Tail Decay and Collision Barriers for Polynomial Certificates

Scott Schubr

October 25, 2026

Abstract

We study the degree at which polynomial square integrals recover all isolated negative atoms of a real signed measure. Fix the number of atoms $k \geq 2$, their separation from the remainder $d > 0$, and a Gaussian moment rate $c > 0$. For normalized atomic weights $1/k$, remainder variation at most R, and a fixed Gaussian moment budget, the worst degree for recovering k negative directions is asymptotic to

$$\frac{1}{8cd^2}\left[\log\Big(R/\Delta^{2(k-1)}\Big)\right]^2$$

as the minimum atom separation Δ tends to zero, allowing arbitrary jointly varying $R \geq 1$. The upper bound uses classical Laguerre estimates and Lagrange cardinals; a vanishing-jet subspace gives the matching lower bound. For an exponential rather than Gaussian budget, the same obstruction shows that the collision power in an earlier sufficient bound is necessary on the logarithmic scale, for every fixed k. For every weaker tail $e^{c|x|^p}$, $0 < p < 1$, we give a fixed-budget positive background that hides an isolated negative atom from every polynomial square above an explicit contrast, and extend this construction to compact negative measures. The classical kernel, orthogonal-polynomial and Stieltjes-class ingredients are credited explicitly; the conclusions concern degree and negative index, not computational complexity.

1 The recovery problem and its normalization

Let $\mathcal{P}_N$ be the real polynomials of degree at most N. For a real finite signed measure μ with all absolute moments, its square form is

$$\mathcal{Q}_\mu(P) = \int P(x)^2 \, d\mu(x).$$

The negative index on $\mathcal{P}_N$ is the largest dimension of a subspace on which this form is strictly negative away from zero. It equals the number of strictly negative eigenvalues of the original moment matrix $(\int x^{i+j}\, d\mu)_{0\leq i,j\leq N}$, independently of the polynomial basis. A null vector is not a negative direction. Write

$$n_k(\mu) = \inf\{N \geq 0 : \operatorname{ind}_-(\mathcal{Q}_\mu|_{\mathcal{P}_N}) \geq k\}, \qquad \inf \varnothing = \infty. \tag{1}$$

Throughout, degree means the degree of P, rather than of P^2.

Fix $k \geq 2$ and $B, c, d > 0$, and put $m = k - 1$. Let $C > e^{cd^2}$. For $R \geq 1$ and $0 < \Delta \leq 1$, define $\mathcal{G}_k(R, \Delta)$ by

$$\mu = -\frac{1}{k}\sum_{j=1}^{k} \delta_{b_j} + \nu, \qquad b_j \in [-B, B], \qquad |b_i - b_j| \geq \Delta \quad (i \neq j), \tag{2}$$

$$\operatorname{supp}\nu \subset \bigcap_{j=1}^{k} \{|x - b_j| \geq d\}, \qquad \|\nu\|_{\mathrm{TV}} \leq R, \qquad \int e^{cx^2}\, d|\nu|(x) \leq CR. \tag{3}$$

The upper class allows a signed remainder. Set

$$D_k^G(R,\Delta) = \sup_{\mu\in\mathcal{G}_k(R,\Delta)} n_k(\mu), \qquad X = R/\Delta^{2m}. \tag{4}$$

Only sufficiently small Δ will enter our limit; the constructions below make the class nonempty there for every $R \geq 1$. The background-only moment convention in (3) is part of the definition. It gives the full bound $\int e^{cx^2}\, d|\mu| \leq e^{cB^2} + CR$, without charging the translated negative atoms to an implicit allowance.

Theorem 1.1 (Sharp Gaussian joint law)**.** *For every fixed k, B, c, d, C as above,*

$$\frac{D_k^G(R,\Delta)}{[\log(R/\Delta^{2(k-1)})]^2} \longrightarrow \frac{1}{8cd^2} \quad \text{as } \Delta \downarrow 0, \quad \text{with arbitrary jointly varying } R \geq 1. \tag{5}$$

Precisely, the limit holds along every sequence $\Delta_\ell \to 0$, $R_\ell \geq 1$. The upper bound constructs a common strictly negative k-dimensional polynomial subspace on the original measure. For certified computable input parameters and atom positions, the basis can be chosen with rational coefficients without evaluating the moments of μ. If ν is positive, the recovered index is exactly k.

This is a collision limit with k fixed. It asserts neither a result uniform in growing k nor the same leading constant at a fixed non-small separation as $R \to \infty$. Section 5 proves a logarithmic collision law under an exponential budget, using the earlier upper bound of [3]. Section 6 supplies explicit permanent failure under every subexponential budget $e^{c|x|^p}$, $0 < p < 1$.

The underlying approximation and moment tools are classical. In particular, the Gaussian certificate reduces to a half-line Laguerre estimate under $v = x^2$; the sine perturbation in the last section belongs to classical generalized-gamma Stieltjes classes [5, 6, 7]. Our quantitative statements combine these tools with a full-index obstruction: before a specified degree, an entire subspace of codimension $k-1$ remains nonnegative. This is stronger than failure of selected cardinal polynomials, and it is unchanged by a different polynomial basis.

2 Evaluation kernels and vanishing jets

The finite-rank spectral criterion provides a useful classical interface between kernels and the full negative index. Kernel matrices for finite mass modifications and the associated rank-update formulas are developed, for example, in [1, 2]. We give the elementary index identity to distinguish recovery of every negative dimension from a single scalar test.

Proposition 2.1 (Finite-section kernel criterion)**.** *In a fixed basis $\phi_0, \dots, \phi_N$ of $\mathcal{P}_N$, let $G > 0$ be the Gram matrix of a positive reference measure τ. For nodes b_i and positive weights w_i, set*

$$E_{ij} = \phi_j(b_i), \qquad W = \operatorname{diag}(w_i), \qquad K = W^{1/2} E G^{-1} E^{\mathsf{T}} W^{1/2}.$$

For $R > 0$, the original moment matrix of $R\tau - \sum_i w_i \delta_{b_i}$ is $A = RG - E^{\mathsf{T}} W E$. Its negative index is the number of eigenvalues of K strictly greater than R; its nullity is the number equal to R, counting multiplicities.

Proof. Congruence by $G^{-1/2}$ changes A into $RI - B^{\mathsf{T}}B$, where $B = W^{1/2} E G^{-1/2}$. The nonzero eigenvalues of $B^{\mathsf{T}}B$ and $BB^{\mathsf{T}} = K$ agree, with multiplicity. Additional zero eigenvalues in either rectangular product cannot affect either comparison with $R > 0$. Sylvester's law of inertia proves the assertion. □

For distinct nodes and $N \geq k-1$, E has full row rank. Recovery of all k negative directions is then equivalent to the smallest eigenvalue of K exceeding R, rather than its trace or a single diagonal entry being large. This criterion requires a positive reference Gram; it cannot be applied by treating an arbitrary signed remainder as positive. The uniform estimates below do not assume that its eigenvalues have already been computed.

We first isolate the elementary common mechanism. Let ρ be a positive probability measure of infinite support in $[D^2, \infty)$, $D > 0$, with all moments. Write p_j for its real orthonormal polynomials and

$$K_{\rho,l}(t,t) = \sum_{j=0}^{l} p_j(t)^2, \qquad K_l = K_{\rho,l}(0,0).$$

For a positive measure ω, the evaluation kernel has the variational description

$$K_{\omega,l}(t,t) = \sup_{0 \neq Q \in \mathcal{P}_l} \frac{|Q(t)|^2}{\int Q^2 \, d\omega}. \tag{6}$$

Indeed, expansion in any orthonormal basis and Cauchy–Schwarz give the upper bound, with equality for the kernel polynomial. Infinite positive support makes all finite polynomial norms nondegenerate.

Lemma 2.2 (Uniform evaluation inside a gap)**.** *For $0 \leq t < D^2$,*

$$K_{v\rho,l}(t,t) \leq D^{-2} K_{\rho,l}(t,t) \leq D^{-2} K_l, \tag{7}$$

where $v\rho$ denotes the measure $v \, d\rho(v)$. Let τ_* be the even probability obtained by lifting $v \sim \rho$ to $x = \pm\sqrt{v}$ with equal sign probabilities. For every $Q \in \mathcal{P}_q$ and $|b| < D$,

$$|Q(b)|^2 \leq (1 + b^2/D^2) K_{\lfloor q/2 \rfloor} \int Q^2 \, d\tau_*. \tag{8}$$

Proof. The first inequality of (7) follows from $\int H^2 v \, d\rho \geq D^2 \int H^2 \, d\rho$ and (6). Every zero of p_j lies in the convex hull of the support, hence in $[D^2, \infty)$; this is the classical zero-location theorem [4, §18.2(vi)]. For completeness, the usual sign-change argument proves it: if a degree-j orthogonal polynomial had fewer than j sign changes in the open interval containing its positive support, the product of the sign-changing factors, of degree less than j, would have a nonzero integral against it, contradicting orthogonality. Infinite support excludes concentration on its finitely many zeros. In particular, writing its zeros as $r_i \geq D^2$ gives

$$\frac{|p_j(t)|}{|p_j(0)|} = \prod_i (1 - t/r_i) \leq 1, \qquad 0 \leq t < D^2.$$

Sum the squared inequalities to finish (7).

Write $Q(x) = E(x^2) + xH(x^2)$ and $l = \lfloor q/2 \rfloor$. Then $\deg E \leq l$ and $\deg H \leq l$, the latter possibly one smaller. The even lift gives

$$\int Q^2 \, d\tau_* = \int E^2 \, d\rho + \int v H^2 \, d\rho.$$

The squared evaluation norms of the two terms at b are bounded by K_l and $b^2 D^{-2} K_l$, respectively. Applying Cauchy–Schwarz on the direct sum of these two polynomial Hilbert spaces proves (8). If $H = 0$ the same bound remains valid. □

Lemma 2.3 (Vanishing-jet barrier). *Fix $k \geq 2$, $m = k-1$, real $z_1, \dots, z_k$ with $Z = \max_j |z_j| > 0$, and $0 < \varepsilon \leq \varepsilon_0$ with $\varepsilon_0 Z < D$. Suppose τ is a positive measure supported on $\{|x| \geq D\}$ and $\tau \geq \eta\tau_*$ for some $\eta > 0$, where τ_* is as in Lemma 2.2. Put*

$$\mu = R\tau - \frac{1}{k}\sum_{j=1}^{k} \delta_{\varepsilon z_j}.$$

For $N \geq m$ let $l = \lfloor (N-m)/2 \rfloor$. If

$$\varepsilon^{2m} Z^{2m} (1 + \varepsilon_0^2 Z^2/D^2) K_l \leq R\eta D^{2m}, \tag{9}$$

the square form on the full space $\mathcal{P}_N$ has negative index at most m. For $N < m$ that conclusion holds without any condition.

Proof. In $\mathcal{P}_N$ consider

$$V_m = \{P : P^{(j)}(0) = 0,\ 0 \leq j < m\}.$$

For $N \geq m$, this has codimension m and consists exactly of $P = x^m Q$, $\deg Q \leq N-m$. Its positive contribution satisfies

$$R\int P^2\, d\tau \geq R\eta D^{2m} \int Q^2\, d\tau_*.$$

Lemma 2.2 bounds the magnitude of its negative contribution by

$$\frac{1}{k}\sum_j (\varepsilon z_j)^{2m} Q(\varepsilon z_j)^2 \leq \varepsilon^{2m} Z^{2m} (1 + \varepsilon_0^2 Z^2/D^2) K_l \int Q^2\, d\tau_*.$$

Thus (9) makes the entire restriction to V_m nonnegative, including every mixed term. A strictly negative subspace of dimension $m+1$ must intersect a codimension-m subspace nontrivially, which is impossible here. If $N < m$, the ambient space already has dimension $N+1 \leq m$. □

The signed form itself need not be even in Lemma 2.3. Parity is used only for the positive reference norm in Lemma 2.2. Condition (9) allows equality: a null direction does not supply the missing strictly negative dimension.

3 Classical Laguerre estimates

We use the ordinary Laguerre polynomials

$$L_n(u) = \sum_{j=0}^{n} \binom{n}{j} \frac{(-u)^j}{j!}, \qquad \int_0^\infty L_n(u) L_h(u) e^{-u}\, du = \delta_{nh}. \tag{10}$$

Their classical envelope [4, Eq. 18.14.8] is

$$|L_n(u)| \leq e^{u/2}, \qquad u \geq 0. \tag{11}$$

The normalization $L_n(0) = 1$ is important. We need only the following elementary leading growth outside the positive half-line, rather than a full Plancherel–Rotach expansion.

Lemma 3.1 (External-point growth). *For every fixed $a > 0$, as $n \to \infty$,*

$$\log L_n(-a) = 2\sqrt{an} + O_a(\log(n+2)), \tag{12}$$

$$\log\left(\sum_{j=0}^{n} L_j(-a)^2\right) = 4\sqrt{an} + O_a(\log(n+2)). \tag{13}$$

Proof. All coefficients in $L_n(-a)$ are positive. Since $\binom{n}{j} \le n^j/j!$,

$$L_n(-a) \le \sum_{j\ge 0} \frac{(na)^j}{(j!)^2} \le e^{2\sqrt{na}}.$$

For the last inequality, compare the even coefficients of $e^{2\sqrt{na}}$: the inequality $\binom{2j}{j} \le 4^j$ implies $1/(j!)^2 \le 4^j/(2j)!$.

For a lower bound take $j = \lfloor \sqrt{na} \rfloor$. For large n, $1 \le j \le n/2$, and

$$\binom{n}{j} = \frac{n^j}{j!} \prod_{h=0}^{j-1} (1 - h/n) \ge \frac{n^j}{j!} e^{-j^2/n},$$

because $\log(1-t) \ge -2t$ on $[0, 1/2]$. The logarithm of this single term of (10) is at least

$$j \log(na) - 2\log(j!) - j^2/n = 2\sqrt{na} + O_a(\log(n+2)),$$

by Stirling's logarithmic formula and $j = \sqrt{na} + O(1)$. This proves (12). The kernel sum is bounded below by its last term and above by $(n+1)e^{4\sqrt{an}}$, proving (13). □

In particular, for fixed $a, A > 0$ and $X \to \infty$, the inequality $L_n(-a)^2 > AX$ is satisfied by

$$n = \left\lceil \frac{(1+\theta)(\log X)^2}{16a} \right\rceil \tag{14}$$

for any fixed $\theta > 0$ and all sufficiently large X. In the opposite direction, taking l to be the integer part of $(1-\theta)(\log X)^2/(16a)$, $0 < \theta < 1$, makes the sum in (13) at most AX for all sufficiently large X. The thresholds here may depend on fixed a, A, θ, but not on a decomposition of X into contrast and separation.

4 Proof of the Gaussian joint law

4.1 A common negative space for every admissible geometry

Fix $0 < \gamma < c$. Let

$$\Lambda_j(x) = \prod_{i \ne j} \frac{x - b_i}{b_j - b_i}$$

be the Lagrange cardinals for the atoms in (2). They satisfy $\Lambda_j(b_i) = \delta_{ij}$ and

$$|\Lambda_j(x)| \le A\Delta^{-m}(1 + |x|)^m, \tag{15}$$

with A depending only on k, B; for example the sum of absolute coefficients is at most $(1+B)^m\Delta^{-m}$. Define original-variable polynomials

$$f_j(x) = \Lambda_j(x) \frac{L_n(\gamma[(x-b_j)^2 - d^2])}{L_n(-\gamma d^2)}. \tag{16}$$

Their denominators are positive by (10). They have degree at most $2n+m$ and interpolate $f_j(b_i) = \delta_{ij}$.

On $\operatorname{supp}\nu$, all arguments of the numerator Laguerre polynomials are nonnegative. Equations (11) and (15) imply

$$e^{-cx^2/2}|f_j(x)| \le \frac{A\Delta^{-m}e^{-\gamma d^2/2}}{L_n(-\gamma d^2)}(1+|x|)^m e^{[\gamma(x-b_j)^2-cx^2]/2}.$$

There is a finite constant $M = M(\gamma, c, k, B)$ bounding the last two factors uniformly in $x \in \mathbb{R}$ and $|b_j| \le B$. Indeed, the exponent is at most $-(c-\gamma)x^2/2+\gamma B|x|+\gamma B^2/2$, whose negative quadratic coefficient pays the polynomial multiplier. Thus one fixed $A_1 > 0$ gives

$$\int f_j^2\, d|\nu| \le \frac{A_1 R\Delta^{-2m}}{L_n(-\gamma d^2)^2}. \tag{17}$$

The constant absorbs C and the displayed fixed Gaussian factor; it is independent of R, Δ, n and of the particular admissible geometry.

For a real vector $t = (t_1, \dots, t_k)$, the atomic contribution to $\mathcal{Q}_\mu(\sum_j t_j f_j)$ is exactly $-\sum_j t_j^2/k$. For the complete remainder, including its mixed terms, Cauchy–Schwarz and (17) give

$$\mathcal{Q}_\mu\left(\sum_j t_j f_j\right) \le \left[-\frac{1}{k} + \frac{kA_1 R\Delta^{-2m}}{L_n(-\gamma d^2)^2}\right]\sum_j t_j^2. \tag{18}$$

Choose n so that

$$L_n(-\gamma d^2)^2 > 2k^2 A_1 X. \tag{19}$$

The full span is then strictly negative with margin $1/(2k)$. Its dimension is k, also directly seen from interpolation. By (14), for each fixed $\theta > 0$ it suffices, for large X, to take $n = \lceil (1+\theta)(\log X)^2/(16\gamma d^2)\rceil$. Consequently

$$\limsup_{\substack{\Delta\to 0\\ R\ge 1}} \frac{D_k^G(R,\Delta)}{(\log X)^2} \le \frac{1}{8\gamma d^2}. \tag{20}$$

Here and below the notation means the bound along every allowed sequence. First let $\theta \downarrow 0$, then $\gamma \uparrow c$ after the limit. This proves the upper coefficient in (5).

If ν is positive, the space of polynomials vanishing at all k atoms has codimension at most k and nonnegative square form. The same intersection argument as in Lemma 2.3 prevents an index larger than k. No such upper-index statement is made for signed remainders.

4.2 Rational coefficients and a paid perturbation margin

Remark 4.1 (Retaining the actual geometry). In (15) one may retain the individual quantities

$$d_j = \prod_{i\ne j}|b_j - b_i|^{-2}, \qquad S_{\mathrm{geom}} = \sum_j d_j$$

instead of bounding each by Δ^{-2m}. The same proof gives $\int f_j^2\, d|\nu| \le A_1 R d_j/L_n(-\gamma d^2)^2$, with a fixed A_1. The triangle inequality followed by weighted Cauchy–Schwarz yields

$$\left\|\sum_j t_j f_j\right\|^2_{L^2(|\nu|)} \le \left(\sum_j t_j^2\right)\left(\sum_j \|f_j\|^2_{L^2(|\nu|)}\right) \le \frac{A_1 R S_{\mathrm{geom}}}{L_n(-\gamma d^2)^2}\sum_j t_j^2.$$

Thus $L_n(-\gamma d^2)^2 > 2kA_1 R S_{\mathrm{geom}}$ is a sufficient common strict-negative condition. This is a routine finite-geometry refinement of the upper bound; no matching lower theorem for irregular multiscale clusters follows from the equally spaced lower family.

The strict margin above also gives a rational basis when the declared data are certified computable. Let $J = e^{cB^2} + CR$, so that $\int e^{cx^2}\, d|\mu| \le J$. For a coefficient error polynomial $e(x) = \sum_{h=0}^{2n+m} \epsilon_h x^h$, the triangle inequality gives

$$\|e\|_{L^2(|\mu|)} \le \sqrt{J} \sum_h |\epsilon_h| M_h, \qquad M_0 = 1, \quad M_h = (h/(ce))^{h/2} \quad (h \ge 1). \tag{21}$$

This follows by maximizing $|x|^{2h} e^{-cx^2}$. The same finite sum bounds the norms of the known polynomials f_j. Choose a computable common bound $F \ge \max_j \|f_j\|_{L^2(|\mu|)}$. Round each polynomial to a rational g_j with error norm at most $\delta > 0$. For $\|t\|_2 = 1$,

$$\left| \mathcal{Q}_\mu\left(\sum_j t_j g_j\right) - \mathcal{Q}_\mu\left(\sum_j t_j f_j\right) \right| \le 2kF\delta + k\delta^2.$$

Taking the right side below $1/(4k)$ preserves strict negativity and linear independence. Both (19) and the coefficient bounds can be certified from the stated data; no original moments are needed. The construction uses rounded coefficients.

4.3 A fixed background obstructing the last negative direction

Put $z_j = j - (k+1)/2$, $1 \le j \le k$, so their minimum spacing is one and $Z = \max_j |z_j| = m/2 > 0$. Fix $\varepsilon_0 > 0$ so small that $Z\varepsilon_0 < B$, $\varepsilon_0 \le 1$, and, with $D = d + Z\varepsilon_0$,

$$e^{cD^2} < C. \tag{22}$$

Fix also $s > 0$ with $cs^2 < 1$. Let U have density e^{-u} on $[0, \infty)$ and let

$$X_* = \pm\sqrt{D^2 + s^2 U} \tag{23}$$

with independent equal signs; write τ_* for its law. Its support is $|x| \ge D$ and

$$E_s := \int e^{cx^2}\, d\tau_* = \frac{e^{cD^2}}{1 - cs^2} < \infty. \tag{24}$$

Mix this law with the endpoint probability $\tau_D = (\delta_D + \delta_{-D})/2$. For example, with $E_0 = e^{cD^2}$, choose

$$\eta = \min\left(1, \frac{C - E_0}{2(E_s - E_0)}\right) > 0, \qquad \tau = \eta\tau_* + (1 - \eta)\tau_D.$$

Then $\int e^{cx^2}\, d\tau \le C$. The background law and all these parameters are fixed before either R or ε varies.

For $R \ge 1$ and $0 < \varepsilon \le \varepsilon_0$, set

$$\mu_{R,\varepsilon} = R\tau - \frac{1}{k} \sum_{j=1}^{k} \delta_{\varepsilon z_j}. \tag{25}$$

Its minimum atom spacing is $\Delta = \varepsilon$ and $D - \varepsilon Z \ge d$, so it belongs to $\mathcal{G}_k(R, \varepsilon)$ with the same fixed budget.

The pushforward ρ of (23) under $v = x^2$ has orthonormal polynomials $L_j((v - D^2)/s^2)$. With $a = D^2/s^2$,

$$K_l = \sum_{j=0}^{l} L_j(-a)^2, \qquad \log K_l = 4\sqrt{al} + O_a(\log(l+2)). \tag{26}$$

Lemma 2.3 says that the full degree-N index is at most m when

$$K_{\lfloor (N-m)/2 \rfloor} \le A_2 X, \qquad A_2 = \frac{\eta D^{2m}}{Z^{2m}(1+\varepsilon_0^2 Z^2/D^2)} > 0, \quad X = R/\varepsilon^{2m}. \tag{27}$$

For fixed $0 < \theta < 1$, take

$$l = \left\lfloor \frac{(1-\theta)(\log X)^2}{16a} \right\rfloor, \qquad N = m + 2l.$$

For large X, (26) gives $\log K_l \le \sqrt{1-\theta}\log X + O_a(\log\log X)$. The fixed constant A_2 is therefore paid in (27). Since $n_k(\mu_{R,\varepsilon}) > N$,

$$\liminf_{\substack{\varepsilon\to 0 \\ R\ge 1}} \frac{D_k^G(R,\varepsilon)}{(\log X)^2} \ge \frac{s^2}{8D^2}, \tag{28}$$

after $\theta \downarrow 0$. Each law used to prove this inequality was fixed. Now let $s^2 \uparrow 1/c$ and then $\varepsilon_0 \downarrow 0$, so that $D \downarrow d$, after taking the indicated limits. Every sequence $\varepsilon \to 0$ eventually lies in the domain of each such fixed family. Thus the lower coefficient is at least $1/(8cd^2)$, completing the proof of Theorem 1.1.

4.4 The single-atom constant

Corollary 4.2. *Fix $c, d > 0$ and $C > e^{cd^2}$. Among $\mu = -\delta_0 + \nu$ with $\operatorname{supp}\nu \subset \{|x| \ge d\}$, $\|\nu\|_{\mathrm{TV}} \le R$, and $\int e^{cx^2}\, d|\nu| \le CR$, the supremum $D_1^G(R)$ of the least negative-square degree satisfies*

$$\frac{D_1^G(R)}{(\log R)^2} \longrightarrow \frac{1}{8cd^2} \qquad (R \to \infty).$$

Proof. The direct polynomial $P_n(x) = L_n(c(x^2 - d^2))/L_n(-cd^2)$ has $P_n(0) = 1$. Its weighted envelope on the remainder support gives

$$\int P_n^2\, d\mu \le -1 + \frac{CRe^{-cd^2}}{L_n(-cd^2)^2}.$$

Lemma 3.1 proves the upper coefficient. For the lower, use (23) with $D = d$ and the same fixed endpoint mixture. For $\mu = R\tau_* - \delta_0$, odd parts vanish at zero and have positive norm. Christoffel extremality gives the exact threshold

$$n_1(\mu) = 2\min\{l : K_l > R\}, \qquad K_l = \sum_{j=0}^{l} L_j(-d^2/s^2)^2.$$

Equality $K_l = R$ supplies a null square only. For the mixture, domination by $\eta\tau_*$ gives $n_1(R\tau - \delta_0) \ge n_1(\eta R\tau_* - \delta_0)$. Invert (13) and let $s^2 \uparrow 1/c$ after the limit. □

The corollary emphasizes the classical half-line reduction behind the Gaussian result. The joint theorem additionally requires uniform control of arbitrary translated geometries and necessity of the full k-dimensional recovery cost.

5 Exponential tails: necessity of the collision power

The earlier exponential-tail theorem [3] proves the following sufficient bound. For fixed k, B, c, d, C and positive atomic weights bounded above and below, if

$$\mu = -\sum_j w_j \delta_{b_j} + \nu, \quad \operatorname{supp}\nu \subset \bigcap_j \{|x - b_j| \geq d\}, \quad \int e^{c|x|}\, d|\mu| \leq C(1+R),$$

with $R \geq 1$, $b_j \in [-B, B]$ and minimum separation at least $0 < \Delta \leq 1$, then

$$n_k(\mu) \leq A R^{\alpha} \Delta^{-2m\alpha} \log \frac{R+2}{\Delta}, \qquad \alpha = \frac{\pi}{2cd}, \quad m = k-1. \tag{29}$$

The constant depends only on the fixed data. A signed remainder is allowed; the index is exactly k if the remainder is positive. We cite this upper theorem rather than repeat its critical-weight approximation proof.

Let $\mathcal{E}_k(R, \Delta)$ be precisely this class with weights fixed at $w_j = 1/k$, $B > 0$ and $C > e^{cd}$, and let D_k^E be the supremum of n_k. The budget in this section is the displayed full exponential variation budget, in contrast to (3).

Theorem 5.1 (Exponential collision barrier)**.** *For every fixed $k \geq 2$, $B, c, d > 0$ and $C > e^{cd}$,*

$$\frac{\log D_k^E(R, \Delta)}{\log(R/\Delta^{2(k-1)})} \longrightarrow \frac{\pi}{2cd} \quad (\Delta \downarrow 0), \tag{30}$$

along every sequence with arbitrary $R \geq 1$. More specifically, for each $0 < \beta < \pi/(2cd)$ there are $\varepsilon_0, A_\beta > 0$ and one fixed positive even probability τ such that, for every $R \geq 1$ and $0 < \varepsilon \leq \varepsilon_0$,

$$\mu_{R,\varepsilon} = R\tau - \frac{1}{k}\sum_{j=1}^{k} \delta_{\varepsilon(j-(k+1)/2)} \in \mathcal{E}_k(R, \varepsilon), \qquad n_k(\mu_{R,\varepsilon}) \geq A_\beta (R/\varepsilon^{2m})^{\beta}. \tag{31}$$

5.1 The classical continuous dual Hahn input

We record the normalization needed for the proof. For $a > 0$, the continuous dual Hahn probability W_a with parameters $(a, 1/2, 1/2)$ has density, for $u > 0$,

$$\frac{2u \tanh(\pi u)|\Gamma(a+iu)|^2}{\Gamma(a+1/2)^2}. \tag{32}$$

Its polynomial variable is u^2. In the standard normalization [4, §18.25],

$$S_n(u^2) = (a+1/2)_n^2 {}_3F_2\left(\begin{matrix}-n, a+iu, a-iu \\ a+1/2, a+1/2\end{matrix}; 1\right), \tag{33}$$

$$\|S_n\|_{W_a}^2 = (n!)^2 (a+1/2)_n^2, \qquad S_n(-a^2) = (a+1/2)_n^2. \tag{34}$$

These are the classical orthogonality data used also in [3]. Stirling's formula on a vertical line gives the tail

$$\frac{2u \tanh(\pi u)|\Gamma(a+iu)|^2}{\Gamma(a+1/2)^2} \asymp_a u^{2a} e^{-\pi u} \quad (u \to \infty).$$

Thus for the even lift $x = \pm s\sqrt{u^2 + a^2}$ the exponential moment at rate c is finite whenever $cs < \pi$. Its squared-variable kernel at $x^2 = 0$ is exactly

$$K_l = \sum_{j=0}^{l} \left[\frac{(a+1/2)_j}{j!}\right]^2 \sim \frac{l^{2a}}{2a\Gamma(a+1/2)^2}. \tag{35}$$

The evaluation $x^2 = 0$ corresponds to $u^2 = -a^2$, explaining why the scale s disappears from this kernel. The asymptotic follows from the gamma ratio and summation of j^{2a-1}; $a > 0$ makes that sum divergent with the stated power. In particular $K_l \le K_0(l+1)^{2a}$ for a fixed finite $K_0 > 0$ and all $l \ge 0$.

5.2 Proof of Theorem 5.1

Fix $\beta < \alpha$. Use the centered grid above and $Z = m/2$. Choose $0 < \varepsilon_0 \le 1$ with $\varepsilon_0 Z \le B$ such that

$$D = d + \varepsilon_0 Z, \qquad e^{cD} < C, \qquad \frac{cD}{\pi} < \frac{1}{2\beta}.$$

Next fix a, s satisfying

$$\frac{cD}{\pi} < a < \frac{1}{2\beta}, \qquad s = D/a.$$

The CDH lift τ_* has support $|x| \ge D$ and finite moment $E_s = \int e^{c|x|}\, d\tau_*$ because $cs < \pi$. Mix it with τ_D using one fixed $\eta > 0$ so that

$$\tau = \eta\tau_* + (1-\eta)\tau_D, \qquad \int e^{c|x|}\, d\tau \le C.$$

For example take $\eta = \min(1, (C - e^{cD})/[2(E_s - e^{cD})])$. The denominator is positive since τ_* is not concentrated at the endpoints. For every allowed R, ε, the background is at distance at least $D - Z\varepsilon \ge d$ from each atom. Its exponential contribution is at most CR, while the total contribution of the normalized negative atoms is at most $e^{c\varepsilon_0 Z} < C$. This verifies the exact class budget in (31).

Apply Lemma 2.3 and (35). Put

$$T = R/\varepsilon^{2m}, \quad \gamma = \frac{1}{2a} > \beta, \quad q = \left[\frac{\eta D^{2m}}{Z^{2m}(1 + \varepsilon_0^2 Z^2/D^2)K_0}\right]^{\gamma} > 0.$$

Since $l + 1 \le N + 1$, the condition $N + 1 \le qT^\gamma$ makes the degree-N index at most m, with the dimension argument covering $N < m$. If $qT^\gamma \ge 4$, choose $N = \lfloor qT^\gamma/2 \rfloor$; then $N + 1 \le qT^\gamma$ and $n_k > N \ge qT^\gamma/2 - 1$, so in fact $n_k \ge N + 1 > qT^\gamma/2$. If $qT^\gamma < 4$, the trivial bound $n_k \ge 1$ gives $n_k \ge qT^\gamma/4$. Therefore globally

$$n_k(\mu_{R,\varepsilon}) \ge \frac{q}{4}T^\gamma \ge \frac{q}{4}T^\beta,$$

where $T \ge 1$ because $R \ge 1$ and $\varepsilon \le 1$. This proves (31) with a constant independent of both variables.

For any sequence $\Delta \to 0$, each fixed-β construction eventually applies, yielding a logarithmic lower ratio at least β. The upper bound (29) gives a ratio at most α: indeed $\log((R+2)/\Delta) = O(\log(R/\Delta^{2m}))$, and its logarithm is negligible compared with $\log(R/\Delta^{2m}) \to \infty$. Let $\beta \uparrow \alpha$ after these limits to obtain (30).

The domain $\varepsilon \le \varepsilon_0$ depends on β. The claim uses the specified small-separation regime. Nor does the theorem determine an endpoint power bound with uniform constants, the optimal logarithmic factor, or a leading multiplicative constant. It does show that the inverse-separation power in (29) cannot be replaced by a merely logarithmic cost in the uniform recovery problem.

6 Subexponential tails and permanent masking

Finite recovery can fail entirely when the exponential hypothesis is weakened. We give an explicit construction at any prescribed gap and budget, followed by a compact-negative-measure extension. The generalized gamma sine perturbation is classical; see the Stieltjes-class discussions in [5, 6, 7]. The argument below additionally computes a positive external evaluation bound at the specified gap.

Theorem 6.1 (An isolated atom invisible to all polynomial squares)**.** *Fix $0 < p < 1$, $c, d > 0$ and $C > e^{cd^p}$. There are a positive even probability τ, supported on $|x| \geq d$, and an explicit finite $R_0 > 0$ such that*

$$\int e^{c|x|^p}\, d\tau \leq C, \qquad \int P^2\, d(R\tau - \delta_0) \geq 0 \quad \textit{for every real polynomial } P$$

whenever $R \geq \max(1, R_0)$. The background is fixed before R varies; it has all moments. The signed measures have remainder variation R and full moment budget at most $1 + CR$, yet $n_1 = \infty$.

6.1 Parameters, budget and the classical zero moments

Set $\beta = p/2 \in (0, 1/2)$ and $\theta = \pi\beta \in (0, \pi/2)$. Choose $a > c$ so that

$$e^{cd^p} \left(\frac{a}{a-c}\right)^{1/\beta} \leq C. \tag{36}$$

For example, any $a > c/[1 - (e^{cd^p}/C)^\beta]$ suffices. Define

$$f(u) = e^{-au^\beta}, \qquad Z_a = \frac{\Gamma(1/\beta)}{\beta a^{1/\beta}}, \qquad dW(u) = Z_a^{-1} f(u)\, du \quad (u > 0), \tag{37}$$

$$h(u) = \sin(a \tan\theta\, u^\beta). \tag{38}$$

Let τ be the law of $\pm\sqrt{U + d^2}$, where $U \sim W$ and the sign is independent and fair. All polynomial moments are finite. Since $(u + d^2)^\beta \leq u^\beta + d^{2\beta}$,

$$\int e^{c|x|^p}\, d\tau \leq e^{cd^p} \frac{Z_{a-c}}{Z_a} = e^{cd^p} \left(\frac{a}{a-c}\right)^{1/\beta} \leq C. \tag{39}$$

For every integer $n \geq 0$, the complex gamma integral gives

$$\int_0^\infty u^n f(u) h(u)\, du = \frac{\Gamma((n+1)/\beta)}{\beta} \operatorname{Im}[a(1 - i\tan\theta)]^{-(n+1)/\beta} = 0. \tag{40}$$

The complex number inside the negative power has argument $-\theta$; the resulting sine is $\sin((n+1)\pi) = 0$. All integrals are absolutely convergent. Thus $W(1 + th)$, $-1 \leq t \leq 1$, is a classical family of positive probability measures with the same moments. Moment indeterminacy alone will not be used as a substitute for an evaluation bound at the specific external point needed below.

6.2 An explicit external Stieltjes transform

Lemma 6.2. *For every $z > 0$,*

$$S_z := \int_0^\infty \frac{f(u)h(u)}{u + z}\, du = \pi \exp\left(-\frac{az^\beta}{\cos\theta}\right) > 0. \tag{41}$$

Proof. Put $A = a(1 - i\tan\theta) = ae^{-i\theta}/\cos\theta$. Use the branch $0 \le \arg w \le \pi$ for w^β and the function $F(w) = e^{-Aw^\beta}/(w+z)$. Throughout this sector,

$$\operatorname{Re}(Aw^\beta) \ge a|w|^\beta,$$

because $-\theta + \beta \arg w \in [-\theta, 0]$. Rotate the positive ray counterclockwise to angle $0 < \phi < \pi$. There is no enclosed pole; the outer arc tends to zero by the displayed exponential decay and the small arc at zero tends to zero. Therefore

$$\int_0^\infty \frac{e^{-Au^\beta}}{u+z}\,du = \int_0^\infty \frac{e^{-Ae^{i\beta\phi}v^\beta}}{v+ze^{-i\phi}}\,dv. \tag{42}$$

As $\phi \uparrow \pi$, the numerator tends to the real function $g(v) = e^{-av^\beta/\cos\theta}$ and the denominator tends to $v - z - i0$. The boundary value of the integral on the right is

$$\mathrm{PV}\int_0^\infty \frac{g(v)}{v-z}\,dv + i\pi g(z).$$

Here the boundary passage can be verified without an interchange at a singularity. Outside a fixed neighborhood of z, use dominated convergence with the uniform exponential decay. Inside it, write the numerator as its value at z plus its difference from that value. These numerators converge in C^1 on that neighborhood. The difference term has the usual convergent principal-value limit, whereas integrating the constant term explicitly by a logarithm gives the jump $+i\pi$. The displacement of the real part of the pole is $O((\pi-\phi)^2)$ and its imaginary part is positive of order $\pi - \phi$, so the same local subtraction gives this upper-half-plane boundary sign. Take imaginary parts in (42) to obtain (41). □

6.3 A uniform all-degree evaluation bound

For any real polynomial Q, the quotient $(Q(u) - Q(-z))/(u+z)$ is a polynomial. Equations (40) and (41) consequently imply

$$\int_0^\infty \frac{Q(u)f(u)h(u)}{u+z}\,du = S_z Q(-z).$$

Cauchy–Schwarz and $|h| \le 1$ give

$$\begin{aligned} S_z^2 Q(-z)^2 &\le \left(\int Q^2 f\,du\right)\left(\int \frac{fh^2}{(u+z)^2}\,du\right) \\ &\le Z_a^2 z^{-2}\int Q^2\,dW. \end{aligned}$$

Hence the degree-independent bound

$$\int Q^2\,dW \ge \lambda_z Q(-z)^2, \qquad \lambda_z = \left[\frac{\pi z}{Z_a}e^{-az^\beta/\cos\theta}\right]^2 > 0 \tag{43}$$

holds for every polynomial. This is an explicit conservative lower bound for a Christoffel value.

For $z = d^2$, split an arbitrary real polynomial as $P(x) = Q(x^2 - d^2) + xH(x^2 - d^2)$. Symmetry gives

$$\int P^2\,d\tau = \int [Q(u)^2 + (u+d^2)H(u)^2]\,dW(u) \ge \lambda_{d^2} P(0)^2. \tag{44}$$

Theorem 6.1 follows with

$$R_0 = \lambda_{d^2}^{-1} = \left[\frac{Z_a}{\pi d^2} e^{ad^p/\cos(\pi p/2)}\right]^2, \tag{45}$$

since $\mathcal{Q}_{R\tau-\delta_0}(P) \geq (R\lambda_{d^2} - 1)P(0)^2 \geq 0$. This proves an all-degree obstruction directly, rather than extrapolating from finite moment matrices.

Corollary 6.3 (Compact negative measures)**.** *Keep the background τ of Theorem 6.1. Fix $0 \leq r < d$ and a finite positive measure κ supported on $[-r, r]$, of mass w. Set $z_- = d^2 - r^2 > 0$ and*

$$\lambda_* = (\pi/Z_a)^2 \min\left\{z_-^2 e^{-2az_-^\beta/\cos\theta}, d^4 e^{-2ad^p/\cos\theta}\right\} > 0. \tag{46}$$

Then every polynomial square has nonnegative integral against $R\tau - \kappa$ whenever $R \geq 2w/\lambda_*$. Moreover,

$$\int e^{c|x|^p}\, d|R\tau - \kappa| \leq CR + we^{cr^p}.$$

Proof. For $|b| \leq r$ put $z = d^2 - b^2 \in [z_-, d^2]$. The function $\log(z^2 e^{-2az^\beta/\cos\theta})$ has derivative $2/z - 2a\beta z^{\beta-1}/\cos\theta$, which changes sign at most once, from positive to negative. Its minimum on this interval is therefore at an endpoint, so $\lambda_z \geq \lambda_*$. For the split in (44), equation (43) gives

$$\begin{aligned} |P(b)|^2 &\leq 2[Q(-z)^2 + b^2 H(-z)^2] \\ &\leq \frac{2}{\lambda_*}\int [Q(u)^2 + b^2 H(u)^2]\, dW(u) \leq \frac{2}{\lambda_*}\int P^2\, d\tau, \end{aligned}$$

because $b^2 < d^2 \leq u + d^2$. Integrate this bound against κ. The final moment estimate follows from (39) and its support bound. The supports of τ and κ are separated by at least $d - r$. □

The negative measure may be atomic, absolutely continuous or singular continuous. The result supplies admissible examples; it does not say that every measure satisfying a subexponential bound is moment-indeterminate, or that every negative atom under such a bound is invisible. The threshold (45) can become very large near $p = 1$ or the endpoint budget. The mechanism is consistent with classical moment indeterminacy, rather than a new determinacy criterion.

7 Finite reproduction and scope

The accompanying standard-library script `verify_finite_controls.py` performs exact rational checks; it requires no supplied moment table or external dependency. Run

```
python verify_finite_controls.py
python -O verify_finite_controls.py
```

Its checks use explicit conditions, which are not removed by optimized Python execution. For the Gaussian lower example $X = \pm\sqrt{1+U}$, U exponential, it independently reconstructs

$$\mathbb{E}X^{2j} = \sum_{h=0}^{j} \binom{j}{h} h!, \qquad \mathbb{E}X^{2j+1} = 0,$$

expands $L_n(x^2 - 1)$, and verifies its original-moment Gram matrix. It compares full original Hankel inertia with the exact kernel threshold, including strict-negative and exact-null cases, and contracts the normalized kernel squares against those original moments.

The same law supplies finite clustered-atom examples for $k = 2, 3, 4$. For those examples the script pays a stated finite kernel bound, checks the entire vanishing-jet Gram block and the full signed moment index, and includes a zero-background negative control. The support is $|x| \geq 1$ and the distance to the clustered atoms is at least $1 - \max_j |b_j| > 0$. They illustrate the index logic rather than reproduce the CDH asymptotic construction. Finite arithmetic tests prove neither the asymptotic theorems nor the all-degree masking identity.

The Gaussian conclusion determines an exact leading minimax constant in its declared joint collision limit. The exponential conclusion determines the logarithmic power, leaving its leading constant and logarithmic factor unresolved. The subexponential construction marks a different boundary: under the weaker hypotheses, the least certificate degree can be infinite. None of these statements implies a running-time bound or a sign conclusion for a particular spectral or theta measure without first verifying that measure's hypotheses.

Disclosure. AI tools assisted with mathematical development, source preparation and verification. The author is responsible for the statements and proofs. The author declares no competing interests.

References

[1] A. M. Delgado, L. Fernández, T. E. Pérez, and M. A. Piñar, Multivariate orthogonal polynomials and modified moment functionals, *SIGMA* **12** (2016), 090. https://doi.org/10.3842/SIGMA.2016.090.

[2] B. Beckermann, M. Putinar, E. B. Saff, and N. Stylianopoulos, *Perturbations of Christoffel–Darboux kernels. I: Detection of outliers*, arXiv:1812.06560v2, 2019. https://arxiv.org/abs/1812.06560v2.

[3] S. Schubr, *Sharp Degree Bounds for Signed-Measure Certificates*, 2026. https://research.luminaryoddities.com/sharp-degree-bounds-signed-measure-certificates/v/1.

[4] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain (eds.), *NIST Digital Library of Mathematical Functions*, Chapter 18, especially §§18.2, 18.14 and 18.25. https://dlmf.nist.gov/.

[5] G. D. Lin, Recent developments on the moment problem, *Journal of Statistical Distributions and Applications* **4** (2017), article 5. https://doi.org/10.1186/s40488-017-0059-2.

[6] C. Kleiber, The generalized lognormal distribution and the Stieltjes moment problem, *Journal of Theoretical Probability* **27** (2014), 1167–1177. https://arxiv.org/abs/1301.1277.

[7] J. Stoyanov and L. Tolmatz, New Stieltjes classes involving generalized gamma distributions, *Statistics & Probability Letters* **69** (2004), 213–219.