Strong permanent inequalities for three-row bipartite and triangle-free supports

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Abstract

For every complex $3 \times q$ contraction X, we prove that the permanent of $\left(\begin{smallmatrix} I_3 & X \\ X^* & I_q \end{smallmatrix}\right)$ is at least the permanent of its entrywise squared modulus. The proof retains all square-minor orders and controls the coupled third-order cancellations through the row Gram matrix. It gives an explicit sum of nonnegative lower bounds and includes arbitrary phases, zero entries and singular endpoints. Combining this theorem with three pentagon blow-up cases and an exact finite graph classification proves the same strong inequality for every positive semidefinite matrix with triangle-free support on at most seven vertices, after diagonal scaling. Two previously unpublished blow-up cases are proved here, one analytically and one by an explicit positive integer polynomial certificate. Reproducibility supplements contain the certificate and finite classification verifier. The unrestricted Chollet conjecture is not settled.

1 Main results and relation to earlier work

For a Hermitian matrix A, write $\mathcal{P}(A) = \operatorname{per}(A)$ and $\mathcal{Q}(A) = \operatorname{per}(|A|^2)$, with absolute value and square entrywise. Write $d(A) = \prod_i a_{ii}$. Support graphs record off-diagonal positions allowed to be nonzero; a host edge may have weight zero. A correlation matrix is positive semidefinite with diagonal one.

Theorem 1. *For every finite $q \geq 1$ and complex $3 \times q$ matrix X with $\|X\|_{\mathrm{op}} \leq 1$, set*

$$A = \begin{pmatrix} I_3 & X \\ X^* & I_q \end{pmatrix}.$$

Then $\mathcal{Q}(A) \leq \mathcal{P}(A)$. Consequently every complex positive semidefinite matrix supported in a complete bipartite graph $K_{p,q}$ with $\min(p, q) \leq 3$ satisfies

$$\mathcal{Q}(A) \leq d(A)\mathcal{P}(A). \tag{1}$$

Theorem 2. *Every complex positive semidefinite matrix of order at most seven whose off-diagonal support is triangle-free satisfies* (1)*.*

The first theorem is an all-order statement with one bounded bipartition. The second is an order-bounded consequence using additional nonbipartite hosts. The proof of Theorem 1 uses the full operator contraction.

Chollet's permanent conjecture concerns $\operatorname{per}(A \circ B) \leq \operatorname{per}(A)\operatorname{per}(B)$ for positive semidefinite inputs [1]. Its normalized self-case is $\mathcal{Q}(A) \leq \mathcal{P}(A)^2$. The stronger self-inequality proved here fails for unrestricted positive semidefinite matrices [2]. Graph-based structural methods for the conjecture are developed by Pant and Singh [3]. The author's earlier papers treat cycles and single

chords [4], and overlapping cycles in two- and three-chord seven-cycle supports [5]. The present paper extends the bipartite side to every $K_{3,q}$ and completes the remaining nonbipartite maximal triangle-free hosts of order seven. The previously published adjacent-doubled pentagon case is cited explicitly below; its certificate is not represented as a new result.

2 Exact expansions for three rows

Let $X = (x_{aj})$, with rows $a = 0, 1, 2$. Put

$$G = XX^*, \quad r_a = G_{aa}, \quad g_{ab} = G_{ab}, \quad w_{aj} = |x_{aj}|^2, \quad c_j = \sum_a w_{aj}, \quad S = \sum_{a,j} w_{aj}, \quad V = \sum_{a,j} w_{aj}^2.$$

For $a < b$ let c denote the unique other row, and set

$$D_{ab} = \sum_j w_{aj} w_{bj}, \qquad H_{ab} = \sum_j w_{cj} x_{aj} \overline{x_{bj}}, \qquad T = \sum_j w_{0j} w_{1j} w_{2j}.$$

Here $0 \preceq G \preceq I_3$, and $0 \le r_a, c_j \le 1$.

For row and column subsets I, J of equal size write $X[I, J]$ for the square submatrix. A permutation of the block matrix chooses the same number of vertices in each part to cross. The two cross-part bijections are independent. Summing them gives

$$\mathcal{P}(A) = \sum_{k=0}^{3} \sum_{|I|=|J|=k} |\operatorname{per} X[I, J]|^2, \tag{2}$$

$$\mathcal{Q}(A) = \sum_{k=0}^{3} \sum_{|I|=|J|=k} \operatorname{per}(W[I, J])^2, \tag{3}$$

where $W = (w_{aj})$ and the empty permanent is one. A nonexistent minor order contributes zero. Thus the formulas include $q < 3$.

Let F_2, F_3 denote the order-two and order-three sums in (2).

Lemma 3. *For arbitrary complex X, without a contraction assumption,*

$$F_2 = \sum_{a<b} \big(r_a r_b + |g_{ab}|^2 - 2D_{ab}\big), \tag{4}$$

$$F_3 = \operatorname{per}(G) - 2 \sum_{a<b} r_c D_{ab} - 4\Re \sum_{a<b} \overline{g_{ab}} H_{ab} + 12T. \tag{5}$$

Proof. Expanding each two-by-two permanent and summing distinct column pairs proves (4). For clarity we derive the collision multiplicities in (5). Expand $\operatorname{per}(XX^*)$ by column assignments and group by column multiplicities. A multiset with multiplicities n_j contributes the squared modulus of the permanent of X with those repeated columns, divided by $\prod_j n_j!$. This follows directly by collecting the equal labelled assignments.

Distinct columns contribute F_3. For columns j, j, k with $j \ne k$, the repeated-column permanent is $2 \sum_a x_{ak} \prod_{b \ne a} x_{bj}$. After division by 2!, its contribution is twice the squared modulus of that sum. A triple column contributes $6 \prod_a w_{aj}$. Allowing $k = j$ in the double-column sum adds $18 \prod_a w_{aj}$. Hence

$$F_3 = \operatorname{per}(G) - 2 \sum_{j,k} \left| \sum_a x_{ak} \prod_{b \ne a} x_{bj} \right|^2 + 12T.$$

On expanding the square, its diagonal part is $\sum_{a\le b} r_c D_{ab}$. Its cross part is $2\Re\sum_{a<b}\overline{g_{ab}}H_{ab}$: summing the k factor gives g_{ab} and the j factor gives $\overline{H_{ab}}$. This proves (5), including the conjugations and the coefficient 12. □

Define the size-k matching sum

$$Z_k = \sum_{|I|=|J|=k} \operatorname{per}(W[I,J]).$$

Inclusion–exclusion on repeated columns yields

$$Z_2 = \sum_{a<b}(r_a r_b - D_{ab}), \qquad Z_3 = r_0 r_1 r_2 - \sum_{a<b} r_c D_{ab} + 2T. \tag{6}$$

Every square submatrix B of W has permanent between zero and one: its permanent is at most the product of its row sums, which are at most one. Therefore

$$E = \sum_{k=2}^{3} \sum_{|I|=|J|=k} \operatorname{per}(W[I,J])\big(1 - \operatorname{per}(W[I,J])\big) \ge 0. \tag{7}$$

Using (2)–(6), $\operatorname{per}(G) = r_0 r_1 r_2 + \sum r_c |g_{ab}|^2 + 2\Re(g_{01}g_{12}g_{20})$, and $S - V = \sum_j c_j(1-c_j) + 2\sum D_{ab}$ gives the exact identity

$$\begin{aligned}\mathcal{P}(A) - \mathcal{Q}(A) = E &+ \sum_j c_j(1-c_j) + \sum_{a<b}(1-r_c)D_{ab} \\ &+ \sum_{a<b}(1+r_c)|g_{ab}|^2 + 2\Re(g_{01}g_{12}g_{20}) \\ &- 4\Re\sum_{a<b}\overline{g_{ab}}H_{ab} + 10T.\end{aligned} \tag{8}$$

No independent choice of phases has entered this identity.

3 Controlling the coupled cancellations

Lemma 4. *For the quantities above,*

$$2\Re(g_{01}g_{12}g_{20}) \ge -\frac{1}{3}\sum_{a<b}|g_{ab}|^2, \qquad |H_{ab}|^2 \le r_c T. \tag{9}$$

Proof. Since $0 \preceq G \preceq I$, the operator norm of $G - I/2$ is at most $1/2$, so each $|g_{ab}| \le 1/2$. For $u, v, z \in [0, 1/2]$,

$$u^2 + v^2 + z^2 \ge 3(uvz)^{2/3} \ge 6uvz.$$

This proves the first inequality. Weighted Cauchy–Schwarz proves the second:

$$|H_{ab}|^2 = \left|\sum_j \sqrt{w_{cj}}\big(\sqrt{w_{cj}}\,x_{aj}\overline{x_{bj}}\big)\right|^2 \le \Big(\sum_j w_{cj}\Big)\Big(\sum_j w_{cj}w_{aj}w_{bj}\Big) = r_c T.$$

□

Put $a_c = 2/3 + r_c > 0$. Completing a complex square gives

$$a_c|g_{ab}|^2 - 4\Re(\overline{g_{ab}}H_{ab}) = a_c\left|g_{ab} - \frac{2H_{ab}}{a_c}\right|^2 - \frac{4|H_{ab}|^2}{a_c}.$$

For $0 \le r_c \le 1$, $4r_c/(2/3 + r_c) \le 12/5$. Lemma 4 and (8) consequently imply the explicit bound

$$\begin{aligned}\mathcal{P}(A) - \mathcal{Q}(A) \ge E + \sum_j c_j(1-c_j) + \sum_{a<b}(1-r_c)D_{ab} \\ + \sum_{a<b}\left(\frac{2}{3} + r_c\right)\left|g_{ab} - \frac{2H_{ab}}{2/3 + r_c}\right|^2 + \frac{14}{5}T \ge 0.\end{aligned} \tag{10}$$

Indeed the three completed squares consume at most $36T/5$ from the $10T$ in (8). All remaining terms are nonnegative. This proves the normalized assertion of Theorem 1. Zero rows, repeated singular values and operator-norm-one endpoints require no division by a vanishing quantity. A smaller bipartition can be padded by zero rows, adding isolated unit diagonal vertices to the block matrix. Diagonal scaling is treated in Section 6.

4 Two pentagon blow-up lemmas

We supply the additional nonbipartite cases needed for Theorem 2. For a graph H and edge weights w_e, let

$$Z_H(w) = \sum_{M \text{ matching}} m_M, \quad m_M = \prod_{e\in M} w_e, \qquad L_H = Z_H(w) - Z_H(w^2).$$

The triangle-free vertex-star bound for a correlation matrix is

$$s_v = 1 - \sum_{e\ni v} w_e \ge 0. \tag{11}$$

It follows by taking the Schur complement in the principal star matrix $\left(\begin{smallmatrix} 1 & z^* \\ z & I \end{smallmatrix}\right)$. The proofs in this section use only $w_e \ge 0$ and (11).

On a triangle-free seven-vertex graph no two long cycles are disjoint. For each unoriented cycle C, let $h_C = (\prod_{e\in C} w_e)^{1/2}$ and let $Z_{\bar C}$ be the matching function of its induced vertex complement. Expanding permutations into fixed points, transpositions and at most one longer cycle, and pairing its two orientations, gives

$$\mathcal{P}(A) - \mathcal{Q}(A) \ge L_H - \sum_C E_C, \qquad E_C = 2h_C(1+h_C)Z_{\bar C}(w). \tag{12}$$

Here we used $\Re H_C \ge -h_C$ and $Z_{\bar C}(w^2) \le Z_{\bar C}(w)$. Thus arbitrary common phases are covered. This is a Sachs-type cycle expansion; a complex gain-graph formulation appears in [6].

4.1 A tripled part

Lemma 5. *Let H_t have edges $05, 06, 15, 16, 25, 26, 34, 36, 45$. Under (11), every unit-diagonal Hermitian matrix supported in H_t satisfies $\mathcal{P}(A) - \mathcal{Q}(A) \ge L_{H_t}/700$.*

Proof. The core B is $K_{2,3}$ with hubs $5,6$ and leaves $0,1,2$, and an external path $5,4,3,6$. Each simple cycle uses two of the four internally disjoint hub-to-hub paths. There are exactly three four-cycles and three five-cycles.

Write $u_j = w_{5j}$, $v_j = w_{6j}$ and $h_{jk} = \sqrt{u_j u_k v_j v_k}$. The hub sums are at most one, so the singleton core loss is at least $2\sum_{j<k}(u_j u_k + v_j v_k) \geq 4\sum h_{jk}$. For opposite matching weights $p = u_j v_k$, $q = u_k v_j$, Cauchy–Schwarz gives $\sqrt{p} + \sqrt{q} \leq 1$. With $h = \sqrt{pq}$,

$$p(1-p) + q(1-q) - 2h(1-h) = (p+q-2h)(1-p-q-2h) \geq 0. \tag{13}$$

Each size-two core matching belongs to one leaf pair, and there are no larger core matchings. Consecutive cycle weights sum to at most one, which implies $h_{jk} \leq 1/4$. Hence

$$L_B \geq \sum_{j<k}(6h_{jk} - 2h_{jk}^2) \geq \frac{11}{2}\sum_{j<k} h_{jk}.$$

Put $z = w_{34}$. Each four-cycle complement has matching function $1 + z$. Therefore their total cost in (12) is at most $(5/11)R_B$, where $R_B = (1+z)L_B$. Each term of R_B charges a full matching at most once, with external-edge pattern empty or 34. Adding the latter edge charges $zm(1-m) \leq zm(1-zm)$; the empty core matching has zero loss.

For a five-cycle, adjacent inequalities give $h \leq 2^{-5/2}$ and

$$U_C := \sum_{e \in C} w_e(1-w_e) \geq \sum_i w_i w_{i+1} \geq 5h^{4/5}.$$

Thus $2h(1+h) \leq \gamma U_C$, where

$$\gamma = \frac{\sqrt{2}}{5} + \frac{1}{20} < \frac{233}{700}.$$

The ratio bound follows by maximizing $(2/5)h^{1/5}(1+h)$ at the product cap. The rational comparison is $\sqrt{2} < 99/70$, or $9800 < 9801$. Every five-cycle complement consists of two isolated leaves, so its cost is at most $(233/700)U_C$ and charges only singletons.

An external path singleton lies in all three five-cycles and has no nonzero cluster charge; its coefficient is $699/700$. A core singleton receives at most $5/11 + 233/700 = 6063/7700 < 699/700$. A matching of size at least two receives only the cluster coefficient $5/11$. Therefore total cost is at most $(699/700)L_{H_t}$. Equation (12) proves the lemma, with zero-product cases included directly. □

4.2 Two nonadjacent doubled parts

Lemma 6. *Let H_n have ordered edges*

$$05, 06, 15, 16, 24, 26, 34, 36, 45. \tag{14}$$

Under (11)*, every unit-diagonal Hermitian matrix supported in H_n satisfies $\mathcal{P}(A) - \mathcal{Q}(A) \geq L_{H_n}/50$.*

Proof. There are two four-cycles with edge-index lists $(0,1,2,3)$ and $(4,5,6,7)$, and four five-cycles with lists $(0,1,4,5,8)$, $(0,1,6,7,8)$, $(2,3,4,5,8)$ and $(2,3,6,7,8)$. These are all cycles.

For a cyclic edge order e_i define

$$A_C = \Big(\sum_i w_i w_{i+1}\Big) Z_{\bar{C}}, \qquad B_C = J_C Z_{\bar{C}}, \quad J_C = \sum_{I \text{ selected}} m_I(1-m_I).$$

For a four-cycle select its two opposite pairs once each; for a five-cycle select its five distance-two pairs $\{e_i, e_{i+2}\}$. Consecutive bounds give $h_C \le 2^{-|C|/2}$. AM–GM yields $E_C \le (5/8)A_C$ for a four-cycle. Identity (13) gives $J_C \ge 2h_C(1-h_C)$, hence $E_C \le (5/3)B_C$ there.

For a five-cycle put $m_i = w_i w_{i+2}$ and $q_i = \sqrt{m_i m_{i+1}}$. Cauchy–Schwarz gives

$$\sqrt{m_i} + \sqrt{m_{i+1}} \le \sqrt{(w_i + w_{i+1})(w_{i+2} + w_{i+3})} \le 1.$$

Applying (13) to m_i, m_{i+1}, summing and dividing by two gives

$$J_C \ge \sum_i q_i(1-q_i) \ge \frac{3}{4}\sum_i q_i \ge \frac{15}{4} h_C^{4/5}.$$

The resulting ratio is at most $(8/15)h^{1/5}(1+h)$, whose endpoint is $(4\sqrt{2}+1)/15 < 4/9$. The last comparison is $288 < 289$. Thus $E_C \le (4/9)B_C$ for each five-cycle.

Set $x_C = 147/250$ for each four-cycle and $x_C = 1$ for each five-cycle, and $\beta_4 = 5/3$, $\beta_5 = 4/9$. Then

$$E_C \le (1-x_C)\frac{5}{8}A_C + x_C\beta_{|C|}B_C,$$

with the angle term zero for five-cycles. An internal selected pair I and an outside matching T determine a matching $M = I \cup T$ and charge at most its full loss, since $m_I(1-m_I)m_T \le m_M(1-m_M)$. For a fixed cycle this representation is unique. Define

$$c_M = \sum_{C:\, M \text{ eligible}} 3600 x_C \beta_{|C|}, \qquad A_{\mathrm{mix}} = \sum_{|C|=4} 2250(1-x_C)A_C.$$

Eligibility means exactly two edges of M lie on C, form a selected pair, and every other edge has both endpoints outside C.

The accompanying `nonadjacent_certificate.json` specifies the exact identity

$$\sum_{1\le|M|\le2} (3528 - c_M)m_M(1-m_M) - A_{\mathrm{mix}} = \sum_{j=1}^{119} N_j w^{a_j} \prod_{v\in V_j} s_v. \tag{15}$$

Every N_j is a positive integer, every exponent in a_j is nonnegative integral, and V_j is a multiset of at most two vertices; total degree is at most four. The certificate's `monomial`, `slacks` and `integer_coefficient` fields are respectively a_j, V_j, N_j, with no multiplier. Its metadata specify (14), all six cycles, mixtures, scale 3600 and budget 3528.

The supplied `verify_nonadjacent_certificate.py` reconstructs the graph cycles, all 38 matchings, selected pairs, complementary matchings and charges. Exact rational expansion requires coefficientwise equality in (15), and checks $c_M \le 3528$ for size-three matchings (the maximum is 3528). This is a finite positive integer identity. Products of defining slacks are a standard positivity framework [7]; here the actual identity establishes positivity directly on all closed faces, without invoking an existence theorem for strictly positive polynomials.

The right side of (15) is nonnegative by (11). Applying this bound to sizes one and two and the charge bound to size three proves $3600\sum_C E_C \le 3528 L_{H_n}$. Equation (12) completes the proof. Every estimate includes zero weights. □

5 Finite classification and triangle-free coverage

Lemma 7. *Up to isomorphism there are six maximal triangle-free graphs on seven vertices: $K_{1,6}$, $K_{2,5}$, $K_{3,4}$, and the three pentagon blow-ups with part sizes $(2,2,1,1,1)$, $(2,1,2,1,1)$ and $(3,1,1,1,1)$ in cyclic order.*

Proof. We use exhaustive finite enumeration, supplied as `classify_trianglefree.py`. Here is its complete search principle. Given a labelled triangle-free graph on vertices $0, \dots, n-1$, add vertex n with each possible independent subset of the old vertices as its neighborhood. A triangle through the new vertex is absent exactly when that subset is independent. Deleting the largest vertex shows inductively that every labelled triangle-free graph is generated exactly once. Continue to seven vertices. A resulting graph is maximal exactly when every nonedge has a common neighbor: this is precisely the condition that adding that edge creates a triangle.

The enumeration produces 133501 labelled triangle-free graphs and 1743 maximal ones. For isomorphism testing it compares all degree-preserving vertex permutations, and obtains the six classes above. In the displayed order their labelled multiplicities are $7, 21, 35, 630, 630, 420$, summing to 1743. Explicit representatives of the last two classes are H_n, H_t. A representative of the adjacent-doubled class is the graph with cycle $01, 12, 23, 34, 45, 56, 60$ and chords $03, 15, 26$. The executable supplement embeds the six representatives and checks that each maximal output matches exactly one representative. Thus it verifies both exhaustion and absence of duplicate classes, without depending on graph-library labels or a floating computation. □

Proof of Theorem 2. Normalize the diagonal, with zero diagonals handled in Section 6, and pad isolated unit diagonal vertices to order seven if necessary. Extend the support graph to a maximal triangle-free graph by repeatedly adding an edge that creates no triangle. The procedure terminates and Lemma 7 lists its possible outcomes. Newly added host edges receive matrix entry zero, which changes neither matrix nor permanents.

Each of the three bipartite maximal graphs is covered by Theorem 1; $K_{1,6}$ and $K_{2,5}$ are included by zero-row padding. The nonadjacent-doubled and tripled-part hosts are covered by Lemmas 6 and 5. For the adjacent-doubled host, the author's previously published three-chord theorem [5] proves the stronger explicit gap $L_H/50$ for the cycle with chords $03, 15, 26$. Graph relabeling preserves both permanents. These six cases prove the claim. This last step is a stated dependency on the published theorem and its electronic certificate, rather than a new proof of that predecessor. □

The three nonbipartite cases in fact give a uniform normalized gap $L_H/700$ for every nonbipartite triangle-free support through order seven: a maximal extension retains an existing odd cycle and is therefore one of those three classes.

6 Scaling, consequences and limitations

For a positive semidefinite matrix with strictly positive diagonal, diagonal normalization gives $A = DCD$, where $D_{ii} = \sqrt{a_{ii}}$ and C is a correlation matrix. Consequently

$$\mathcal{P}(A) = d(A)\mathcal{P}(C), \qquad \mathcal{Q}(A) = d(A)^2\mathcal{Q}(C).$$

The normalized theorems therefore prove (1). If $a_{ii} = 0$, positivity gives $|a_{ij}|^2 \le a_{ii}a_{jj} = 0$ for every j; the resulting zero row forces both permanents to vanish. These cases include every singular endpoint.

For correlation matrices in the proved classes, $\mathcal{Q}(A) \ge 1$ and $\mathcal{P}(A) \ge \mathcal{Q}(A)$, so $\mathcal{Q}(A) \le \mathcal{P}(A)^2$. The result controls the self-case on the specified supports. It does not give Bapat–Sunder domination with an arbitrary second positive semidefinite factor, nor strong contraction for $K_{4,4}$, supports with triangles, arbitrary rank-six matrices of order seven, or unrestricted matrix order.

Reproducibility and acknowledgment

The three-row theorem is entirely analytic. The electronic supplements contain the 119-term integer certificate and verifier for Lemma 6, and the exhaustive classification verifier for Lemma 7. They require only the Python standard library. Finite verification proves the displayed identities and enumerations; the reductions to these statements are given above. AI tools assisted with drafting and verification. The author is responsible for the statements, proofs, attribution and final text.

References

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