The nonlinear minimax exponent for Darcy inference at anisotropic minima, with power-law and saddle critical points

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Luminary Oddities

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Abstract

We estimate a linear functional of a scalar conductivity from Gaussian white-noise observations of its Darcy potential. For a reference potential with one nondegenerate minimum, Liu, Chang and Hu (arXiv:2611.01290) showed that the linearized minimax rate is governed by the effective dimension of the Hessian at the minimum, its trace over its largest eigenvalue. Their Conjecture 6.1 asserts that, when this dimension is at most four and their transport compatibility condition fails, the linearized exponent is also the local exponent of the nonlinear experiment on shrinking full Sobolev neighborhoods, for all sufficiently large smoothness s. We prove this conjecture. On each sufficiently small fixed Sobolev ball the exponent is $\min\{(2s+T)/(4s+8), 1/2\}$, where T is the infimum of the effective dimension over the ball; T lies strictly below the reference value, so every fixed ball is strictly harder than the limit. The upper bound uses a Fourier-regularized transport correction that is uniform over prediction pilots. The singular time function of the adjoint transport equation also answers three further problems of theirs in specific models. Under interior support constraints with nonzero score at the minimum, root-sample estimation is possible exactly when some complete lower block of Hessian eigenvalues has sum exceeding four times its largest member. At the radial power-law minimum of even order p the critical dimension becomes $2p$. At a two-block linear saddle the larger block dimension decides whether a square-integrable representer exists. The first two statements also hold nonlinearly at large smoothness, the second for radial conductivities.

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1 Introduction

1.1 The question

How accurately can a weighted average of a conductivity be estimated from a noisy observation of the potential that it induces? Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be smooth, bounded and connected. Let g, f_0 be smooth up to the boundary and bounded below by positive constants. For a scalar conductivity f write u_f for the solution of

$$L_f u_f = g, \qquad L_f = \operatorname{div}(f\nabla), \qquad u_f|_{\partial\Omega} = 0.$$

The data are the Gaussian white-noise observation with mean u_f and noise level $N^{-1/2}$ in $L^2(\Omega)$. Fix nonzero $\psi \in C_c^\infty(\Omega)$ and set $\Phi(f) = \int_\Omega f\psi$. Liu, Chang and Hu [3], henceforth LCH, treat this problem when the reference $u_0 = u_{f_0}$ has one critical point, a nondegenerate minimum x_0 in the interior (their Assumption 2.1), over the full Sobolev ball

$$\mathcal{F}_\eta = \{f = f_0 + h : \|h\|_{H^s(\Omega)} \leq \eta,\ \inf_\Omega f > 0\},$$

which they denote $\mathcal{F}_s(f_0, \eta)$. For small fixed $\eta > 0$ and the smoothness used below, each u_f has a unique minimum x_f and positive Hessian H_f there. Write $\lambda_f = \lambda_{\max}(H_f)$, $T_f = \operatorname{tr}(H_f)/\lambda_f$ and $T_-(\eta) = \inf_{f \in \mathcal{F}_\eta} T_f$. The number T_f is the effective dimension of LCH, and anisotropy refers to the eigenvalues of H_f. Use absolute risk

$$\mathcal{R}_N(\mathcal{F}_\eta) = \inf_{\widehat{\Phi}} \sup_{f \in \mathcal{F}_\eta} \mathbb{E}_f |\widehat{\Phi} - \Phi(f)|, \qquad e(\mathcal{F}_\eta) = \liminf_{N \to \infty} \frac{\log(1/\mathcal{R}_N(\mathcal{F}_\eta))}{\log N}.$$

The LCH condition (C) [3, Definition 2.2] holds at a smooth reference if $\psi(x_f) = 0$ and the integral of $-\psi$ is independent of the complete forward trajectory. Here each trajectory is parametrized by $\dot{x} = \nabla u_f(x)$, approaches x_f as $t \to -\infty$, and ends on $\partial\Omega$. LCH relate this condition to the efficient information analysis of Nickl and Paternain [4]. In the linearized experiment at f_0, LCH Theorem 3.1 gives the minimax rate $N^{-1/2}$ if (C) holds, and otherwise

$$a_N(s, T_{f_0}) = \begin{cases} N^{-(2s+T_{f_0})/(4s+8)}, & T_{f_0} < 4, \\ \sqrt{\log N / N}, & T_{f_0} = 4, \\ N^{-1/2}, & T_{f_0} > 4. \end{cases}$$

In the nonlinear experiment, at a reference failing (C) with $T_{f_0} \le 4$, their Theorem 3.3(b) gives the lower bound $\mathcal{R}_N(\mathcal{F}_\eta) \gtrsim a_N(s, T_{f_0})$, and Theorem 3.3(c) gives $\mathcal{R}_N(\mathcal{F}_\eta) \gtrsim N^{-1/2}$ in every case. The rate is $N^{-1/2}$ when $T_{f_0} > 4$ (their Theorem 3.7), and at the canonical isotropic model in dimensions two to four their Corollary 3.8 identifies the limit of the local exponent as $\eta \downarrow 0$. For anisotropic references below the elbow $T_{f_0} = 4$, LCH name the minimax rate on full neighborhoods as the main question of their Section 6 and conjecture the following. In it, $N \to \infty$ is taken first, at fixed η, in the definition of e, and then $\eta \downarrow 0$.

Conjecture 6.1 (Liu, Chang and Hu [3, §6])**.** *Under Assumption 2.1 with $T_{f_0} \le 4$, if* (C) *fails at f_0, then for all sufficiently large s,*

$$\lim_{\eta \downarrow 0} e(\mathcal{F}_\eta) = \min\left\{ \frac{2s + T_{f_0}}{4s + 8}, \frac{1}{2} \right\}.$$

1.2 Main results

Theorem 1 proves Conjecture 6.1; any $s > 2d + 4$ suffices. It also gives the exponent on each fixed ball: if (C) fails at f_0, then for every sufficiently small fixed $\eta > 0$,

$$e(\mathcal{F}_\eta) = \min\left\{ \frac{2s + T_-(\eta)}{4s + 8}, \frac{1}{2} \right\},$$

and (1) gives the corresponding upper bound on the risk. Corollary 2 shows that $T_-(\eta) < T_{f_0}$. Hence, when (C) fails and $T_{f_0} \le 4$, every fixed full neighborhood has a strictly smaller exponent than the conjectured limit, which is reached only as the radius shrinks. This extends the strict inequality of LCH Corollary 3.8 from the canonical model to every such reference. LCH remark after the conjecture that the truncation of the transport representer used in the linearized analysis would have to be made uniform over the data-dependent pilot. Sections 2 and 3 make it uniform, using only a fixed Sobolev bound on the pilot.

LCH then list further problems for their nondegenerate theory [3, §6]. The second bullet concerns several, degenerate or saddle-type critical points, which arise for instance for sources of variable sign. The third concerns the minimax rate under interior support constraints or known boundary values, for anisotropic references with $\psi(x_0) \neq 0$; their Section 2.1 singles out the interior-support case. Parts II–IV answer these problems in the following cases.

Interior support (Part II). Let the perturbations be supported in a fixed compact $K \subset \Omega$ with $x_0 \in \operatorname{int} K$, and let $\psi(x_0) \neq 0$. Theorems 3 and 4 show that the rate is $N^{-1/2}$ exactly when $\mathcal{D}(H) > 4$, where $\mathcal{D}(H)$ is the largest effective dimension of a complete lower block of eigenvalues of $H = H_{f_0}$ (Section 5). This holds in the linearized experiment for every $s \geq 0$ and in the nonlinear experiment for sufficiently large s, and Theorem 7 gives an upper bound below the threshold. This answers the third bullet for interior support constraints at the level of the root-sample threshold; the case $\psi(x_0) = 0$ is LCH Remark 3.9. For example, in dimension six with Hessian eigenvalues $1, 1, 1, 1, 1, 10$ one has $T_{f_0} = 3/2$ and $\mathcal{D}(H) = 5$, so the rate on full neighborhoods is slower than root-sample while the interior-support rate is root-sample (Section 5).

A power-law degenerate minimum (Part III). For the radial reference $u_0 = (|x|^p - 1)/p$ on the unit ball, with $p \geq 4$ even, the Hessian vanishes at the minimum. Theorems 8 and 9 give the rate $N^{-(2s+d)/(4s+4p)}$ for $d < 2p$, $\sqrt{\log N/N}$ for $d = 2p$ and $N^{-1/2}$ for $d > 2p$, in the linearized experiment for every $s \geq 0$ and in the nonlinear experiment over radial conductivities for sufficiently large s. The critical dimension is therefore $2p$. This answers the degenerate case of the second bullet for this radial family.

A two-block saddle (Part IV). Let $f_0 = 1$ and let the reference gradient equal $(\lambda x, -\mu y)$ on a product neighborhood of an interior critical point in $\mathbb{R}^m \times \mathbb{R}^n$, with ψ nonzero at that point and perturbations supported near it. Theorem 11 shows that an L^2 representer exists exactly when $\max(m, n) > 4$, for all $\lambda, \mu > 0$. Otherwise the risk multiplied by $\sqrt{N}$ diverges, in the linearized experiment for every s and in the nonlinear experiment for $s > (m + n)/2 + 1$. For $\lambda = \mu$ and $m \neq n$, Theorem 12 gives the sharp linearized rate, governed by $M = \max(m, n)$, with matching nonlinear lower bounds. The reference may have other critical points, and Section 18 gives an example with positive source and 3^n nondegenerate critical points. This answers the saddle-type case of the second bullet for this two-block model.

1.3 Mechanism and organization

In each geometry the representer potential solves $b \cdot \nabla w = \psi$ with $b = \nabla u_0$, and its singular part is a multiple of the time spent near the critical point. The square integrability of the second derivatives of that time function fixes a critical dimension, and below it the exponent takes the form $(2s + D)/(4s + 4\kappa)$, where D is the volume growth and κ the derivative growth of the time function. This form is sharp in Parts I, III and IV and an upper bound in Part II. In Part I the sharp quantity on a fixed ball is $T_-(\eta)$, the infimum of the effective dimension over the ball, and T_{f_0} enters as its limit:

geometry	critical condition	reference
nondegenerate minimum, full support	$T_{f_0} > 4$	Part I, [3]
nondegenerate minimum, interior support	$\mathcal{D}(H) > 4$	Part II
power-law minimum of order p	$d > 2p$	Part III
two-block saddle	$\max(m, n) > 4$	Part IV

Part I (Sections 2–4) proves Theorem 1 and Corollary 2. Parts II (Sections 5–9), III (Sections 10–13) and IV (Sections 14–18) each open with their setting and results, and reuse the shell and commutator estimates of Part I where they apply.

Edition note. This second edition adds Parts II–IV, which consolidate the first editions of three Luminary Research Archive articles [8, 5, 6], to the proof of Conjecture 6.1 in the first edition [7].

Part I. The anisotropic minimum: Conjecture 6.1

This part determines the polynomial exponent on each sufficiently small fixed full Sobolev ball. The controlling effective dimension is the infimum over the ball, and the conjectured local exponent follows as the radius tends to zero. A perturbation in a top Hessian eigendirection shows that this infimum is strictly smaller than the effective dimension of the reference (Corollary 2), so every fixed ball has a strictly smaller exponent than the limit. The setting, the ball $\mathcal{F}_\eta$, the quantities T_f and $T_-(\eta)$, the exponent e and condition (C) are those of Section 1.1.

Theorem 1. *For all sufficiently large s and sufficiently small fixed $\eta > 0$,*

$$\mathcal{R}_N(\mathcal{F}_\eta) \lesssim \begin{cases} N^{-(2s+T_-(\eta))/(4s+8)}, & T_-(\eta) < 4, \\ \sqrt{\log N/N}, & T_-(\eta) = 4, \\ N^{-1/2}, & T_-(\eta) > 4. \end{cases} \tag{1}$$

If (C) fails at f_0, the exact exponent on each such ball is

$$e(\mathcal{F}_\eta) = \min\left\{\frac{2s + T_-(\eta)}{4s+8}, \frac{1}{2}\right\}.$$

In particular, if $T_{f_0} \le 4$ and (C) fails there, then

$$\lim_{\eta \downarrow 0} e(\mathcal{F}_\eta) = \min\left\{\frac{2s + T_{f_0}}{4s+8}, \frac{1}{2}\right\},$$

which is LCH Conjecture 6.1.

For $T_-(\eta) > 4$, the upper bound and LCH Theorem 3.3(c) give the root-N rate. At $T_-(\eta) = 4$, the exponent is $1/2$ and the displayed upper bound carries a logarithmic factor. The constants in the risk bounds depend on the fixed ball, and the exact exponent describes the power of N on that ball. The proof applies, for example, whenever $s > 2d + 4$.

The proof uses the LCH pilot and exact conditional bias identity. The new issue is to regularize the transport solution uniformly over pilot fields with only a fixed Sobolev bound. We first obtain a uniform Fourier-shell estimate and an integrable gradient for the singular solution. Then we split the commutator with the Fourier cutoff. Low coefficient frequencies use the shell estimate. High coefficient frequencies are paired directly with the small pilot error in supremum norm. The integrable gradient also controls the nonlinear remainder.

2 Uniform bounds for characteristic solutions

Fix an integer $k > d/2$ with $k \ge 2$, and take $s > d/2 + k$. Additional strict inequalities on s appear below. We use LCH Lemma S4.1. On a fixed compact regular sublevel domain $K \Subset \Omega$, the characteristic solution w_p of

$$b_p \cdot \nabla w_p = \psi, \qquad b_p = \nabla u_p, \qquad w_p|_{\partial\Omega} = 0$$

is supported in a fixed compact subset of the interior of K. The backward flow stays in K and reaches a uniformly convex ball about the minimum in uniformly bounded time. We may apply the following bounds to smooth pilots, since smooth conductivities in the ball are dense in it in H^s: approximate the perturbation by smooth ones after a contraction toward the center. The finite derivative estimates below hold uniformly. In particular $\|u_p\|_{H^{s+1}(\Omega)}$ and $\|b_p\|_{C^k}$ are bounded, whereas $\|u_p - u_0\|_{C^2} = O(\eta)$.

Use the following notation with the pilot index suppressed. Write x_p for the minimum, H for the Hessian, $Q = \operatorname{tr} H$, $\lambda = \lambda_{\max}(H)$, and $T = Q/\lambda$. In a common-radius ball B about x_p, $Db \geq mI$ with $m > 0$ uniform. Denote the flow of $-b$ by X_t. For $t \geq 0$ and $x, y \in B$,

$$\begin{aligned} |X_t x - x_p| &\leq e^{-mt}|x - x_p|, \\ \det DX_t(x) &\asymp e^{-Qt}, \\ |X_t x - X_t y| &\geq c\, e^{-\lambda t}|x - y|. \end{aligned} \tag{2}$$

The first bound is the differential inequality from convexity. Liouville's identity gives the second because, along each orbit, the time integral of $|\operatorname{div} b - Q|$ is uniformly bounded. For the last, $Db(z) - H = O(|z - x_p|)$ and the differential inequality for the separation has an integrable error, uniformly in initial points. Every point in $B \setminus \{x_p\}$ is $X_r\theta$ for a unique $\theta \in \partial B$ and exit time $r \geq 0$ from the ball under the forward flow. Its volume element is comparable to $e^{-Qr}\, dr\, d\sigma(\theta)$. The comparison follows from (2) and the uniform transversality of b on ∂B.

Set $c_p = \psi(x_p)$ and, in B, define the particular solution

$$v_p(x) = \int_0^\infty [\psi(X_t x) - c_p]\, dt.$$

It satisfies $b \cdot \nabla v_p = \psi - c_p$. Indeed, $v_p(X_\tau x) - v_p(x) = -\int_0^\tau [\psi(X_t x) - c_p]\, dt$. Its C^k norm is uniformly bounded: the zeroth-order integrand decays by (2). The first derivative of X_t decays as e^{-mt}, and the higher variational equations have the same contracting propagator and forcing terms that are products of at least two lower derivatives. Induction gives the same exponential bound up to order k. On compact sets separated from the minimum, w_p has uniformly bounded derivatives through order k. Choose a fixed regular level surface of u_0 between the support of ψ and ∂K, high enough to contain the convex balls on its inner side. The characteristic integral ends there because the forward flow points out of the sublevel set and cannot return. In the region between a smaller convex ball and this surface, $\nabla u_0 \cdot b_p$ is uniformly positive by C^1 closeness and compactness. The forward flow cannot enter the smaller ball from outside, so the travel time is uniformly bounded. The same positivity gives transversality at the surface. Ordinary variational and implicit function bounds give the required derivatives. The same bounds apply to the values of w_p and v_p on ∂B.

Set $F = w_p - v_p$ on $B \setminus \{x_p\}$. Then $b \cdot \nabla F = c_p$. In particular, with the preceding flow coordinates,

$$F(X_r\theta) = F(\theta) - c_p r.$$

For $1 \leq j \leq k$, the forward variational and exit-time equations give

$$|D^j F(X_r\theta)| \leq C_j e^{j\lambda r}, \qquad |F(X_r\theta)| \leq C(1 + r). \tag{3}$$

Indeed, the derivative of the forward flow along a segment of length t in B is at most $Ce^{\lambda t}$ by the integrable deviation of Db from H. In a higher variational equation the forcing products of total order j are bounded by $Ce^{j\lambda t}$. Integrating them against the first-order propagator gives the same bound at the endpoint for $j \geq 2$. Implicit differentiation of the exit condition, whose time derivative has a uniform positive lower bound, gives $C_j e^{j\lambda r}$ for each derivative of the exit time and exit position of order j. The chain rule for the bounded sphere data gives (3).

Choose a real smooth cutoff ζ inside B, equal to one near x_p, by translating a fixed function. Put $q_p = \zeta F$ on $\mathbb{R}^d$ by zero extension and $w_p^{\mathrm{reg}} = w_p - q_p$. The latter has uniformly bounded C^k norm and compact support inside K. We claim

$$\|q_p\|_{W^{1,1}(\mathbb{R}^d)} \leq C, \tag{4}$$

$$\int_{R \leq |\xi| \leq 2R} |\widehat{q}_p(\xi)|^2\, d\xi \leq C R^{-T_p}, \qquad R \geq 1. \tag{5}$$

For (4), integrate (3) in exit coordinates. The integral for the gradient converges uniformly because $Q - \lambda$ is the sum of the other positive eigenvalues and is bounded below when $d \geq 2$. This classical gradient is also the weak gradient across the point x_p. Indeed the first bound of (2) gives $r \lesssim 1+|\log|x-x_p||$, and the integration-by-parts cutoff error tends to zero at rate $O(\epsilon^{d-1}(1+|\log\epsilon|))$ in a ball of radius ϵ. We also have $\|q_p\|_2 \leq C$ from the logarithmic growth bound.

For (5) use finite differences, as in LCH Lemma S2.11. Here we give the uniform estimate with the finite number k of derivatives. Let $E_t = X_t(B)$. Choose fixed c_0 such that $X_{c_0}(B)$ lies in the half-radius ball. Apply X_t to $X_{c_0}(B)$ and ∂B. The separation bound in (2) gives $\operatorname{dist}(E_{t+c_0}, \partial E_t) \gtrsim e^{-\lambda t}$. For a difference step z, choose large t by $|z| = c_1 e^{-\lambda t}$ with c_1 uniform and sufficiently small. Then all shifts by jz, $0 \leq j \leq k$, of E_t lie in E_{t-c_0}, where $\zeta = 1$ for sufficiently large t. Adding $c_p t$ to each shifted value leaves the kth difference unchanged. Bounding its square by a fixed multiple of the sum of the squared shifted values and translating each integral bounds the contribution on E_t by a constant times

$$\int_{E_{t-c_0}} |F(x) + c_p t|^2 \, dx \leq C e^{-Qt} \int_B |F(y) + c_p c_0|^2 \, dy \leq C e^{-Qt}.$$

For an outside point of exit time $r \leq t$, the segment of length at most $k|z|$ stays outside E_{r+c_0} after reducing c_1 if needed. The same separation estimate gives this because the point lies on ∂E_r, and the segment is shorter than the distance to E_{r+c_0}. For bounded r and away from B, the regular derivatives alone suffice. Thus the outer squared difference is bounded by

$$C|z|^{2k} \left(1 + \int_0^t e^{(2k\lambda - Q)r} \, dr\right) \leq C|z|^T.$$

The final constant is uniform since $T \leq d < 2k$ and λ is bounded below. Also $e^{-Qt} \asymp |z|^T$ uniformly. Steps outside this small-$|z|$ range are bounded using the uniform L^2 norm of q_p. Plancherel and averaging the difference bounds over z in a ball of radius R^{-1} prove (5): the average of $|e^{i\xi\cdot z} - 1|^{2k}$ is bounded below on $R \leq |\xi| \leq 2R$ by scaling and compactness of the unit annulus.

3 A uniformly regularized pilot correction

Take two real cutoffs centered at the pilot minimum, with $\zeta_2 = 1$ on a neighborhood of $\operatorname{supp}\zeta$ and ζ_2 compactly supported in B. Let P_R have a real even symbol $\mu(\xi/R)$, where $\mu \in C_c^\infty$ equals one on the unit ball and vanishes outside the ball of radius two. Put

$$w_{p,R} = w_p^{\mathrm{reg}} + \zeta_2 P_R q_p, \qquad z_{p,R} = L_p w_{p,R}.$$

All supports lie in K. The convolution kernel of P_R has uniformly bounded L^1 norm. By (4),

$$\|\nabla w_{p,R}\|_1 \leq C. \tag{6}$$

The second-order product estimate and the upper bounds on $p, \nabla p$ give $\|z_{p,R}\|_2 \leq C\|w_{p,R}\|_{H^2}$. Write $T_* = T_-(\eta)$. The shell sum in (5) yields

$$\|z_{p,R}\|_2 \lesssim \begin{cases} R^{2-T_*/2}, & T_* < 4, \\ (1+\log R)^{1/2}, & T_* = 4, \\ 1, & T_* > 4, \end{cases} \tag{7}$$

uniformly in p. In each dyadic annulus the contribution to the squared H^2 norm is at most $CR_j^{4-T_*}$, with R_j its radius, and only $R_j \lesssim R$ contribute. The low frequencies use $\|q_p\|_2 \leq C$.

Write $b = b_p$ and use $\widetilde{b}$ for its compact extension obtained by a fixed smooth cutoff equal to one near the supports of ζ_2 for all p. Then $\|\widetilde{b}\|_{H^s(\mathbb{R}^d)} \le C$. For the adjoint defect $r_{p,R} = \psi - b \cdot \nabla w_{p,R}$, the exact decomposition on Ω is

$$r_{p,R} = \zeta_2(I - P_R)(\widetilde{b} \cdot \nabla q_p) + \zeta_2[P_R, \widetilde{b} \cdot \nabla]q_p - (b \cdot \nabla\zeta_2)P_R q_p. \tag{8}$$

Here $[P_R, \widetilde{b}\cdot\nabla] = P_R(\widetilde{b}\cdot\nabla) - (\widetilde{b}\cdot\nabla)P_R$. The weak gradient in (4) makes these identities valid across the minimum. The decomposition follows by expanding $b \cdot \nabla(\zeta_2 P_R q_p)$, and using $b \cdot \nabla q_p = \psi - b \cdot \nabla w_p^{\mathrm{reg}}$, supported where $\zeta_2 = 1$. The derivative of the outer cutoff is supported at positive uniform distance from q_p, so the last term is $O(R^{-L})$ in supremum norm for every fixed L by the Schwartz kernel bound and $\|q_p\|_1 \le C$. The exact right hand side

$$\widetilde{b} \cdot \nabla q_p = c_p\zeta + F\, b \cdot \nabla\zeta$$

has bounded H^k norm, because the derivative of ζ is supported away from the minimum. Thus the first term in (8), tested against $h = f - p$, is bounded by CR^{-s-k} using $\|\zeta_2 h\|_{H^s} \le C$ and the frequency support of $I - P_R$ away from zero.

To treat the commutator, split $\widetilde{b} = b_{\mathrm{lo}} + b_{\mathrm{hi}}$ by a smooth frequency cutoff equal to one on $|\omega| \le R/32$ and supported on $|\omega| \le R/16$. The low coefficient piece uses the argument of LCH Lemma S2.12. Its Fourier kernel, aside from transform constants, is

$$i\widehat{b}_{\mathrm{lo}}(\xi - \vartheta) \cdot \vartheta\, [\mu(\xi/R) - \mu(\vartheta/R)].$$

The factor $|\vartheta|\,|\mu(\xi/R) - \mu(\vartheta/R)|$ is at most $C|\xi - \vartheta|$. For a nonzero low-piece kernel, both input and output frequencies lie between fixed multiples of R. This follows from $|\xi - \vartheta| \le R/16$ and the constant regions of μ. Young's convolution inequality, $\||\omega|\widehat{b}_{\mathrm{lo}}(\omega)\|_1 \le C$ by $s > d/2+1$, and (5) give an L^2 bound of $CR^{-T_p/2}$. Its H^{-s} bound is therefore $CR^{-s-T_p/2}$. Pairing against $\zeta_2 h$, extended by zero from the interior, preserves this order.

For the high coefficient piece, Cauchy–Schwarz in Fourier space gives $\|b_{\mathrm{hi}}\|_\infty \le CR^{d/2-s}$. The two terms are bounded individually in L^1 by $C\|b_{\mathrm{hi}}\|_\infty\|\nabla q_p\|_1$. For the first term apply the L^1 convolution bound after multiplication. For the second use $\nabla P_R q_p = P_R \nabla q_p$ and apply the convolution bound before multiplication. Thus each term is $O(R^{d/2-s})$ in L^1. We pair this part against the small supremum norm of h on K. Since $k > T_p/2$, the result is

$$|\langle h, r_{p,R}\rangle| \le C\big(R^{-s-T_*/2} + R^{d/2-s}\|h\|_{L^\infty(K)}\big). \tag{9}$$

This estimate requires only the fixed H^s budget on the pilot field. The high coefficient frequencies need no uniform higher Sobolev bounds.

4 Risk and the fixed-neighborhood exponent

Use the split white-noise prediction pilot of LCH Lemma S4.2 with $\mathcal{G} = \mathcal{F}_\eta$ and the countable smooth dense subset above. The resulting p takes countably many values and

$$\Big(\mathbb{E}_f \delta_p^q\Big)^{1/q} \le C_q N^{-(s+1)/(2s+d+2)}, \qquad \delta_p = \|u_f - u_p\|_2, \quad q \ge 1.$$

With the independent second observation Y_2 at noise level $\sqrt{2/N}$, use

$$\widehat{\Phi} = \langle p, \psi\rangle + Y_2(z_{p,R}) - \langle u_p, z_{p,R}\rangle.$$

Countable selection ensures measurability of this use of the second observation. If a nonrandomized estimator is desired, condition on the original observation to remove the independent randomization used to split the noise. Jensen's inequality preserves the risk.

For $v = u_f - u_p$ and $h = f - p$, LCH Lemma S4.3 gives the exact conditional bias

$$-\langle h, r_{p,R}\rangle + \int_\Omega h\,\nabla v \cdot \nabla w_{p,R}.$$

This follows by testing $L_p v = -\operatorname{div}(h\nabla u_f)$ against $w_{p,R}$. By (6), the last term is bounded by $C\|h\|_{L^\infty(K)}\|\nabla v\|_{L^\infty(K)}$ uniformly in R.

Fix $\sigma > d/2$ independent of s. The local transport stability estimate and interpolation in LCH Lemma S4.4 and (S4.4) give

$$\|h\|_{L^\infty(K)} \le C\,\delta_p^{(s-1)(1-\sigma/s)/(s+1)}, \qquad \|\nabla v\|_{L^\infty(K)} \le C\,\delta_p^{(s-\sigma)/(s+1)}.$$

We used $H^\sigma(K) \hookrightarrow L^\infty(K)$ and $H^{1+\sigma}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$. The pilot moments then yield

$$\mathbb{E}_f\|h\|_{L^\infty(K)} \lesssim N^{-A},$$
$$\mathbb{E}_f\big[\|h\|_{L^\infty(K)}\|\nabla v\|_{L^\infty(K)}\big] \lesssim N^{-A-B}, \qquad A = \frac{(s-1)(1-\sigma/s)}{2s+d+2}, \quad B = \frac{s-\sigma}{2s+d+2}.$$

The moment bounds also apply to exponents below one by Jensen. Besides $s > d/2 + k$ and $s > \sigma$, we require $A + B > 1/2$ and $A > d/(2s+4)$. These hold for all sufficiently large s since $A, B \to 1/2$ as $s \to \infty$. For the sufficient condition $s > 2d + 4$, choose $\sigma = d/2 + 1$ and $k = \max(2, \lfloor d/2 \rfloor + 1)$. Then $s - 1 \ge 7s/8$, $\sigma < s/4$ and $2s + d + 2 < 5s/2$, giving $A > 21/80$, $B > 3/10$ and $d/(2s+4) < 1/4$. Use $R = N^{1/(2s+4)}$. Since $T_* \le d$, $A > d/(2s+4)$ implies

$$R^{d/2-s}N^{-A} \le R^{-s-T_*/2}.$$

The conditional Gaussian fluctuation has expected absolute value bounded by $\sqrt{2/N}\|z_{p,R}\|_2$. Combining (7), (9) and the nonlinear remainder bound now gives (1) in all three cases.

It remains to match the polynomial exponent. First, failure of (C) persists under small perturbation of the smooth reference in this class. If $\psi(x_0) \neq 0$, this follows from continuity of the critical point. Otherwise use the values of $F = w_p - v_p$ on the translated sphere ∂B. At the reference the profile is nonconstant, and when $\psi(x_p) = 0$ these sphere values are exactly the limits of w_p along the respective backward trajectories. The values depend continuously on p for the above uniform local geometry. For w_p, use its finite forward exit integral up to the fixed transverse surface. For v_p, use continuous dependence of the backward flow on each fixed time interval and the uniform exponential tail. Hence two distinct sphere values at the reference persist for sufficiently small η. This proves the persistence, including at nearby pilots where $\psi(x_p) \neq 0$.

The map $p \mapsto T_p$ is continuous in H^s on this ball, by the regularity and continuity of u_p in H^{s+1} and the nondegenerate minimum. If $T_* < 4$, approximate its infimum arbitrarily closely by smooth strict interior points p of the ball. Around each such p there is a smaller full H^s ball contained in $\mathcal{F}_\eta$. Its smooth reference satisfies LCH Assumption 2.1 and fails (C). For the single critical point required by Assumption 2.1, C^1-closeness of u_p (LCH Lemma S4.1(a)) to u_0, whose gradient is bounded below on $\overline{\Omega} \setminus B$ (by compactness in the interior and by the Hopf lemma at $\partial\Omega$), gives a single critical point of u_p on $\overline{\Omega}$, the minimum $x_p \in B$. Theorem 3.3(b) of LCH applies on sufficiently small balls about the new center p, and lower bounds over a small ball hold over every larger ball (LCH, Section 2.1). For $T_p < 4$ this bounds $e(\mathcal{F}_\eta)$ above by $(2s + T_p)/(4s + 8)$. Letting these T_p approach T_* gives the stated exponent upper bound. If $T_* \ge 4$, the parametric lower bound of Theorem 3.3(c) suffices for the exponent. This proves equality of the exponents for fixed η. Finally $T_-(\eta) \to T_{f_0}$ as $\eta \downarrow 0$ by the same continuity, proving the conjectured limit.

Corollary 2. *For each sufficiently small $\eta > 0$, one has $T_-(\eta) < T_{f_0}$. If (C) fails at f_0 and $T_{f_0} \le 4$, the exponent on every such fixed ball is strictly smaller than the local limit in Theorem 1.*

Proof. Let a be a unit eigenvector for the largest eigenvalue of H_{f_0}. Choose $V \in C_c^\infty(\Omega)$ equal to $(a \cdot (x - x_0))^2/2$ near x_0. By the transport inverse in LCH Lemma S2.2 there is a smooth perturbation h in $H^s(\Omega)$ such that

$$\operatorname{div}(h \nabla u_0) = -L_{f_0} V.$$

Indeed, apply the inverse to $-L_{f_0}V$, which belongs to every Sobolev space. For small $t > 0$ put $f_t = f_0 + th$. Substitution into the forward equation gives

$$L_{f_t}(u_{f_t} - u_0 - tV) = -t^2 \operatorname{div}(h \nabla V).$$

The uniform H^{s-1} to H^{s+1} Dirichlet inverse bound from LCH Lemma S4.1 and Sobolev embedding give an $O(t^2)$ error in C^2. Since $\nabla V(x_0) = 0$ and the Hessian stays uniformly positive near x_0, the minimum moves by $O(t^2)$. Consequently $H_{f_t} = H_{f_0} + taa^{\mathsf{T}} + O(t^2)$. Write $Q_0 = \operatorname{tr} H_{f_0}$ and $\lambda_0 = \lambda_{\max}(H_{f_0})$. The trace increases by $t + O(t^2)$. The largest eigenvalue is $\lambda_0 + t + O(t^2)$ by the variational formula: the Rayleigh quotient of $H_{f_0} + taa^{\mathsf{T}}$ is at most $\lambda_0 + t$ and equals it at a. This holds for any unit top eigenvector a, also when λ_0 has multiplicity greater than one. Hence

$$T_{f_t} = T_{f_0} - \frac{Q_0 - \lambda_0}{\lambda_0^2} t + O(t^2).$$

Since $d \ge 2$ and H_{f_0} is positive definite, $Q_0 > \lambda_0$. Choose $t > 0$ small enough that $f_t \in \mathcal{F}_\eta$ and $T_{f_t} < T_{f_0}$. The exponent comparison now follows from Theorem 1. □

For s as in Theorem 1, Corollary 2 extends the strict inequality that LCH Corollary 3.8 proves at the canonical model for $d \in \{2, 3\}$ to every reference satisfying LCH Assumption 2.1 with $T_{f_0} \le 4$ at which (C) fails, including $T_{f_0} = 4$. Since $\mathcal{R}_N(\mathcal{F}_\eta)$ is nondecreasing in η, it holds for every larger fixed ball as well.

Part II. Interior support: the root-sample threshold

LCH, Section 6, third bullet, asks for the minimax rate under interior support constraints or known boundary values, for anisotropic references with $\psi(x_0) \neq 0$. Their Remark 3.9 treats interior perturbations in the linearized model when $\psi(x_0) = 0$, where the parametric rate holds, and isotropic Hessians, where $a_N(s, d)$ persists. This part answers the question for interior support constraints at the level of the root-sample threshold, at every nondegenerate minimum with nonzero score, and so complements Remark 3.9: with known interior support the adjoint equation needs to hold only near the permitted perturbations, so a singular adjoint may be built from a subset of the slow Hessian directions, and the threshold becomes the largest effective dimension across complete slow eigenvalue blocks. The criterion contains Remark 3.9 as its $\psi(x_0) = 0$ branch and recovers its isotropic threshold through $\mathcal{D}(H) = d$.

5 Experiment and result

Keep the domain, source and reference of Section 1.1. Let $K \Subset \Omega$ be compact with $x_0 \in \operatorname{int}(K)$. Fix $\psi \in C_c^\infty(\Omega)$ and target $\Phi(f) = \int f\psi$. The perturbations satisfy

$$\mathcal{H}_s(K, \eta) = \{h \in H^s(\Omega) : \operatorname{supp} h \subset K,\ \|h\|_{H^s(\Omega)} \le \eta\}.$$

The nonlinear white-noise experiment has mean u_f and noise level $N^{-1/2}$, with $f = f_0 + h$ and η small enough to ensure positivity. The linearized experiment has mean $Ah = -L_{f_0}^{-1} \operatorname{div}(h \nabla u_0)$ with the same noise and target $\int h\psi$. Risk is minimax mean absolute error in each experiment.

Let $H = D^2 u_0(x_0)$ have distinct eigenvalues $0 < \lambda_1 < \cdots < \lambda_b$ with multiplicities $m_1, \ldots, m_b$. At each spectral cut set

$$Q_j = \sum_{i \le j} m_i \lambda_i, \qquad \mathcal{D}(H) = \max_{1 \le j \le b} \frac{Q_j}{\lambda_j}.$$

The full effective dimension of LCH is $T_{f_0} = Q_b / \lambda_b$. The quantity $\mathcal{D}(H)$ also tests the lower spectral blocks; only cuts after complete blocks are needed.

Theorem 3. *In the linearized experiment, an L^2 representer for the functional over L^2 perturbations supported in K exists if and only if*

$$\psi(x_0) = 0 \qquad \text{or} \qquad \mathcal{D}(H) > 4. \tag{10}$$

If a representer exists the linearized risk is $O(N^{-1/2})$. If (10) fails, the linearized risk multiplied by $\sqrt{N}$ diverges to infinity. The latter statement holds for every $s \ge 0$ and $\eta > 0$.

The branch $\psi(x_0) = 0$ is LCH Remark 3.9, and for H a multiple of the identity $\mathcal{D}(H) = d$ recovers their isotropic threshold.

Theorem 4. *Assume $\psi(x_0) \neq 0$. If $\mathcal{D}(H) > 4$, the nonlinear risk on $f_0 + \mathcal{H}_s(K, \eta)$ is of order $N^{-1/2}$ for sufficiently large s and small fixed $\eta > 0$. If $\mathcal{D}(H) \le 4$, the nonlinear risk multiplied by $\sqrt{N}$ diverges to infinity on every sufficiently small fixed interior Sobolev ball, for $s > d/2$.*

Thus the case $\psi(x_0) \neq 0$ of the interior-support question of LCH, Section 6, has an exact root-sample criterion, and Theorem 7 below gives an upper bound governed by a retained block below that threshold. The result uses the freedom to choose an adjoint solution on a region containing K. Such a solution can be singular along the subspace of discarded fast directions.

For example, take dimension six, with five eigenvalues equal to 1 and one equal to 10. The full effective dimension is $3/2$ and $\mathcal{D}(H) = 5$. On an ellipsoid $\{x^\top H x < 1\}$ the reference $f_0 = 1$ and constant source $g = \operatorname{tr} H$ give $u_0 = (x^\top H x - 1)/2$. For $\psi(0) \neq 0$, the full-neighborhood rate is slower than root-sample by LCH Theorem 3.3(b). The fixed interior-support model is root-sample by Theorem 4. Locally an admissible singular part of the adjoint is a multiple of the logarithm of the radius in the five slow coordinates.

6 A local Sobolev transport criterion

Let $b = \nabla u_0$. The positive-spectrum normal form of Sternberg and Chen [9, 1], in the form of LCH Lemma S2.14, provides smooth local coordinates at the minimum in which b is conjugate to a triangular polynomial field V with components

$$V_j(y) = \lambda_j y_j + P_j(y_1, \ldots, y_{j-1}), \qquad y_j \in \mathbb{R}^{m_j}.$$

A resonant monomial in the jth block uses only strictly slower variables, since its weighted degree is λ_j and its ordinary degree is at least two. In particular the divergence of V restricted to its first j blocks is the constant Q_j. Denote the restriction by $V^{[j]}$.

Lemma 5. *There is a local H^2 solution of $V \cdot \nabla W = 1$ near the origin if and only if $\mathcal{D}(H) > 4$.*

Proof. For necessity suppose that $Q_j \le 4\lambda_j$ for every j. We first give the reduction for the field on its first j blocks, when

$$V^{[j]} \cdot \nabla U = -aU, \quad a > 0, \quad \text{or} \quad V^{[j]} \cdot \nabla U = 1.$$

The backward flow X_t leaves a sufficiently small ball invariant and contracts toward zero. Its volume Jacobian is $e^{-Q_j t}$. The fastest block has the exact affine formula

$$(X_t y)_j = e^{-\lambda_j t} y_j + G_t(y_1, \dots, y_{j-1}),$$

and the other blocks do not depend on y_j. For H^2 solutions the equation integrates along this flow: composing an H^1 function with the smooth flow on each finite time interval gives the weak chain rule, by smooth approximation locally, and integration in time then gives the identity in L^2 on a smaller ball. In the homogeneous equation, $U(X_t y) = e^{at} U(y)$; in the constant-source equation, $U(X_t y) = U(y) - t$. Taking two derivatives in any pair of fastest variables gives

$$(\partial_{y_j^\mu} \partial_{y_j^\nu} U)(X_t y) = e^{(2\lambda_j + a)t}\, \partial_{y_j^\mu} \partial_{y_j^\nu} U(y)$$

in the first case, and the same formula with $a = 0$ in the second. The fast dependence of X_t is affine, so the identity follows directly from the Sobolev chain rule.

For a fixed $t_0 > 0$, the annular sets $X_{\ell t_0}(B \setminus X_{t_0} B)$, $\ell \ge 0$, partition a sufficiently small starting ball B up to the point 0 and boundaries of measure zero. If one fastest second derivative had positive squared integral on the first set, its integral on subsequent sets would be multiplied by $e^{(4\lambda_j + 2a - Q_j)\ell t_0}$, with $a = 0$ for constant source. The sum diverges. Thus every fastest second derivative vanishes in B.

On a smaller product box the solution is affine in y_j,

$$U(y) = A(y_{<j}) \cdot y_j + B_0(y_{<j}),$$

with A, B_0 locally H^2. The affine assertion follows by slicing, and the coefficients inherit H^2 by integration against fixed smooth test functions of the fast variables extracting the constant and linear coefficients. Matching the linear coefficient in the equation yields

$$V^{[j-1]} \cdot \nabla A = -(a + \lambda_j) A$$

for the homogeneous equation, and the same formula with $a = 0$ for constant source. By induction on j, every H^2 solution to the homogeneous equation with $a > 0$ vanishes. Indeed the affine slope vanishes by the induction hypothesis and then B_0 satisfies the reduced homogeneous equation. In zero dimensions the equation is $0 = -aU$. In the constant-source equation the slope therefore vanishes at each stage, so one reduces to a constant-source equation on the next smaller block. In zero dimensions this is impossible. This proves necessity, including equality.

For sufficiency choose j with $Q_j > 4\lambda_j$. A local solution is the negative of the forward exit time from a small sphere for $V^{[j]}$, viewed as a function of only the first j blocks. Indeed the forward flow decreases the exit time at unit speed. Its H^2 property follows from Lemma 6 below. Dependence on the other variables is constant on a product neighborhood. □

Lemma 6. *Let B_s be a C^3 field near zero, with $B_s(0) = 0$ and symmetric positive derivative H_s at zero. Put $Q_s = \operatorname{tr} H_s$ and $\lambda_s = \lambda_{\max}(H_s)$. If $Q_s > 4\lambda_s$, the negative forward exit time ω from a sufficiently small sphere is in H^2 inside that sphere and satisfies $B_s \cdot \nabla \omega = 1$. The bounds are uniform under common C^3 and positive spectral bounds and a common positive gap $Q_s - 4\lambda_s$.*

Proof. On a small uniform ball the symmetric part of DB_s is uniformly positive. Its backward flow contracts. The time integral of the deviation of DB_s from H_s along a trajectory in the ball is uniformly bounded, by exponential contraction and the Lipschitz bound. In exit coordinates (t, θ), with θ on the sphere, the volume element is bounded by $Ce^{-Q_s t}\, dt\, d\sigma(\theta)$ by the Jacobian formula and transversality. The forward variational equations bound derivatives of orders one and two by, respectively, $Ce^{\lambda_s t}$ and $Ce^{2\lambda_s t}$, as in (3). Implicit differentiation at the uniformly transverse exit gives the same bounds for the exit time. Its value is t. The resulting squares are integrable by $Q_s > 4\lambda_s$. To identify the weak derivatives at zero, smoothly flatten the exit-time variable above level L_0 over a transition interval of length one, with bounded first and second derivatives of the flattening. These functions are constant near zero and converge in H^2 by the integral bounds and dominated convergence. The transport equation passes to the limit in distributions. □

7 A uniform slow-coordinate factor

The preceding exact normal form suffices for a fixed smooth reference. A nonlinear upper bound also needs a uniformly bounded representer at a data-dependent pilot. Direct use of a smooth normal form could cause small denominators when nearby Hessians cross resonances. We construct a factor using only the separation of the retained and discarded eigenvalues. It is uniformly C^k for any fixed $k \geq 2$ when s is sufficiently large.

Fix a spectral cut of H. At nearby Hessians retain the same number of smallest eigenvalues. If the cut is proper, there is a uniform positive gap between the retained and discarded blocks. By the uniform regularity in LCH Lemma S4.1, for sufficiently large fixed s, all nearby fields $b_p = \nabla u_p$ have a uniform C^{M+2} bound for any fixed integer M used below, and $\|u_p - u_0\|_{C^2} = O(\eta)$. Hence the retained eigenvalues at x_p are within $O(\eta)$ of those at x_0, by continuity of eigenvalues. In particular, if the cut satisfies $Q_j > 4\lambda_j$ at the reference, then for η small every pilot's retained block satisfies

$$Q_s(p) - 4\lambda_s(p) \;\geq\; \tfrac{1}{2}\,(Q_j - 4\lambda_j) > 0, \tag{11}$$

which is the common gap required in Lemma 6. We always center coordinates at the minimum and diagonalize the Hessian orthogonally. Bounds are uniform under any choice of orthonormal bases for spectral blocks.

Fix $m > 0$ smaller than one half of a uniform lower spectral bound, and $\Lambda > 0$ larger than a uniform upper spectral bound. Fix the desired regularity $k \geq 2$ and choose $M \geq k + 1$ so $(M + 1)m > (k + 1)\Lambda$. For each p, make polynomial coordinate changes of orders 2 through M in the retained output components. At order r these remove every degree-r monomial in a retained component that contains at least one discarded input. For a monomial $y^\alpha e_i$ in a coordinate change, its linear effect on that coefficient is multiplication by $\alpha \cdot \lambda - \lambda_i$, as follows by differentiating the change along the linear vector field and subtracting the linear output. The denominator is bounded below by the positive spectral gap, since one input is discarded, the output is retained, and all eigenvalues are positive. Solve at each degree and leave earlier degrees untouched. Taylor's theorem and the common finite differentiability bound give the transformed field b with retained part of the form

$$b_s(y) = B_s(y_s) + R(y), \qquad |D^\alpha R(y)| \leq C|y|^{M+1-|\alpha|} \quad (|\alpha| \leq k). \tag{12}$$

Here $y = (y_s, y_f)$ are retained and discarded coordinates, and B_s is the restriction of the retained components to $y_f = 0$. The changes and their inverses have uniform bounds on a fixed small chart. At zero the derivative is still the diagonal Hessian. If no block is discarded, one may take the identity factor and $B_s = b_p$ in centered rotated coordinates.

In the proper-cut case shrink the chart so that the symmetric part of Db is bounded below by m and the backward flow X_t stays in the chart on a common small ball. It satisfies $|X_t y| \le e^{-mt}|y|$ and $\|D^j X_t\| \le C_j e^{-mt}$ for $1 \le j \le k$ by induction in the backward variational equations. Extend B_s to all retained coordinates by cutting off only its nonlinear part in a small ball. This preserves its derivative at zero and yields a forward flow φ_t^s with $\|DB_s\| \le \Lambda$ and bounded derivatives through order $k+1$. Shrinking the support permits the bound by Λ because the nonlinear part vanishes to second order and its derivative to first order. The forward variational equations give $\|D^j \varphi_t^s\| \le C_j e^{j\Lambda t}$ for $1 \le j \le k+1$.

Let $\pi_s y = y_s$ and set $P_t(y) = \varphi_t^s(\pi_s X_t y)$. The exact time derivative is

$$\partial_t P_t(y) = -D\varphi_t^s(\pi_s X_t y)\, R(X_t y).$$

Indeed the two terms involving B_s cancel by the flow identity $B_s(\varphi_t^s(z)) = D\varphi_t^s(z) B_s(z)$. By (12), the chain rule and backward contraction, derivatives of $R(X_t y)$ through order $j \le k$ are bounded by $C|y|^{M+1-j} e^{-(M+1)mt}$. Multiplying by the forward derivative bounds shows that $\partial_t P_t$ is integrable in C^k. Let $\pi = \lim_{t\to\infty} P_t$. It is uniformly C^k, with $\pi(y) = y_s + O(|y|^{M+1})$ and $D\pi(y) = D\pi_s + O(|y|^M)$. In particular $y \mapsto (\pi(y), y_f)$ is a uniformly invertible C^k change of coordinates on a smaller ball. For fixed $\tau \ge 0$ the semigroup identity in the defining limit gives

$$\pi(X_\tau y) = \varphi_{-\tau}^s(\pi(y)).$$

It follows that $D\pi\, b = B_s \circ \pi$. Localizing to a smaller neighborhood, only the unextended retained field is used. This construction gives the required factor uniformly with no nonresonance hypothesis among the retained eigenvalues.

8 Adjoint representers and risk

First work with the reference alone. An L^2 function z is a representer for supported perturbations if and only if $w = L_{f_0}^{-1} z \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies $b \cdot \nabla w = \psi$ almost everywhere on K. This follows by integration by parts against h, and by the Dirichlet elliptic H^2 estimate for the inverse. In particular the equation is necessary on a neighborhood of x_0. The smooth particular solution with right hand side $\psi - \psi(x_0)$ is the integral of $\psi(X_t x) - \psi(x_0)$ for $0 \le t < \infty$, with X_t the backward gradient flow in the original coordinates. After subtracting it and dividing by $\psi(x_0)$ when this value is nonzero, a representer would produce a local H^2 solution with constant source. Smooth conjugacy preserves local H^2, so Lemma 5 proves the necessity of (10).

We construct the sufficient representers simultaneously for the reference and the nonlinear pilot family. Choose a fixed regular compact sublevel of u_0 with K in its interior, contained in Ω. The proof of LCH Lemma S4.1 shows that for small enough η it is invariant under the backward flows of b_p and every point enters a common small neighborhood of the minimum in uniformly bounded time. Indeed outside that neighborhood the gradients are uniformly bounded below and u_p decreases along each backward trajectory. The strictly outward gradient on the sublevel boundary prevents backward escape. By further contracting within the local ball, choose one fixed time t_1 so that this backward flow sends a neighborhood U of K into a ball where the factor of Section 7 is defined and where its output is in the exit-time domain of Lemma 6. Compose the factor with the local polynomial chart and with this fixed-time flow. Write the result as $\Pi_p(x)$. It satisfies

$$D\Pi_p(x) b_p(x) = B_s(\Pi_p(x)), \qquad x \in U,$$

by the factor relation and the fact that a fixed-time flow pushes its vector field to itself.

Let v_p be the preceding backward integral with $\psi(x_p)$ subtracted. On U it has bounded C^2 norm, uniformly in p. Indeed split off the fixed time until entry to the convex ball. Beyond that time the backward first and second variational derivatives decay exponentially, so the differentiated integrals converge uniformly. It satisfies $b_p \cdot \nabla v_p = \psi - \psi(x_p)$. If the spectral cut inequality holds, use ω_p from Lemma 6 and take

$$w_p = \chi\{v_p + \psi(x_p)\,\omega_p \circ \Pi_p\},$$

where χ is a fixed smooth function supported in U and equal to one near K. By the uniform gap (11), the bounds of Lemma 6 hold uniformly over the pilots. The second term is therefore uniformly H^2 on U, because the factor can be completed by the discarded coordinates to a uniformly controlled C^2 diffeomorphism, before composition with the fixed-time flow. In those coordinates, the time function is independent of the discarded variables. Product integration on a bounded coordinate box gives the H^2 bound, and composition with the diffeomorphisms preserves it. The weak chain rule and the transport equations show that $b_p \cdot \nabla w_p = \psi$ on K. If $\psi(x_0) = 0$ for the linearized problem, simply take $w_0 = \chi v_0$. In every sufficient case $\|w_p\|_{H^2}$ and hence $\|\nabla w_p\|_{L^1(K)}$ and $\|L_p w_p\|_2$ are bounded uniformly in p. The exact linear estimator against this representer gives the stated linearized upper bound.

For the nonlinear parametric upper bound apply the split white-noise pilot of LCH Lemma S4.2 to the supported class. This only uses a countable prediction-dense subset of the class, so no approximation by smooth functions supported in K is needed. By that lemma, $\delta_p = \|u_f - u_p\|_2$ has all fixed positive moments of the order of $N^{-(s+1)/(2s+d+2)}$ to the corresponding powers. Take the second-sample one-step estimate using $z_p = L_p w_p$. The exact testing identity of LCH Lemma S4.3 gives conditional bias

$$\int_\Omega (f - p)\,\nabla(u_f - u_p) \cdot \nabla w_p$$

because $f - p$ is supported in K. Conditional standard deviation is $\sqrt{2/N}\|z_p\|_2$. For a fixed $\sigma > d/2$ use LCH Lemma S4.4 and (S4.4) on the larger compact sublevel, Sobolev embedding, and the uniform $L^1(K)$ bound on ∇w_p. The conditional bias in absolute value is at most

$$C\|f - p\|_{L^\infty(K)}\|\nabla(u_f - u_p)\|_{L^\infty(K)} \le C\delta_p^{[(s-1)(1-\sigma/s)+(s-\sigma)]/(s+1)}.$$

The expected value of this bound is $o(N^{-1/2})$ for sufficiently large s, since its exponent after including the pilot rate tends to one. This gives the claimed uniform parametric risk upper bound. The independent randomization of the split can be removed by conditioning on the original observation and using Jensen's inequality.

Finally suppose (10) fails. The ratio

$$\frac{|\int h\psi|}{\|Ah\|_2}, \qquad h \in C_c^\infty(\operatorname{int} K), \quad h \neq 0,$$

is unbounded. Otherwise the functional $Ah \mapsto \int h\psi$ would extend boundedly to the L^2 closure of that range and have an L^2 representer. Its weak transport equation on $\operatorname{int} K$ would contradict Lemma 5 near x_0. The denominators are nonzero: if $\operatorname{div}(hb) = 0$, then $h(X_t x)\det DX_t(x) = h(x)$, and the left-hand side tends to zero by boundedness of h and eventual exponential contraction. Hence $h(x) = 0$. For any fixed smooth direction with a prescribed large ratio, compare zero to the perturbation $t_N h$, where $t_N = c/(\sqrt{N}\|Ah\|_2)$ and $c > 0$ is a small absolute constant. It belongs to any fixed positive radius ball for all sufficiently large N. The linear Gaussian Kullback–Leibler divergence is $c^2/2$. In the nonlinear experiment it tends to the same value by differentiability of u_{f_0+th} at zero with derivative Ah, which follows by subtracting the elliptic equations and taking

the limit in the variational inverse. The two-point testing inequality therefore lower bounds $\sqrt{N}$ times the minimax risk by a positive constant times the chosen ratio, for all sufficiently large N depending on the direction. Letting the ratio be arbitrarily large proves the divergence in both experiments. When $\psi(x_0) \neq 0$ and the spectral criterion is satisfied, the same two-point argument with any one smooth supported direction of nonzero functional gives the ordinary parametric lower bound. This proves Theorems 3 and 4.

9 An upper bound below the root-sample threshold

The slow factor also improves the nonparametric upper bound. Choose any fixed spectral cut retaining at least two directions, possibly the full set, and let r be the number retained. In the nonlinear interior model write

$$T_* = \inf_{p \in f_0 + \mathcal{H}_s(K,\eta)} \frac{\sum_{i=1}^{r} \lambda_i(p)}{\lambda_r(p)},$$

where the eigenvalues at the pilot minimum are ordered with multiplicity. By taking η small, a proper reference cut stays separated from the discarded block.

Theorem 7. *For sufficiently large s and small fixed η, the nonlinear interior minimax absolute risk has the upper bounds*

$$\begin{cases} O(N^{-(2s+T_*)/(4s+8)}), & T_* < 4, \\ O(\sqrt{\log N/N}), & T_* = 4, \\ O(N^{-1/2}), & T_* > 4. \end{cases}$$

In particular one may choose a cut attaining $\mathcal{D}(H)$ at the reference. Its T_* tends to $\mathcal{D}(H)$ as $\eta \downarrow 0$.

Proof. Take an integer $k > d/2$, $k \geq 2$, and use the uniformly C^k factor of Section 7. Its construction only needs the spectral gap across a proper cut, so it applies also when the retained trace is at most four times its largest eigenvalue. Use the negative slow exit time ω_p on a common small ball. The proof of Lemma 6, with the higher variational equations through order k, gives at exit time t

$$|\omega_p| = t, \qquad |D^j \omega_p| \leq C_j e^{j\lambda_s t} \quad (1 \leq j \leq k),$$

and the volume element is bounded by $Ce^{-Q_s t}\, dt\, d\sigma$. In particular ω_p lies in L^2 and $W^{1,1}$ uniformly, since $Q_s - \lambda_s$ is bounded below when at least two eigenvalues are retained. Smooth flattening at large exit time verifies the weak gradient.

Use the fixed-time extension of Π_p to U containing K. Let χ_1 and χ_2 be smooth and compactly supported in U, with $\chi_1 = 1$ near K and $\chi_2 = 1$ on a neighborhood of $\operatorname{supp} \chi_1$. Put $q = \chi_1\, \omega_p \circ \Pi_p$ and extend by zero to $\mathbb{R}^d$. Then q is uniformly in $L^2 \cap W^{1,1}$ by the controlled factor coordinates, and

$$\int_{R \leq |\xi| \leq 2R} |\widehat{q}(\xi)|^2\, d\xi \leq CR^{-Q_s/\lambda_s}. \tag{13}$$

This is the finite-difference argument proving (5) in Section 2, applied to the slow backward flow in the retained coordinates with (Q, λ) replaced by (Q_s, λ_s). The only additions are these. Uniform Lipschitz control of Π_p keeps the slow coordinates of all shifted points $x + \ell z$, $0 \leq \ell \leq k$, with $\Pi_p(x) \in E_t$ inside E_{t-c_0}. On the inner set one changes variables to the slow and complementary fast coordinates at each shifted point; the fast cross-sections are bounded, so the inner integral is again $Ce^{-Q_s t}$, and the term from writing $q = \chi_1(\omega_p \circ \Pi_p + t) - t\chi_1$ contributes at most $Ct^2|z|^{2k}e^{-Q_s t}$. The outer integral is bounded by $C|z|^{Q_s/\lambda_s}$ since $2k > d \geq Q_s/\lambda_s$.

For a smooth Fourier cutoff P_R as in LCH Lemma S2.12, use the representer potential

$$w_{p,R} = \chi_1 v_p + \psi(x_p)\chi_2 P_R q.$$

It has uniformly bounded gradient in L^1 since $q \in W^{1,1}$. The shell bound controls $\|L_p w_{p,R}\|_2$ by $CR^{2-T_*/2}$ if $T_* < 4$, $C(1+\log R)^{1/2}$ if $T_* = 4$, and a constant if $T_* > 4$, by summation of dyadic H^2 shells. On the support of $h = f - p$ the transport derivative of the potential is $\psi - \psi(x_p)$ from its smooth part, so the defect equals $\psi(x_p)$ times

$$(I - P_R)(b_p \cdot \nabla q) + [P_R, b_p \cdot \nabla]q,$$

with b_p extended by a smooth cutoff beyond the support of χ_2 and all expressions restricted to K. The first term uses $b_p \cdot \nabla q = \chi_1 + (b_p \cdot \nabla \chi_1)(\omega_p \circ \Pi_p)$. The χ_1 contribution is rapidly decreasing. The second summand is L^1 and supported a positive distance from K, so its convolution is also rapidly decreasing on K by the Schwartz kernel estimate. The commutator is treated by the low/high coefficient split of Section 3, with (13) in place of (5); it gives the bound (9) with T_* as defined here. The pilot argument of Section 4 bounds the supremum norm in expectation by CN^{-A} and the nonlinear remainder by $o(N^{-1/2})$ through the uniform gradient L^1 bound. With $A > d/(2s+4)$ and $R = N^{1/(2s+4)}$ the three cases follow. Continuity of the minimum-shifted Hessian yields the final limit for T_*. □

Part III. A power-law degenerate minimum

LCH, Section 6, second bullet, lists degenerate critical points among the further problems for their nondegenerate theory. This part answers it for one radial model in that degenerate class: the even power-law minimum on the unit ball, sharply in the linearized experiment and, for sufficiently smooth radial conductivities, in the nonlinear experiment. The exponential flow at a nondegenerate minimum is replaced by an explicit power-law flow, the transition takes place at dimension $2p$, and the critical logarithm is forced by perturbations on disjoint annuli. Setting $p = 2$ in the rates recovers the isotropic rate $a_N(s,d)$ and critical dimension four of LCH.

10 Experiment and results

Let $\Omega = B_1(0) \subset \mathbb{R}^d$ with $d \geq 2$. Fix an even integer $p \geq 4$ and put

$$f_0 = 1, \qquad u_0(x) = \frac{|x|^p - 1}{p}, \qquad b(x) = \nabla u_0(x) = |x|^{p-2}x, \qquad g(x) = (d+p-2)|x|^{p-2}.$$

The conductivity equation is $\operatorname{div}(f\nabla u_f) = g$, with $u_f = 0$ on the boundary. The reference has a unique minimum at the center with zero Hessian and radial gradient of order $p-1$, and the source vanishes at the center to order $p-2$, exactly as the power law requires. This model therefore lies in the degenerate class of the second bullet rather than under LCH Assumption 2.1.

Fix $\psi \in C_c^\infty(\Omega)$ with $\psi(0) \neq 0$. In the linearized experiment the observation is Gaussian white noise in $L^2(\Omega)$ with mean

$$Ah = -\Delta_D^{-1} \operatorname{div}(hb)$$

and noise level $N^{-1/2}$. Here Δ_D^{-1} is the Dirichlet inverse of the Laplacian. The target is $\int_\Omega h\psi$, for $\|h\|_{H^s(\Omega)} \leq \eta$. The forward operator is bounded from L^2 to L^2 since b is bounded and the

Dirichlet energy estimate applies. In the nonlinear experiment the mean is u_f and the target is $\Phi(f) = \int_\Omega f\psi$. All risks below are minimax mean absolute errors. Write

$$\mathfrak{r}_N(s,d,p) = \begin{cases} N^{-(2s+d)/(4s+4p)}, & d < 2p, \\ \sqrt{\log N/N}, & d = 2p, \\ N^{-1/2}, & d > 2p. \end{cases}$$

Theorem 8. *For every $s \geq 0$ and fixed $\eta > 0$ the linearized risk is of order $\mathfrak{r}_N(s,d,p)$. The lower bounds hold even when h is radial and supported in any fixed ball about zero contained in Ω.*

Theorem 9. *For sufficiently large s and small fixed $\eta > 0$, the nonlinear risk over radial positive f with $\|f-1\|_{H^s} \leq \eta$ is of order $\mathfrak{r}_N(s,d,p)$. The nonlinear lower bounds hold also over the unrestricted ball of positive conductivities.*

The adjoint range condition for efficient Darcy inference is studied by Nickl and Paternain [4]. Koers, Szabó and van der Vaart [2] use linear inverse methods in nonlinear PDE recovery, including the Darcy problem. Here the reciprocal conductivity makes the radial observation map exactly linear.

11 An adjoint cutoff with small dual bias

Lemma 10. *Fix $s \geq 0$. Let β have bounded C^2 norm and support in a fixed compact subset of the unit ball. For all sufficiently small $a > 0$ there is a potential $w_a \in H^2 \cap H_0^1(\Omega)$ such that*

$$\left| \int h(\beta - b \cdot \nabla w_a) \right| \leq C a^{s+d/2} \|h\|_{H^s(\Omega)}, \tag{14}$$

$$\|\Delta w_a\|_2 \leq C \begin{cases} a^{d/2-p}, & d < 2p, \\ (1 + \log(1/a))^{1/2}, & d = 2p, \\ 1, & d > 2p. \end{cases} \tag{15}$$

The constants are uniform in β with the indicated support and norm bounds.

Proof. Write $\nu = p - 2 > 0$. In polar coordinates, the punctured-domain adjoint solution is

$$w(r\vartheta) = -\int_r^1 t^{-\nu-1} \beta(t\vartheta)\, dt.$$

It vanishes near the outer boundary. Its derivatives through order two satisfy $|D^j w(x)| \leq C|x|^{-\nu-j}$ for $0 \leq j \leq 2$. The radial derivatives follow by differentiation of the lower endpoint, and angular derivatives cost at most $|x|^{-1}$ per Euclidean derivative, with uniformly bounded angular derivatives of the integrand through order two. Multiplying w by a fixed radial cutoff dilated to equal zero for $r < a$ and one for $r > 2a$ gives a preliminary potential $\widetilde{w}_a$. Its defect $\beta - b \cdot \nabla \widetilde{w}_a$ is supported in $|x| \leq 2a$ and uniformly bounded, by the derivative and size bounds above. Integrating the bound on $D^2 w$ outside $2a$ and its cutoff analogue on $a \leq r \leq 2a$ gives precisely (15) for $\widetilde{w}_a$.

Choose an integer L with $L + 1 > s - d/2$. We correct every moment of the defect of order at most L. For each multi-index α in that range,

$$\int y^\alpha |y|^\nu y \cdot \nabla V(y)\, dy = -(d + \nu + |\alpha|) \int |y|^\nu y^\alpha V(y)\, dy.$$

For the moment map take the fixed test functions $V_\gamma(y) = \chi(y)|y|^\nu y^\gamma$, where $\chi \geq 0$ is smooth, supported in a fixed annulus inside $|y| < 2$ and positive on an open subset. The matrix of moments

of $|y|^\nu y \cdot \nabla V_\gamma$ differs from an invertible polynomial Gram matrix by multiplication by an invertible diagonal matrix. Now

$$b(x) \cdot \nabla \left(a^{-\nu} V_\gamma(x/a) \right) = |x/a|^\nu (x/a) \cdot \nabla V_\gamma(x/a).$$

Hence adding an appropriate bounded linear combination of these scaled potentials to $\widetilde{w}_a$ cancels the desired moments. The coefficients are uniformly bounded because the preliminary defect in y coordinates is uniformly bounded and compactly supported. The added Laplacian has L^2 norm at most $Ca^{d/2-p}$, preserving (15).

The corrected defect has the form $\rho_a(x/a)$ with ρ_a supported in the fixed ball of radius two, uniformly bounded and with all moments through degree L zero. Therefore $|\widehat{\rho_a}(\xi)| \le C|\xi|^{L+1}$ for $|\xi| \le 1$, and Plancherel bounds its L^2 norm at high frequencies. The homogeneous H^{-s} norm is uniformly bounded because $2(L+1) - 2s + d > 0$. Scaling gives a bound $Ca^{s+d/2}$ for the defect on $\mathbb{R}^d$. Testing against any bounded Sobolev extension of h gives (14). □

In the linearized model, $\langle Ah, \Delta w_a \rangle = \int h\, b \cdot \nabla w_a$ by the weak Dirichlet equation. Use the linear observation against Δw_a with $\beta = \psi$ and $a = N^{-1/(2s+2p)}$. The bias (14) and noise (15) give the upper bound of Theorem 8 in all three regimes, for every $s \ge 0$, since $\beta = \psi$ is smooth.

12 Lower bounds on one scale and at the critical dimension

For a smooth radial template ζ supported in $\{1/2 < |x| < 1\}$, put

$$V(\rho) = -\int_\rho^\infty t^{p-1} \zeta(t)\, dt, \qquad \rho \ge 0,$$

where $\zeta(t)$ is its radial profile. The function V is smooth and radial, supported in the unit ball, and possibly constant on the inner ball. For $h(x) = A_0 \zeta(x/a)$ with $0 < a < 1$, the linearized mean Ah equals $-A_0 a^p V(x/a)$, by its radial derivative and the boundary condition. In particular,

$$\|Ah\|_2 = |A_0| a^{p+d/2} \|V\|_2, \qquad \|h\|_{H^s(\Omega)} \le C|A_0| a^{d/2-s}. \tag{16}$$

A template with nonzero spatial integral satisfies $|\int h\psi| \gtrsim |A_0| a^d$ as $a \downarrow 0$ because $\psi(0) \ne 0$.

If $d < 2p$, take $A_0 = \gamma a^{s-d/2}$ and $a = N^{-1/(2s+2p)}$, with $\gamma > 0$ fixed and sufficiently small. Then the Sobolev bound holds, the Gaussian Kullback–Leibler divergence from zero is bounded by a small constant, and the functional gap is of order $N^{-(2s+d)/(4s+4p)}$. For $d > 2p$ one fixed smooth radial supported direction of nonzero functional, at amplitude $\gamma/\sqrt{N}$, gives the ordinary parametric lower bound.

For $d = 2p$ choose the template so

$$\int_0^\infty t^{p-1} \zeta(t)\, dt = 0, \qquad \psi(0) \int_{\mathbb{R}^d} \zeta(x)\, dx > 0.$$

The powers $p-1$ and $d-1$ differ, so a linear combination of two disjoint radial bumps with opposite coefficients satisfies the first condition and gives the second after fixing the sign. Now V also vanishes on the inner ball. Let $a_j = a_0 4^{-j}$ with $a_0 > 0$ small and

$$H_j(x) = a_j^{-p-d/2} \zeta(x/a_j).$$

The means AH_j have disjoint annular supports and the same positive L^2 norm. The functions H_j satisfy $\int H_j \psi \ge c > 0$ by the template condition and $d = 2p$, uniformly in j after choosing a_0 small.

Also $\int |H_j||\psi| \le C$ and $\|H_j\|_{H^s} \le C a_j^{-s-p}$. Set

$$J_N = \left\lfloor \frac{\log N}{4(s+p)\log 4} \right\rfloor, \qquad h_N = \frac{\gamma}{\sqrt{N J_N}} \sum_{j=1}^{J_N} H_j.$$

For large N, disjointness of the observation means gives $N\|Ah_N\|_2^2 \le C\gamma^2$. The Sobolev triangle inequality and geometric summation give $\|h_N\|_{H^s} \le C\gamma N^{-1/4}/\sqrt{J_N}$. The functional gap is at least $c\gamma\sqrt{J_N/N}$. Choosing γ small and applying the two-point Gaussian testing bound proves the critical lower bound. All these perturbations can be supported in any fixed interior ball about zero by choosing fixed templates or the largest scale accordingly.

13 The nonlinear radial experiment

For a positive radial conductivity put $\theta = 1/f$. The radial solution satisfies

$$\frac{d}{dr} u_f(r) = r^{p-1}\theta(r), \qquad u_f(1) = 0.$$

Indeed the flux $r^{d-1} f(r) u_f'(r)$ integrates the radial source to r^{d+p-2}, with zero center flux in the regular weak solution. Thus the Gaussian observation is exactly linear in θ. The target is the smooth nonlinear functional $G(\theta) = \int \psi/\theta$. The reciprocal map is bounded on the uniformly positive Sobolev class for $s > d/2$ and small fixed radius, by the Sobolev algebra and smooth composition bounds. It maps sufficiently small radial Sobolev balls at one into comparable balls. In particular the reciprocal class has uniformly bounded H^s norm, is uniformly positive, and its forward mean has bounded H^{s+1} norm. For the last assertion use $\nabla u_f = b\theta$ with smooth b and the Dirichlet trace.

Use two independent observations from Gaussian noise splitting. On the first, take an approximate minimum-distance prediction pilot q in the reciprocal class, by the construction of LCH Lemma S4.2, which only requires the uniform H^{s+1} bound on forward means. A bounded compactly supported Sobolev extension and a Fourier cutoff at $J \asymp N^{1/(2s+d+2)}$ give an L^2 prediction estimate with root mean square error $O(N^{-(s+1)/(2s+d+2)})$. The cutoff bias is $O(J^{-s-1})$ and the smoothed white noise has squared L^2 expectation $O(J^d/N)$ by the kernel's Hilbert–Schmidt norm. Its fixed moments have the same orders to their corresponding powers by Gaussian moment bounds. Choose a measurable approximate nearest forward mean in a fixed countable dense collection from the reciprocal class. The triangle inequality preserves all fixed moment bounds. Writing $v = u_f - u_{1/q}$ and $\delta = \|v\|_2$, they give

$$\left(\mathbb{E}_f \delta^\ell\right)^{1/\ell} \le C_\ell N^{-(s+1)/(2s+d+2)} \qquad (\ell > 0). \tag{17}$$

We also need an L^2 bound for the parameter error $h = \theta - q$. If $k = \lfloor s \rfloor \ge 1$ and a is small, then

$$\|h\|_2 \le C\big(a^k + a^{-(p-1)}\|\nabla v\|_2\big). \tag{18}$$

To see this, outside the ball of radius a, use $\nabla v = bh$ to control h by its weighted norm. Inside B_a, approximate h on B_{2a} by a polynomial of degree at most $k-1$, with L^2 error at most $Ca^k\|h\|_{H^k(B_{2a})}$, by the polynomial Poincaré inequality on the unit ball and a rescaling. The polynomial's L^2 norm on B_a is bounded by a fixed multiple of its norm on $B_{2a} \setminus B_a$ by equivalence of norms on the fixed polynomial space. This gives the same weighted bound inside the ball up to the approximation error. Since $\|h\|_{H^k}$ is uniformly bounded, (18) follows.

Interpolation from the uniform H^{s+1} bound gives $\|\nabla v\|_2 \le C\delta^{s/(s+1)}$. Optimizing a in (18) gives

$$\|h\|_2 \le C\delta^{sk/((s+1)(k+p-1))}.$$

This holds for small δ and extends to the remaining values by the uniform bound. Together with (17) it implies $\mathbb{E}_f\|h\|_2^2 = o(N^{-1/2})$ for sufficiently large s, since the exponent of N^{-1} in the bound is

$$\frac{2sk}{(k+p-1)(2s+d+2)},$$

which tends to one.

The exact expansion of G is

$$G(\theta) - G(q) = \int h\beta_q + O(\|h\|_2^2), \qquad \beta_q = -q^{-2}\psi,$$

uniformly on the reciprocal class. For $s > d/2 + 2$, Sobolev embedding gives the functions β_q uniform C^2 bounds and common compact support; this is the hypothesis of Lemma 10, and it is contained in the large-s hypothesis of Theorem 9. Use Lemma 10 with $\beta = \beta_q$ and $a = N^{-1/(2s+2p)}$. The L^2 representer for the cutoff term in the reciprocal experiment is $-\Delta w_a$ because $\nabla v = bh$ and both potentials have zero boundary trace. Starting at $G(q)$, add the second-sample residual observation against this representer. Countable pilot selection ensures measurability. Its conditional bias is bounded by $Ca^{s+d/2} + C\|h\|_2^2$, and its conditional noise has standard deviation at most $\sqrt{2/N}\|\Delta w_a\|_2$. These bounds give the asserted upper rate after taking expectation. Conditional expectation given the unsplit observation removes auxiliary randomization without increasing risk.

For the nonlinear lower bounds put $\theta = 1 + h$ for the same radial perturbations h as in the linearized proof, and $f = 1/(1 + h)$. For fixed sufficiently large s and sufficiently small γ, these conductivities belong to the desired Sobolev ball by the composition bound. The observed mean difference is exactly $-Ah$. The linear functional gap in the reciprocal target has the same leading magnitude, since

$$G(1 + h) - G(1) = -\int h\psi + \int \frac{h^2\psi}{1 + h}.$$

In each of the three lower-bound constructions $\|h\|_\infty \to 0$ and $\int |h||\psi| \le C|\int h\psi|$. The latter follows by single-template scaling or disjointness and the common orientation of the critical annular templates. The nonlinear remainder is therefore negligible relative to the gap. The same Gaussian two-point bounds finish the proof.

Part IV. A two-block saddle

LCH, Section 6, second bullet, also lists saddle-type critical points among the further problems for their nondegenerate theory. This part answers it for one exact local model, the two-block linear saddle: an exact representer criterion for all positive block rates, and sharp linearized rates with matching nonlinear lower bounds at equal rates, for either block larger. At a saddle the local adjoint may accumulate the time spent near the critical point through either the expanding or the contracting coordinates, and the larger block sets the critical dimension.

14 Setting and results

Let $\Omega \subset \mathbb{R}^{m+n}$ be smooth, bounded and connected, with $m, n \ge 1$. Work at constant conductivity $f_0 = 1$. Let u_0 be a smooth reference potential with zero Dirichlet boundary condition and known

source $g = \Delta u_0$. Suppose that in a product neighborhood of zero, in orthonormal coordinates $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$,

$$b(x, y) = \nabla u_0(x, y) = (\lambda x, -\mu y), \qquad \lambda, \mu > 0.$$

The potential may have other critical points outside this neighborhood. Let K be a fixed compact subset of the product neighborhood with $0 \in \operatorname{int} K$. Fix $\psi \in C_c^\infty(\Omega)$ with $\psi(0) \neq 0$. The parameter perturbations satisfy $\operatorname{supp} h \subset K$ and $\|h\|_{H^s(\Omega)} \leq \eta$. The linearized Gaussian white-noise mean is $Ah = -\Delta_D^{-1} \operatorname{div}(hb)$, the noise level is $N^{-1/2}$, and the target is $\int h\psi$. In the nonlinear experiment the white-noise mean is u_f with $\operatorname{div}(f \nabla u_f) = g$ and zero trace, where $f = 1 + h > 0$, and the target is $\int f\psi$. Risks are minimax mean absolute errors.

Theorem 11. *For arbitrary positive λ, μ, the supported linearized problem has an L^2 representer if and only if $\max(m, n) > 4$. If $\max(m, n) \leq 4$, the minimax risk times $\sqrt{N}$ diverges in the linearized experiment for every $s \geq 0, \eta > 0$, and in the nonlinear experiment for $s > (m+n)/2+1$ and small fixed $\eta > 0$.*

Theorem 12. *Suppose $\lambda = \mu$ and $m \neq n$, and put $M = \max(m, n)$. For all $s \geq 0$ and fixed $\eta > 0$, the linearized risk has the sharp order*

$$\begin{cases} N^{-(2s+M)/(4s+8)}, & M < 4, \\ \sqrt{\log N / N}, & M = 4, \\ N^{-1/2}, & M > 4. \end{cases}$$

These orders are also lower bounds for the nonlinear experiment over the supported Sobolev class when $s > (m + n)/2 + 1$ and $\eta > 0$ is sufficiently small.

Flow reversal. It suffices to prove Theorem 12 for $m > n$. Replacing b by $-b$ exchanges the roles of the blocks, so the expanding block becomes the y block, of dimension n. In the linearized experiment $A_{-b}h = -A_b h$ has the same norm, and the representer equation $-b \cdot \nabla w = \psi$ is the same as $b \cdot \nabla(-w) = \psi$. The risk is therefore invariant, and the case $n > m$ follows with n in place of m. In the nonlinear lower bounds the same identity $L_f(u_f - u_0) = \Delta A h$ of Section 17 is used, and only $\|Ah\|_2$ enters. Below we write the proofs for $m > n$.

15 Local H^2 representers at a saddle

By the Dirichlet inverse and integration by parts, a representer has the form Δw with $w \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $b \cdot \nabla w = \psi$ almost everywhere on K. Necessity already concerns the weak equation near zero.

For the upper part, suppose first that $m > 4$. Write $r = |x|$ and choose an exit radius R larger than the maximal r on K. Extend ψ smoothly by zero and use the auxiliary linear saddle on the whole coordinate space. Its forward exit potential for $r > 0$ is

$$w(x, y) = -\frac{1}{\lambda} \int_r^R \psi(tx/r, (r/t)^{\mu/\lambda} y) \, \frac{dt}{t}.$$

Following the flow changes only the lower endpoint in its equivalent time parametrization, so $b \cdot \nabla w = \psi$. With $c = \psi(0)$ this potential differs from $(c/\lambda) \log r$ by a bounded function, since $|\psi(tx/r, (r/t)^{\mu/\lambda} y) - c| \leq C(t + (r/t)^{\mu/\lambda} |y|)$. Derivatives of orders one and two in x cost at most Cr^{-1} and Cr^{-2}, respectively. Indeed each derivative of x/r costs one power of r^{-1}, while derivatives of $(r/t)^{\mu/\lambda}$ carry the same power and the factor $(r/t)^{\mu/\lambda}$. Angular derivatives of the integrand carry

t and derivatives in y carry $(r/t)^{\mu/\lambda}$, both integrable with dt/t. The lower endpoint derivatives obey the same bounds. First and second derivatives in y are bounded, and the mixed derivative is $O(r^{-1})$. For $m > 4$ these estimates give H^2 across $r = 0$ by cutting off near that subspace and taking the limit. More explicitly, fill the inner ball in x with $(c/\lambda)\log\epsilon$ and join it smoothly at $r \asymp \epsilon$. The derivative errors are supported on a set of volume $O(\epsilon^m)$ and have squared integral $O(\epsilon^{m-4})$ at order two. The smoothened potentials converge in H^2. Multiplying by a smooth cutoff equal to one near K gives a Dirichlet representer. If instead $n > 4$, reverse the direction of the flow and use the exit in y, changing the sign of ψ.

For necessity assume that $m, n \le 4$ and an H^2 solution exists near the saddle. Average it over the rotations $O(m) \times O(n)$ on a smaller product ball. This preserves H^2 and gives the same equation with the averaged score, whose value at zero is still $c \neq 0$. Write the averaged solution and score as $W(r,t)$ and $\Psi(r,t)$ for the two radii. On the positive quadrant W is locally H^2 in two variables, hence continuous, so its values at points are defined, and

$$\lambda r W_r - \mu t W_t = \Psi.$$

For almost every fixed $t_0 > 0$ in a compact radial collar, $\int_0^{r_0} |W_r(r,t_0)|^2 r^{m-3}\,dr < \infty$. For $m \ge 2$ this follows from the tangential Hessian component W_r/r. For $m = 1$ rotation averaging makes the corresponding one-dimensional slice even, so its first derivative has zero trace at zero and the one-dimensional Hardy inequality gives the same integral. For almost every fixed $r_0 > 0$ the analogous bound holds for the t derivative. By Cauchy–Schwarz these slices grow at most $O(1 + \sqrt{|\log r|})$ as $r \downarrow 0$, and similarly in t, because $\int_r^{r_0} v^{3-m}\,dv$ is bounded for $m < 4$ and logarithmic for $m = 4$.

The weak transport equation integrates in the interior along almost every characteristic of invariant label $z = r^\mu t^\lambda$, by changing to the smooth flow coordinates and using the one-dimensional weak chain rule. The three properties used now each hold off a null set: the r-slice bound at t_0, the t-slice bound at r_0, and the integrated transport identity along the characteristic. Choose the pair (r_0, t_0) in the intersection of the corresponding full-measure sets, and then the labels z from the full-measure set of characteristics; for this fixed pair the slice bounds hold on the segments $\{t = t_0\}$ and $\{r = r_0\}$ simultaneously, and by continuity of W the endpoint values are the pointwise values. Now join the points with $t = t_0$ and $r = r_0$ along the characteristic of label z. As $z \downarrow 0$, the starting r and ending t tend to zero, and the transit time is $\tau = (\lambda\mu)^{-1}\log(1/z) + O(1)$. The variation of W between the two endpoints is $c\tau + O(1)$ because $|\Psi - c| \le C(r + t)$ and the integral of $r + t$ along the transit is uniformly bounded. This contradicts the slice bounds, which give an endpoint difference $o(\log(1/z))$.

The nonexistence of a local solution implies that $|\int h\psi|/\|Ah\|_2$ is unbounded over fixed smooth directions supported in the interior of K. Otherwise the target on the range of A would have a bounded linear functional extension and thus an L^2 representer, whose weak equation would hold near zero. Directions in the kernel with nonzero target already give the same lower-bound claim: their linearized means are equal, and at perturbation amplitude $t = \gamma_0/\sqrt{N}$ with γ_0 arbitrary their nonlinear mean gaps are $o(t)$, so the Kullback–Leibler divergence $N \cdot o(t)^2$ tends to zero. For nonzero denominator, scale a fixed direction by $c_1/(\sqrt{N}\|Ah\|_2)$, with $c_1 > 0$ small. Its Gaussian Kullback–Leibler divergence is bounded and its root-sample target gap can be arbitrarily large. For a zero-denominator direction one instead uses an arbitrarily large fixed multiple of $N^{-1/2}$, with vanishing divergence. The same arguments hold in the nonlinear model because along each fixed smooth direction its mean is differentiable at zero with derivative Ah. A fixed supported smooth direction of nonzero target also gives the ordinary parametric lower bound in either experiment. This proves Theorem 11.

16 An upper bound by a normal moment correction

Suppose $\lambda = \mu$ and $m > n$. Use the exit potential w from the expanding x block. Let $\chi_0(x/\epsilon)$ be a smooth radial cutoff equal to zero for $r \le \epsilon$ and one for $r \ge 2\epsilon$. Put

$$\widetilde{w}_\epsilon = \frac{c}{\lambda}\log\epsilon + \chi_0(x/\epsilon)\left(w - \frac{c}{\lambda}\log\epsilon\right).$$

The defect $\rho_\epsilon = \psi - b\cdot\nabla\widetilde{w}_\epsilon$ is supported in the tube $r \le 2\epsilon$. It and its first two derivatives in y are bounded uniformly on a fixed smaller product chart containing K. To see this, use

$$\rho_\epsilon = (1-\chi_0(x/\epsilon))\psi - (b\cdot\nabla\chi_0(x/\epsilon))\left(w - \frac{c}{\lambda}\log\epsilon\right).$$

The second term is on $r \asymp \epsilon$, where the quantity in parentheses and its first two y derivatives are bounded. Let η_K be a smooth cutoff equal to one near K, supported in this product chart. By the derivative estimates for w, the Laplacian of $\eta_K\widetilde{w}_\epsilon$ has L^2 norm bounded by

$$C\begin{cases}\epsilon^{m/2-2}, & m<4,\\ (1+\log(1/\epsilon))^{1/2}, & m=4,\\ 1, & m>4.\end{cases} \tag{19}$$

The leading integral is $\int_\epsilon^R r^{m-5}\,dr$ for its squared second-derivative norm. Cutoff terms where r is very small and a derivative falls on η_K are controlled by the same estimate and local integrability of $\log r$.

Fix $L \ge 0$ with $L+1 > s - m/2$. For each multi-index α in the x block of order at most L, define the scaled moment

$$F_\alpha(y) = \int_{\mathbb{R}^m} X^\alpha \rho_\epsilon(\epsilon X, y)\,dX.$$

We use a correction $U(x/\epsilon, y)$ to the potential. Its moments $M_\alpha(y) = \int X^\alpha U(X,y)\,dX$ need to satisfy

$$-\lambda(m + |\alpha| + y\cdot\nabla_y)M_\alpha(y) = F_\alpha(y).$$

This identity for the moments of the transport derivative follows by integration by parts in X. One solution is

$$M_\alpha(y) = -\lambda^{-1}\int_0^1 v^{m+|\alpha|-1}F_\alpha(vy)\,dv.$$

The moments and their first two y derivatives are uniformly bounded. Choose fixed smooth functions supported in the X ball of radius two, dual to the finite set of moments $\int X^\alpha(\cdot)\,dX$. They can be obtained by inverting a polynomial Gram matrix with a positive smooth interior weight. Use their linear combination with coefficients $M_\alpha(y)$ for U. Taking $w_\epsilon = \eta_K(\widetilde{w}_\epsilon + U(x/\epsilon, y))$ gives the Dirichlet potential. The added term has Laplacian norm at most $C\epsilon^{m/2-2}$, preserving (19).

On K the corrected defect equals the scaled, uniformly bounded, compactly supported X function with all its moments through L zero, for every y. Its homogeneous $H^{-s}(\mathbb{R}^m)$ norm in the physical x variable is $O(\epsilon^{s+m/2})$ uniformly in y. This follows by Taylor expansion of its Fourier transform at zero and Plancherel at high frequencies, then scaling. Extend h by zero using its interior support. Its $L^2_y H^s_x$ norm is at most $C\|h\|_{H^s(\Omega)}$ by an ambient Sobolev extension and Fourier comparison. Cauchy–Schwarz in y on the bounded chart then bounds the paired defect by $C\epsilon^{s+m/2}$.

The estimator pairs the linearized observation with Δw_ϵ. Its bias is the paired defect and its Gaussian standard deviation is $N^{-1/2}\|\Delta w_\epsilon\|_2$. Choose $\epsilon = N^{-1/(2s+4)}$ and apply (19). This proves all upper bounds in Theorem 12.

Remark 13. The moment step extends to $\lambda \neq \mu$: the equation $-(\lambda(m+|\alpha|)+\mu\, y\cdot\nabla_y)M_\alpha = F_\alpha$ is solved by $M_\alpha(y) = -\mu^{-1}\int_0^1 v^{\lambda(m+|\alpha|)/\mu-1}F_\alpha(vy)\,dv$, and the rest of this section is unchanged. Hence the upper bounds of Theorem 12 hold for all positive block rates with $m > n$, and by the flow reversal of Section 14 with $M = \max(m,n)$ in place of m whenever $m \neq n$; the equal-rate hypothesis enters only in the transit lower bounds below.

17 Lower bounds from interior saddle transits

Write $r = |x|$ and $t = |y|$. Choose a single smooth radial cutoff χ equal to one near zero and supported at small fixed radii, so the product of its supports lies inside $\operatorname{int} K$. Choose $\vartheta \in C_c^\infty(1,2)$ with

$$\int \vartheta(z)\,\frac{dz}{z} = 0, \qquad \int \vartheta(z)\log z\,\frac{dz}{z} \neq 0. \tag{20}$$

If $m = 4$, also impose $\int \vartheta(z)z^{-2}\,dz/z = 0$. These conditions are compatible by linear independence of the indicated functions on the interval.

For small $\epsilon > 0$ use the smooth supported perturbation

$$H_\epsilon(x,y) = r^{-m}t^{-n}\vartheta(rt/\epsilon)\chi(r)\chi(t).$$

On its support both radii are bounded below by positive multiples of ϵ. The product rt is invariant under the saddle flow. The prefactor $r^{-m}t^{-n}$ absorbs the divergence of b. Consequently the divergence source $S_\epsilon = \operatorname{div}(H_\epsilon b)$ is supported only on the two outer transition collars of χ. There it is a sum of two sources, one thin in x and one thin in y. More precisely, for all sufficiently small ϵ,

$$S_\epsilon = S_x^\epsilon + S_y^\epsilon, \qquad S_x^\epsilon(x,y) = \epsilon^{-m}S_x(x/\epsilon, y), \qquad S_y^\epsilon(x,y) = \epsilon^{-n}S_y(x, y/\epsilon),$$

for fixed smooth compactly supported profiles. Indeed in the x-thin term t stays on the transition collar of χ, so $r = O(\epsilon)$ and $\chi(r) = 1$. The opposite term is symmetric. Each source is radial in its thin coordinates and has zero integral in those coordinates for every fixed other coordinate. The cancellation follows by the change $z = rt/\epsilon$ and the first condition in (20).

We record the Sobolev norm bound

$$\|H_\epsilon\|_{H^s(\Omega)} \le C_s\epsilon^{-s-m/2}. \tag{21}$$

For derivatives of orders a in x and b' in y, a pointwise bound is $Cr^{-m-a}t^{-n-b'}$ on the same invariant band. Use polar measure and $t = \epsilon z/r$. Its squared integral is bounded by

$$C\epsilon^{-n-2b'}\int_{c_0\epsilon}^{C_0} r^{n+2b'-m-2a}\,\frac{dr}{r}.$$

For $a+b' \le k$ this is $O(\epsilon^{-m-2k})$: when exponents coincide the endpoint bound acquires a logarithm, but the common exponent is at most $n+2k < m+2k$. All integer $k \ge 0$ are allowed. Interpolating consecutive integer bounds gives (21) for real $s \ge 0$.

To bound the observation norm, solve in the thin variables $\Delta_x V_x^\epsilon = S_x^\epsilon$ and $\Delta_y V_y^\epsilon = S_y^\epsilon$ using the radial inverse. Zero mass in the thin coordinates makes each inverse compactly supported in the same outer thin ball: its radial derivative is the interior mass divided by a power of radius, and vanishes outside the source. Choose its constant value there to be zero. For the two profiles this gives

$$V_x^\epsilon(x,y) = \epsilon^{2-m}V_x(x/\epsilon, y), \qquad V_y^\epsilon(x,y) = \epsilon^{2-n}V_y(x, y/\epsilon)$$

with fixed smooth compact profiles. By Dirichlet H^2 regularity, duality gives

$$\|AH_\epsilon\|_2 \le C(\|V_x^\epsilon\|_2 + \|V_y^\epsilon\|_2) \le C(\epsilon^{2-m/2} + \epsilon^{2-n/2}). \tag{22}$$

When $m = 4$, the additional moment in (20) also makes V_x^ϵ zero inside the inner thin radius, uniformly for y on its collar. Specifically, for a zero-mass radial four-dimensional source $F(r)$, the inverse which vanishes outside the source has inner constant $-\frac{1}{2}\int F(r)r\,dr$. Substituting this source into the formula gives exactly a multiple of the additional moment. Thus the x-thin potentials of sufficiently separated geometric scales have disjoint supports.

It remains to show a nonvanishing signal under the zero marginal. Let $\Psi(r,t)$ be the angular average of ψ. After changing variables to $z = rt/\epsilon$, the signal is a fixed positive angular constant times

$$\int_1^2 \vartheta(z)\frac{dz}{z}\, I(\epsilon z), \qquad I(P) = \int_0^\infty \chi(r)\chi(P/r)\Psi(r, P/r)\,\frac{dr}{r}.$$

As $P \downarrow 0$, $I(P) = c\log(1/P) + C_1 + o(1)$. For the constant part of Ψ this follows by splitting the integral at $\sqrt{P}$: on either outer half the small-coordinate cutoff equals one and the remaining integral is logarithmic plus a fixed constant. For the varying part write

$$\Psi(r,t) - c = (\Psi(r,0) - c) + (\Psi(0,t) - c) + O(rt).$$

The two single-coordinate terms have limits by dominated convergence, after changing $t = P/r$ for the second. The remaining term is bounded by $O(P\log(1/P))$. The expansion therefore holds uniformly when $P = \epsilon z$ with $1 \le z \le 2$. By (20), $\int H_\epsilon\psi$ tends to a nonzero constant. Fix the orientation of ϑ to make this limit positive.

For $m < 4$, choose $\epsilon = N^{-1/(2s+4)}$ and compare zero to $h = \gamma\epsilon^{s+m/2}H_\epsilon$ with fixed small $\gamma > 0$. Formula (21) gives the Sobolev constraint and (22) gives bounded small Gaussian divergence. The target gap is of order $\epsilon^{s+m/2}$, as required.

If $m = 4$, choose $\epsilon_j = \epsilon_0 B^{-j}$ with a fixed $B > 1$ large enough for the critical $V_x^{\epsilon_j}$ to have disjoint supports. The other potentials have summable norm since $n < 4$. Hence

$$\left\| A\sum_{j=1}^{J} H_{\epsilon_j} \right\|_2 \le C(\sqrt{J} + 1)$$

by the same Dirichlet duality argument. Let $J = \lfloor \log N/[4(s+2)\log B]\rfloor$ and use

$$h = \frac{\gamma}{\sqrt{NJ}}\sum_{j=1}^{J} H_{\epsilon_j}.$$

Its Sobolev norm is $O(\gamma N^{-1/4}/\sqrt{J})$ by (21), and its Gaussian divergence is bounded by $C\gamma^2$. Each summand has a uniformly positive target gap for small fixed ϵ_0, so the total gap is at least $c_2\gamma\sqrt{J/N}$. The two-point Gaussian bound gives the critical lower rate. For $m > 4$ the ordinary parametric lower bound was proved in Theorem 11.

The nonlinear lower bounds use the same alternatives when $s > (m+n)/2+1$ and the Sobolev radius is sufficiently small. For $L_f = \operatorname{div}(f\nabla)$ one has exactly

$$L_f(u_f - u_0) = -\operatorname{div}(hb) = \Delta Ah.$$

The Dirichlet operator ΔL_f^{-1} is uniformly bounded on L^2 because $f = 1 + h$ is close to 1 in $W^{1,\infty}$. For instance factor $L_f\Delta_D^{-1}$ as the identity plus an L^2 operator of small norm using the Dirichlet H^2 estimate, and invert by a Neumann series. By self-adjointness and Dirichlet integration by parts, duality gives $\|u_f - u_0\|_2 \le C\|Ah\|_2$. Thus the same Gaussian divergence bounds apply up to constants, and the functional gap is unchanged. With the flow reversal of Section 14 this proves Theorem 12.

18 A positive-source example with several critical points

For $m > n$ choose a smooth even function V on the line with $V'' \geq -1$, quadratic behavior $V(y) = V(0) - y^2/2$ near zero, two equal minima at $\pm a$ of value zero, and no other critical points besides zero. It can be constructed by taking $V'' = -1 + \alpha$ with α smooth, even, zero near zero, increasing for positive y where it transitions, and equal to a constant greater than one for all sufficiently large $|y|$. Integrate with $V'(0) = 0$ and shift the minimum to zero. This gives a confining double well with nondegenerate minima. Put

$$U(x, y) = \tfrac{1}{2}|x|^2 + \sum_{j=1}^{n} V(y_j), \qquad \Omega = \{U < L\}, \qquad u_0 = U - L,$$

where $L > nV(0)$. The domain is bounded and $\partial\Omega$ is smooth since every critical value of U is at most $nV(0)$. It is connected: one can first reduce x to zero and descend each y_j to one of the minima. Any two such minima can be joined by changing one coordinate at a time through its barrier while all others remain at their minima. These paths stay below L. The source is

$$g = \Delta U = m + \sum_{j=1}^{n} V''(y_j) \geq m - n > 0.$$

There are 3^n critical points, including the saddle at zero with $b = (x, -y)$ on a product neighborhood. Thus the nonparametric and critical linearized rates of Theorem 12 occur at an interior saddle of a reference with strictly positive source and several nondegenerate critical points on a smooth connected domain.

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