The nonlinear minimax exponent for Darcy inference at an anisotropic minimum

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Abstract

We prove Liu, Chang and Hu's Conjecture 6.1 on estimating linear functionals of a scalar conductivity from Gaussian white-noise observations of its Darcy potential. The conjecture concerns a reference potential with a unique nondegenerate minimum, effective dimension at most four, and a functional that fails the transport compatibility condition. When this condition fails and the Sobolev smoothness s is sufficiently large, we determine the polynomial minimax exponent on every sufficiently small fixed Sobolev ball. It is the smaller of $1/2$ and $(2s + T_*)/(4s+8)$, where T_* is the infimum over the ball of the ratio of the Hessian trace to its largest eigenvalue at the minimum. The infimum is strictly smaller than the reference ratio. Thus when the reference ratio is at most four, every sufficiently small fixed ball has a strictly smaller exponent than the local limit. The proof uses a Fourier-regularized transport correction at a prediction pilot. An integrable gradient controls the nonlinear remainder, and a frequency split in the commutator uses the pilot's small supremum-norm error.

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1 Introduction and main result

How fast can one estimate a linear functional of a conductivity from a noisy observation of its Darcy potential? Liu, Chang and Hu [1], henceforth LCH, identified the minimax rate in the linearized experiment at a reference whose potential has a single nondegenerate minimum. In the nonlinear experiment they proved lower bounds at an incompatible reference, sharp rates on isotropic classes, and a parametric rate on full neighborhoods above the critical effective dimension. Their Conjecture 6.1 asks for the local polynomial exponent on full neighborhoods at or below that dimension.

Theorem 1 answers this question by giving the exponent on each sufficiently small fixed ball. The controlling effective dimension is the infimum over the ball. The conjectured local exponent then follows as the radius tends to zero. A perturbation in a top Hessian eigendirection shows that this infimum is strictly smaller than the effective dimension of the reference (Corollary 2).

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be smooth, bounded and connected. Let g, f_0 be smooth up to the boundary and bounded below by positive constants. For a scalar conductivity f write u_f for the solution of

$$L_f u_f = g, \qquad L_f = \operatorname{div}(f\nabla), \qquad u_f|_{\partial\Omega} = 0.$$

The reference $u_0 = u_{f_0}$ has one critical point, a nondegenerate minimum x_0 in the interior. The data are the Gaussian white-noise observation with mean u_f and noise level $N^{-1/2}$ in $L^2(\Omega)$. Fix

nonzero $\psi \in C_c^\infty(\Omega)$, set $\Phi(f) = \int_\Omega f\psi$, and take the full Sobolev ball

$$\mathcal{F}_\eta = \big\{ f = f_0 + h : \|h\|_{H^s(\Omega)} \le \eta,\ \inf_\Omega f > 0 \big\}.$$

For small fixed $\eta > 0$ and the smoothness used below, each u_f has a unique minimum x_f and positive Hessian H_f there. Write $\lambda_f = \lambda_{\max}(H_f)$, $T_f = \mathrm{tr}(H_f)/\lambda_f$ and $T_-(\eta) = \inf_{f\in\mathcal{F}_\eta} T_f$. Anisotropy here refers to the eigenvalues of H_f. Use absolute risk

$$\mathcal{R}_N(\mathcal{F}_\eta) = \inf_{\widehat{\Phi}} \sup_{f\in\mathcal{F}_\eta} \mathbb{E}_f|\widehat{\Phi} - \Phi(f)|, \qquad e(\mathcal{F}_\eta) = \liminf_{N\to\infty} \frac{\log(1/\mathcal{R}_N(\mathcal{F}_\eta))}{\log N}.$$

The LCH condition (C) holds at a smooth reference if $\psi(x_f) = 0$ and the integral of $-\psi$ is independent of the complete forward trajectory. Here each trajectory is parametrized by $\dot{x} = \nabla u_f(x)$, approaches x_f as $t \to -\infty$, and ends on $\partial\Omega$. At a reference failing this condition, their Theorem 3.3(b) gives the nonlinear lower bound of order $N^{-(2s+T_f)/(4s+8)}$ when $T_f < 4$, and of order $\sqrt{\log N/N}$ when $T_f = 4$. In the local question, $N \to \infty$ at fixed η in the definition of e, and then $\eta \downarrow 0$.

Theorem 1. *For all sufficiently large s and sufficiently small fixed $\eta > 0$,*

$$\mathcal{R}_N(\mathcal{F}_\eta) \ \lesssim\ \begin{cases} N^{-(2s+T_-(\eta))/(4s+8)}, & T_-(\eta) < 4, \\ \sqrt{\log N/N}, & T_-(\eta) = 4, \\ N^{-1/2}, & T_-(\eta) > 4. \end{cases} \tag{1}$$

If (C) fails at f_0, the exact exponent on each such ball is

$$e(\mathcal{F}_\eta) = \min\left\{ \frac{2s + T_-(\eta)}{4s+8}, \frac{1}{2} \right\}.$$

In particular, if $T_{f_0} \le 4$ and (C) fails there, then

$$\lim_{\eta\downarrow 0} e(\mathcal{F}_\eta) = \min\left\{ \frac{2s + T_{f_0}}{4s+8}, \frac{1}{2} \right\},$$

as in LCH Conjecture 6.1.

For $T_-(\eta) > 4$, the upper bound and LCH Theorem 3.3(c) give the root-N rate. At $T_-(\eta) = 4$, the exponent is $1/2$ and the displayed upper bound has a logarithmic factor. The constants in the risk upper bounds may depend on the fixed ball. The exact exponent describes the power of N on that ball. The proof applies, for example, whenever $s > 2d + 4$.

The proof uses the LCH pilot and exact conditional bias identity. The new issue is to regularize the transport solution uniformly over pilot fields with only a fixed Sobolev bound. We first obtain a uniform Fourier-shell estimate and an integrable gradient for the singular solution. Then we split the commutator with the Fourier cutoff. Low coefficient frequencies use the shell estimate. High coefficient frequencies are paired directly with the small pilot error in supremum norm. The integrable gradient also controls the nonlinear remainder.

2 Uniform bounds for characteristic solutions

Fix an integer $k > d/2$ with $k \ge 2$, and take $s > d/2 + k$. Additional strict inequalities on s appear below. We use LCH Lemma S4.1. On a fixed compact regular sublevel domain $K \Subset \Omega$, the characteristic solution w_p of

$$b_p \cdot \nabla w_p = \psi, \qquad b_p = \nabla u_p, \qquad w_p|_{\partial\Omega} = 0$$

is supported in a fixed compact subset of the interior of K. The backward flow stays in K and reaches a uniformly convex ball about the minimum in uniformly bounded time. We may apply the following bounds to smooth pilots, since smooth conductivities in the ball are dense in it in H^s: approximate the perturbation by smooth ones after a contraction toward the center. The finite derivative estimates below hold uniformly. In particular $\|u_p\|_{H^{s+1}(\Omega)}$ and $\|b_p\|_{C^k}$ are bounded, whereas $\|u_p - u_0\|_{C^2} = O(\eta)$.

Use the following notation with the pilot index suppressed. Write x_p for the minimum, H for the Hessian, $Q = \operatorname{tr} H$, $\lambda = \lambda_{\max}(H)$, and $T = Q/\lambda$. In a common-radius ball B about x_p, $Db \geq mI$ with $m > 0$ uniform. Denote the flow of $-b$ by X_t. For $t \geq 0$ and $x, y \in B$,

$$\begin{aligned} |X_t x - x_p| &\leq e^{-mt}|x - x_p|, \\ \det DX_t(x) &\asymp e^{-Qt}, \\ |X_t x - X_t y| &\geq c\, e^{-\lambda t}|x - y|. \end{aligned} \tag{2}$$

The first bound is the differential inequality from convexity. Liouville's identity gives the second because, along each orbit, the time integral of $|\operatorname{div} b - Q|$ is uniformly bounded. For the last, $Db(z) - H = O(|z - x_p|)$ and the differential inequality for the separation has an integrable error, uniformly in initial points. Every point in $B \setminus \{x_p\}$ is $X_r\theta$ for a unique $\theta \in \partial B$ and exit time $r \geq 0$ from the ball under the forward flow. Its volume element is comparable to $e^{-Qr}\, dr\, d\sigma(\theta)$. The comparison follows from (2) and the uniform transversality of b on ∂B.

Set $c_p = \psi(x_p)$ and, in B, define the particular solution

$$v_p(x) = \int_0^\infty [\psi(X_t x) - c_p]\, dt.$$

It satisfies $b \cdot \nabla v_p = \psi - c_p$. Indeed, $v_p(X_\tau x) - v_p(x) = -\int_0^\tau [\psi(X_t x) - c_p]\, dt$. Its C^k norm is uniformly bounded: the zeroth-order integrand decays by (2). The first derivative of X_t decays as e^{-mt}, and the higher variational equations have the same contracting propagator and forcing terms that are products of at least two lower derivatives. Induction gives the same exponential bound up to order k. On compact sets separated from the minimum, w_p has uniformly bounded derivatives through order k. Choose a fixed regular level surface of u_0 between the support of ψ and ∂K, high enough to contain the convex balls on its inner side. The characteristic integral ends there because the forward flow points out of the sublevel set and cannot return. In the region between a smaller convex ball and this surface, $\nabla u_0 \cdot b_p$ is uniformly positive by C^1 closeness and compactness. The forward flow cannot enter the smaller ball from outside, so the travel time is uniformly bounded. The same positivity gives transversality at the surface. Ordinary variational and implicit function bounds give the required derivatives. The same bounds apply to the values of w_p and v_p on ∂B.

Set $F = w_p - v_p$ on $B \setminus \{x_p\}$. Then $b \cdot \nabla F = c_p$. In particular, with the preceding flow coordinates,

$$F(X_r\theta) = F(\theta) - c_p r.$$

For $1 \leq j \leq k$, the forward variational and exit-time equations give

$$|D^j F(X_r\theta)| \leq C_j e^{j\lambda r}, \qquad |F(X_r\theta)| \leq C(1 + r). \tag{3}$$

Indeed, the derivative of the forward flow along a segment of length t in B is at most $Ce^{\lambda t}$ by the integrable deviation of Db from H. In a higher variational equation the forcing products of total order j are bounded by $Ce^{j\lambda t}$. Integrating them against the first-order propagator gives the same bound at the endpoint for $j \geq 2$. Implicit differentiation of the exit condition, whose time derivative has a uniform positive lower bound, gives $C_j e^{j\lambda r}$ for each derivative of the exit time and exit position of order j. The chain rule for the bounded sphere data gives (3).

Choose a real smooth cutoff ζ inside B, equal to one near x_p, by translating a fixed function. Put $q_p = \zeta F$ on $\mathbb{R}^d$ by zero extension and $w_p^{\mathrm{reg}} = w_p - q_p$. The latter has uniformly bounded C^k norm and compact support inside K. We claim

$$\|q_p\|_{W^{1,1}(\mathbb{R}^d)} \le C, \tag{4}$$

$$\int_{R\le|\xi|\le 2R} |\widehat{q_p}(\xi)|^2 \, d\xi \le C R^{-T_p}, \qquad R \ge 1. \tag{5}$$

For (4), integrate (3) in exit coordinates. The integral for the gradient converges uniformly because $Q - \lambda$ is the sum of the other positive eigenvalues and is bounded below when $d \ge 2$. This classical gradient is also the weak gradient across the point x_p. Indeed the first bound of (2) gives $r \lesssim 1 + |\log|x - x_p||$, and the integration-by-parts cutoff error tends to zero at rate $O(\epsilon^{d-1}(1 + |\log \epsilon|))$ in a ball of radius ϵ. We also have $\|q_p\|_2 \le C$ from the logarithmic growth bound.

For (5) use finite differences, as in LCH Lemma S2.11. Here we give the uniform estimate with the finite number k of derivatives. Let $E_t = X_t(B)$. Choose fixed c_0 such that $X_{c_0}(B)$ lies in the half-radius ball. Apply X_t to $X_{c_0}(B)$ and ∂B. The separation bound in (2) gives $\operatorname{dist}(E_{t+c_0}, \partial E_t) \gtrsim e^{-\lambda t}$. For a difference step z, choose large t by $|z| = c_1 e^{-\lambda t}$ with c_1 uniform and sufficiently small. Then all shifts by jz, $0 \le j \le k$, of E_t lie in E_{t-c_0}, where $\zeta = 1$ for sufficiently large t. Adding $c_p t$ to each shifted value leaves the kth difference unchanged. Bounding its square by a fixed multiple of the sum of the squared shifted values and translating each integral bounds the contribution on E_t by a constant times

$$\int_{E_{t-c_0}} |F(x) + c_p t|^2 \, dx \le C e^{-Qt} \int_B |F(y) + c_p c_0|^2 \, dy \le C e^{-Qt}.$$

For an outside point of exit time $r \le t$, the segment of length at most $k|z|$ stays outside E_{r+c_0} after reducing c_1 if needed. The same separation estimate gives this because the point lies on ∂E_r, and the segment is shorter than the distance to E_{r+c_0}. For bounded r and away from B, the regular derivatives alone suffice. Thus the outer squared difference is bounded by

$$C|z|^{2k} \left(1 + \int_0^t e^{(2k\lambda - Q)r} \, dr\right) \le C|z|^T.$$

The final constant is uniform since $T \le d < 2k$ and λ is bounded below. Also $e^{-Qt} \asymp |z|^T$ uniformly. Steps outside this small-$|z|$ range are bounded using the uniform L^2 norm of q_p. Plancherel and averaging the difference bounds over z in a ball of radius R^{-1} prove (5): the average of $|e^{i\xi\cdot z} - 1|^{2k}$ is bounded below on $R \le |\xi| \le 2R$ by scaling and compactness of the unit annulus.

3 A uniformly regularized pilot correction

Take two real cutoffs centered at the pilot minimum, with $\zeta_2 = 1$ on a neighborhood of $\operatorname{supp}\zeta$ and ζ_2 compactly supported in B. Let P_R have a real even symbol $\mu(\xi/R)$, where $\mu \in C_c^\infty$ equals one on the unit ball and vanishes outside the ball of radius two. Put

$$w_{p,R} = w_p^{\mathrm{reg}} + \zeta_2 P_R q_p, \qquad z_{p,R} = L_p w_{p,R}.$$

All supports lie in K. The convolution kernel of P_R has uniformly bounded L^1 norm. By (4),

$$\|\nabla w_{p,R}\|_1 \le C. \tag{6}$$

The second-order product estimate and the upper bounds on $p, \nabla p$ give $\|z_{p,R}\|_2 \le C\|w_{p,R}\|_{H^2}$. Write $T_* = T_-(\eta)$. The shell sum in (5) yields

$$\|z_{p,R}\|_2 \lesssim \begin{cases} R^{2-T_*/2}, & T_* < 4, \\ (1+\log R)^{1/2}, & T_* = 4, \\ 1, & T_* > 4, \end{cases} \tag{7}$$

uniformly in p. In each dyadic annulus the contribution to the squared H^2 norm is at most $CR_j^{4-T_*}$, with R_j its radius, and only $R_j \lesssim R$ contribute. The low frequencies use $\|q_p\|_2 \le C$.

Write $b = b_p$ and use $\widetilde{b}$ for its compact extension obtained by a fixed smooth cutoff equal to one near the supports of ζ_2 for all p. Then $\|\widetilde{b}\|_{H^s(\mathbb{R}^d)} \le C$. For the adjoint defect $r_{p,R} = \psi - b\cdot\nabla w_{p,R}$, the exact decomposition on Ω is

$$r_{p,R} = \zeta_2(I - P_R)(\widetilde{b}\cdot\nabla q_p) + \zeta_2[P_R, \widetilde{b}\cdot\nabla]q_p - (b\cdot\nabla\zeta_2)P_R q_p. \tag{8}$$

Here $[P_R, \widetilde{b}\cdot\nabla] = P_R(\widetilde{b}\cdot\nabla) - (\widetilde{b}\cdot\nabla)P_R$. The weak gradient in (4) makes these identities valid across the minimum. The decomposition follows by expanding $b\cdot\nabla(\zeta_2 P_R q_p)$, and using $b\cdot\nabla q_p = \psi - b\cdot\nabla w_p^{\mathrm{reg}}$, supported where $\zeta_2 = 1$. The derivative of the outer cutoff is supported at positive uniform distance from q_p, so the last term is $O(R^{-L})$ in supremum norm for every fixed L by the Schwartz kernel bound and $\|q_p\|_1 \le C$. The exact right hand side

$$\widetilde{b}\cdot\nabla q_p = c_p\zeta + F\, b\cdot\nabla\zeta$$

has bounded H^k norm, because the derivative of ζ is supported away from the minimum. Thus the first term in (8), tested against $h = f - p$, is bounded by CR^{-s-k} using $\|\zeta_2 h\|_{H^s} \le C$ and the frequency support of $I - P_R$ away from zero.

To treat the commutator, split $\widetilde{b} = b_{\mathrm{lo}} + b_{\mathrm{hi}}$ by a smooth frequency cutoff equal to one on $|\omega| \le R/32$ and supported on $|\omega| \le R/16$. The low coefficient piece uses the argument of LCH Lemma S2.12. Its Fourier kernel, aside from transform constants, is

$$i\widehat{b}_{\mathrm{lo}}(\xi - \vartheta)\cdot\vartheta\,\big[\mu(\xi/R) - \mu(\vartheta/R)\big].$$

The factor $|\vartheta|\,|\mu(\xi/R) - \mu(\vartheta/R)|$ is at most $C|\xi - \vartheta|$. For a nonzero low-piece kernel, both input and output frequencies lie between fixed multiples of R. This follows from $|\xi - \vartheta| \le R/16$ and the constant regions of μ. Young's convolution inequality, $\||\omega|\widehat{b}_{\mathrm{lo}}(\omega)\|_1 \le C$ by $s > d/2+1$, and (5) give an L^2 bound of $CR^{-T_p/2}$. Its H^{-s} bound is therefore $CR^{-s-T_p/2}$. Pairing against $\zeta_2 h$, extended by zero from the interior, preserves this order.

For the high coefficient piece, Cauchy–Schwarz in Fourier space gives $\|b_{\mathrm{hi}}\|_\infty \le CR^{d/2-s}$. The two terms are bounded individually in L^1 by $C\|b_{\mathrm{hi}}\|_\infty\|\nabla q_p\|_1$. For the first term apply the L^1 convolution bound after multiplication. For the second use $\nabla P_R q_p = P_R\nabla q_p$ and apply the convolution bound before multiplication. Thus each term is $O(R^{d/2-s})$ in L^1. We pair this part against the small supremum norm of h on K. Since $k > T_p/2$, the result is

$$|\langle h, r_{p,R}\rangle| \le C\big(R^{-s-T_*/2} + R^{d/2-s}\|h\|_{L^\infty(K)}\big). \tag{9}$$

This estimate requires only the fixed H^s budget on the pilot field. The high coefficient frequencies need no uniform higher Sobolev bounds.

4 Risk and the fixed-neighborhood exponent

Use the split white-noise prediction pilot of LCH Lemma S4.2 with $\mathcal{G} = \mathcal{F}_\eta$ and the countable smooth dense subset above. The resulting p takes countably many values and

$$\left(\mathbb{E}_f \delta_p^q\right)^{1/q} \le C_q N^{-(s+1)/(2s+d+2)}, \qquad \delta_p = \|u_f - u_p\|_2, \quad q \ge 1.$$

With the independent second observation Y_2 at noise level $\sqrt{2/N}$, use

$$\widehat{\Phi} = \langle p, \psi\rangle + Y_2(z_{p,R}) - \langle u_p, z_{p,R}\rangle.$$

Countable selection ensures measurability of this use of the second observation. If a nonrandomized estimator is desired, condition on the original observation to remove the independent randomization used to split the noise. Jensen's inequality preserves the risk.

For $v = u_f - u_p$ and $h = f - p$, LCH Lemma S4.3 gives the exact conditional bias

$$-\langle h, r_{p,R}\rangle + \int_\Omega h\, \nabla v \cdot \nabla w_{p,R}.$$

This follows by testing $L_p v = -\operatorname{div}(h\nabla u_f)$ against $w_{p,R}$. By (6), the last term is bounded by $C\|h\|_{L^\infty(K)}\|\nabla v\|_{L^\infty(K)}$ uniformly in R.

Fix $\sigma > d/2$ independent of s. The local transport stability estimate and interpolation in LCH Lemma S4.4 and (S4.4) give

$$\|h\|_{L^\infty(K)} \le C\, \delta_p^{(s-1)(1-\sigma/s)/(s+1)}, \qquad \|\nabla v\|_{L^\infty(K)} \le C\, \delta_p^{(s-\sigma)/(s+1)}.$$

We used $H^\sigma(K) \hookrightarrow L^\infty(K)$ and $H^{1+\sigma}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$. The pilot moments then yield

$$\begin{aligned} \mathbb{E}_f\|h\|_{L^\infty(K)} &\lesssim N^{-A}, \\ \mathbb{E}_f\big[\|h\|_{L^\infty(K)}\|\nabla v\|_{L^\infty(K)}\big] &\lesssim N^{-A-B}, \qquad A = \frac{(s-1)(1-\sigma/s)}{2s+d+2}, \quad B = \frac{s-\sigma}{2s+d+2}. \end{aligned}$$

The moment bounds also apply to exponents below one by Jensen. Besides $s > d/2 + k$ and $s > \sigma$, we require $A + B > 1/2$ and $A > d/(2s+4)$. These hold for all sufficiently large s since $A, B \to 1/2$ as $s \to \infty$. For the sufficient condition $s > 2d + 4$, choose $\sigma = d/2 + 1$ and $k = \max(2, \lfloor d/2\rfloor + 1)$. Then $s - 1 \ge 7s/8$, $\sigma < s/4$ and $2s + d + 2 < 5s/2$, giving $A > 21/80$, $B > 3/10$ and $d/(2s+4) < 1/4$. Use $R = N^{1/(2s+4)}$. Since $T_* \le d$, $A > d/(2s+4)$ implies

$$R^{d/2-s} N^{-A} \le R^{-s-T_*/2}.$$

The conditional Gaussian fluctuation has expected absolute value bounded by $\sqrt{2/N}\|z_{p,R}\|_2$. Combining (7), (9) and the nonlinear remainder bound now gives (1) in all three cases.

It remains to match the polynomial exponent. First, failure of (C) is open near the smooth reference in this class. If $\psi(x_0) \neq 0$, it follows from continuity of the critical point. Otherwise use the values of $F = w_p - v_p$ on the translated sphere ∂B. At the reference the profile is nonconstant, and when $\psi(x_p) = 0$ these sphere values are exactly the limits of w_p along the respective backward trajectories. The values depend continuously on p for the above uniform local geometry. For w_p, use its finite forward exit integral up to the fixed transverse surface. For v_p, use continuous dependence of the backward flow on each fixed time interval and the uniform exponential tail. Hence two distinct sphere values at the reference persist for sufficiently small η. This proves openness, including at nearby pilots where $\psi(x_p) \neq 0$.

The map $p \mapsto T_p$ is continuous in H^s on this ball, by the regularity and continuity of u_p in H^{s+1} and the nondegenerate minimum. If $T_* < 4$, approximate its infimum arbitrarily closely by smooth strict interior points p of the ball. Around each such p there is a smaller full H^s ball contained in $\mathcal{F}_\eta$. Its smooth reference satisfies LCH Assumption 2.1 and fails (C). For $T_p < 4$, their Theorem 3.3(b) bounds $e(\mathcal{F}_\eta)$ above by $(2s + T_p)/(4s + 8)$. Letting these T_p approach T_* gives the stated exponent upper bound. If $T_* \geq 4$, the parametric lower bound of Theorem 3.3(c) suffices for the exponent. This proves equality of the exponents for fixed η. Finally $T_-(\eta) \to T_{f_0}$ as $\eta \downarrow 0$ by the same continuity, proving the conjectured limit.

Corollary 2. *For each sufficiently small $\eta > 0$, one has $T_-(\eta) < T_{f_0}$. If (C) fails at f_0 and $T_{f_0} \leq 4$, the exponent on every such fixed ball is strictly smaller than the local limit in Theorem 1.*

Proof. Let a be a unit eigenvector for the largest eigenvalue of H_{f_0}. Choose $V \in C_c^\infty(\Omega)$ equal to $(a \cdot (x - x_0))^2/2$ near x_0. By the transport inverse in LCH Lemma S2.2 there is a smooth perturbation h in $H^s(\Omega)$ such that

$$\operatorname{div}(h \nabla u_0) = -L_{f_0} V.$$

Indeed, apply the inverse to $-L_{f_0}V$, which belongs to every Sobolev space. For small $t > 0$ put $f_t = f_0 + th$. Substitution into the forward equation gives

$$L_{f_t}(u_{f_t} - u_0 - tV) = -t^2 \operatorname{div}(h \nabla V).$$

The uniform H^{s-1} to H^{s+1} Dirichlet inverse bound from LCH Lemma S4.1 and Sobolev embedding give an $O(t^2)$ error in C^2. Since $\nabla V(x_0) = 0$ and the Hessian stays uniformly positive near x_0, the minimum moves by $O(t^2)$. Consequently $H_{f_t} = H_{f_0} + taa^{\mathsf{T}} + O(t^2)$. Write $Q_0 = \operatorname{tr} H_{f_0}$ and $\lambda_0 = \lambda_{\max}(H_{f_0})$. The trace increases by $t + O(t^2)$ and the largest eigenvalue is $\lambda_0 + t + O(t^2)$ by the variational formula for that eigenvalue. Hence

$$T_{f_t} = T_{f_0} - \frac{Q_0 - \lambda_0}{\lambda_0^2} t + O(t^2).$$

Since $d \geq 2$ and H_{f_0} is positive definite, $Q_0 > \lambda_0$. Choose $t > 0$ small enough that $f_t \in \mathcal{F}_\eta$ and $T_{f_t} < T_{f_0}$. The exponent comparison now follows from Theorem 1. □

Declaration. OpenAI Codex assisted with deriving and checking proofs, locating references, and revising the exposition. The author assumes responsibility for all content.

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