers. Take the second-sample one-step estimate using $z_p = L_p w_p$. The exact testing identity of LCH Lemma S4.3 gives conditional bias

$$\int_\Omega (f-p)\,\nabla(u_f - u_p)\cdot\nabla w_p$$

because $f - p$ is supported in K. Conditional standard deviation is $\sqrt{2/N}\|z_p\|_2$. For a fixed $\sigma > d/2$ use LCH Lemma S4.4 and (S4.4) on the larger compact sublevel, Sobolev embedding, and the uniform $L^1(K)$ bound on ∇w_p. The conditional bias in absolute value is at most

$$C\|f-p\|_{L^\infty(K)}\|\nabla(u_f-u_p)\|_{L^\infty(K)} \le C\delta_p^{[(s-1)(1-\sigma/s)+(s-\sigma)]/(s+1)}.$$

The expected value of this bound is $o(N^{-1/2})$ for sufficiently large s, since its exponent after including the pilot rate tends to one. This gives the claimed uniform parametric risk upper bound. The independent randomization of the split can be removed by conditioning on the original observation and using Jensen's inequality.

Finally suppose (10) fails. The ratio

$$\frac{|\int h\psi|}{\|Ah\|_2}, \qquad h \in C_c^\infty(\operatorname{int} K), \quad h \neq 0,$$

is unbounded. Otherwise the functional $Ah \mapsto \int h\psi$ would extend boundedly to the L^2 closure of that range and have an L^2 representer. Its weak transport equation on $\operatorname{int} K$ would contradict Lemma 5 near x_0. The denominators are nonzero: if $\operatorname{div}(hb) = 0$, then $h(X_t x)\det DX_t(x) = h(x)$, and the left-hand side tends to zero by boundedness of h and eventual exponential contraction. Hence $h(x) = 0$. For any fixed smooth direction with a prescribed large ratio, compare zero to the perturbation $t_N h$, where $t_N = c/(\sqrt{N}\|Ah\|_2)$ and $c > 0$ is a small absolute constant. It belongs to any fixed positive radius ball for all sufficiently large N. The linear Gaussian Kullback–Leibler divergence is $c^2/2$. In the nonlinear experiment it tends to the same value by differentiability of u_{f_0+th} at zero with derivative Ah, which follows by subtracting the elliptic equations and taking the limit in the variational inverse. The two-point testing inequality therefore lower bounds $\sqrt{N}$ times the minimax risk by a positive constant times the chosen ratio, for all sufficiently large N depending on the direction. Letting the ratio be arbitrarily large proves the divergence in both experiments. When $\psi(x_0) \neq 0$ and the spectral criterion is satisfied, the same two-point argument with any one smooth supported direction of nonzero functional gives the ordinary parametric lower bound. This proves Theorems 3 and 4.

9 An upper bound below the root-sample threshold

The slow factor also improves the nonparametric upper bound. Choose any fixed spectral cut retaining at least two directions, possibly the full set, and let r be the number retained. In the nonlinear interior model write

$$T_* = \inf_{p \in f_0 + \mathcal{H}_s(K,\eta)} \frac{\sum_{i=1}^r \lambda_i(p)}{\lambda_r(p)},$$

where the eigenvalues at the pilot minimum are ordered with multiplicity. By taking η small, a proper reference cut stays separated from the discarded block.

Theorem 7. *For sufficiently large s and small fixed η, the nonlinear interior minimax absolute risk has the upper bounds*

$$\begin{cases} O(N^{-(2s+T_*)/(4s+8)}), & T_* < 4, \\ O(\sqrt{\log N/N}), & T_* = 4, \\ O(N^{-1/2}), & T_* > 4. \end{cases}$$

In particular one may choose a cut attaining $\mathcal{D}(H)$ at the reference. Its T_* tends to $\mathcal{D}(H)$ as $\eta \downarrow 0$.

Proof. Take an integer $k > d/2$, $k \geq 2$, and use the uniformly C^k factor of Section 7. Its construction only needs the spectral gap across a proper cut, so it applies also when the retained trace is at most four times its largest eigenvalue. Use the negative slow exit time ω_p on a common small ball. The proof of Lemma 6, with the higher variational equations through order k, gives at exit time t

$$|\omega_p| = t, \qquad |D^j \omega_p| \leq C_j e^{j\lambda_s t} \quad (1 \leq j \leq k),$$

and the volume element is bounded by $Ce^{-Q_s t}\, dt\, d\sigma$. In particular ω_p lies in L^2 and $W^{1,1}$ uniformly, since $Q_s - \lambda_s$ is bounded below when at least two eigenvalues are retained. Smooth flattening at large exit time verifies the weak gradient.

Use the fixed-time extension of Π_p to U containing K. Let χ_1 and χ_2 be smooth and compactly supported in U, with $\chi_1 = 1$ near K and $\chi_2 = 1$ on a neighborhood of $\operatorname{supp} \chi_1$. Put $q = \chi_1\, \omega_p \circ \Pi_p$ and extend by zero to $\mathbb{R}^d$. Then q is uniformly in $L^2 \cap W^{1,1}$ by the controlled factor coordinates, and

$$\int_{R \leq |\xi| \leq 2R} |\widehat{q}(\xi)|^2\, d\xi \leq CR^{-Q_s/\lambda_s}. \tag{13}$$

This is the finite-difference argument proving (5) in Section 2, applied to the slow backward flow in the retained coordinates with (Q, λ) replaced by (Q_s, λ_s). The only additions are these. Uniform Lipschitz control of Π_p keeps the slow coordinates of all shifted points $x + \ell z$, $0 \leq \ell \leq k$, with $\Pi_p(x) \in E_t$ inside E_{t-c_0}. On the inner set one changes variables to the slow and complementary fast coordinates at each shifted point; the fast cross-sections are bounded, so the inner integral is again $Ce^{-Q_s t}$, and the term from writing $q = \chi_1(\omega_p \circ \Pi_p + t) - t\chi_1$ contributes at most $Ct^2|z|^{2k}e^{-Q_s t}$. The outer integral is bounded by $C|z|^{Q_s/\lambda_s}$ since $2k > d \geq Q_s/\lambda_s$.

For a smooth Fourier cutoff P_R as in LCH Lemma S2.12, use the representer potential

$$w_{p,R} = \chi_1 v_p + \psi(x_p)\chi_2 P_R q.$$

It has uniformly bounded gradient in L^1 since $q \in W^{1,1}$. The shell bound controls $\|L_p w_{p,R}\|_2$ by $CR^{2-T_*/2}$ if $T_* < 4$, $C(1 + \log R)^{1/2}$ if $T_* = 4$, and a constant if $T_* > 4$, by summation of dyadic H^2 shells. On the support of $h = f - p$ the transport derivative of the potential is $\psi - \psi(x_p)$ from its smooth part, so the defect equals $\psi(x_p)$ times

$$(I - P_R)(b_p \cdot \nabla q) + [P_R, b_p \cdot \nabla] q,$$

with b_p extended by a smooth cutoff beyond the support of χ_2 and all expressions restricted to K. The first term uses $b_p \cdot \nabla q = \chi_1 + (b_p \cdot \nabla \chi_1)(\omega_p \circ \Pi_p)$. The χ_1 contribution is rapidly decreasing. The second summand is L^1 and supported a positive distance from K, so its convolution is also rapidly decreasing on K by the Schwartz kernel estimate. The commutator is treated by the low/high coefficient split of Section 3, with (13) in place of (5); it gives the bound (9) with T_* as defined here. The pilot argument of Section 4 bounds the supremum norm in expectation by CN^{-A} and

the nonlinear remainder by $o(N^{-1/2})$ through the uniform gradient L^1 bound. With $A > d/(2s+4)$ and $R = N^{1/(2s+4)}$ the three cases follow. Continuity of the minimum-shifted Hessian yields the final limit for T_*. □

When $\operatorname{tr} H < 4\lambda_{\min}(H)$, Part V determines the interior-support rate below the threshold: it is $N^{-(2s+d)/(4s+8)}$, the rate of an isotropic minimum, in the linearized experiment for every $s \geq 0$ and in the nonlinear experiment for $s > 2d+4$, and for anisotropic H it is polynomially faster than the bound of Theorem 7 (Theorem 26 and Corollary 27).

Part III. A power-law degenerate minimum

LCH, Section 6, second bullet, lists degenerate critical points among the further problems for their nondegenerate theory. This part answers it for one radial model in that degenerate class: the even power-law minimum on the unit ball, sharply in the linearized experiment and, for sufficiently smooth radial conductivities, in the nonlinear experiment. The exponential flow at a nondegenerate minimum is replaced by an explicit power-law flow, the transition takes place at dimension $2p$, and the critical logarithm is forced by perturbations on disjoint annuli. Setting $p = 2$ in the rates recovers the isotropic rate $a_N(s, d)$ and critical dimension four of LCH.

10 Experiment and results

Let $\Omega = B_1(0) \subset \mathbb{R}^d$ with $d \geq 2$. Fix an even integer $p \geq 4$ and put

$$f_0 = 1, \qquad u_0(x) = \frac{|x|^p - 1}{p}, \qquad b(x) = \nabla u_0(x) = |x|^{p-2}x, \qquad g(x) = (d+p-2)|x|^{p-2}.$$

The conductivity equation is $\operatorname{div}(f\nabla u_f) = g$, with $u_f = 0$ on the boundary. The reference has a unique minimum at the center with zero Hessian and radial gradient of order $p-1$, and the source vanishes at the center to order $p-2$, exactly as the power law requires. This model therefore lies in the degenerate class of the second bullet rather than under LCH Assumption 2.1.

Fix $\psi \in C_c^\infty(\Omega)$ with $\psi(0) \neq 0$. In the linearized experiment the observation is Gaussian white noise in $L^2(\Omega)$ with mean

$$Ah = -\Delta_D^{-1} \operatorname{div}(hb)$$

and noise level $N^{-1/2}$. Here Δ_D^{-1} is the Dirichlet inverse of the Laplacian. The target is $\int_\Omega h\psi$, for $\|h\|_{H^s(\Omega)} \leq \eta$. The forward operator is bounded from L^2 to L^2 since b is bounded and the Dirichlet energy estimate applies. In the nonlinear experiment the mean is u_f and the target is $\Phi(f) = \int_\Omega f\psi$. All risks below are minimax mean absolute errors. Write

$$\mathfrak{r}_N(s, d, p) = \begin{cases} N^{-(2s+d)/(4s+4p)}, & d < 2p, \\ \sqrt{\log N/N}, & d = 2p, \\ N^{-1/2}, & d > 2p. \end{cases}$$

Theorem 8. *For every $s \geq 0$ and fixed $\eta > 0$ the linearized risk is of order $\mathfrak{r}_N(s, d, p)$. The lower bounds hold even when h is radial and supported in any fixed ball about zero contained in Ω.*

Theorem 9. *For sufficiently large s and small fixed $\eta > 0$, the nonlinear risk over radial positive f with $\|f-1\|_{H^s} \leq \eta$ is of order $\mathfrak{r}_N(s, d, p)$. The nonlinear lower bounds hold also over the unrestricted ball of positive conductivities.*

The adjoint range condition for efficient Darcy inference is studied by Nickl and Paternain [6]. Koers, Szabó and van der Vaart [4] use linear inverse methods in nonlinear PDE recovery, including the Darcy problem. Here the reciprocal conductivity makes the radial observation map exactly linear.

11 An adjoint cutoff with small dual bias

Lemma 10. *Fix $s \geq 0$. Let β have bounded C^2 norm and support in a fixed compact subset of the unit ball. For all sufficiently small $a > 0$ there is a potential $w_a \in H^2 \cap H_0^1(\Omega)$ such that*

$$\left| \int h(\beta - b \cdot \nabla w_a) \right| \leq C a^{s+d/2} \|h\|_{H^s(\Omega)}, \tag{14}$$

$$\|\Delta w_a\|_2 \leq C \begin{cases} a^{d/2-p}, & d < 2p, \\ (1 + \log(1/a))^{1/2}, & d = 2p, \\ 1, & d > 2p. \end{cases} \tag{15}$$

The constants are uniform in β with the indicated support and norm bounds.

Proof. Write $\nu = p - 2 > 0$. In polar coordinates, the punctured-domain adjoint solution is

$$w(r\vartheta) = -\int_r^1 t^{-\nu-1} \beta(t\vartheta)\, dt.$$

It vanishes near the outer boundary. Its derivatives through order two satisfy $|D^j w(x)| \leq C|x|^{-\nu-j}$ for $0 \leq j \leq 2$. The radial derivatives follow by differentiation of the lower endpoint, and angular derivatives cost at most $|x|^{-1}$ per Euclidean derivative, with uniformly bounded angular derivatives of the integrand through order two. Multiplying w by a fixed radial cutoff dilated to equal zero for $r < a$ and one for $r > 2a$ gives a preliminary potential $\widetilde{w}_a$. Its defect $\beta - b \cdot \nabla \widetilde{w}_a$ is supported in $|x| \leq 2a$ and uniformly bounded, by the derivative and size bounds above. Integrating the bound on $D^2 w$ outside $2a$ and its cutoff analogue on $a \leq r \leq 2a$ gives precisely (15) for $\widetilde{w}_a$.

Choose an integer L with $L + 1 > s - d/2$. We correct every moment of the defect of order at most L. For each multi-index α in that range,

$$\int y^\alpha |y|^\nu y \cdot \nabla V(y)\, dy = -(d + \nu + |\alpha|) \int |y|^\nu y^\alpha V(y)\, dy.$$

For the moment map take the fixed test functions $V_\gamma(y) = \chi(y)|y|^\nu y^\gamma$, where $\chi \geq 0$ is smooth, supported in a fixed annulus inside $|y| < 2$ and positive on an open subset. The matrix of moments of $|y|^\nu y \cdot \nabla V_\gamma$ differs from an invertible polynomial Gram matrix by multiplication by an invertible diagonal matrix. Now

$$b(x) \cdot \nabla \left(a^{-\nu} V_\gamma(x/a) \right) = |x/a|^\nu (x/a) \cdot \nabla V_\gamma(x/a).$$

Hence adding an appropriate bounded linear combination of these scaled potentials to $\widetilde{w}_a$ cancels the desired moments. The coefficients are uniformly bounded because the preliminary defect in y coordinates is uniformly bounded and compactly supported. The added Laplacian has L^2 norm at most $Ca^{d/2-p}$, preserving (15).

The corrected defect has the form $\rho_a(x/a)$ with ρ_a supported in the fixed ball of radius two, uniformly bounded and with all moments through degree L zero. Therefore $|\widehat{\rho_a}(\xi)| \leq C|\xi|^{L+1}$ for $|\xi| \leq 1$, and Plancherel bounds its L^2 norm at high frequencies. The homogeneous H^{-s} norm is uniformly bounded because $2(L+1) - 2s + d > 0$. Scaling gives a bound $Ca^{s+d/2}$ for the defect on $\mathbb{R}^d$. Testing against any bounded Sobolev extension of h gives (14). □

In the linearized model, $\langle Ah, \Delta w_a \rangle = \int h\, b \cdot \nabla w_a$ by the weak Dirichlet equation. Use the linear observation against Δw_a with $\beta = \psi$ and $a = N^{-1/(2s+2p)}$. The bias (14) and noise (15) give the upper bound of Theorem 8 in all three regimes, for every $s \geq 0$, since $\beta = \psi$ is smooth.

12 Lower bounds on one scale and at the critical dimension

For a smooth radial template ζ supported in $\{1/2 < |x| < 1\}$, put

$$V(\rho) = -\int_{\rho}^{\infty} t^{p-1} \zeta(t)\, dt, \qquad \rho \geq 0,$$

where $\zeta(t)$ is its radial profile. The function V is smooth and radial, supported in the unit ball, and possibly constant on the inner ball. For $h(x) = A_0 \zeta(x/a)$ with $0 < a < 1$, the linearized mean Ah equals $-A_0 a^p V(x/a)$, by its radial derivative and the boundary condition. In particular,

$$\|Ah\|_2 = |A_0| a^{p+d/2} \|V\|_2, \qquad \|h\|_{H^s(\Omega)} \leq C |A_0| a^{d/2-s}. \tag{16}$$

A template with nonzero spatial integral satisfies $|\int h\psi| \gtrsim |A_0| a^d$ as $a \downarrow 0$ because $\psi(0) \neq 0$.

If $d < 2p$, take $A_0 = \gamma a^{s-d/2}$ and $a = N^{-1/(2s+2p)}$, with $\gamma > 0$ fixed and sufficiently small. Then the Sobolev bound holds, the Gaussian Kullback–Leibler divergence from zero is bounded by a small constant, and the functional gap is of order $N^{-(2s+d)/(4s+4p)}$. For $d > 2p$ one fixed smooth radial supported direction of nonzero functional, at amplitude $\gamma/\sqrt{N}$, gives the ordinary parametric lower bound.

For $d = 2p$ choose the template so

$$\int_0^{\infty} t^{p-1} \zeta(t)\, dt = 0, \qquad \psi(0) \int_{\mathbb{R}^d} \zeta(x)\, dx > 0.$$

The powers $p-1$ and $d-1$ differ, so a linear combination of two disjoint radial bumps with opposite coefficients satisfies the first condition and gives the second after fixing the sign. Now V also vanishes on the inner ball. Let $a_j = a_0 4^{-j}$ with $a_0 > 0$ small and

$$H_j(x) = a_j^{-p-d/2} \zeta(x/a_j).$$

The means AH_j have disjoint annular supports and the same positive L^2 norm. The functions H_j satisfy $\int H_j \psi \geq c > 0$ by the template condition and $d = 2p$, uniformly in j after choosing a_0 small. Also $\int |H_j||\psi| \leq C$ and $\|H_j\|_{H^s} \leq C a_j^{-s-p}$. Set

$$J_N = \left\lfloor \frac{\log N}{4(s+p)\log 4} \right\rfloor, \qquad h_N = \frac{\gamma}{\sqrt{N J_N}} \sum_{j=1}^{J_N} H_j.$$

For large N, disjointness of the observation means gives $N\|Ah_N\|_2^2 \leq C\gamma^2$. The Sobolev triangle inequality and geometric summation give $\|h_N\|_{H^s} \leq C\gamma N^{-1/4}/\sqrt{J_N}$. The functional gap is at least $c\gamma\sqrt{J_N/N}$. Choosing γ small and applying the two-point Gaussian testing bound proves the critical lower bound. All these perturbations can be supported in any fixed interior ball about zero by choosing fixed templates or the largest scale accordingly.

13 The nonlinear radial experiment

For a positive radial conductivity put $\theta = 1/f$. The radial solution satisfies

$$\frac{d}{dr} u_f(r) = r^{p-1} \theta(r), \qquad u_f(1) = 0.$$

Indeed the flux $r^{d-1}f(r)u_f'(r)$ integrates the radial source to r^{d+p-2}, with zero center flux in the regular weak solution. Thus the Gaussian observation is exactly linear in θ. The target is the smooth nonlinear functional $G(\theta) = \int \psi/\theta$. The reciprocal map is bounded on the uniformly positive Sobolev class for $s > d/2$ and small fixed radius, by the Sobolev algebra and smooth composition bounds. It maps sufficiently small radial Sobolev balls at one into comparable balls. In particular the reciprocal class has uniformly bounded H^s norm, is uniformly positive, and its forward mean has bounded H^{s+1} norm. For the last assertion use $\nabla u_f = b\theta$ with smooth b and the Dirichlet trace.

Use two independent observations from Gaussian noise splitting. On the first, take an approximate minimum-distance prediction pilot q in the reciprocal class, by the construction of LCH Lemma S4.2, which only requires the uniform H^{s+1} bound on forward means. A bounded compactly supported Sobolev extension and a Fourier cutoff at $J \asymp N^{1/(2s+d+2)}$ give an L^2 prediction estimate with root mean square error $O(N^{-(s+1)/(2s+d+2)})$. The cutoff bias is $O(J^{-s-1})$ and the smoothed white noise has squared L^2 expectation $O(J^d/N)$ by the kernel's Hilbert–Schmidt norm. Its fixed moments have the same orders to their corresponding powers by Gaussian moment bounds. Choose a measurable approximate nearest forward mean in a fixed countable dense collection from the reciprocal class. The triangle inequality preserves all fixed moment bounds. Writing $v = u_f - u_{1/q}$ and $\delta = \|v\|_2$, they give

$$\left(\mathbb{E}_f \delta^\ell\right)^{1/\ell} \le C_\ell N^{-(s+1)/(2s+d+2)} \qquad (\ell > 0). \tag{17}$$

We also need an L^2 bound for the parameter error $h = \theta - q$. If $k = \lfloor s \rfloor \ge 1$ and a is small, then

$$\|h\|_2 \le C\big(a^k + a^{-(p-1)}\|\nabla v\|_2\big). \tag{18}$$

To see this, outside the ball of radius a, use $\nabla v = bh$ to control h by its weighted norm. Inside B_a, approximate h on B_{2a} by a polynomial of degree at most $k-1$, with L^2 error at most $Ca^k\|h\|_{H^k(B_{2a})}$, by the polynomial Poincaré inequality on the unit ball and a rescaling. The polynomial's L^2 norm on B_a is bounded by a fixed multiple of its norm on $B_{2a} \setminus B_a$ by equivalence of norms on the fixed polynomial space. This gives the same weighted bound inside the ball up to the approximation error. Since $\|h\|_{H^k}$ is uniformly bounded, (18) follows.

Interpolation from the uniform H^{s+1} bound gives $\|\nabla v\|_2 \le C\delta^{s/(s+1)}$. Optimizing a in (18) gives

$$\|h\|_2 \le C\delta^{sk/((s+1)(k+p-1))}.$$

This holds for small δ and extends to the remaining values by the uniform bound. Together with (17) it implies $\mathbb{E}_f\|h\|_2^2 = o(N^{-1/2})$ for sufficiently large s, since the exponent of N^{-1} in the bound is

$$\frac{2sk}{(k+p-1)(2s+d+2)},$$

which tends to one.

The exact expansion of G is

$$G(\theta) - G(q) = \int h\beta_q + O(\|h\|_2^2), \qquad \beta_q = -q^{-2}\psi,$$

uniformly on the reciprocal class. For $s > d/2 + 2$, Sobolev embedding gives the functions β_q uniform C^2 bounds and common compact support; this is the hypothesis of Lemma 10, and it is contained in the large-s hypothesis of Theorem 9. Use Lemma 10 with $\beta = \beta_q$ and $a = N^{-1/(2s+2p)}$. The L^2 representer for the cutoff term in the reciprocal experiment is $-\Delta w_a$ because $\nabla v = bh$ and both potentials have zero boundary trace. Starting at $G(q)$, add the second-sample residual

observation against this representer. Countable pilot selection ensures measurability. Its conditional bias is bounded by $Ca^{s+d/2} + C\|h\|_2^2$, and its conditional noise has standard deviation at most $\sqrt{2/N}\|\Delta w_a\|_2$. These bounds give the asserted upper rate after taking expectation. Conditional expectation given the unsplit observation removes auxiliary randomization without increasing risk.

For the nonlinear lower bounds put $\theta = 1 + h$ for the same radial perturbations h as in the linearized proof, and $f = 1/(1 + h)$. For fixed sufficiently large s and sufficiently small γ, these conductivities belong to the desired Sobolev ball by the composition bound. The observed mean difference is exactly $-Ah$. The linear functional gap in the reciprocal target has the same leading magnitude, since

$$G(1 + h) - G(1) = -\int h\psi + \int \frac{h^2\psi}{1 + h}.$$

In each of the three lower-bound constructions $\|h\|_\infty \to 0$ and $\int |h||\psi| \le C|\int h\psi|$. The latter follows by single-template scaling or disjointness and the common orientation of the critical annular templates. The nonlinear remainder is therefore negligible relative to the gap. The same Gaussian two-point bounds finish the proof.

Part IV. A two-block saddle

LCH, Section 6, second bullet, also lists saddle-type critical points among the further problems for their nondegenerate theory. This part answers it for one exact local model, the two-block linear saddle, in the supported linearized experiment. It gives an exact representer criterion, the sharp linearized rate for all positive block rates and all block dimensions at every smoothness $s > 0$, and the exponent of the rate at $s = 0$, with matching nonlinear lower bounds at sufficient smoothness. At a saddle the local adjoint accumulates the time spent near the critical point. A trajectory enters thin in the expanding block and leaves thin in the contracting block, and the rate is decided by how the two ends of the transit share this cost.

14 Setting and results

Let $\Omega \subset \mathbb{R}^{m+n}$ be smooth, bounded and connected, with $m, n \ge 1$. Work at constant conductivity $f_0 = 1$. Let u_0 be a smooth reference potential with zero Dirichlet boundary condition and known source $g = \Delta u_0$. For $a > 0$ write $B(a) = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : |x| < a,\ |y| < a\}$ in orthonormal coordinates, and suppose that for some $a_0 > 0$

$$b(x, y) = \nabla u_0(x, y) = (\lambda x, -\mu y) \quad \text{on } B(8a_0) \subset \Omega, \qquad \lambda, \mu > 0.$$

The potential may have other critical points outside $B(8a_0)$. Let $K \subset B(a_0)$ be compact with $0 \in \operatorname{int} K$, and fix $r_1 \in (0, a_0]$ with $B(4r_1) \subset \operatorname{int} K$; r_1 is decreased finitely often below. Fix $\psi \in C_c^\infty(\Omega)$ with $\psi_0 := \psi(0) \ne 0$; replacing ψ by $-\psi$ changes no risk, so we assume $\psi_0 > 0$. The perturbations range over the class $\mathcal{H}_s(K, \eta)$ of Section 5. The linearized experiment observes $Y = Ah + N^{-1/2}\mathbb{W}$ with $Ah = -\Delta_D^{-1} \operatorname{div}(hb)$ and $\mathbb{W}$ white noise on $L^2(\Omega)$, and the target is $\int h\psi$. In the nonlinear experiment the white-noise mean is u_f with $\operatorname{div}(f\nabla u_f) = g$ and zero trace, where $f = 1 + h > 0$, and the target is $\int f\psi$. Risks are minimax mean absolute errors. We write $\mathcal{R}_N$ for the linearized risk over $\mathcal{H}_s(K, \eta)$ and $\mathcal{R}_N^{\mathrm{nl}}$ for the nonlinear one, and put

$$r = |x|, \qquad \rho = |y|, \qquad \Gamma = \frac{\mu}{\lambda}, \qquad \Theta = m\lambda - n\mu = \operatorname{div} b, \qquad z = r^\mu \rho^\lambda.$$

The clock z is invariant along the flow of b. In this part C, c denote positive constants depending only on the fixed model and on fixed auxiliary functions, never on N or on the scale parameters.

Theorem 11. *For arbitrary positive λ, μ, the supported linearized problem has an L^2 representer if and only if $\max(m,n) > 4$. If $\max(m,n) \le 4$, the minimax risk times $\sqrt{N}$ diverges in the linearized experiment for every $s \ge 0, \eta > 0$, and in the nonlinear experiment for $s > (m+n)/2+1$ and small fixed $\eta > 0$.*

Effective dimension. For $m, n \in \{1,2,3\}$ put

$$M = M(m,n,\lambda,\mu) = \begin{cases} m + \dfrac{n}{4}\Big((4-m) - (4-n)\dfrac{\mu}{\lambda}\Big)_+, & m\lambda \ge n\mu, \\ n + \dfrac{m}{4}\Big((4-n) - (4-m)\dfrac{\lambda}{\mu}\Big)_+, & m\lambda \le n\mu, \end{cases} \tag{19}$$

where $t_+ = \max(t,0)$. We call M the effective dimension of the saddle. Both lines give $\max(m,n)$ when $m\lambda = n\mu$, and the formula is symmetric: $M(m,n,\lambda,\mu) = M(n,m,\mu,\lambda)$.

Theorem 12. *Let $\lambda, \mu > 0$ and $\eta > 0$.*

(a) If $m, n \le 3$, and either $s > 0$, or $s = 0$ and $m\lambda \ne n\mu$, then $\mathcal{R}_N \asymp N^{-(2s+M)/(4s+8)}$.

(b) If $m, n \le 3$, $s = 0$ and $m\lambda = n\mu$, then $cN^{-M/8}(\log N)^{-(4-M)/8} \le \mathcal{R}_N \le CN^{-M/8}$.

(c) If $\max(m,n) = 4$ and $s \ge 0$, then $\mathcal{R}_N \asymp \sqrt{\log N / N}$.

In particular, for $m, n \le 3$ and every $s \ge 0$ the exponent $\lim_{N\to\infty} \log(1/\mathcal{R}_N)/\log N$ equals $(2s + M)/(4s+8)$. The orders in (a) and (c) are also lower bounds for $\mathcal{R}_N^{\mathrm{nl}}$ when $s > (m+n)/2+1$ and $\eta > 0$ is sufficiently small. If $\max(m,n) > 4$, then $\mathcal{R}_N \asymp N^{-1/2}$ by Theorem 11.

Remark 13. For all positive rates with $\max(m,n) \le 3$, Proposition 21 and Remark 16 give the single-block upper bound $N^{-(2s+\max(m,n))/(4s+8)}$. By Corollary 14, for $s > 0$, and for $s = 0$ when $m\lambda \ne n\mu$, this bound is sharp exactly when one block has both the larger product of dimension and rate and the smaller product of rate and four minus dimension, in particular whenever $\lambda = \mu$ and either $s > 0$ or $m \ne n$. Outside that region $M > \max(m,n)$, so for every $s \ge 0$ the exponent improves by $(M - \max(m,n))/(4s+8) > 0$, and the box certificate of Proposition 22 attains it.

Corollary 14. *For $m, n \le 3$ one has $\max(m,n) \le M < 4$, and $M = \max(m,n)$ if and only if either*

$$m\lambda \ge n\mu \quad \text{and} \quad (4-m)\lambda \le (4-n)\mu, \qquad \text{or} \qquad n\mu \ge m\lambda \quad \text{and} \quad (4-n)\mu \le (4-m)\lambda.$$

In particular, for $m = n$,

$$M = m + \frac{m(4-m)}{4}\Big(1 - \frac{\min(\lambda,\mu)}{\max(\lambda,\mu)}\Big),$$

which exceeds m whenever $\lambda \ne \mu$.

Theorem 15. *Let $\lambda = \mu$, $m = n \le 3$ and $s = 0$. Then $\mathcal{R}_N \asymp N^{-m/8}(\log N)^{-(4-m)/8}$.*

Theorem 15 shows that the lower bound of Theorem 12(b) is attained when $\lambda = \mu$ and $m = n$; for equal rates and equal dimensions there is no additional logarithm when $s > 0$. For the planar saddle $m = n = 1$ with $\mu = \lambda/4$ the rate is $N^{-(2s+25/16)/(4s+8)}$ for every $s \ge 0$, instead of $N^{-(2s+1)/(4s+8)}$; for $m = n = 2$, $\mu = \lambda/2$ and $s = 1$ it is $N^{-3/8}$ instead of $N^{-1/3}$. Together with Theorem 11, these statements give the order of the linearized risk at every exact two-block saddle for every $s > 0$ and, at $s = 0$, whenever $\max(m,n) \ge 4$, $m\lambda \ne n\mu$ or $(\lambda, m) = (\mu, n)$; in the remaining case they give its exponent.

Remark 16 (Flow reversal). Replacing b by $-b$ changes A into $-A$, so the linearized experiment is equivalent and every risk is unchanged; the representer equation $-b \cdot \nabla w = \psi$ is the same as $b \cdot \nabla(-w) = \psi$. The field $-b = (-\lambda x, \mu y)$ is the two-block saddle in which the y block, of dimension n, expands at rate μ and the x block contracts at rate λ. Hence every statement proved for (m, n, λ, μ) holds for (n, m, μ, λ) with the blocks exchanged. In the nonlinear lower bounds the identity $L_f(u_f - u_0) = \Delta A h$ of Section 17 is used, and only $\|Ah\|_2$ enters. The effective dimension (19) is invariant under the exchange. We assume $m\lambda \geq n\mu$ whenever this is convenient.

15 Local H^2 representers at a saddle

By the Dirichlet inverse and integration by parts, a representer has the form Δw with $w \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $b \cdot \nabla w = \psi$ almost everywhere on K. Necessity already concerns the weak equation near zero.

For the upper part, suppose first that $m > 4$. Write $r = |x|$ and choose an exit radius R larger than the maximal r on K. Extend ψ smoothly by zero and use the auxiliary linear saddle on the whole coordinate space. Its forward exit potential for $r > 0$ is

$$w(x, y) = -\frac{1}{\lambda} \int_r^R \psi(tx/r, (r/t)^{\mu/\lambda} y) \, \frac{dt}{t}.$$

Following the flow changes only the lower endpoint in its equivalent time parametrization, so $b \cdot \nabla w = \psi$. With $\psi_0 = \psi(0)$ this potential differs from $(\psi_0/\lambda) \log r$ by a bounded function, since $|\psi(tx/r, (r/t)^{\mu/\lambda} y) - \psi_0| \leq C(t + (r/t)^{\mu/\lambda}|y|)$. Derivatives of orders one and two in x cost at most Cr^{-1} and Cr^{-2}, respectively. Indeed each derivative of x/r costs one power of r^{-1}, while derivatives of $(r/t)^{\mu/\lambda}$ carry the same power and the factor $(r/t)^{\mu/\lambda}$. Angular derivatives of the integrand carry t and derivatives in y carry $(r/t)^{\mu/\lambda}$, both integrable with dt/t. The lower endpoint derivatives obey the same bounds. First and second derivatives in y are bounded, and the mixed derivative is $O(r^{-1})$. For $m > 4$ these estimates give H^2 across $r = 0$ by cutting off near that subspace and taking the limit. More explicitly, fill the inner ball in x with $(\psi_0/\lambda) \log \epsilon$ and join it smoothly at $r \asymp \epsilon$. The derivative errors are supported on a set of volume $O(\epsilon^m)$ and have squared integral $O(\epsilon^{m-4})$ at order two. The smoothened potentials converge in H^2. Multiplying by a smooth cutoff equal to one near K gives a Dirichlet representer. If instead $n > 4$, reverse the direction of the flow and use the exit in y, changing the sign of ψ.

For necessity assume that $m, n \leq 4$ and an H^2 solution exists near the saddle. Average it over the rotations $O(m) \times O(n)$ on a smaller product ball. This preserves H^2 and gives the same equation with the averaged score, whose value at zero is still $\psi_0 \neq 0$. Write the averaged solution and score as $W(r, t)$ and $\Psi(r, t)$ for the two radii. On the positive quadrant W is locally H^2 in two variables, hence continuous, so its values at points are defined, and

$$\lambda r W_r - \mu t W_t = \Psi.$$

For almost every fixed $t_0 > 0$ in a compact radial collar, $\int_0^{r_0} |W_r(r, t_0)|^2 r^{m-3} \, dr < \infty$. For $m \geq 2$ this follows from the tangential Hessian component W_r/r. For $m = 1$ rotation averaging makes the corresponding one-dimensional slice even, so its first derivative has zero trace at zero and the one-dimensional Hardy inequality gives the same integral. For almost every fixed $r_0 > 0$ the analogous bound holds for the t derivative. By Cauchy–Schwarz these slices grow at most $O(1 + \sqrt{|\log r|})$ as $r \downarrow 0$, and similarly in t, because $\int_r^{r_0} v^{3-m} \, dv$ is bounded for $m < 4$ and logarithmic for $m = 4$.

The weak transport equation integrates in the interior along almost every characteristic of invariant label $z = r^\mu t^\lambda$, by changing to the smooth flow coordinates and using the one-dimensional weak chain rule. The three properties used now each hold off a null set: the r-slice bound at t_0, the

t-slice bound at r_0, and the integrated transport identity along the characteristic. Choose the pair (r_0, t_0) in the intersection of the corresponding full-measure sets, and then the labels z from the full-measure set of characteristics; for this fixed pair the slice bounds hold on the segments $\{t = t_0\}$ and $\{r = r_0\}$ simultaneously, and by continuity of W the endpoint values are the pointwise values. Now join the points with $t = t_0$ and $r = r_0$ along the characteristic of label z. As $z \downarrow 0$, the starting r and ending t tend to zero, and the transit time is $\tau = (\lambda\mu)^{-1} \log(1/z) + O(1)$. The variation of W between the two endpoints is $\psi_0 \tau + O(1)$ because $|\Psi - \psi_0| \le C(r + t)$ and the integral of $r + t$ along the transit is uniformly bounded. This contradicts the slice bounds, which give an endpoint difference $o(\log(1/z))$.

The nonexistence of a local solution implies that $|\int h\psi| / \|Ah\|_2$ is unbounded over fixed smooth directions supported in the interior of K. Otherwise the target on the range of A would have a bounded linear functional extension and thus an L^2 representer, whose weak equation would hold near zero. Directions in the kernel with nonzero target already give the same lower-bound claim: their linearized means are equal, and at perturbation amplitude $t = \gamma_0/\sqrt{N}$ with γ_0 arbitrary their nonlinear mean gaps are $o(t)$, so the Kullback–Leibler divergence $N \cdot o(t)^2$ tends to zero. For nonzero denominator, scale a fixed direction by $\gamma_1/(\sqrt{N}\|Ah\|_2)$, with $\gamma_1 > 0$ small. Its Gaussian Kullback–Leibler divergence is bounded and its root-sample target gap can be arbitrarily large. For a zero-denominator direction one instead uses an arbitrarily large fixed multiple of $N^{-1/2}$, with vanishing divergence. The same arguments hold in the nonlinear model because along each fixed smooth direction its mean is differentiable at zero with derivative Ah. A fixed supported smooth direction of nonzero target also gives the ordinary parametric lower bound in either experiment. This proves Theorem 11.

16 Certificates and upper bounds

16.1 Estimators from potentials and thin defects

Lemma 17. *Let $w \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\widehat{T} := \langle Y, \Delta w \rangle$. Then $\mathbb{E}_h \widehat{T} = \langle h, b \cdot \nabla w \rangle$ for every $h \in L^2(\Omega)$ supported in K, and $\widehat{T} - \mathbb{E}_h \widehat{T}$ is centered Gaussian with variance $N^{-1}\|\Delta w\|_2^2$. Consequently*

$$\mathcal{R}_N \le \sup_{h \in \mathcal{H}_s(K,\eta)} |\langle h, \psi - b \cdot \nabla w \rangle| + N^{-1/2} \|\Delta w\|_2.$$

Proof. The function $v = Ah \in H_0^1(\Omega)$ is the weak solution of $\int \nabla v \cdot \nabla \varphi = -\int hb \cdot \nabla \varphi$ for $\varphi \in H_0^1(\Omega)$. Green's formula for $w \in H^2 \cap H_0^1$ gives $\langle v, \Delta w \rangle = -\int \nabla v \cdot \nabla w = \int hb \cdot \nabla w$. The noise part is $N^{-1/2}\langle \mathbb{W}, \Delta w \rangle$, and a centered Gaussian variable has mean absolute value at most its standard deviation. □

We call w a *certificate* and $\psi - b \cdot \nabla w$ on K its *defect*.

Let $\eta_0 \in C_c^\infty(B(16a_0))$ equal one on $B(8a_0)$, put $\tilde{\psi} = \eta_0 \psi$ extended by zero, and use the linear field $(\lambda x, -\mu y)$ on all of $\mathbb{R}^{m+n}$. With $R_0 = 32a_0$ let

$$w_1(x,y) = -\frac{1}{\lambda} \int_r^{R_0} \tilde{\psi}\Big(\frac{tx}{r}, \Big(\frac{r}{t}\Big)^{\Gamma} y\Big) \frac{dt}{t}, \qquad w_2(x,y) = \frac{1}{\mu} \int_\rho^{R_0} \tilde{\psi}\Big(\Big(\frac{\rho}{t}\Big)^{1/\Gamma} x, \frac{ty}{\rho}\Big) \frac{dt}{t}$$

for $r > 0$ and $\rho > 0$ respectively. The first is the exit potential of Section 15; the second is the exit potential of the reversed field for the score $-\tilde{\psi}$, which we call the entrance potential.

Lemma 18. *On $B(8a_0)$ one has $b \cdot \nabla w_1 = \psi$ where $r > 0$ and $b \cdot \nabla w_2 = \psi$ where $\rho > 0$. Moreover*

$$|w_1 - \tfrac{\psi_0}{\lambda} \log r| \le C, \quad |\partial_x w_1| \le \tfrac{C}{r}, \quad |\partial_x^2 w_1| \le \tfrac{C}{r^2}, \quad |\partial_x \partial_y w_1| \le \tfrac{C}{r}, \quad |\partial_y w_1| + |\partial_y^2 w_1| \le C,$$

and the same bounds hold for w_2 with $-\frac{\psi_0}{\mu}\log\rho$ in place of $\frac{\psi_0}{\lambda}\log r$ and the roles of (x,r) and (y,ρ) exchanged. The function $\Xi := w_1 - w_2$ satisfies $b\cdot\nabla\Xi = 0$ where $r\rho > 0$, and

$$\Big|\Xi - \frac{\psi_0}{\lambda\mu}\log z\Big| \le C_1, \qquad |\nabla\Xi| \le C\Big(1+\frac{1}{r}+\frac{1}{\rho}\Big), \qquad |D^2\Xi| \le C\Big(1+\frac{1}{r^2}+\frac{1}{\rho^2}\Big).$$

Proof. The statements for w_1 are the estimates of Section 15, which hold for arbitrary positive rates: the integrand is constant along the flow, so only the lower limit moves, and the derivative bounds follow by differentiating under the integral. The statements for w_2 follow by Remark 16. The identity $\frac{\psi_0}{\lambda}\log r + \frac{\psi_0}{\mu}\log\rho = \frac{\psi_0}{\lambda\mu}\log z$ gives the bounds for Ξ. □

Lemma 19. *Let $L \ge -1$ be an integer with $L+1 > s - m/2$, let $0 < \epsilon \le a_0$, and let $e(x,y) = \tilde e(x/\epsilon, y)$ on $B(8a_0)$, where for every y the function $\tilde e(\cdot,y)$ is supported in $\{|X|\le 2\}$, is bounded by $\mathcal{B}(y)$ and has vanishing moments $\int X^\alpha \tilde e(X,y)\,dX = 0$ for $|\alpha| \le L$. Then*

$$\sup_{h\in\mathcal{H}_s(K,\eta)} |\langle h, e\rangle| \le C\eta\,\epsilon^{s+m/2}\|\mathcal{B}\|_{L^2(\{|y|<8a_0\})}.$$

Proof. Extend h by zero; then $\|h\|_{H^s(\mathbb{R}^{m+n})} \le C\|h\|_{H^s(\Omega)}$, and by Plancherel $\int \|h(\cdot,y)\|^2_{\dot H^s(\mathbb{R}^m)}dy \le \|h\|^2_{H^s(\mathbb{R}^{m+n})}$. Plancherel in x and Cauchy–Schwarz in y give $|\langle h,e\rangle| \le \|h\|_{H^s}(\int\|e(\cdot,y)\|^2_{\dot H^{-s}}dy)^{1/2}$, and scaling gives $\|e(\cdot,y)\|_{\dot H^{-s}} = \epsilon^{s+m/2}\|\tilde e(\cdot,y)\|_{\dot H^{-s}}$. The Fourier transform of $\tilde e(\cdot,y)$ is bounded by $C\mathcal{B}(y)$, and by Taylor's formula and the vanishing moments by $C\mathcal{B}(y)|\xi|^{L+1}$ for $|\xi|\le 1$. Since $2(L+1) - 2s + m > 0$, the low frequencies contribute a finite multiple of $\mathcal{B}(y)^2$, and the high frequencies at most $\|\tilde e(\cdot,y)\|^2_{L^2} \le C\mathcal{B}(y)^2$. □

For $s = 0$ one may take $L = -1$, so no moment condition is needed.

Lemma 20 (Moment correction)**.** *Let L, ϵ and $\tilde e$ be as in Lemma 19 without the moment condition, and assume that for some $P \in (0, 8a_0]$*

$$\tilde e(X,y) = 0 \ \text{ for } |y| \ge P, \qquad |\partial_y^j \tilde e(X,y)| \le C_0 P^{-j} \quad (j = 0,1,2).$$

Put $\varpi_\alpha = \lambda(m+|\alpha|)/\mu$. There are fixed $\phi_\alpha \in C_c^\infty(\{|X|<2\})$ and C^2 functions $\mathcal{M}_\alpha$ such that $U(X,y) = \sum_{|\alpha|\le L}\phi_\alpha(X)\mathcal{M}_\alpha(y)$ and $U_\epsilon(x,y) = U(x/\epsilon, y)$ satisfy:

- *(i) for every y, $X \mapsto (e - b\cdot\nabla U_\epsilon)(\epsilon X, y)$ is supported in $\{|X|\le 2\}$ and has vanishing moments of all orders $|\alpha| \le L$;*
- *(ii) $|\partial_y^j \mathcal{M}_\alpha(y)| \le CC_0P^{-j}\min\{1, (P/|y|)^{\varpi_\alpha + j}\}$ for $j \le 2$;*
- *(iii) $|b\cdot\nabla U_\epsilon(x,y)| \le CC_0\min\{1,(P/|y|)^{\varpi_0}\}$;*
- *(iv) if $2\lambda m > n\mu$ and $\epsilon \le P$, then $\|\Delta U_\epsilon\|_{L^2(B(8a_0))} \le CC_0\,\epsilon^{m/2-2}P^{n/2}$.*

Proof. Let $F_\alpha(y) = \int X^\alpha \tilde e(X,y)\,dX$, so $|\partial_y^j F_\alpha| \le CC_0P^{-j}$ and $F_\alpha = 0$ for $|y| \ge P$. Choose ϕ_α with $\int X^\gamma\phi_\alpha\,dX = \delta_{\alpha\gamma}$ for $|\alpha|,|\gamma| \le L$, by inverting the Gram matrix of the monomials against a positive smooth weight on $\{|X|<2\}$, and put

$$\mathcal{M}_\alpha(y) = -\frac{1}{\mu}\int_0^1 v^{\varpi_\alpha - 1}F_\alpha(vy)\,dv.$$

Since $(\varpi_\alpha + y\cdot\nabla)\int_0^1 v^{\varpi_\alpha-1}F(vy)\,dv = \int_0^1 \frac{d}{dv}[v^{\varpi_\alpha}F(vy)]\,dv = F(y)$, we have $\lambda(m+|\alpha|)\mathcal{M}_\alpha + \mu\, y\cdot\nabla\mathcal{M}_\alpha = -F_\alpha$. Now $b\cdot\nabla U_\epsilon = [\lambda X\cdot\nabla_X U - \mu\, y\cdot\nabla_y U](x/\epsilon, y)$, and integration by parts gives $\int X^\alpha X\cdot\nabla_X U\,dX = -(m+|\alpha|)\mathcal{M}_\alpha$, so the α-moment of $(b\cdot\nabla U_\epsilon)(\epsilon\,\cdot, y)$ is $-\lambda(m+|\alpha|)\mathcal{M}_\alpha -$

$\mu\, y\cdot\nabla\mathcal{M}_\alpha = F_\alpha$. This proves (i). For (ii), $|\partial_y^j\mathcal{M}_\alpha(y)| \le \mu^{-1}\int_0^1 v^{\varpi_\alpha-1+j}|\partial^j F_\alpha(vy)|\,dv$, and the integrand vanishes for $v > P/|y|$. Part (iii) follows from (ii) and $y\cdot\nabla\mathcal{M}_\alpha = -\varpi_\alpha\mathcal{M}_\alpha - F_\alpha/\mu$. For (iv), ΔU_ϵ is supported in $\{|x|\le 2\epsilon\}$ and bounded by $C\sum_\alpha(\epsilon^{-2}|\mathcal{M}_\alpha| + |\partial_y^2\mathcal{M}_\alpha|)$. By (ii) and $2\varpi_0 > n$, $\int_{|y|<8a_0}\min\{1,(P/|y|)^{2\varpi_0}\}dy \le CP^n$, and the second derivatives contribute at most $C\epsilon^m P^{n-4} \le C\epsilon^{m-4}P^n$. □

16.2 The tube certificate

Let $\chi_0 \in C^\infty(\mathbb{R}^m)$ be radial with $\chi_0 = 0$ on $\{|X|\le 1\}$ and $\chi_0 = 1$ on $\{|X|\ge 2\}$. For $0<\epsilon\le a_0$ put

$$\tilde w_1 = \frac{\psi_0}{\lambda}\log\epsilon + \chi_0(x/\epsilon)\Big(w_1 - \frac{\psi_0}{\lambda}\log\epsilon\Big), \qquad E_1 = \psi - b\cdot\nabla\tilde w_1 \quad \text{on } B(8a_0).$$

Since $b\cdot\nabla[\chi_0(x/\epsilon)] = \lambda(X\cdot\nabla\chi_0)(x/\epsilon)$, the defect $E_1 = \tilde e_1(x/\epsilon, y)$ is supported in $\{r\le 2\epsilon\}$, with $\tilde e_1(X,y) = (1-\chi_0(X))\psi(\epsilon X, y) - \lambda X\cdot\nabla\chi_0(X)\,(w_1(\epsilon X,y) - \frac{\psi_0}{\lambda}\log\epsilon)$. By Lemma 18,

$$\begin{aligned} &|\partial_y^j\tilde e_1| \le C\ (j\le 2), \qquad |\tilde w_1 - \tfrac{\psi_0}{\lambda}\log\epsilon| \le C \ \text{ on } \{r\le 2\epsilon\},\\ &|\nabla\tilde w_1| \le C(1+\max(r,\epsilon)^{-1}), \qquad |D^2\tilde w_1| \le C(1+\max(r,\epsilon)^{-2}). \end{aligned} \tag{20}$$

Fix $\eta_K \in C_c^\infty(B(2a_0))$ with $\eta_K = 1$ on $B(a_0)\supset K$, and $\zeta\in C_c^\infty(\{|y|<8a_0\})$ with $\zeta = 1$ on $\{|y|\le 2a_0\}$.

Proposition 21 (Tube certificate)**.** *Let $m\le 4$, $n\ge 1$, $\lambda,\mu>0$, $s\ge 0$ and $0<\epsilon\le a_0$. Let U_ϵ be given by Lemma 20 applied to $\zeta(y)\tilde e_1$ with $P = 8a_0$. The certificate $w_\epsilon = \eta_K(\tilde w_1 + U_\epsilon)$ has bias at most $C\eta\epsilon^{s+m/2}$ and*

$$\|\Delta w_\epsilon\|_2 \le C\begin{cases}\epsilon^{m/2-2}, & m<4,\\ (1+\log(1/\epsilon))^{1/2}, & m=4.\end{cases} \tag{21}$$

Proof. The function $\zeta\tilde e_1$ vanishes for $|y|\ge 8a_0$ and has bounded y-derivatives, so Lemma 20 applies. On $K\subset B(a_0)$ we have $\zeta = \eta_K = 1$, so the defect is $\zeta E_1 - b\cdot\nabla U_\epsilon$, which satisfies Lemma 19 with $\mathcal{B}\le C$. By (20), the leading part of the energy is $\int_{B(2a_0)}|D^2\tilde w_1|^2 \le C(1+\epsilon^{m-4} + \int_\epsilon^{2a_0} r^{m-5}dr)$. The terms in which derivatives fall on η_K are bounded by $C|\nabla\tilde w_1| + C|\tilde w_1|$ on $\operatorname{supp}\nabla\eta_K$. There $|\tilde w_1| \le C(1+|\log\max(r,\epsilon)|)$, and by (20) these terms contribute at most

$$C\Big(1+\int_0^{2a_0}\max(r,\epsilon)^{-2}r^{m-1}dr\Big)^{1/2} \le C\big(1+\epsilon^{(m-2)/2} + |\log\epsilon|^{1/2}\big),$$

which is at most the right side of (21). Finally $\|\Delta U_\epsilon\|_2 \le C\epsilon^{m/2-2}$ by Lemma 20(ii). □

The rates enter Proposition 21 only through the moment equation of Lemma 20.

16.3 The box certificate

Fix a convex $\mathcal{S}\in C^\infty(\mathbb{R})$ with $\mathcal{S}(g) = -g$ for $g\le -1$, $\mathcal{S}(g) = 0$ for $g\ge 1$ and $-1\le\mathcal{S}'\le 0$; for instance $\mathcal{S}(g) = \int_g^\infty\chi_1$ for a smooth nonincreasing χ_1 equal to 1 on $(-\infty,-1]$ and to 0 on $[1,\infty)$, with $\chi_1(t)+\chi_1(-t) = 1$. For a constant Ξ_* the function $w_2 - \mathcal{S}(\Xi-\Xi_*)$ equals $w_1-\Xi_*$ where $\Xi\le\Xi_*-1$ and equals w_2 where $\Xi\ge\Xi_*+1$. It is a smoothed minimum of w_2 and $w_1-\Xi_*$, and it solves the transport equation exactly, because $\mathcal{S}(\Xi-\Xi_*)$ is invariant along the flow.

Proposition 22 (Box certificate)**.** *Let $m,n\le 3$ and $2m\lambda > n\mu$. There is C_2 such that for all $s\ge 0$ and $0<\epsilon\le P\le a_0/C_2$ there is a certificate w with*

$$\sup_{h\in\mathcal{H}_s(K,\eta)}|\langle h,\psi - b\cdot\nabla w\rangle| \le C\eta\,\epsilon^{s+m/2}P^{n/2}, \qquad \|\Delta w\|_2 \le C\big(1+\epsilon^{m/2-2}P^{n/2} + (\epsilon^\Gamma P)^{n/2-2}\big).$$

For $s=0$ its defect is bounded by C and supported in $\{r\le 2\epsilon,\ \rho\le C_2P\}$.

Proof. Let $\Xi_* = \frac{\psi_0}{\lambda\mu}\log(\epsilon^\mu P^\lambda)$, $\tilde\Xi = \tilde w_1 - w_2$, $\Xi_\epsilon = \tilde\Xi - \Xi_*$, $\rho_* = \epsilon^\Gamma P$ and

$$V = w_2 - \mathcal{S}(\Xi_\epsilon) \quad \text{on } B(8a_0).$$

Regions. For $r \ge 2\epsilon$ we have $\tilde\Xi = \Xi$, and Lemma 18 gives $\Xi_\epsilon = \frac{\psi_0}{\lambda\mu}\log\frac{z}{\epsilon^\mu P^\lambda} + O(1)$. For $r \le 2\epsilon$, (20) gives $\Xi_\epsilon = \frac{\psi_0}{\mu}\log\frac{\rho}{P} + O(1)$. Hence there is C_3 with

$$\begin{aligned} \{\Xi_\epsilon < 1\} &\subset \{\rho \le C_3\rho_* r^{-\Gamma},\ r \ge 2\epsilon\} \cup \{\rho \le C_3 P,\ r \le 2\epsilon\}, \\ \{\Xi_\epsilon > -1\} &\subset \{\rho \ge C_3^{-1}\min(\rho_* r^{-\Gamma}, P)\}. \end{aligned} \tag{22}$$

In particular $V = \tilde w_1 - \Xi_*$ near $\{y = 0\}$ and $V = w_2$ near $\{x = 0\} \cap \{\rho \ge C_3 P\}$, so $V \in C^2(B(8a_0))$.

Defect. Since $b\cdot\nabla w_2 = \psi$ and $b\cdot\nabla\tilde\Xi = -E_1$, $\psi - b\cdot\nabla V = -\mathcal{S}'(\Xi_\epsilon)E_1 =: \hat E$. By (22) it is supported in $\{r \le 2\epsilon,\ \rho \le C_3 P\}$ and bounded by $|E_1| \le C$; take $C_2 \ge C_3$. In scaled form $\hat E = \hat e(x/\epsilon, y)$ with $\hat e = -\mathcal{S}'(\Xi_\epsilon)\tilde e_1$. Where $\mathcal{S}''(\Xi_\epsilon) \ne 0$ we have $|\Xi_\epsilon| < 1$, hence $\rho \asymp P$, and there $|\partial_y\Xi_\epsilon| \le C/P$ and $|\partial_y^2\Xi_\epsilon| \le C/P^2$. Therefore $|\partial_y^j \hat e| \le CP^{-j}$ for $j \le 2$, and Lemma 20 applies with C_3P in place of P. Put $w = \eta_K(V + U_\epsilon)$, with $U_\epsilon = 0$ when $s = 0$. On K the defect $\hat E - b\cdot\nabla U_\epsilon$ satisfies Lemma 19 with $\mathcal{B}(y) \le C\min\{1, (P/|y|)^{\varpi_0}\}$, and $\|\mathcal{B}\|_2 \le CP^{n/2}$ because $2\varpi_0 = 2\lambda m/\mu > n$. This is the bias bound.

Energy. We have $D^2V = D^2w_2 - \mathcal{S}'(\Xi_\epsilon)D^2\tilde\Xi - \mathcal{S}''(\Xi_\epsilon)\nabla\tilde\Xi\otimes\nabla\tilde\Xi$. On $\{\Xi_\epsilon \le -1\}$ this is $D^2\tilde w_1$, on $\{\Xi_\epsilon \ge 1\}$ it is D^2w_2, and in between it is bounded by $|D^2\tilde w_1| + |D^2w_2| + C|\nabla\tilde\Xi|^2$. Hence

$$|D^2V|^2 \le C\big(1 + \max(r,\epsilon)^{-4}\mathbf{1}_{\{\Xi_\epsilon<1\}} + \rho^{-4}\mathbf{1}_{\{\Xi_\epsilon>-1\}}\big).$$

By (22) and $m - 4 - n\Gamma < 0$,

$$\int_{\{\Xi_\epsilon<1\}} \max(r,\epsilon)^{-4} \le C\epsilon^{m-4}P^n + C\int_{2\epsilon}^{8a_0} r^{m-5}(\rho_* r^{-\Gamma})^n\,dr \le C\epsilon^{m-4}P^n,$$

since $\rho_*^n\epsilon^{m-4-n\Gamma} = \epsilon^{m-4}P^n$. Since $n < 4$ and $\rho_* r^{-\Gamma} \ge P$ exactly when $r \le \epsilon$,

$$\int_{\{\Xi_\epsilon>-1\}} \rho^{-4} \le C\int_0^{8a_0} r^{m-1}\big(\rho_*^{n-4}r^{\Gamma(4-n)} + P^{n-4}\mathbf{1}_{\{r\le\epsilon\}}\big)dr \le C\big(\rho_*^{n-4} + \epsilon^m P^{n-4}\big),$$

and $\epsilon^m P^{n-4} \le \epsilon^{m-4}P^n$. The terms in which derivatives fall on η_K live on $\operatorname{supp}\nabla\eta_K$, where $r \ge a_0$ or $\rho \ge a_0$. Where $\rho \ge a_0$ we have $\Xi_\epsilon \ge 1$ once C_2 is large, so $V = w_2$ with bounded value and first derivatives. Where $r \ge a_0$ and $\rho < a_0$, (22) gives three ranges. For $\rho \le c\rho_*$, $V = w_1 - \Xi_*$ has bounded first derivatives, and the constant $|\Xi_*| \le C\log(1/\epsilon)$ multiplies functions supported on a set of measure $C\rho_*^n$. The transition range $c\rho_* < \rho < C\rho_*$ has measure $C\rho_*^n$, and there $|\nabla V| \le C/\rho_*$. On $C\rho_* \le \rho < a_0$, $V = w_2$ and $|\nabla w_2| \le C/\rho$, which contributes at most $C(1+\rho_*^{(n-2)/2} + |\log\rho_*|^{1/2})$. All these terms are at most $C(1 + \rho_*^{n/2-2})$. Together with Lemma 20(iv) this proves the energy bound. □

16.4 Averaging along the critical band

Proposition 23 (Averaged certificate)**.** *Let $\lambda = \mu$, $m = n \le 3$ and $s = 0$. For all sufficiently small $Z > 0$ there is a certificate w with*

$$\|\psi - b\cdot\nabla w\|_{L^2(K)} \le CZ^{m/2}(\log(1/Z))^{-1/2}, \qquad \|\Delta w\|_2 \le C(1 + Z^{m/2-2}).$$

Proof. Here $\Gamma = 1$. Let $P_k = 2^{-k}a_0/C_2$ and $\epsilon_k = Z/P_k$ for $0 \le k \le k_Z$, where k_Z is the largest integer with $\epsilon_{k_Z} \le P_{k_Z}$; then $k_Z + 1 \ge c\log(1/Z)$. Let w_k be the box certificate of Proposition 22 with $(\epsilon, P) = (\epsilon_k, P_k)$ and $s = 0$, and put $w = (k_Z+1)^{-1}\sum_k w_k$. Each w_k has $\epsilon_k^{\Gamma} P_k = Z$ and $\epsilon_k^{m/2-2} P_k^{m/2} = Z^{m/2-2}P_k^2 \le CZ^{m/2-2}$, so the triangle inequality gives the energy bound. The defect of w is the average of the defects $\hat{E}_k$, with $|\hat{E}_k| \le C\mathbf{1}_{\mathcal{Q}_k}$ and $\mathcal{Q}_k = \{r \le 2\epsilon_k,\ \rho \le C_2 P_k\}$. Since $|\mathcal{Q}_k \cap \mathcal{Q}_l| \le C\min(\epsilon_k,\epsilon_l)^m \min(P_k,P_l)^m = CZ^m 2^{-m|k-l|}$,

$$\Big\| \sum_k \hat{E}_k \Big\|_2^2 \le C \sum_{k,l} |\mathcal{Q}_k \cap \mathcal{Q}_l| \le C(k_Z+1)Z^m,$$

and dividing by $(k_Z+1)^2$ gives the defect bound. □

16.5 Proofs of the upper bounds

Write $\delta_N = N^{-1/2}$ and, when $m\lambda \ge n\mu$, $\Upsilon = (4-m) - (4-n)\Gamma$, so that $M = m + n\Upsilon_+/4$.

Proof of the upper bounds in Theorems 12 and 15. For Theorem 12(a) and (b) assume $m\lambda \ge n\mu$ by Remark 16, and use Lemma 17 with $\epsilon = \delta_N^{1/(s+2)}$. If $\Upsilon \le 0$, Proposition 21 gives $\mathcal{R}_N \le C\epsilon^{s+m/2} + C\delta_N\epsilon^{m/2-2} \le CN^{-(2s+m)/(4s+8)}$; this covers (b), where $\Theta = 0$ and $m \ge n$ give $\Upsilon = 4(n-m)/n \le 0$ and $M = m$. If $\Upsilon > 0$, then $2m\lambda > n\mu$, and Proposition 22 with $P = \epsilon^{\Upsilon/4}$ gives

$$\mathcal{R}_N \le C\big(\epsilon^{s+m/2}P^{n/2} + \delta_N + \delta_N\epsilon^{m/2-2}P^{n/2} + \delta_N(\epsilon^{\Gamma}P)^{n/2-2}\big).$$

In powers of δ_N the bias has exponent $\frac{s+m/2}{s+2} + \frac{n\Upsilon}{8(s+2)}$, the first energy term $1 - \frac{4-m}{2(s+2)} + \frac{n\Upsilon}{8(s+2)}$, and the second $1 - \frac{(4-n)(4\Gamma+\Upsilon)}{8(s+2)}$. All three equal $\frac{2s+M}{2(s+2)}$ because $4\Gamma + \Upsilon = 4 - m + n\Gamma$ and $4 - M = (4-n)(4-m+n\Gamma)/4$; since $M < 4$ the term δ_N is smaller. Hence $\mathcal{R}_N \le CN^{-(2s+M)/(4s+8)}$. For part (c) assume $m = 4$; then Proposition 21 with $\epsilon = N^{-1/(2s+4)}$ gives $\mathcal{R}_N \le C\epsilon^{s+2} + CN^{-1/2}(\log N)^{1/2}$. For Theorem 15, Proposition 23 with $Z = (N^{-1}\log N)^{1/4}$ gives

$$\mathcal{R}_N \le CZ^{m/2}(\log(1/Z))^{-1/2} + CN^{-1/2}(1 + Z^{m/2-2}) \le CN^{-m/8}(\log N)^{-(4-m)/8}.$$ □

Proof of Corollary 14. Assume $m\lambda \ge n\mu$, so $\Gamma \le m/n$. If $\Upsilon \le 0$ then $M = m$, and $\Upsilon \le 0$ forces $m \ge n$: otherwise $\lambda > \mu$ and $(4-m)\lambda > (4-n)\mu$. If $\Upsilon > 0$ then $M > m$ and

$$M - n = \frac{4-n}{4\lambda}\,\Theta \ge 0,$$

with equality only for $\Theta = 0$; then $\Upsilon > 0$ forces $m < n$, $M = n = \max(m,n)$, and $(4-n)\mu = (4-n)m\lambda/n \le (4-m)\lambda$, which is the second displayed alternative. So $M \ge \max(m,n)$ with equality exactly in the displayed cases, using Remark 16 when $m\lambda \le n\mu$. Also $4 - M = 4 - m > 0$ if $\Upsilon \le 0$ and $4 - M = (4-n)(4-m+n\Gamma)/4 > 0$ if $\Upsilon > 0$. For $m = n$ and $\lambda > \mu$, $\Upsilon = (4-m)(1-\mu/\lambda) > 0$, which gives the displayed formula. □

17 Lower bounds from interior saddle transits

17.1 Transit alternatives

Fix $\vartheta \in C_c^\infty((1,2))$ and a cutoff $\chi \in C^\infty([0,\infty))$ with $0 \le \chi \le 1$, $\chi = 1$ on $[0,1]$ and $\chi = 0$ on $[2,\infty)$, such that

$$\int \vartheta(z)\,\frac{dz}{z} = 0, \qquad \int \vartheta(z)\log z\,\frac{dz}{z} < 0. \tag{23}$$

For $P_1, R_1 \in (0, r_1]$ and a band level $z_0 > 0$ put

$$\epsilon_x = (z_0 P_1^{-\lambda})^{1/\mu}, \qquad \epsilon_y = (z_0 R_1^{-\mu})^{1/\lambda}, \qquad R_* = z_0^{1/(\lambda+\mu)},$$
$$H_{z_0}(x,y) = r^{-m}\rho^{-n}\,\vartheta(z/z_0)\,\chi(r/R_1)\,\chi(\rho/P_1). \tag{24}$$

The trajectories in the band $z \approx z_0$ cross the entry collar $\{P_1 \le \rho \le 2P_1\}$ at $r \approx \epsilon_x$ and the exit collar $\{R_1 \le r \le 2R_1\}$ at $\rho \approx \epsilon_y$, and R_* is the radius at which $r = \rho$ on the band. Separate collar radii are what unequal rates require.

Proposition 24. *There are $c_1 \in (0,1)$ and $C, c_0 > 0$ such that, after decreasing r_1, the following holds whenever $\epsilon_x \le c_1 \min(P_1, R_1)$ and $\epsilon_y \le c_1 \min(P_1, R_1)$. The function H_{z_0} belongs to $C_c^\infty(B(4r_1))$, and with $\mathcal{E}_{\mathrm{in}} = \epsilon_x^{-s-m/2} P_1^{-n/2}$ and $\mathcal{E}_{\mathrm{out}} = \epsilon_y^{-s-n/2} R_1^{-m/2}$:*

(a) $\int H_{z_0}\psi \ge c_0$;

(b) the observation norm satisfies

$$\|AH_{z_0}\|_2 \le C(\epsilon_x^{2-m/2} P_1^{-n/2} + \epsilon_y^{2-n/2} R_1^{-m/2}); \tag{25}$$

(c) if $m\lambda \ne n\mu$, then $\|H_{z_0}\|_2 \le C(\epsilon_x^{-m/2} P_1^{-n/2} + \epsilon_y^{-n/2} R_1^{-m/2})$; if $m\lambda = n\mu$, then $\|H_{z_0}\|_2 \le C(\log(R_1/\epsilon_x))^{1/2} \epsilon_x^{-m/2} P_1^{-n/2}$;

(d) for $s > 0$, with $\mathfrak{t} = \log(R_1/\epsilon_x)$ and $\mathfrak{q} = \max(\epsilon_x/R_*, R_*/R_1)$,

$$\|H_{z_0}\|_{H^s(\Omega)} \le C(1 + \mathfrak{t}\,\mathfrak{q}^s)(\mathcal{E}_{\mathrm{in}} + \mathcal{E}_{\mathrm{out}}); \tag{26}$$

(e) if $m = 4$ and $\int \vartheta(z) z^{-2/\mu} dz/z = 0$, then $\operatorname{div}(H_{z_0} b) = \Delta_x V_x + \Delta_y V_y$ with $\|V_x\|_2 \le C P_1^{-n/2}$, $\|V_y\|_2 \le C\epsilon_y^{2-n/2} R_1^{-m/2}$ and $\operatorname{supp} V_x \subset \{c\epsilon_x \le r \le C\epsilon_x,\ P_1 \le \rho \le 2P_1\}$; the same holds with the blocks exchanged if $n = 4$ and $\int \vartheta(z) z^{-2/\lambda} dz/z = 0$.

Proof. Write $H = H_{z_0}$. On the support of $\vartheta(z/z_0)$ we have $z_0 \le z \le 2z_0$, so $\rho \le 2P_1$ forces $r \ge 2^{-\lambda/\mu}\epsilon_x$ and $r \le 2R_1$ forces $\rho \ge 2^{-\mu/\lambda}\epsilon_y$. Hence H is smooth and $\operatorname{supp} H \subset B(4r_1)$.

Divergence. Since $b \cdot \nabla r = \lambda r$ and $b \cdot \nabla \rho = -\mu\rho$, we have $\operatorname{div}(r^{-m}\rho^{-n} b) = 0$ and $b \cdot \nabla z = 0$. Therefore

$$\operatorname{div}(Hb) = r^{-m}\rho^{-n}\vartheta(z/z_0)\Big[\lambda \frac{r}{R_1}\chi'\Big(\frac{r}{R_1}\Big)\chi\Big(\frac{\rho}{P_1}\Big) - \mu\frac{\rho}{P_1}\chi\Big(\frac{r}{R_1}\Big)\chi'\Big(\frac{\rho}{P_1}\Big)\Big] =: S_{\mathrm{out}} + S_{\mathrm{in}}.$$

On $\operatorname{supp} S_{\mathrm{in}}$ we have $P_1 \le \rho \le 2P_1$ and $r \le 2^{1/\mu}\epsilon_x \le R_1$ for c_1 small, so $\chi(r/R_1) = 1$ there; likewise $\chi(\rho/P_1) = 1$ on $\operatorname{supp} S_{\mathrm{out}}$. In the variables $X = x/\epsilon_x$, $Y = y/P_1$ one has $z/z_0 = |X|^\mu |Y|^\lambda$, hence

$$S_{\mathrm{in}}(x,y) = \epsilon_x^{-m} P_1^{-n} \Sigma_{\mathrm{in}}(X,Y), \qquad \Sigma_{\mathrm{in}}(X,Y) = -\mu |X|^{-m} |Y|^{1-n} \vartheta(|X|^\mu |Y|^\lambda) \chi'(|Y|),$$

a fixed smooth compactly supported function, radial in X. Its X-integral vanishes for every Y, since the substitution $t = |X|^\mu |Y|^\lambda$ turns it into a multiple of $\int \vartheta(t)\, dt/t = 0$. The same holds for $S_{\mathrm{out}} = \epsilon_y^{-n} R_1^{-m} \Sigma_{\mathrm{out}}(x/R_1, y/\epsilon_y)$ in the y variable.

(b). Let $v_{\mathrm{in}}(\cdot, Y)$ be the radial solution of $\Delta_X v = \Sigma_{\mathrm{in}}(\cdot, Y)$ vanishing for large $|X|$. The source has zero mass, so the radial derivative $r^{1-m}\int_0^r \Sigma_{\mathrm{in}} t^{m-1} dt$ vanishes outside it, and v_{in} is smooth, bounded and supported in a fixed ball. Put $V_x(x,y) = \epsilon_x^{2-m} P_1^{-n} v_{\mathrm{in}}(x/\epsilon_x, y/P_1)$, so that $\Delta_x V_x = S_{\mathrm{in}}$ and $\|V_x\|_2 = C\epsilon_x^{2-m/2} P_1^{-n/2}$, and define V_y symmetrically, with $\|V_y\|_2 = C\epsilon_y^{2-n/2} R_1^{-m/2}$. Both are compactly supported in Ω, so by the Dirichlet H^2 estimate, for $\varphi \in L^2(\Omega)$,

$$|\langle AH, \varphi\rangle| = |\langle V_x, \Delta_x \Delta_D^{-1}\varphi\rangle + \langle V_y, \Delta_y \Delta_D^{-1}\varphi\rangle| \le C(\|V_x\|_2 + \|V_y\|_2)\|\varphi\|_2.$$

(e). For a zero-mass radial source $F(r)$ in $\mathbb{R}^4$, the inverse that vanishes outside the source equals $-\frac12\int_0^\infty F(r)r\,dr$ inside it, by Fubini applied to $-\int_0^\infty r^{-3}\int_0^r F(t)t^3\,dt\,dr$. For $F=\Sigma_{\mathrm{in}}(\cdot,Y)$ the substitution $t=|X|^\mu|Y|^\lambda$ turns this value into a multiple of $\int\vartheta(t)t^{-2/\mu}dt/t=0$. So $v_{\mathrm{in}}(\cdot,Y)$ vanishes inside and outside the source, which lies in $\{c\le|X|\le C\}$.

(a). Let $\Psi(r,\rho)$ be the average of ψ over $O(m)\times O(n)$ and $\omega_{m,n}=|S^{m-1}||S^{n-1}|$. Changing ρ to z at fixed r, with $d\rho/\rho=dz/(\lambda z)$,

$$\int H\psi=\frac{\omega_{m,n}}{\lambda}\int\vartheta(z/z_0)I(z)\frac{dz}{z},\qquad I(z)=\int_0^\infty\chi\Big(\frac{r}{R_1}\Big)\chi\Big(\frac{\rho(r,z)}{P_1}\Big)\Psi(r,\rho(r,z))\frac{dr}{r},$$

with $\rho(r,z)=(zr^{-\mu})^{1/\lambda}$. For $\Psi\equiv\psi_0$ the r-integral splits at $r=(z/P_1^\lambda)^{1/\mu}$ and at R_1: below the first point only the second cutoff varies, above R_1 only the first, and in between both equal one. For c_1 small the two outer ranges do not overlap, and $I(z)=\frac{\psi_0}{\mu}\log\frac{R_1^\mu P_1^\lambda}{z}+\psi_0(1+\frac{\lambda}{\mu})\int_1^2\chi(t)\frac{dt}{t}$ exactly. By (23) the constant terms integrate to zero, and the main term is

$$-\frac{\omega_{m,n}\psi_0}{\lambda\mu}\int\vartheta(t)\log t\,\frac{dt}{t}=:2c_0>0.$$

For general Ψ, $|\Psi(r,\rho)-\psi_0|\le C(r+\rho)$, and $\int(r+\rho)\,dr/r\le C(R_1+P_1)$ along the band, because $\rho(r,z)$ decreases geometrically in r. Hence $|\int H\psi-2c_0|\le Cr_1$, which is at most c_0 after decreasing r_1.

(c). On the support $|H|^2\le Cr^{-2m}\rho^{-2n}$. Polar coordinates and $\rho\mapsto z$ give

$$\|H\|_2^2\le C\int_{z_0}^{2z_0}z^{-n/\lambda}\frac{dz}{z}\int_{c\epsilon_x}^{2R_1}r^{-\Theta/\lambda}\frac{dr}{r}.$$

If $\Theta>0$ the r-integral is at most $C\epsilon_x^{-\Theta/\lambda}$, giving $Cz_0^{-n/\lambda}\epsilon_x^{n\Gamma-m}=C\epsilon_x^{-m}P_1^{-n}$. If $\Theta<0$ it is at most $CR_1^{-\Theta/\lambda}$, giving $C\epsilon_y^{-n}R_1^{-m}$. If $\Theta=0$ it is at most $C\log(R_1/\epsilon_x)$, and $z_0^{-n/\lambda}=\epsilon_x^{-m}P_1^{-n}$.

(d). Fix $\phi\in C_c^\infty(\mathbb{R})$ with $\sum_{j\in\mathbb{Z}}\phi(\cdot-j)=1$, and put $H_j=\phi(\log r-j)H$; at most $\mathfrak{t}+C$ indices contribute. On $\operatorname{supp}H_j$ one has $r\asymp R=e^j$ and $\rho\asymp P=(z_0R^{-\mu})^{1/\lambda}$. Each x-derivative of any factor of H_j, cutoffs included, costs at most C/r, and each y-derivative at most C/ρ. Since $|\operatorname{supp}H_j|\le CR^mP^n$, we get $\|H_j\|_{H^k}\le C_kR^{-m/2}P^{-n/2}\min(R,P)^{-k}$ for integers k. The interpolation inequality $\|f\|_{H^s}\le\|f\|_{H^k}^{1-\theta}\|f\|_{H^{k+1}}^{\theta}$, $s=k+\theta$, gives $\|H_j\|_{H^s}\le Ce^{a(j)}$ with

$$a(j)=-\frac{m}{2}j-\frac{n}{2}\log P-s\min(j,\log P),\qquad\log P=\frac{\log z_0-\mu j}{\lambda}.$$

The function a is the maximum of two affine functions of j, with slope $\varsigma_1=-\Theta/(2\lambda)-s$ where $R<P$ and $\varsigma_2=-\Theta/(2\lambda)+s\Gamma$ where $R>P$; thus $\varsigma_2-\varsigma_1=s(1+\Gamma)$. At the two ends of the band $e^{a(j)}$ is comparable to $\mathcal{E}_{\mathrm{in}}$ and $\mathcal{E}_{\mathrm{out}}$, and the kink lies at $R\asymp R_*$. If $\varsigma_1<0<\varsigma_2$, the sum $\sum_j e^{a(j)}$ is bounded by two geometric series, hence by $C(\mathcal{E}_{\mathrm{in}}+\mathcal{E}_{\mathrm{out}})$. If $\varsigma_1\ge0$, then $\varsigma_2\ge s$; the terms with $R>R_*$ form a geometric series bounded by $C\mathcal{E}_{\mathrm{out}}$, and each of the at most $\mathfrak{t}+C$ terms with $R\le R_*$ is at most $e^{a(\mathrm{kink})}\le C\mathcal{E}_{\mathrm{out}}(R_*/R_1)^s$. The case $\varsigma_2\le0$ is symmetric, with the factor $(\epsilon_x/R_*)^s$. Summing over j proves (26). □

17.2 Two-point bounds

Lemma 25. *Let $h\in\mathcal{H}_s(K,\eta)$ with $N\|Ah\|_2^2\le1$. Then $\mathcal{R}_N\ge c|\int h\psi|$. If moreover $s>(m+n)/2+1$ and $\|h\|_{H^s}\le\eta_1$ for a sufficiently small η_1, then $\mathcal{R}_N^{\mathrm{nl}}\ge c|\int h\psi|$ over every class with $\eta\ge\eta_1$.*

Proof. Compare h with 0. The Kullback–Leibler divergence of the two Gaussian experiments is $N\|Ah\|_2^2/2$, and Le Cam's two-point bound [13, Chapter 2] gives the first claim. For the second, let $L_f = \operatorname{div}(f\nabla)$ with $f = 1 + h$. Then exactly

$$L_f(u_f - u_0) = -\operatorname{div}(hb) = \Delta Ah.$$

The Dirichlet operator ΔL_f^{-1} is uniformly bounded on L^2 because f is close to 1 in $W^{1,\infty}$, which follows from $\|h\|_{H^s} \le \eta_1$ and $s > (m+n)/2 + 1$ by the Sobolev embedding: factor $L_f\Delta_D^{-1}$ as the identity plus an L^2 operator of small norm using the Dirichlet H^2 estimate, and invert by a Neumann series. By self-adjointness and Dirichlet integration by parts, duality gives $\|u_f - u_0\|_2 \le C\|Ah\|_2$. So the divergence is at most $CN\|Ah\|_2^2$, and the two targets differ by $\int h\psi$. □

17.3 Proofs of the lower bounds

Throughout, $\delta_N = N^{-1/2}$ and $\gamma \in (0, 1]$ is a small fixed constant.

Proof of the lower bound in Theorem 12(a). By Remark 16 assume $\Theta = m\lambda - n\mu \ge 0$, and let $\Upsilon = (4-m) - (4-n)\Gamma$ and $\epsilon_x = \delta_N^{1/(s+2)}$.

Case $\Upsilon \le 0$, so $M = m$. Let $P_1 = r_1$ and $R_1 = r_1(\epsilon_x/r_1)^{\tau_1}$, where $\tau_1 = \max\{0, 1 - 2s/(\Gamma(n+2s) - m)\}$ if $\Gamma(n+2s) > m$ and $\tau_1 = 0$ otherwise. Then $z_0 = \epsilon_x^\mu P_1^\lambda$ and $\epsilon_y \asymp \epsilon_x^{\Gamma(1-\tau_1)}$, and

$$\Gamma(1-\tau_1)(n+2s) + m\tau_1 \le m + 2s, \qquad \Gamma(1-\tau_1)(4-n) - m\tau_1 \ge 4 - m, \qquad \tau_1 < \frac{\Gamma}{1+\Gamma}. \quad (27)$$

The first inequality holds by the choice of τ_1. The second holds for all τ_1 up to $\tau_{\max} = (\Gamma(4-n) - (4-m))/(\Gamma(4-n) + m) \ge 0$. If $\tau_1 > 0$ then $\tau_1 = \tau_{\min} = 1 - 2s/(\Gamma(n+2s) - m)$, and $\tau_{\min} \le \tau_{\max}$ because

$$(\tau_{\max} - \tau_{\min})(\Gamma(n+2s) - m)(\Gamma(4-n) + m) = (2s+4)(m - n\Gamma) \ge 0,$$

with both factors on the left positive. For the third, the left side of the first inequality decreases in τ_1, and $\tau^\circ = \Gamma/(1+\Gamma)$ satisfies it strictly, since this amounts to $\Gamma n < m + 2s$, which follows from $\Theta \ge 0$ when $s > 0$ and from $\Theta > 0$ when $s = 0$. By the first two inequalities the exit terms in Proposition 24(b)–(d) are at most constant multiples of the entry terms. By the third, ϵ_y/R_1, ϵ_x/P_1 and $\mathfrak{q}$ are at most N^{-c}, so the hypotheses hold and $\mathfrak{t}\,\mathfrak{q}^s \le C$; for $s = 0$ part (c) applies with $\Theta > 0$. Hence $h = \gamma\epsilon_x^{s+m/2}H_{z_0}$ has $\|h\|_{H^s} \le C\gamma$ and $\|Ah\|_2 \le C\gamma\epsilon_x^{s+2} = C\gamma\delta_N$, and Lemma 25 gives $\mathcal{R}_N \ge c\gamma N^{-(2s+m)/(4s+8)}$.

Case $\Upsilon > 0$. Let $R_1 = r_1$ and $P_1 = r_1(\epsilon_x/r_1)^{\Upsilon/4}$; since $\Upsilon \le 4 - m < 4$, $\epsilon_x/P_1 \le N^{-c}$. Then $\epsilon_y \asymp \epsilon_x^\Gamma P_1 \asymp \epsilon_x^{(4-m+n\Gamma)/4}$, and

$$\epsilon_x^{-(2s+m)}P_1^{-n} \asymp \epsilon_x^{-(2s+M)}, \qquad \epsilon_x^{4-m}P_1^{-n} \asymp \epsilon_x^{4-M} \asymp \epsilon_y^{4-n},$$

using $4 - M = (4-n)(4-m+n\Gamma)/4$. Moreover $(2s+M) - (2s+n)(4-m+n\Gamma)/4 = (4+2s)(m - n\Gamma)/4 \ge 0$, so $\mathcal{E}_{\text{out}}^2 \le C\mathcal{E}_{\text{in}}^2$. Both ends of the band are at distance at least $c\log N$ from the kink, so $\mathfrak{t}\,\mathfrak{q}^s \le C$; for $s = 0$ part (c) applies with $\Theta > 0$. With $h = \gamma\epsilon_x^{s+M/2}H_{z_0}$ we get $\|h\|_{H^s} \le C\gamma$ and $\|Ah\|_2 \le C\gamma\epsilon_x^{s+M/2}\epsilon_x^{(4-M)/2} = C\gamma\delta_N$, and Lemma 25 gives $\mathcal{R}_N \ge c\gamma N^{-(2s+M)/(4s+8)}$.

For $s > (m+n)/2 + 1$ the same alternatives with $\gamma \le \eta_1/C$ give the nonlinear bound. □

Proof of the lower bounds in Theorem 12(b) and Theorem 15. Here $s = 0$ and $\Theta = 0$, so $M = \max(m, n)$; Theorem 15 is the case $\lambda = \mu$, $m = n$. By Remark 16 assume $m \ge n$, so $\Gamma = m/n \ge 1$ and $M = m$. Take $P_1 = R_1 = r_1$; then $z_0 = \epsilon_x^\mu r_1^\lambda$ and $\epsilon_y \asymp \epsilon_x^\Gamma$. Since $\Gamma(4-n) = m(4-n)/n \ge 4 - m$ exactly when $m \ge n$, the exit term in Proposition 24(b) is at most a constant multiple of the

entry term, so $\|AH_{z_0}\|_2 \le C\epsilon_x^{2-m/2}$, and part (c) with $\Theta = 0$ gives $\|H_{z_0}\|_2 \le C\epsilon_x^{-m/2}(\log(1/\epsilon_x))^{1/2}$. Choose $\epsilon_x^2 = \delta_N(\log(1/\delta_N))^{1/2}$ and $h = \gamma\epsilon_x^{m/2}(\log(1/\epsilon_x))^{-1/2}H_{z_0}$. Then $\|h\|_2 \le C\gamma$, $\|Ah\|_2 \le C\gamma\delta_N$, and $\int h\psi \ge c\gamma\epsilon_x^{m/2}(\log(1/\epsilon_x))^{-1/2} \ge c\gamma N^{-m/8}(\log N)^{-(4-m)/8}$. □

Proof of the lower bound in Theorem 12(c). By Remark 16 assume $m = 4$. Impose on ϑ, besides (23), the condition $\int \vartheta z^{-2/\mu}dz/z = 0$, and if $n = 4$ also $\int \vartheta z^{-2/\lambda}dz/z = 0$; this is possible because 1, $\log z$, $z^{-2/\mu}$ and $z^{-2/\lambda}$ are linearly independent on $(1, 2)$ unless the last two coincide. Take $P_1 = R_1 = r_1$ and band levels with $\epsilon_x^{(j)} = c'r_1\mathsf{B}^{-j}$, $1 \le j \le J$, for a small fixed c' and a large fixed B. By Proposition 24(e) the potentials $V_x^{(j)}$ have norms at most C and, for B large, pairwise disjoint supports. The potentials $V_y^{(j)}$ have norms at most $C(\epsilon_y^{(j)})^{(4-n)/2}$, summable in j when $n < 4$, and when $n = 4$ they have norms at most C and disjoint supports. The duality argument of Proposition 24(b), applied to the sum of the sources, gives

$$\Big\|A\sum_{j\le J} H^{(j)}\Big\|_2 \le C\Big(\Big\|\sum_j V_x^{(j)}\Big\|_2 + \Big\|\sum_j V_y^{(j)}\Big\|_2\Big) \le C(\sqrt{J}+1).$$

By Proposition 24(c),(d), $\|H^{(j)}\|_{H^s} \le C(\epsilon_x^{(j)})^{-C'}$ for a fixed $C' > 0$. Choose $J = \lfloor \tau_0 \log N \rfloor$ with $\tau_0 > 0$ so small that $\sum_{j\le J}\|H^{(j)}\|_{H^s} \le N^{1/4}$, and put $h = \gamma(NJ)^{-1/2}\sum_{j\le J} H^{(j)}$. Then $\|h\|_{H^s} \le C\gamma N^{-1/4}$, $N\|Ah\|_2^2 \le C\gamma^2$, and $\int h\psi \ge c_0\gamma\sqrt{J/N} \ge c\gamma\sqrt{\log N/N}$ by Proposition 24(a). The nonlinear statement follows from Lemma 25, since $\|h\|_{H^s} \to 0$. □

18 A positive-source example with several critical points

Let $\lambda, \mu > 0$ with $m\lambda > n\mu$. Choose a smooth even function V on the line with $V'' \ge -1$, quadratic behavior $V(y) = V(0) - y^2/2$ near zero, two equal minima at $\pm a$ of value zero, and no other critical points besides zero. It can be constructed by taking $V'' = -1 + \alpha$ with α smooth, even, zero near zero, increasing for positive y where it transitions, and equal to a constant greater than one for all sufficiently large $|y|$. Integrate with $V'(0) = 0$ and shift the minimum to zero. This gives a confining double well with nondegenerate minima. Put

$$U(x, y) = \tfrac{\lambda}{2}|x|^2 + \mu\sum_{j=1}^n V(y_j), \qquad \Omega = \{U < L\}, \qquad u_0 = U - L,$$

where $L > n\mu V(0)$. The domain is bounded and $\partial\Omega$ is smooth since every critical value of U is at most $n\mu V(0)$. It is connected: one can first reduce x to zero and descend each y_j to one of the minima. Any two such minima can be joined by changing one coordinate at a time through its barrier while all others remain at their minima. These paths stay below L. The source is

$$g = \Delta U = m\lambda + \mu\sum_{j=1}^n V''(y_j) \ge m\lambda - n\mu > 0.$$

There are 3^n critical points, including the saddle at zero with $b = (\lambda x, -\mu y)$ on a product neighborhood. Thus the linearized rates of Theorem 12 occur at an interior saddle of a reference with strictly positive source and several nondegenerate critical points on a smooth connected domain. For $m = n$ and $\lambda > \mu$ this includes the regime $M > \max(m, n)$ of Corollary 14; for example $m = n = 2$, $\lambda = 2\mu$ and $s = 1$ give the rate $N^{-3/8}$.

19 Numerical confirmation

For radial ψ, the operator A^*A, the Sobolev form and the target are invariant under $O(m)\times O(n)$, so the linear minimax risk over an ellipsoid is attained on radial h. In a finite-dimensional discretization with Sobolev Gram matrix $\mathbf{S}$, mass matrix $\mathbf{M}$, discretized forward map $\mathbf{A}$ and target vector $\mathbf{t}$, the minimum over linear estimators of the worst-case squared bias plus variance is

$$\mathcal{R}_N^{\mathrm{lin}} = \mathbf{t}^\top \left(\mathbf{S} + N\,\mathbf{A}^\top\mathbf{M}\mathbf{A}\right)^{-1}\mathbf{t},$$

by the Woodbury identity after minimizing $(\mathbf{t}-\mathbf{A}^\top\mathbf{M}v)^\top\mathbf{S}^{-1}(\mathbf{t}-\mathbf{A}^\top\mathbf{M}v)+N^{-1}v^\top\mathbf{M}v$ over v. Its square root has the order of the minimax risk [3, 2]. We discretize $\Omega = B^m(1)\times B^n(1)$ with $b=(x,-\mu y)$, $K=\{r,\rho\le\frac12\}$ and $\psi=1$ on K by tensor-product linear finite elements in $(\log r,\log\rho)\in[-U,0]^2$, with exact exponential weights, and compute $\mathcal{R}_N^{\mathrm{lin}}$ for $s\in\{0,1\}$ up to $N=10^{28}$. The table reports the local exponent $e_N=-\frac12\,d\log\mathcal{R}_N^{\mathrm{lin}}/d\log N$, taken as a difference quotient between the two values of N in the last column (scripts `radial_risk.py` and `run_numerics.py` in `verification/`).

m	n	μ/λ	s	single block	predicted	observed	$\log_{10}N$
2	1	1	1	0.3333	0.3333	0.3327	14–16
2	2	1	1	0.3333	0.3333	0.3345	14–16
3	1	1	1	0.4167	0.4167	0.4182	14–16
2	2	1/2	1	0.3333	0.3750	0.3751	24–28
2	2	1/4	1	0.3333	0.3958	0.3962	24–28
1	1	1/4	1	0.2500	0.2969	0.2969	20–24
2	1	1/2	1	0.3333	0.3438	0.3438	24–28
3	1	5	1	0.4167	0.4250	0.4252	24–28
3	3	1/2	1	0.4167	0.4479	0.4483	24–28
2	2	1/2	0	0.2500	0.3125	0.3125	20–24
2	1	1	0	0.2500	0.2500	0.2499	14–16

The column "single block" is $(2s+\max(m,n))/(4s+8)$ and "predicted" is $(2s+M)/(4s+8)$. The observed exponents agree with the prediction and separate clearly from the single-block value whenever the two differ. For $\lambda=\mu$ and $m=n$ at $s=0$, a risk asymptotic to a constant multiple of the order in Theorem 15 has local exponent $e_N=\frac{m}{8}+\frac{4-m}{8\log N}+o(1/\log N)$. At the geometric midpoints this gives 0.2584, 0.2572 and 0.2564 for $m=n=2$ on the ranges $\log_{10}N\in[12,14]$, $[14,16]$ and $[16,18]$, where the observed values are 0.2592, 0.2577 and 0.2562. For $m=n=1$ it gives 0.1375, 0.1359, 0.1346 and 0.1336 on the four ranges from $[12,14]$ to $[18,20]$, where the observed values are 0.1368, 0.1353, 0.1341 and 0.1332. For $m=n=1$ the discrete systems are solved only to relative residuals between 10^{-6} and $5\cdot10^{-4}$, so the entries for $m=n=1$ carry about three reliable digits. For $m=2$, $n=1$ and $s=0$, which has no logarithm, the local exponents stay within 0.0004 of 1/4 for $N\ge10^{10}$.

Part V. Interior support below the root-sample threshold

Part II decides when interior-support estimation is root-sample and, below that threshold, bounds the risk through a spectral block of slow directions (Theorem 7). This part determines the rate below the threshold when the trace of the Hessian is at most four times its smallest eigenvalue. A different singular part of the representer, the time elapsed since the trajectory left a small Euclidean ball about the minimum, attains the local lower bound of LCH. In this range the interior-support rate is therefore the rate of an isotropic minimum in dimension d, whatever the anisotropy of

the Hessian, and this answers the interior-support case of LCH, Section 6, third bullet, for these references. The setting, the class $\mathcal{H}_s(K,\eta)$, the quantities Q_j, $\mathcal{D}(H)$ and T_{f_0}, and the risk are those of Section 5.

20 The rate when the trace is at most four times the smallest eigenvalue

Write $\lambda_1 = \lambda_{\min}(H)$ for the smallest eigenvalue of $H = D^2u_0(x_0)$. Since $\operatorname{tr} H \geq d\lambda_1$, the condition $\operatorname{tr} H \leq 4\lambda_1$ forces $d \leq 4$, and for $d = 4$ it forces $H = \lambda_1 I$, the isotropic case of LCH Remark 3.9. We therefore take $d \in \{2,3\}$, and write $e_d(s) = (2s+d)/(4s+8)$.

Theorem 26. *Let $d \in \{2,3\}$, and assume $\psi(x_0) \neq 0$ and $\operatorname{tr} H \leq 4\lambda_1$.*

(a) In the linearized experiment, for every $s \geq 0$ and $\eta > 0$ there are constants $0 < c \leq C$ such that, for all large N, the minimax risk over $\mathcal{H}_s(K,\eta)$ is at least $cN^{-e_d(s)}$ and at most

$$CN^{-e_d(s)} \quad \text{if } \operatorname{tr} H < 4\lambda_1, \qquad C\Big(\frac{N}{\log N}\Big)^{-e_d(s)} \quad \text{if } \operatorname{tr} H = 4\lambda_1.$$

(b) If $\operatorname{tr} H < 4\lambda_1$, then for $s > 2d+4$ and sufficiently small fixed $\eta > 0$ the nonlinear risk on $f_0 + \mathcal{H}_s(K,\eta)$ is bounded above and below by constant multiples of $N^{-e_d(s)}$.

In particular the linearized interior-support risk has polynomial exponent exactly $e_d(s)$ whenever $\operatorname{tr} H \leq 4\lambda_1$.

The lower bounds are LCH Proposition S3.14(ii) in the linearized experiment and LCH Theorem 3.3(a) in the nonlinear one; both use an even conductivity bump of width $N^{-1/(2s+4)}$ at x_0, which is supported in K [5]. The new content is the matching upper bound. For an isotropic Hessian $H = \lambda_1 I$ with $d \leq 3$ the hypothesis holds, and the theorem recovers the isotropic interior rate $a_N(s,d)$ of LCH Remark 3.9.

Corollary 27. *Assume $\psi(x_0) \neq 0$, $\operatorname{tr} H \leq 4\lambda_1$, and that H is not a multiple of the identity. Then $d \in \{2,3\}$ and*

$$T_{f_0} \;\leq\; \mathcal{D}(H) \;<\; d.$$

Hence the interior-support exponent $e_d(s)$ is strictly larger than $(2s+\mathcal{D}(H))/(4s+8)$, which is the limit as $\eta \downarrow 0$ of the exponent in Theorem 7 at the cut attaining $\mathcal{D}(H)$, and strictly larger than the linearized full-neighborhood exponent $(2s+T_{f_0})/(4s+8)$ of LCH Theorem 3.1, which LCH Theorem 3.3(b) shows to be unimprovable in the nonlinear experiment.

Proof. If $d = 4$ the hypothesis forces $H = \lambda_1 I$, which is excluded, so $d \in \{2,3\}$. Each ratio Q_j/λ_j of Section 5 is a sum of $m_1 + \cdots + m_j \leq d$ eigenvalues divided by the largest of them, so it is at most d, with equality only if j is the last cut and all eigenvalues coincide. Thus $\mathcal{D}(H) < d$, and $T_{f_0} \leq \mathcal{D}(H)$ because T_{f_0} is the ratio at the last cut. Condition (C) fails since $\psi(x_0) \neq 0$, and $T_{f_0} < d \leq 3$, so LCH Theorems 3.1 and 3.3(b) apply with the exponent $(2s+T_{f_0})/(4s+8)$. □

For example, in the plane with Hessian eigenvalues 1 and 2 one has $\operatorname{tr} H = 3 < 4$ and $T_{f_0} = \mathcal{D}(H) = 3/2$. The linearized full-neighborhood rate is $N^{-(4s+3)/(8s+16)}$, whereas under interior support the rate is $N^{-(s+1)/(2s+4)}$. In the plane the hypothesis reads $\lambda_{\max} \leq 3\lambda_{\min}$; in three dimensions it reads $\lambda_2 + \lambda_3 \leq 3\lambda_1$ for the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$ counted with multiplicity.

The mechanism is the following. A representer potential solves $b \cdot \nabla w = \psi$ only on K, so its singular part may be any function that increases at unit speed along the flow. Instead of the exit

time to the boundary of Ω, which the full-neighborhood problem forces, we use the time since the trajectory left the ball $B_\epsilon(x_0)$. Its second derivatives on the set of points that left the ball between times t and $t+1$ are at most $e^{-2\lambda_1 t}\epsilon^{-2}$, and this set has volume of order $\epsilon^d e^{(\operatorname{tr} H)t}$. Under the hypothesis the resulting energies $\epsilon^{d-4}e^{(\operatorname{tr} H-4\lambda_1)t}$ are summable in t, so the energy is that of a single ball of radius ϵ. The only violation of the transport equation is a bounded defect inside the ball, and after cancelling finitely many of its moments it costs $\epsilon^{s+d/2}$, exactly as for an isotropic bump.

20.1 Entry times into a small ball

The following hypotheses are stated for one gradient field; they hold uniformly for the reference and for the pilot fields $b_p = \nabla u_p$ used in Sections 8 and 9.

Entry data. Let $\mathcal{U} \subset \Omega$ be a compact set with $K \subset \operatorname{int}\mathcal{U}$. We say that a field $b = \nabla u$ on a neighborhood of $\mathcal{U}$ has *entry data* $(\Lambda_0, \ell, r_*, t_0)$ at a point $x_* \in \operatorname{int} K$ if $b \in C^2$ with $\|b\|_{C^2} \le \Lambda_0$, $b(x_*) = 0$, and:

(E1) $B_* = \overline{B}_{r_*}(x_*) \subset \mathcal{U}$, and on B_* the symmetric matrix Db has all eigenvalues at least $\ell > 0$;

(E2) $\mathcal{U}$ is invariant under the backward flow X_t (the flow of $-b$), every backward trajectory from $\mathcal{U}$ lies in $B_{r_*/2}(x_*)$ after time t_0, and every forward trajectory spends total time at most t_0 in $\mathcal{U} \setminus B_*$.

Write $H_* = Db(x_*)$, $\lambda_* = \lambda_{\min}(H_*)$ and $Q_* = \operatorname{tr} H_*$.

For the reference these hold with $x_* = x_0$ and $\mathcal{U}$ the fixed regular compact sublevel of u_0 from Section 8. On a small ball $Db = D^2u_0$ is close to H. Outside it $|\nabla u_0|$ is bounded below on $\mathcal{U}$, so u_0 changes at a rate bounded below along every trajectory there; since u_0 is monotone along trajectories and bounded on $\mathcal{U}$, this bounds the total time spent. The outward gradient on the boundary of the sublevel set gives backward invariance. For the pilots, LCH Lemma S4.1 gives uniform C^3 bounds on u_p and $\|u_p - u_0\|_{C^3} = O(\eta)$ for sufficiently large s, so the same data serve for all pilots, with $x_* = x_p$, once η is small; this is the uniform entry statement already used in Section 8.

On B_*, $b(x') = \int_0^1 Db(x_* + \sigma(x' - x_*))\, d\sigma\, (x' - x_*)$ gives $(x' - x_*) \cdot b(x') \ge \ell |x' - x_*|^2$. Hence $|X_t x - x_*|$ is strictly decreasing while $X_t x \in B_* \setminus \{x_*\}$, B_* is backward invariant, and a forward trajectory that leaves B_* does not return to it. For $0 < \epsilon \le r_*/4$ and $x \in \mathcal{U} \setminus \{x_*\}$ let $\tau_\epsilon(x) \in \mathbb{R}$ be the unique time with $|X_{\tau_\epsilon(x)} x - x_*| = \epsilon$, where X_t for $t < 0$ is the forward flow. It is the backward entry time into $B_\epsilon(x_*)$ for $x \notin B_\epsilon(x_*)$ and minus the forward exit time for $x \in B_\epsilon(x_*)$. Since $\tau_\epsilon(X_{-t}x) = \tau_\epsilon(x) + t$,

$$b \cdot \nabla \tau_\epsilon = 1 \quad \text{on } \mathcal{U} \setminus \{x_*\}. \tag{28}$$

Lemma 28. *Let b have entry data $(\Lambda_0, \ell, r_*, t_0)$ at x_*, and assume $Q_* \le 4\lambda_*$. There are $\epsilon_0 > 0$ and C such that for $0 < \epsilon \le \epsilon_0$:*

(i) $\tau_\epsilon \in C^2(\mathcal{U} \setminus \{x_*\})$ and $0 \le \tau_\epsilon \le \lambda_*^{-1} \log(C/\epsilon)$ on $\mathcal{U} \setminus B_\epsilon(x_*)$;

(ii)

$$\int_{\mathcal{U}\setminus B_\epsilon(x_*)} (|\nabla \tau_\epsilon|^2 + |D^2 \tau_\epsilon|^2) \le \begin{cases} C\left(1 - e^{-(4\lambda_* - Q_*)}\right)^{-1} \epsilon^{d-4}, & Q_* < 4\lambda_*, \\ C\epsilon^{d-4} \log(1/\epsilon), & Q_* = 4\lambda_*; \end{cases}$$

(iii) the rescaled function $\tilde{\tau}_\epsilon(y) = \tau_\epsilon(x_* + \epsilon y)$ satisfies $\|\tilde{\tau}_\epsilon\|_{C^2(\{1/4 \le |y| \le 2\})} \le C$.

The constants depend only on d, $\mathcal{U}$ and the entry data.

Proof. Write $X = X_t x$ and $r = |x' - x_*|$ for $x' \in B_*$. By Weyl's inequality and $\|D^2 b\| \le \Lambda_0$, the smallest eigenvalue of $Db(x')$ is at least $\lambda_* - \Lambda_0 r$ and $\operatorname{div} b(x') \le Q_* + d\Lambda_0 r$.

Flow bounds. Let $x \in \mathcal{U}$. For $t \ge t_0$ the backward trajectory lies in B_*, where $\frac{d}{dt}|X - x_*|^2 \le -2\ell|X - x_*|^2$; hence $\int_{t_0}^{\infty} |X_t x - x_*|\,dt \le r_*/\ell$. Using this integral in $\frac{d}{dt}|X - x_*|^2 \le -2(\lambda_* - \Lambda_0|X - x_*|)|X - x_*|^2$ and in $\frac{d}{dt}|DX_t v|^2 = -2(DX_t v)^{\mathsf{T}} Db(X)(DX_t v)$, which uses that Db is symmetric, and bounding the first t_0 units of time by $e^{\Lambda_0 t_0}$, we obtain

$$|X_t x - x_*| \le Ce^{-\lambda_* t}, \qquad \|DX_t(x)\| \le Ce^{-\lambda_* t} \qquad (t \ge 0). \tag{29}$$

The second variational equation $\partial_t D^2 X_t = -Db(X)D^2 X_t - D^2 b(X)[DX_t, DX_t]$ and the propagator bound $\|DX_{t-\sigma}(X_\sigma x)\| \le Ce^{-\lambda_*(t-\sigma)}$, which is (29) at the point $X_\sigma x \in \mathcal{U}$, give $\|D^2 X_t(x)\| \le C\int_0^t e^{-\lambda_*(t-\sigma)} e^{-2\lambda_*\sigma}\,d\sigma \le Ce^{-\lambda_* t}$.

Derivatives. Let $x \in \mathcal{U} \setminus B_\epsilon(x_*)$ and $t = \tau_\epsilon(x) \ge 0$. The first bound in (29) at time t gives $\epsilon \le Ce^{-\lambda_* t}$, which is the upper bound on t in (i); $\tau_\epsilon \ge 0$ there by definition. Put $G(x,t) = |X_t x - x_*|^2 - \epsilon^2$. At $(x, \tau_\epsilon(x))$, with $x' = X_t x \in \partial B_\epsilon(x_*)$,

$$G_t = -2(x' - x_*) \cdot b(x'), \quad |G_t| \ge 2\ell\epsilon^2, \qquad \nabla_x G = 2DX_t^{\mathsf{T}}(x' - x_*),$$

and, using $\partial_t X_t = -b(X_t)$ and $\partial_t DX_t = -Db(X_t)DX_t$,

$$\|D_x^2 G\| \le 2\|DX_t\|^2 + 2\epsilon\|D^2 X_t\|, \qquad |\partial_t \nabla_x G| \le C\epsilon\|DX_t\|,$$
$$|G_{tt}| = \big|2|b(x')|^2 + 2(x' - x_*) \cdot Db(x')b(x')\big| \le C\epsilon^2.$$

The implicit function theorem gives $\tau_\epsilon \in C^2$, since $X_t \in C^2$ in x, together with $\nabla\tau_\epsilon = -\nabla_x G / G_t$ and

$$D^2\tau_\epsilon = -G_t^{-1}\big(D_x^2 G + \partial_t\nabla_x G \otimes \nabla\tau_\epsilon + \nabla\tau_\epsilon \otimes \partial_t\nabla_x G + G_{tt}\nabla\tau_\epsilon \otimes \nabla\tau_\epsilon\big).$$

By (29) and $\epsilon \le Ce^{-\lambda_* t}$,

$$|\nabla\tau_\epsilon(x)| \le \frac{Ce^{-\lambda_* t}}{\epsilon}, \qquad |D^2\tau_\epsilon(x)| \le \frac{C}{\epsilon^2}\big(e^{-2\lambda_* t} + \epsilon e^{-\lambda_* t}\big) \le \frac{Ce^{-2\lambda_* t}}{\epsilon^2}. \tag{30}$$

The same computation applies on $B_\epsilon(x_*) \setminus \{x_*\}$, where the forward exit is transverse, so $\tau_\epsilon \in C^2(\mathcal{U} \setminus \{x_*\})$.

Volumes. For an integer $k \ge 0$ let $S_k = \{x \in \mathcal{U} \setminus B_\epsilon(x_*) : k \le \tau_\epsilon(x) < k + 1\}$. If $x \in S_k$, then $y = X_k x \in S_0$ and $x = X_{-k} y$. The set S_0 lies in the image of $\partial B_\epsilon(x_*)$ under the forward flow for times in $[0, 1)$, and $|b(x')| \le \Lambda_0|x' - x_*|$, so $S_0 \subset B_{e^{\Lambda_0}\epsilon}(x_*)$ and $|S_0| \le C\epsilon^d$. Along the forward trajectory from y to x, which lies in $\mathcal{U}$ by backward invariance, the time spent in $\mathcal{U} \setminus B_*$ is at most t_0 by (E2); inside B_* the distance to x_* grows at least like $e^{\ell\sigma}$, so the time spent at distances in $[2^{-i-1}r_*, 2^{-i}r_*]$ is at most $\ell^{-1}\log 2$ and the integral of the distance over the time spent in B_* is at most $2r_*\ell^{-1}\log 2$. Liouville's formula therefore gives $\det DX_{-k}(y) = \exp\int_0^k \operatorname{div} b(X_{-\sigma}y)\,d\sigma \le Ce^{Q_* k}$, and by change of variables

$$|S_k| \le C\epsilon^d e^{Q_* k}. \tag{31}$$

Energy. By (i), $S_k = \emptyset$ for $k > k_\epsilon = \lambda_*^{-1}\log(C/\epsilon)$. By (30) and (31),

$$\int_{\mathcal{U}\setminus B_\epsilon(x_*)} |D^2\tau_\epsilon|^2 \le C\epsilon^{d-4} \sum_{0 \le k \le k_\epsilon} e^{(Q_* - 4\lambda_*)k},$$

which is the bound in (ii) for $D^2\tau_\epsilon$. For the gradient, $Q_* - 2\lambda_* \le 2\lambda_*$ and $e^{\lambda_* k_\epsilon} = C/\epsilon$ give

$$\int_{\mathcal{U}\setminus B_\epsilon(x_*)} |\nabla\tau_\epsilon|^2 \le C\epsilon^{d-2} \sum_{0 \le k \le k_\epsilon} e^{(Q_* - 2\lambda_*)k} \le C\epsilon^{d-2} \sum_{0 \le k \le k_\epsilon} e^{2\lambda_* k} \le C\epsilon^{d-2} e^{2\lambda_* k_\epsilon} \le C\epsilon^{d-4}.$$

Rescaling. The function $\tilde{\tau}_\epsilon$ is the analogous time for the field $\tilde{b}_\epsilon(y) = \epsilon^{-1} b(x_* + \epsilon y)$ and the unit ball. On B_4 one has $|\tilde{b}_\epsilon(y) - H_* y| \le C\epsilon|y|^2$, $\|D\tilde{b}_\epsilon(y) - H_*\| \le C\epsilon|y|$ and $\|D^2\tilde{b}_\epsilon\| \le C\epsilon$, while $y \cdot \tilde{b}_\epsilon(y) \ge \ell|y|^2$. A point of $\{1/4 \le |y| \le 2\}$ reaches the unit sphere in time at most $\ell^{-1}\log 4$ along a trajectory inside B_4, and the crossing is transverse with $y \cdot \tilde{b}_\epsilon(y) \ge \ell$ on the unit sphere. The flow of $\tilde{b}_\epsilon$ and its first two derivatives are bounded on that time interval, so the implicit function theorem gives (iii). □

20.2 The moment-cancelled representer

Fix an integer $L \ge s$, a radial $\zeta \in C_c^\infty(B_{1/2})$ with $\zeta = 1$ on $B_{1/4}$, and functions $\phi_\alpha \in C_c^\infty(B_{1/2})$, $|\alpha| \le L$, with $\int \phi_\alpha y^\beta \, dy = \delta_{\alpha\beta}$ for $|\beta| \le L$; these exist because the Gram matrix of the monomials against a positive bump is invertible, as in Lemma 20. Let $\mathcal{P}_L$ be the space of polynomials of degree at most L and $\mathcal{T}_* p = \operatorname{div}(p\, H_* y) = H_* y \cdot \nabla p + Q_* p$. In orthonormal eigencoordinates of H_*, with eigenvalue vector ν, $\mathcal{T}_* y^\beta = (\beta \cdot \nu + Q_*) y^\beta$, so $\mathcal{T}_*$ is invertible on $\mathcal{P}_L$ with $\|\mathcal{T}_*^{-1}\|$ bounded in terms of d, L and a lower bound for Q_*.

Lemma 29. *Under the hypotheses of Lemma 28 there are $\epsilon_1 \in (0, \epsilon_0]$ and C with the following property. For $0 < \epsilon \le \epsilon_1$ there are coefficients $c_\alpha = c_\alpha(\epsilon)$, $|c_\alpha| \le C$, such that*

$$\omega_\epsilon(x) = \Big(1 - \zeta\big(\tfrac{x - x_*}{\epsilon}\big)\Big)\tau_\epsilon(x) + \sum_{|\alpha| \le L} c_\alpha \, \phi_\alpha\big(\tfrac{x - x_*}{\epsilon}\big)$$

lies in $C^2(\mathcal{U})$, equals τ_ϵ outside $B_{\epsilon/2}(x_*)$, and satisfies:

(i) $\|\omega_\epsilon\|_{H^2(\mathcal{U})}^2 \le C\epsilon^{d-4}$ if $Q_* < 4\lambda_*$, and $\le C\epsilon^{d-4}\log(1/\epsilon)$ if $Q_* = 4\lambda_*$;

(ii) the defect $r_\epsilon = 1 - b \cdot \nabla\omega_\epsilon$ is supported in $\overline{B}_{\epsilon/2}(x_*)$, is bounded by C, and has $\int (x - x_*)^\beta r_\epsilon(x)\, dx = 0$ for $|\beta| \le L$;

(iii) for every $h \in H^s(\Omega)$ supported in K, $|\langle h, r_\epsilon\rangle| \le C\epsilon^{s + d/2}\|h\|_{H^s(\Omega)}$.

The constants depend only on d, s, L, ζ, $\{\phi_\alpha\}$, K, Ω, $\mathcal{U}$ and the entry data.

Proof. The cutoff $1 - \zeta(\cdot/\epsilon)$ vanishes near x_*, so $\omega_\epsilon \in C^2(\mathcal{U})$ by Lemma 28(i). In the variable $y = (x - x_*)/\epsilon$, by (28) and $b(x) \cdot \nabla_x[f((x - x_*)/\epsilon)] = \tilde{b}_\epsilon(y) \cdot \nabla f(y)$,

$$r_\epsilon(x_* + \epsilon y) = \tilde{r}_\epsilon(y) = \zeta(y) + \tilde{\tau}_\epsilon(y)\, \tilde{b}_\epsilon(y) \cdot \nabla\zeta(y) - \sum_\alpha c_\alpha \, \tilde{b}_\epsilon(y) \cdot \nabla\phi_\alpha(y).$$

This is supported in $\overline{B}_{1/2}$ and bounded by Lemma 28(iii) once the c_α are bounded. Integration by parts gives $\int y^\beta \, \tilde{b}_\epsilon \cdot \nabla\phi_\alpha = -\int \phi_\alpha \operatorname{div}(y^\beta \tilde{b}_\epsilon)$, so the moment conditions read $\sum_\alpha \mathcal{A}_{\beta\alpha}(\epsilon) c_\alpha = -\int y^\beta(\zeta + \tilde{\tau}_\epsilon \tilde{b}_\epsilon \cdot \nabla\zeta)$ with $\mathcal{A}_{\beta\alpha}(\epsilon) = \int \phi_\alpha \operatorname{div}(y^\beta \tilde{b}_\epsilon)$. For the linear field $H_* y$ in place of $\tilde{b}_\epsilon$ this matrix is the matrix of $\mathcal{T}_*$ on $\mathcal{P}_L$ in the monomial basis and its dual basis $\{\phi_\alpha\}$, hence invertible with bounded inverse. Since $\|\tilde{b}_\epsilon - H_* y\|_{C^1(B_{1/2})} \le C\epsilon$, the matrix $\mathcal{A}(\epsilon)$ is invertible with bounded inverse for small ϵ, which gives bounded c_α and (ii) after the change of variables.

For (i), $\omega_\epsilon = \tau_\epsilon$ on $\mathcal{U} \setminus B_{\epsilon/2}(x_*)$, whose contribution outside $B_\epsilon(x_*)$ is bounded by Lemma 28(i)–(ii), with $\|\tau_\epsilon\|_{L^2(\mathcal{U})} \le C\log(1/\epsilon)$. On $B_\epsilon(x_*)$, $\omega_\epsilon(x_* + \epsilon y) = (1 - \zeta)\tilde{\tau}_\epsilon + \sum c_\alpha \phi_\alpha$ is bounded in $C^2(B_1)$, which gives a contribution $C\epsilon^{d-4}$ to the squared H^2 norm. Since $d \le 3$, the term $C\log^2(1/\epsilon)$ from the L^2 norm is dominated by ϵ^{d-4}.

For (iii), extend h by zero. Since $\operatorname{supp} h \subset K$, the extension satisfies $\|h\|_{H^s(\mathbb{R}^d)} \le C\|h\|_{H^s(\Omega)}$. Write $\widehat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\, dx$ for the Fourier transform. Plancherel's identity, the Cauchy–Schwarz inequality and the scaling $\widehat{r_\epsilon}(\xi) = \epsilon^d e^{-ix_*\cdot\xi}\widehat{\tilde{r}_\epsilon}(\epsilon\xi)$ give

$$|\langle h, r_\epsilon\rangle| \le (2\pi)^{-d} \||\xi|^s \widehat{h}\|_2 \, \||\xi|^{-s}\widehat{r_\epsilon}\|_2, \qquad \||\xi|^{-s}\widehat{r_\epsilon}\|_2^2 = \epsilon^{2s+d}\int |\xi|^{-2s} |\widehat{\tilde{r}_\epsilon}(\xi)|^2 \, d\xi,$$

the last after the substitution $\xi \mapsto \xi/\epsilon$. For $|\xi| \le 1$, the vanishing moments and Taylor's formula for $e^{-iy\cdot\xi}$ give

$$|\widehat{\tilde{r}}_\epsilon(\xi)| \le \frac{|\xi|^{L+1}}{(L+1)!} \int |y|^{L+1} |\tilde{r}_\epsilon(y)|\, dy \le C|\xi|^{L+1},$$

which is square integrable against $|\xi|^{-2s}$ on the unit ball because $2L+2-2s+d>0$. For $|\xi|>1$ the integral is at most $(2\pi)^d \|\tilde{r}_\epsilon\|_2^2 \le C$. □

20.3 Proof of Theorem 26

Linearized upper bound. The reference field $b = \nabla u_0$ has entry data at $x_* = x_0$ with $H_* = H$, $\lambda_* = \lambda_1$ and $Q_* = \operatorname{tr} H$. Let v_0 be the smooth particular solution of $b \cdot \nabla v_0 = \psi - \psi(x_0)$ from Section 8, the backward integral of $\psi - \psi(x_0)$, which is C^2 on $\mathcal{U}$, and let $\chi \in C_c^\infty(\operatorname{int}\mathcal{U})$ equal one on a neighborhood of K. Put

$$w_\epsilon = \chi(v_0 + \psi(x_0)\omega_\epsilon), \qquad z_\epsilon = L_{f_0} w_\epsilon.$$

Then $w_\epsilon \in C_c^2(\Omega) \subset H^2(\Omega) \cap H_0^1(\Omega)$, and on K one has $b\cdot\nabla w_\epsilon = \psi - \psi(x_0) r_\epsilon$ by Lemma 29(ii). For h supported in K, integration by parts gives $\langle Ah, z_\epsilon\rangle = -\langle \operatorname{div}(hb), w_\epsilon\rangle = \langle h, b\cdot\nabla w_\epsilon\rangle$. The linear estimator $\langle Y, z_\epsilon\rangle$ therefore has error

$$-\psi(x_0)\langle h, r_\epsilon\rangle + N^{-1/2} W(z_\epsilon),$$

where W is the white noise. Since $\|z_\epsilon\|_2 \le C\|w_\epsilon\|_{H^2}$, Lemma 29 bounds its mean absolute error over $\mathcal{H}_s(K,\eta)$ by $C\eta\epsilon^{s+d/2} + CN^{-1/2}\epsilon^{d/2-2}$ if $\operatorname{tr} H < 4\lambda_1$, and by the same expression with the second term multiplied by $(\log(1/\epsilon))^{1/2}$ if $\operatorname{tr} H = 4\lambda_1$. Choosing $\epsilon = N^{-1/(2s+4)}$, respectively $\epsilon = (N/\log N)^{-1/(2s+4)}$, both terms are of the stated order, since $\epsilon^{s+d/2} = \epsilon^{s+2}\epsilon^{d/2-2}$ and $e_d(s) = (s+d/2)/(2s+4)$.

Nonlinear upper bound. Use the split white-noise pilot of LCH Lemma S4.2 on the supported class, as in Section 8, and the one-step estimator of Section 4, $\widehat{\Phi} = \langle p, \psi\rangle + Y_2(z_{p,\epsilon}) - \langle u_p, z_{p,\epsilon}\rangle$, with $z_{p,\epsilon} = L_p w_{p,\epsilon}$ and $w_{p,\epsilon} = \chi(v_p + \psi(x_p)\omega_{p,\epsilon})$ built as above from the pilot field b_p, its minimum x_p, its Hessian H_p and the particular solution v_p of Section 8. For $s > 2d+4$, LCH Lemma S4.1 gives uniform C^3 bounds on u_p and $\|u_p - u_0\|_{C^3} = O(\eta)$, so the entry data are uniform and $\lambda_{\min}(H_p)$, $\operatorname{tr} H_p$ are within $O(\eta)$ of λ_1, $\operatorname{tr} H$, by Weyl's inequality. Since $\operatorname{tr} H < 4\lambda_1$, for η small every pilot satisfies

$$4\lambda_{\min}(H_p) - \operatorname{tr} H_p \;\ge\; \tfrac{1}{2}(4\lambda_1 - \operatorname{tr} H) > 0, \tag{32}$$

so Lemmas 28 and 29 hold with constants uniform over the pilots.

Conditionally on the first observation, LCH Lemma S4.3 gives the exact bias

$$-\psi(x_p)\langle h, r_{p,\epsilon}\rangle + \int_\Omega h\, \nabla(u_f - u_p)\cdot\nabla w_{p,\epsilon}, \qquad h = f - p,$$

because h is supported in K. The conditional standard deviation is $\sqrt{2/N}\|z_{p,\epsilon}\|_2 \le CN^{-1/2}\epsilon^{d/2-2}$, because $\|L_p w\|_2 \le C\|w\|_{H^2}$ with C uniform in p. By Lemma 29(iii) and $\|h\|_{H^s} \le 2\eta$, the first term of the bias is at most $C\eta\epsilon^{s+d/2}$. For the second term,

$$\|\nabla w_{p,\epsilon}\|_{L^1(K)} \le C\|w_{p,\epsilon}\|_{H^1} \le C\epsilon^{d/2-2},$$

and the pilot moments of Section 4 give

$$\mathbb{E}_f\big[\|h\|_{L^\infty(K)}\|\nabla(u_f - u_p)\|_{L^\infty(K)}\big] \le CN^{-A-B}, \qquad A+B > \tfrac{1}{2} \quad (s > 2d+4).$$

Take $\epsilon = N^{-1/(2s+4)}$. Then

$$N^{-A-B}\epsilon^{d/2-2} = N^{-A-B+(4-d)/(4s+8)} \le N^{-e_d(s)},$$

because $e_d(s) + (4-d)/(4s+8) = 1/2 < A + B$. The defect term $C\eta\epsilon^{s+d/2}$ and the standard deviation $CN^{-1/2}\epsilon^{d/2-2}$ are also $O(N^{-e_d(s)})$, uniformly over the class. The randomization of the split is removed as in Section 4.

Lower bounds. In the linearized experiment the lower bound $cN^{-e_d(s)}$ is Proposition S3.14(ii) of LCH, which applies because $x_0 \in \operatorname{int} K$ and $\psi(x_0) \neq 0$. In the nonlinear experiment it is LCH Theorem 3.3(a), valid for $s > d/2$ and small η, whose alternatives $f_0 + h_\epsilon$ have h_ϵ supported in $B_\epsilon(x_0) \subset K$ for large N. This completes the proof. □

20.4 Numerical checks

The scripts `verification/darcylb_*` confirm the main estimates numerically; the proof above does not use them. For linear fields $b = Hy$ with eigenvalues $(1, 2)$ and $(1, 2.6)$ in the plane and $(1, 1.2, 1.5)$ in three dimensions, the energy $\int |D^2\tau|^2$ of the entry time into the unit ball, taken over $\{1 < |y| < R\}$, increases with R toward the finite limit

$$\sum_{i,k} \int_{\{|\theta|=1\}} \frac{|\partial_{ik}\tau(\theta)|^2\,(\theta \cdot H\theta)}{2\lambda_i + 2\lambda_k - \operatorname{tr} H}\, dS(\theta)$$

given by self-similarity when $\operatorname{tr} H < 4\lambda_1$, and at finite R the self-similar formula agrees with direct quadrature. For eigenvalues $(1, 3)$, where $\operatorname{tr} H = 4\lambda_1$, the energy grows by a constant amount on each factor 8 in R, and for eigenvalues $(1, 5)$ its logarithmic slope in R decreases to 0.404, close to $(\operatorname{tr} H - 4\lambda_1)/\lambda_{\max} = 0.4$. For the planar gradient field with cubic terms in `darcylb_v2_nonlinear_entry_energy.py` and Hessian eigenvalues 1 and 2, the rescaled energy $\epsilon^2 \int |D^2\tau_\epsilon|^2$ over $B_{0.4} \setminus B_\epsilon$ increases from 3.41 to 4.68 for $\epsilon = 2^{-3}, \dots, 2^{-9}$, with increments falling from 0.53 to 0.06, in line with Lemma 28(ii). For eigenvalues 1 and 5 it grows at each halving of ϵ by a factor that decreases from 1.54 to 1.34, toward $2^{(\operatorname{tr} H - 4\lambda_1)/\lambda_{\max}} = 1.32$. Exact rational computations check the exponent algebra of the proof, the inequality $A + B > 1/2$ for $s > 2d + 4$, the bound $T_{f_0} \le \mathcal{D}(H) < d$ of Corollary 27 on a grid of spectra, and the invertibility of the moment matrix $\mathcal{A}(\epsilon)$ for that nonlinear field.

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