Strong permanent inequalities on structured supports, with an application to graph Laplacians

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Abstract

For a complex Hermitian positive semidefinite matrix A, the strong permanent inequality $\operatorname{per}(A \circ \overline{A}) \le \operatorname{per}(A) \prod_i a_{ii}$ is the self-conjugate case of the Bapat–Sunder inequality, which Drury showed to fail at order seven. We prove the strong inequality whenever the off-diagonal support of A is bipartite with one part of at most five vertices and the other part of arbitrary size; is a cycle of length at least four or a triangle-free cycle with one chord; or is triangle-free on at most seven vertices. The same inequality then holds hereditarily whenever every block of the support graph is of one of these kinds. The five-row bipartite case combines a three-row identity, a large-entry transfer through Lieb's block inequality and a row-dependent Carlen–Lieb–Loss budget, whose positivity is an exact certificate of 2,097,152 tensor Bernstein coefficients; equality is classified there. The cycle-type supports are handled by allocating matching-partition losses among overlapping cycles, with explicit gaps on chorded seven-cycles from an exact 323-term integer identity. As a consequence, the question of Pant and Singh whether every graph Laplacian satisfies $\operatorname{per}(L \circ L) \le \operatorname{per}(L) \prod_v \deg v$ has an affirmative answer for every graph whose blocks are bipartite graphs, cycles, triangle-free single-chord cycles or triangle-free graphs on at most seven vertices. An exact computation confirms the Laplacian inequality for all 1,915,547 labelled graphs on two to seven vertices without isolated vertices.

Keywords. permanent inequality; Chollet conjecture; Bapat–Sunder inequality; positive semidefinite matrix; graph Laplacian; exact certificate
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1 Introduction

For a square matrix $A = (a_{ij})$ of order n the permanent is $\operatorname{per} A = \sum_{\sigma \in S_n} \prod_i a_{i\sigma(i)}$, with $\operatorname{per}(\varnothing) = 1$. Throughout, $A \succeq 0$ means that A is complex Hermitian positive semidefinite, and

$$\mathcal{P}(A) = \operatorname{per} A, \qquad \mathcal{Q}(A) = \operatorname{per}(A \circ \overline{A}) = \operatorname{per} |A|^2, \qquad \mathrm{d}(A) = \prod_i a_{ii},$$

where $\circ$ is the entrywise product and $|A|^2$ is taken entrywise.

Three permanent inequalities. Chollet asked whether $\operatorname{per}(A \circ B) \le \operatorname{per}(A) \operatorname{per}(B)$ for all $A, B \succeq 0$ [5]; it suffices to treat $B = \overline{A}$, that is $\mathcal{Q}(A) \le \mathcal{P}(A)^2$. Bapat and Sunder asked for the stronger bound $\operatorname{per}(A \circ B) \le \operatorname{per}(A) \prod_i b_{ii}$ [1, p. 117], [2, Conjecture 2]. Its self-conjugate case is the *strong inequality*

$$\mathcal{Q}(A) \le \mathrm{d}(A) \mathcal{P}(A), \tag{1.1}$$

which implies the self-case of Chollet's inequality through permanent dominance $\mathcal{P}(A) \geq \mathrm{d}(A)$ [10]. Shchesnovich showed that the permanent-on-top conjecture, which is stronger still, is false [16], and Drury then gave a rank-two complex correlation matrix of order seven with $\mathcal{P}(A) = 45$ and $\mathcal{Q}(A) = 6185/128$, so that $\mathcal{Q}(A)/(\mathrm{d}(A)\mathcal{P}(A)) = 1237/1152 \approx 1.0738$ [6]; see Zhang's survey [17] and the implication between the two Bapat–Sunder conjectures established by Pioge et al. [13]. Drury's matrix has the support K_7 minus one edge. Inequality (1.1) is therefore a statement about structured classes, and the support graph is the natural structure: Beasley treats sign-compatible factors [3], Rodtes proves Chollet's inequality in order four [14], Sa-nguansin and Rodtes study regions of correlation matrices [15], Li, Zhao and Luo report Chollet's inequality through order six [9], and Pant and Singh prove (1.1) for real symmetric Z-matrices with bipartite support and develop a one-point closure along cut vertices [12].

The question of Pant and Singh. For a graph G the Laplacian L_G is real, symmetric and positive semidefinite, its diagonal is the degree sequence and its off-diagonal support is exactly $E(G)$. Pant and Singh prove (1.1) for L_G whenever every block of G is bipartite or a cycle, and in Section 6 (Concluding Remarks) of [12] they ask, more strongly, whether every graph Laplacian satisfies

$$\mathrm{per}(L_G \circ L_G) \leq \mathrm{per}(L_G) \prod_{v \in V(G)} \deg_G(v). \tag{1.2}$$

The results below give an affirmative answer for the larger block class described in Corollary 1.7, and for every graph on at most seven vertices.

1.1 Main results

A *support graph* lists the off-diagonal positions that may be nonzero; an allowed edge may carry the entry zero. A graph is *bipartite of width p* if it has a bipartition with a part of at most p vertices. The theta graph $\theta(1, b, c)$ is a cycle of length $b + c$ with one chord whose endpoints are at distances b and c along the cycle; it is triangle-free exactly when $b, c \geq 3$.

Theorem 1.1 (Five-row bipartite supports)**.** *Every complex Hermitian positive semidefinite matrix whose off-diagonal support is bipartite of width five satisfies* (1.1)*. The other part has arbitrary finite size, and arbitrary phases, missing edges and singular matrices are included. In particular* (1.1) *holds for every bipartite support on at most eleven vertices.*

Theorem 1.2 (Equality and a capped gap)**.** *Let A be as in Theorem 1.1 with positive diagonal. Equality holds in* (1.1) *exactly when the diagonal normalization of A is a direct sum of 1×1 blocks and rank-one 2×2 correlation blocks. Moreover, in the normalization of Section 2, every five-row contraction with all squared entries at most $7/10$ and all squared row norms at least $1/2$ satisfies $\mathcal{P} - \mathcal{Q} \geq c_H = 2104602658688797/6480000000000000$.*

Theorem 1.3 (Cycles and single-chord cycles)**.** *Inequality* (1.1) *holds for every $A \succeq 0$ whose support is contained in a cycle C_n with $n \geq 4$, or in a triangle-free single-chord cycle $\theta(1, b, c)$, $b, c \geq 3$. On a cycle, the diagonal normalization C of A satisfies $\mathcal{P}(C) - \mathcal{Q}(C) \geq \frac{3}{8} L_{C_n}$, with the matching loss L of* (2.5)*.*

For $n = 3$ the inequality holds for every 3×3 matrix $A \succeq 0$ [2]; see [17].

Let G be the seven-cycle $0, 1, \ldots, 6$ with chords $03, 15, 26$, let G_- omit the chord 26, and let $G_\times$ be the seven-cycle with chords $03, 14$ (Figure 1).

Theorem 1.4 (Chorded seven-cycles)**.** *Let C be a Hermitian matrix with unit diagonal and support contained in $H \in \{G, G_-, G_\times\}$, whose squared entries w_e satisfy $\sum_{e \ni v} w_e \le 1$ at every vertex; every correlation matrix with this support qualifies. Then*

$$\mathcal{P}(C) - \mathcal{Q}(C) \ge \delta_H L_H(w), \qquad \delta_G = \delta_{G_-} = \tfrac{1}{50}, \quad \delta_{G_\times} = \tfrac{1}{12}.$$

Consequently $\mathrm{d}(A)\mathcal{P}(A) - \mathcal{Q}(A) \ge \delta_H \mathrm{d}(A)^2 L_H(|C|^2)$ *for every $A \succeq 0$ with this support, positive diagonal and normalization C, and equality in* (1.1) *holds exactly when the nonzero normalized off-diagonal entries have modulus one and occupy disjoint edges.*

Theorem 1.5 (Triangle-free supports through order seven)**.** *Every $A \succeq 0$ of order at most seven with triangle-free off-diagonal support satisfies* (1.1)*.*

Let $\mathcal{B}$ be the class of graphs consisting of the bipartite graphs of width five, the triangle-free graphs on at most seven vertices, the cycles C_n with $n \ge 4$ and the triangle-free single-chord cycles. Blocks are maximal two-connected subgraphs, bridges counting as blocks K_2.

Theorem 1.6 (Hereditary block class)**.** *Let G be a graph every block of which belongs to $\mathcal{B}$, and let $A \succeq 0$ have support contained in G. Then every principal submatrix D of A satisfies $\mathcal{Q}(D) \le \mathrm{d}(D)\mathcal{P}(D)$.*

Corollary 1.7 (Graph Laplacians)**.** *Let G be a graph every block of which is a bipartite graph, a cycle C_n with $n \ge 3$, a triangle-free single-chord cycle, a triangle-free graph on at most seven vertices. Then L_G satisfies* (1.2)*, and so does every principal submatrix of L_G with its own diagonal product in place of $\prod_v \deg_G(v)$.*

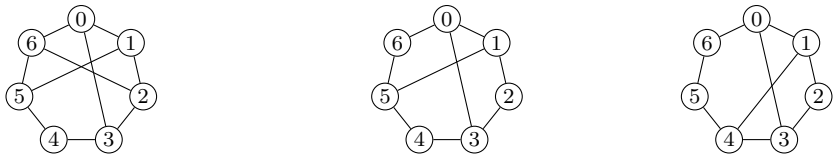
The blocks added to those of Pant and Singh are the nonbipartite triangle-free graphs on at most seven vertices other than cycles, such as the pentagon blow-ups of Section 12, and the single-chord cycles $\theta(1, b, c)$ with $b, c \ge 3$ and b or c even. For complex matrices, Theorem 1.6 allows arbitrary phases on every block; a typical host graph is a $K_{5,40}$ block glued at a vertex to a pentagon blow-up.

Theorem 1.8 (Computer verification through order seven)**.** *For every graph G on at most seven vertices, L_G satisfies* (1.2)*. This is verified by exact integer computation over all labelled graphs:* 1,915,547 *graphs on $2 \le n \le 7$ vertices without isolated vertices, of which* 1,887,284 *have seven vertices; a graph with an isolated vertex gives zero on both sides. Among graphs without isolated vertices, equality holds exactly for perfect matchings, and for odd n the largest ratio of the two sides is* 13/16*; for even $n \ge 4$ the largest ratio short of equality is* 11/15*.*

Theorem 1.8 covers every graph with triangles through order seven, including K_7, the order of Drury's matrix. The complementary inequalities for the ordinary self-case are collected in Section 15: the strong classes give Chollet's inequality for every pair of their members, and an energy criterion gives the ordinary self-case for every bipartite support of width twelve.

1.2 Methods

Two expansions organize the proofs. For bipartite support with normalized cross block X, both permanents are sums over square minors of X (Section 2); complex cancellation appears only inside $\mathcal{P}$. The three-row core (Section 4) is an exact identity with an explicit nonnegative remainder. If some squared entry exceeds 7/10, Lieb's block inequality and a partial-isometry calculation transfer the comparison to a smaller core (Section 5). In the capped domain a four-vertex matching-polytope bound and the square Carlen–Lieb–Loss inequality reduce four rows to fifteen rational corner values



G: chords $03, 15, 26$ gap $L_G/50$, integer identity

G_-: chords $03, 15$ gap $L/50$, subgraph of G

$G_\times$: chords $03, 14$ gap $L/12$, analytic

Figure 1: The three chorded seven-cycles of Theorem 1.4. All three are triangle-free; G represents the pentagon blow-up with part sizes $(2, 2, 1, 1, 1)$.

(Section 6) and five rows to an exact cubic Bernstein certificate on 2048 boxes, after a row-dependent higher-sector estimate and a split of the row domain made before the moments are minimized (Section 7).

For cycle-type supports the permanents are Sachs-type cycle expansions, and every directed cycle is paid for by the matching-partition loss $Z_H(w) - Z_H(w^2)$, a sum of nonnegative terms. The work is an allocation of this loss among overlapping cycles (Sections 9–12); one allocation is an exact integer polynomial identity in the vertex-star slacks. The hereditary class follows from a complex form of the one-point closure of Pant and Singh (Section 13), and the Laplacian corollary from their real closure (Section 14).

2 Normalization and the two expansions

Diagonal scaling. If $A \succeq 0$ has a zero diagonal entry, its row and column vanish and both sides of (1.1) are zero. Otherwise put $E = \operatorname{diag}(\sqrt{a_{11}}, \dots, \sqrt{a_{nn}})$ and $C = E^{-1}AE^{-1}$, a correlation matrix with the same support. Multilinearity gives

$$\mathcal{P}(A) = \mathrm{d}(A)\mathcal{P}(C), \qquad \mathcal{Q}(A) = \mathrm{d}(A)^2\mathcal{Q}(C), \tag{2.1}$$

so (1.1) for A is equivalent to $\mathcal{Q}(C) \le \mathcal{P}(C)$. In this normalization $|c_{ij}| \le 1$.

Bipartite support. If the support is bipartite with parts of sizes p and q, the normalization is

$$A(X) = \begin{pmatrix} I_p & X \\ X^* & I_q \end{pmatrix}, \qquad P = \mathcal{P}(A(X)), \quad Q = \mathcal{Q}(A(X)),$$

and $A(X) \succeq 0$ exactly when $\|X\|_{\mathrm{op}} \le 1$. Write $W = (w_{aj}) = |X|^2$, $r_a = \sum_j w_{aj}$, $c_j = \sum_a w_{aj}$, $R_a = \sum_j w_{aj}^2$, $S = \sum_a r_a$, $\delta_a = r_a - R_a$, $G = XX^*$, $K = X^*X$, $D_{ab} = \sum_j w_{aj}w_{bj}$ and $E_{ab} = \sum_j w_{aj}^2 w_{bj}^2$. Contraction gives the *star conditions*

$$r_a \le 1, \qquad c_j \le 1. \tag{2.2}$$

With $F_k = \sum_{|I|=|J|=k} |\operatorname{per} X[I,J]|^2$ and $Q_k = \sum_{|I|=|J|=k} \operatorname{per}(W[I,J])^2$, a permutation of $A(X)$ moves equally many vertices across the bipartition and the two cross bijections are independent, whence

$$P = \sum_k F_k, \qquad Q = \sum_k Q_k, \qquad F_0 = Q_0 = 1, \quad F_1 = S. \tag{2.3}$$

Every F_k is nonnegative. For a fixed row set I, distinct-column assignments give

$$\sum_{|J|=|I|} \operatorname{per} W[I,J] \le \prod_{a\in I} r_a,$$

and every square minor of W has permanent in $[0,1]$. Expanding the two-row minors,

$$F_2 = e_2(r) + O - 2\sum_{a<b} D_{ab}, \qquad O = \sum_{a<b} |G_{ab}|^2, \tag{2.4}$$

where e_k denotes the elementary symmetric polynomial. Since $2\sum D_{ab} \le S - \sum R_a$, one has $P \ge 1 + \max\{S, e_2(r)\}$.

Cycle expansion and matching loss. For a graph H with edge weights $w_e \in [0,1]$ and a matching M put $m_M = \prod_{e\in M} w_e$ and

$$Z_H(w) = \sum_M m_M, \qquad L_H(w) = Z_H(w) - Z_H(w^2) = \sum_M m_M(1-m_M) \ge 0, \tag{2.5}$$

summing over all matchings of H, the empty one included.

Lemma 2.1 (Cycle-cost bound)**.** *Let H be a graph with no two vertex-disjoint cycles, and let C be a Hermitian matrix with unit diagonal, support contained in H and $w_e = |c_e|^2 \le 1$. For a cycle Y of H let $h_Y = (\prod_{e\in Y} w_e)^{1/2}$ and let $\bar{Y}$ be the subgraph induced outside $V(Y)$. Then*

$$\mathcal{P}(C) - \mathcal{Q}(C) \ge L_H(w) - \sum_Y E_Y, \qquad E_Y = 2h_Y(1+h_Y)Z_{\bar{Y}}(w). \tag{2.6}$$

Proof. A nonzero permutation term consists of fixed points, transpositions along edges and, by hypothesis, at most one cycle of length at least three, which runs along a cycle Y of H. The two orientations of Y contribute conjugate products $H_Y, \overline{H_Y}$ with $|H_Y| = h_Y$. Hence

$$\mathcal{P}(C) = Z_H(w) + 2\sum_Y \Re(H_Y) Z_{\bar{Y}}(w), \qquad \mathcal{Q}(C) = Z_H(w^2) + 2\sum_Y h_Y^2 Z_{\bar{Y}}(w^2).$$

Now use $\Re H_Y \ge -h_Y$ and $Z_{\bar{Y}}(w^2) \le Z_{\bar{Y}}(w)$. □

The bound allows the worst phase on every cycle separately, so it covers all jointly realizable phases. This is a cycle-cover expansion of Sachs type; its complex gain-graph form appears in [11].

3 Shared estimates for bipartite supports

With the notation of Section 2, the exact second-sector identity for Q and the trace identity are

$$Q_2 = e_2(R) + \sum_{a<b} D_{ab}^2 - 2\sum_{a<b} E_{ab}, \tag{3.1}$$

$$C + O = \tfrac{1}{2}\sum_a r_a(1-r_a) + \tfrac{1}{2}\sum_j c_j(1-c_j) + \sum_{j<l} |K_{jl}|^2, \qquad C = \sum_j c_j(1-c_j). \tag{3.2}$$

Since $F_1 - Q_1 = C + 2\sum D_{ab}$ and the later F_k are nonnegative, (2.4) and (3.2) give, for any number of rows under the star conditions,

$$P - Q \ge \tfrac{1}{2}\sum_a r_a(1-r_a) + e_2(r) - \sum_{k\ge 2} Q_k. \tag{3.3}$$

Lemma 3.1 (Matching bound)**.** *Let $z_{ij} \ge 0$ be edge weights of a finite loopless graph with weighted degrees $h_i = \sum_j z_{ij} \le \delta_i$. Then $\sum_{ij} z_{ij}^2 \le \max_M \sum_{ij\in M} \delta_i\delta_j$, the maximum taken over matchings M.*

Proof. Discard zero-degree vertices and set $B_{ij} = z_{ij}/\sqrt{h_i h_j}$, $B_{ii} = 0$. Weighted $2ab \le a^2 + b^2$ gives $|v^T B v| \le \sum_i v_i^2$, so B is a symmetric contraction and $y_{ij} = B_{ij}^2$ satisfies $\sum_j y_{ij} = (B^2)_{ii} \le 1$. For an odd vertex set of size m the compression T of B is a symmetric contraction of trace zero. The convex function $\sum \lambda_i^2$ on the polytope of eigenvalue vectors in $[-1,1]^m$ with zero sum is maximized at a point with at most one interior coordinate; the other $m-1$ coordinates are ± 1 and sum to an even integer, so the remaining coordinate vanishes. Hence $\operatorname{Tr} T^2 \le m-1$ and $\sum_{ij \in E(S)} y_{ij} \le (m-1)/2$. These are the degree and odd-set inequalities of Edmonds's matching polytope [7], so y is a convex combination of matching incidence vectors and $\sum z_{ij}^2 = \sum h_i h_j y_{ij} \le \max_M \sum_M h_i h_j \le \max_M \sum_M \delta_i \delta_j$. □

Column stars give $\sum_{b \ne a} D_{ab} \le \delta_a$, so Lemma 3.1 and (3.1) yield

$$Q_2 \le e_2(R) + M(\delta), \tag{3.4}$$

where $M(\delta)$ is the maximum of $\sum_{ij \in \pi} \delta_i \delta_j$ over matchings π of the row vertices: the three perfect pairings for four rows, the fifteen two-edge matchings for five rows.

The square Carlen–Lieb–Loss inequality [4, Theorem 1.1] states

$$|\operatorname{per} M_k| \le \frac{k!}{k^{k/2}} \prod_{a=1}^{k} \|(M_k)_{a,*}\|_2 \tag{3.5}$$

for $k \times k$ matrices. Under the stars and the cap $w_{aj} \le 7/10$, concentration of each nonnegative row gives

$$0 \le R_a \le f(r_a) \le r_a, \qquad f(t) = \begin{cases} t^2, & 0 \le t \le 7/10, \\ 49/100 + (t - 7/10)^2, & 7/10 \le t \le 1, \end{cases} \tag{3.6}$$

so $R_a \le 29/50$. Applying (3.5) to minors of W and summing over distinct columns,

$$Q_k \le g_k e_k(r) \quad (k = 3,4,5), \qquad (g_3, g_4, g_5) = \left(\tfrac{511}{1000}, \tfrac{2523}{5000}, \tfrac{11}{20}\right), \tag{3.7}$$

because each minor permanent is at most $k!(29/50)^{k/2}/k^{k/2}$; the cap g_4 is exact and the squared margins of g_3, g_5 are $583/600000$ and $992213/19531250000$.

Lemma 3.2 (Three-row collision identity)**.** *For three selected rows, let c denote the third row of a pair $a < b$, and put $g_{ab} = G_{ab}$, $T = \sum_j w_{0j} w_{1j} w_{2j}$ and $H_{ab} = \sum_j w_{cj} x_{aj} \overline{x_{bj}}$. The order-three part of F_3 for these rows is*

$$\operatorname{per} G - 2 \sum_{a<b} r_c D_{ab} - 4 \Re \sum_{a<b} \overline{g_{ab}} H_{ab} + 12T. \tag{3.8}$$

Proof. Expand the Gram permanent by column multisets, each divided by the product of multiplicity factorials. Distinct columns give the order-three part of F_3. Columns j, j, k with $j \ne k$ contribute $2|\sum_a x_{ak} \prod_{b \ne a} x_{bj}|^2$, and a triple column contributes $6 \prod_a w_{aj}$. Extending the double-column sum to $k = j$ adds $18 \prod_a w_{aj}$, so the part equals $\operatorname{per} G - 2 \sum_{j,k} |\sum_a x_{ak} \prod_{b \ne a} x_{bj}|^2 + 12T$. Expanding the square, the diagonal part is $\sum r_c D_{ab}$ and the cross part is $2\Re \sum \overline{g_{ab}} H_{ab}$. The identity is polynomial. □

For a contraction $0 \preceq G \preceq I$, so $|g_{ab}| \le 1/2$, and AM–GM on $[0, 1/2]$ gives

$$2\Re(g_{ab} g_{bc} g_{ca}) \ge -\tfrac{1}{3}(|g_{ab}|^2 + |g_{bc}|^2 + |g_{ca}|^2). \tag{3.9}$$

4 The three-row core

Proposition 4.1. *Every complex $3 \times q$ contraction satisfies $Q \le P$. Explicitly, with $E = \sum_{k=2}^{3} \sum_{|I|=|J|=k} \operatorname{per} W[I,J](1 - \operatorname{per} W[I,J]) \ge 0$ and $a_c = 2/3 + r_c$,*

$$P - Q \ge E + C + \sum_{a<b} (1 - r_c) D_{ab} + \sum_{a<b} a_c \Big| g_{ab} - \frac{2H_{ab}}{a_c} \Big|^2 + \frac{14}{5} T \ge 0. \tag{4.1}$$

Proof. Inclusion–exclusion on repeated columns gives the matching sums $Z_2 = e_2(r) - \sum D_{ab}$ and $Z_3 = r_0 r_1 r_2 - \sum r_c D_{ab} + 2T$ of W. Using (2.4), Lemma 3.2, $\operatorname{per} G = r_0 r_1 r_2 + \sum r_c |g_{ab}|^2 + 2\Re(g_{01} g_{12} g_{20})$ and $S - \sum R_a = C + 2 \sum D_{ab}$, direct substitution yields the exact identity

$$\begin{aligned} P - Q = E + C + \sum_{a<b} (1 - r_c) D_{ab} + \sum_{a<b} (1 + r_c) |g_{ab}|^2 \\ + 2\Re(g_{01} g_{12} g_{20}) - 4\Re \sum_{a<b} \overline{g_{ab}} H_{ab} + 10T. \end{aligned} \tag{4.2}$$

Weighted Cauchy–Schwarz gives $|H_{ab}|^2 \le r_c T$. By (3.9) and completion of the square, $a_c |g_{ab}|^2 - 4\Re(\overline{g_{ab}} H_{ab}) = a_c |g_{ab} - 2H_{ab}/a_c|^2 - 4|H_{ab}|^2/a_c$, and $4r_c/a_c \le 12/5$; the three squares consume at most $36T/5$ of $10T$, proving (4.1). □

Padding by zero rows gives every $p \times q$ contraction with $p \le 3$, so (1.1) holds for bipartite width three.

5 A dimension-independent large-entry transfer

Lemma 5.1. *Let $p \ge 2$ and let X be a complex $p \times q$ contraction. Let D be the core obtained by deleting the row and column of an entry of squared modulus $t \ge 7/10$, and suppose $\mathcal{Q}(A(D)) \le \mathcal{P}(A(D))$. Then*

$$P - Q \ge \tfrac{7}{2000} (1 - t)\, \mathcal{Q}(A(D)) \ge 0.$$

Proof. Place the entry first: $X = \left(\begin{smallmatrix} a & b \\ c & D \end{smallmatrix}\right)$, $t = |a|^2$, $s = 1 - t$, $U_b = \sum_j |b_j|^2 \le s$, $V_b = \sum_i |c_i|^2 \le s$. For $q = 1$ the support is a star, $P - Q = \sum_a w_a(1 - w_a) \ge t(1 - t)$ and $\mathcal{Q}(A(D)) = 1$; assume $q \ge 2$. Partition $A(X)$ into the two endpoints of a and the complementary $A(D)$. Lieb's block inequality [10] and the core hypothesis give

$$P \ge (1 + t) \mathcal{P}(A(D)) \ge (1 + t) \mathcal{Q}(A(D)). \tag{5.1}$$

With $W = |D|^2$ fixed, every minor permanent of the nonnegative rectangle $\left(\begin{smallmatrix} t & u \\ v & W \end{smallmatrix}\right)$ is affine in u and in v, so Q is separately convex in u and v and is bounded by its value at concentrated vectors $u = U_b e_j$, $v = V_b e_i$.

Let $d_{I,J} = \operatorname{per} W[I,J]$ for core subsets of equal size, so that $\|d\|^2 = \mathcal{Q}(A(D))$, and embed d in the Hilbert space with basis $e_{I,J}$ over all pairs of core row and column subsets, $d_{I,J} = 0$ when $|I| \ne |J|$. The creation maps $C_j e_{I,J} = e_{I,J \cup \{j\}}$ for $j \notin J$ (zero otherwise) and $R_i e_{I,J} = e_{I \cup \{i\},J}$ are commuting partial isometries; put $L = R_i C_j$. Expanding the concentrated bordered minors,

$$Q_{\mathrm{conc}} = (1 + t^2)\|d\|^2 + U_b^2 \|C_j d\|^2 + V_b^2 \|R_i d\|^2 + U_b^2 V_b^2 \|Ld\|^2 + 2tU_b V_b \Re\langle d, Ld \rangle,$$

since a minor containing both border indices has permanent $t d_{I,J} + U_b V_b d_{I \setminus \{i\}, J \setminus \{j\}}$. Grouping coordinates by occupation of i, j, the extra form has the 00/11 block $B(U_b, V_b) = \left(\begin{smallmatrix} U_b^2 + V_b^2 + U_b^2 V_b^2 & tU_b V_b \\ tU_b V_b & 0 \end{smallmatrix}\right)$

and diagonal values U_b^2 or V_b^2 on the other types. Its largest eigenvalue $\lambda(U_b, V_b)$ dominates those values and increases in both variables, so $Q \le [1 + t^2 + \lambda(s,s)]\mathcal{Q}(A(D))$. For $M = tsI - B(s,s)$,

$$\det M = ts^2(1 - 3s - s^2), \qquad \operatorname{Tr} M = s(2 - 4s - s^3) \le 2s.$$

For $0 < s \le 3/10$ the entry ts is positive and $1 - 3s - s^2 \ge 1/100$, so M is positive definite with least eigenvalue at least $\det M / \operatorname{Tr} M \ge 7s/2000$. Combining with (5.1) proves the lemma; for $s = 0$ the border terms vanish. □

6 All finite-column four-row supports

Proposition 6.1. *Every complex $4 \times q$ rectangle satisfying the star conditions and $w_{aj} \le 7/10$ satisfies $Q \le P$.*

For each perfect pairing π of the four rows define

$$B_\pi(r, R) = \tfrac{1}{2} \sum_a r_a(1 - r_a) + e_2(r) - e_2(R) - \sum_{\{a,b\} \in \pi} \delta_a \delta_b - g_3 e_3(r) - g_4 e_4(r). \tag{6.1}$$

By (3.3), (3.4) and (3.7), $P - Q \ge \min_\pi B_\pi$. We prove $B_\pi \ge 0$ on the box $0 \le r_a \le 1$, $0 \le R_a \le f(r_a)$.

Proof. Fix π and r. The scalar (6.1) is separately affine in R, so its minimum is at endpoints $R_i \in \{0, f(r_i)\}$. Rows at the zero endpoint are deleted (set to $r_i = R_i = 0$), first in partner pairs, then singly; the order matters.

For zero-marked partners with sums t, u and other rows v, w, the change on deleting the pair is $\frac{1}{2}[t(1-t) + u(1-u)] + (t+u)(v+w) - g_3[tu(v+w) + (t+u)vw] - g_4 tuvw$, the tu term of $e_2(r)$ cancelling the paired δ term. By $2tu \le t + u$ and $2vw \le v + w$ this is at least $\frac{1}{2}[t(1-t) + u(1-u)] + \frac{7257}{20000}(t+u)(v+w) \ge 0$. A remaining zero-marked row t has an f-marked partner p, others u, v; deleting t changes the scalar by $t(1-t)/2 + t\,L$ with $L \ge f(p) + (u+v)[1 - g_3/2 - p(g_3 + g_4/2)]$. For $p \le 1/2$ the bracket is at least $7257/20000$; for $p \ge 1/2$ it is at least $-47/2500$ while $f(p) \ge 1/4$, so $L \ge 531/2500$.

It remains to show $J_\pi(r) = B_\pi(r, f(r)) \ge 0$. Varying $t = r_i$ with partner p, the coefficient of $f(t)$ is at most $p - 2f(p) \le 1/8$, and apart from it the dependence is linear minus $t^2/2$. Hence $J_\pi'' \le -3/4$ on each of $[0, 7/10]$ and $[7/10, 1]$, and separate concavity reduces the minimum to $r_a \in \{0, 7/10, 1\}$. Table 1 lists the minimum over the three pairings for each count pattern; these fifteen values cover all 81 labelled corners. □

Table 1: Exact corner minima of $B_\pi(r, f(r))$; n_0, n_7, n_1 count coordinates equal to $0, 7/10, 1$.

n_0	n_7	n_1	$\min_\pi B_\pi$	n_0	n_7	n_1	$\min_\pi B_\pi$
0	0	4	5401/5000	1	2	1	95291/100000
0	1	3	7133/6250	1	3	0	845327/1000000
0	2	2	551383/500000	2	0	2	609/1250
0	3	1	1350349/1250000	2	1	1	2163/5000
0	4	0	50447677/50000000	2	2	0	2079/5000
1	0	3	6517/5000	3	0	1	0
1	1	2	10661/10000	3	1	0	21/200
				4	0	0	0

Theorem 6.2. *For every finite q, every complex $4 \times q$ contraction satisfies $Q \le P$.*

Proof. In the capped domain apply Proposition 6.1. Otherwise apply Lemma 5.1, whose three-row core hypothesis is Proposition 4.1. □

7 All finite-column five-row supports

In this section there are five rows. The star conditions make W itself a contraction: by weighted Cauchy–Schwarz, $\sum_a |\sum_j w_{aj} v_j|^2 \le \sum_a r_a \sum_j w_{aj} |v_j|^2 \le \sum_j |v_j|^2$. Thus WW^T is a positive semidefinite contraction with diagonal R_a and off-diagonal D_{ab}, so with $U = \sum_a R_a$, the trace inequality and $\sum R_a^2 \ge U^2/5$ give

$$\sum_{a<b} D_{ab}^2 \le \tfrac{1}{2}\Big(U - \sum_a R_a^2\Big), \qquad Q_2 \le \tfrac{3}{10}U^2 + \tfrac{1}{2}U. \tag{7.1}$$

Lemma 7.1 (Third-sector budget)**.** *For a five-row contraction and every $0 \le \theta \le 1$,*

$$P - Q \ge \frac{1-\theta}{2} \sum_a r_a(1-r_a) + e_2(r) + \theta e_3(r) - \sum_{k=2}^{5} Q_k - 2\theta \sum_{a<b} A_{ab} D_{ab}, \qquad A_{ab} = \sum_{c \notin \{a,b\}} r_c.$$

Proof. Summing Lemma 3.2 over row triples, with $T_{abc} = \sum_j w_{aj} w_{bj} w_{cj}$, $T = \sum T_{abc}$ and $H_{ab} = \sum_j (\sum_{c \notin \{a,b\}} w_{cj}) x_{aj} \overline{x_{bj}}$,

$$F_3 = e_3(r) + \sum_{a<b} A_{ab} |g_{ab}|^2 + 2\Re \sum_{a<b<c} g_{ab} g_{bc} g_{ca} - 2 \sum_{a<b} A_{ab} D_{ab} - 4\Re \sum_{a<b} \overline{g_{ab}} H_{ab} + 12T.$$

By (3.9), each pair lying in three triples, the cubic terms total at least $-O$. Weighted Cauchy–Schwarz gives $|H_{ab}|^2 \le A_{ab} \sum_c T_{abc}$, so $A_{ab}|g_{ab}|^2 - 4\Re(\overline{g_{ab}} H_{ab}) \ge -4 \sum_c T_{abc}$ (both sides vanish when $A_{ab} = 0$); summing consumes at most $12T$. With (2.4), $F_2 + \theta F_3 \ge e_2(r) + \theta e_3(r) + (1-\theta)O - 2\sum D_{ab} - 2\theta \sum A_{ab} D_{ab}$. Finally $C + (1-\theta)O \ge (1-\theta)(C+O)$ and (3.2) finish the proof. □

7.1 Row-dependent higher sectors

For a k-row set I and every k-column set J, (3.5) gives $\operatorname{per} W[I,J] \le c_k \prod_{i \in I} \sqrt{R_i}$ with $c_k = k!/k^{k/2}$. Multiplying by the same permanent and summing over J,

$$Q_k \le c_k e_k(r_i \sqrt{R_i}), \qquad k = 3,4,5. \tag{7.2}$$

The tangent bound $\sqrt{R} \le 3/8 + 2R/3$ and the rational constants $(231/200)^2 - 4/3 = 83/120000 > 0$, $c_4 = 3/2$ and $(537/250)^2 - 576/125 = 369/62500 > 0$ give, with $z_i = r_i(3/8 + 2R_i/3)$,

$$Q_3 + Q_4 + Q_5 \le H(r,R) = \tfrac{231}{200} e_3(z) + \tfrac{3}{2} e_4(z) + \tfrac{537}{250} e_5(z). \tag{7.3}$$

The tangent bound restores separate affinity in R while keeping the row moments.

7.2 Choosing the row domain before the moments

For a two-edge matching π of the five rows let $M_\pi(\delta) = \sum_{\{i,j\} \in \pi} \delta_i \delta_j$. If some $r_i \le 1/2$, take $\theta = 0$; (3.4) and (7.3) give $P - Q \ge \min_\pi B_{\mathrm{L},\pi}$ with

$$B_{\mathrm{L},\pi} = \tfrac{1}{2} \sum_i r_i(1-r_i) + e_2(r) - e_2(R) - M_\pi(\delta) - H(r,R). \tag{7.4}$$

If all $r_i \geq 1/2$, the potential $y_i = S/2 - r_i$ is nonnegative, $y_i + y_j = A_{ij}$ and overlap degrees are at most δ_i, so $\sum_{i<j} A_{ij}D_{ij} \leq \sum_i \delta_i(S/2 - r_i)$. Taking $\theta = 3/5$ gives $P - Q \geq \min_\pi B_{\mathrm{H},\pi}$ with

$$B_{\mathrm{H},\pi} = \tfrac{1}{5}\sum_i r_i(1-r_i) + e_2(r) + \tfrac{3}{5}e_3(r) - e_2(R) - M_\pi(\delta) - H(r,R) - \tfrac{6}{5}\sum_i \delta_i(S/2 - r_i). \quad (7.5)$$

Once the region is chosen, each fixed-matching budget is separately affine in every R_i: symmetric functions use distinct indices, matching edges are disjoint, z_i is affine in R_i and the potential term is linear in δ_i. Its minimum over the moment box is therefore attained at endpoints $R_i \in \{0, f(r_i)\}$.

7.3 The exact finite cover

Fix $\pi_0 = \{\{0,1\},\{2,3\}\}$; every two-edge matching is a relabelling. In the low region, a row with $r_i \leq 1/2$ can be moved to row 0 if matched and to row 4 if unmatched. Splitting the other intervals at 7/10 gives sixteen row boxes and thirty-two moment choices, 512 polynomials for each special row. The high cube $[1/2,1]^5$ split at 7/10 gives 1024 polynomials. After endpoint substitution each coordinate degree is at most three, and on each box mapped affinely to $t \in [0,1]^5$

$$B(t) = \sum_{\alpha \in \{0,1,2,3\}^5} \beta_\alpha \prod_{i=0}^{4} \binom{3}{\alpha_i} t_i^{\alpha_i}(1-t_i)^{3-\alpha_i}. \quad (7.6)$$

Appendix A gives the integer formula for every coefficient. The exact results are as follows.

Group	Polynomials	Coefficients	Zeros	Minimum positive
Low, special row 0	512	524288	168	1/20
Low, special row 4	512	524288	176	1/20
High	1024	1048576	0	c_H
Total	2048	2097152	344	1/20

Every coefficient is nonnegative, so every endpoint budget is nonnegative on its box.

Proof of Theorem 1.1. In the capped domain the cover above gives $P \geq Q$. If a squared entry is at least 7/10, Lemma 5.1 applies with its four-row core hypothesis supplied by Theorem 6.2. Padding by zero rows and interchanging the parts gives every width at most five, and (2.1) transfers the result to every $A \succeq 0$. A bipartition of at most eleven vertices has a part of at most five. □

8 Equality for five-row supports

Proof of Theorem 1.2. Every high-region coefficient is at least c_H; the partition of unity, affine interpolation in the moments and matching symmetry give $P - Q \geq c_H$ on the stated region.

For low-region strictness consider the all-upper endpoint $R_i = f(r_i)$ of (7.4). Every row-box vertex has its special row in $\{0, 1/2\}$ and the others in $\{0, 7/10, 1\}$. Direct rational substitution gives the minima in Table 2 by the number k of positive row sums, with $1, 9, 32, 56, 48, 16$ vertices for $k = 0, \ldots, 5$ in each group. Take a point of a low box with at least two positive coordinates. For each positive coordinate choose a positive interval endpoint with positive Bernstein factor at that point, and the zero endpoint for a zero coordinate. The resulting vertex retains two active rows, its coefficient equals its value and is positive by the table, and its basis weight is positive; all other coefficients are nonnegative. For an actual rectangle, $r_i > 0$ implies $R_i > 0$, and the multiaffine interpolation gives the all-upper endpoint the weight $\prod_{r_i>0} R_i/f(r_i) > 0$. Hence each low budget

is positive with two active rows. With one active row, $P - Q = \sum_j w_j(1 - w_j) \geq 3r/10 > 0$ under the cap.

If a squared entry $t \in [7/10, 1)$ occurs, Lemma 5.1 gives $P - Q > 0$ because $\mathcal{Q}(A(D)) \geq 1$. If $t = 1$, the stars isolate the entry, its two-vertex block contributes the factor 2 to both permanents, and one removes it. Equality therefore leaves a zero core: X is a partial unit-modulus matching, and conversely such X gives $P = Q = 2^s$. Scaling translates unit modulus into $|a_{ij}|^2 = a_{ii}a_{jj}$. □

Table 2: Minima of the all-upper low budget at row-box vertices, by the number k of positive row sums.

k	Special row 0	Special row 4
0	0	0
1	0	0
2	3/8	2079/5000
3	12725891826529/14400000000000	12725891826529/14400000000000
4	1071853731000479/864000000000000	18705983227/17280000000
5	360934627338073/518400000000000	323920867338073/518400000000000

9 Reservoirs for cycle-type supports

From here on C is a Hermitian matrix with unit diagonal and support contained in a triangle-free graph H, with $w_e = |c_e|^2$, satisfying the *vertex-star bounds*

$$s_v := 1 - \sum_{e \ni v} w_e \geq 0 \qquad (v \in V(H)). \tag{9.1}$$

Every correlation matrix with triangle-free support satisfies (9.1): a vertex and its neighbours induce a star, whose principal correlation matrix $\left(\begin{smallmatrix} 1 & z^* \\ z & I \end{smallmatrix}\right)$ has Schur complement $1 - \|z\|^2$. In particular $w_e \leq 1$, and adjacent weights satisfy $w_e + w_f \leq 1$.

For a cycle Y of length $\ell \geq 4$ with cyclically ordered edges e_i and weights w_i, multiplying $w_i w_{i+1} \leq 1/4$ gives

$$h_Y \leq 2^{-\ell/2}. \tag{9.2}$$

Define the singleton, angle and pair reservoirs

$$R_Y = \Big(\sum_{e \in Y} w_e(1 - w_e)\Big) Z_{\bar{Y}}(w), \qquad A_Y = \Big(\sum_i w_i w_{i+1}\Big) Z_{\bar{Y}}(w), \qquad B_Y = J_Y Z_{\bar{Y}}(w),$$

where $J_Y = \sum_I m_I(1 - m_I)$ runs over the *selected pairs*: the two opposite-edge pairs if $\ell = 4$, and the ℓ distance-two pairs $I_i = \{e_i, e_{i+2}\}$ if $\ell \geq 5$. Since $w_i(1 - w_i) \geq w_i w_{i+1}$, one has $A_Y \leq R_Y$.

Lemma 9.1 (Pair identity)**.** *If $p, q \geq 0$, $\sqrt{p} + \sqrt{q} \leq 1$ and $h = \sqrt{pq}$, then*

$$p(1 - p) + q(1 - q) - 2h(1 - h) = (p + q - 2h)(1 - p - q - 2h) \geq 0.$$

Proof. Expand; the factors are $(\sqrt{p} - \sqrt{q})^2$ and $1 - (\sqrt{p} + \sqrt{q})^2$. □

Lemma 9.2 (Cycle reservoirs)**.** *Let Y be a cycle of length $\ell \geq 4$ in H, and put $\rho_\ell = \frac{2}{\ell} 2^{-(\ell-4)/2}(1 + 2^{-\ell/2})$. Then ρ_ℓ is decreasing in ℓ and*

$$E_Y \leq \rho_\ell A_Y \leq \rho_\ell R_Y, \qquad E_Y \leq \beta_\ell B_Y \quad (4 \leq \ell \leq 7),$$

with the following rational upper bounds $\alpha_\ell \ge \rho_\ell$ and β_ℓ:

ℓ	4	5	6	7
α_ℓ	5/8	1/3	3/16	1/8
β_ℓ	5/3	4/9	1/4	3/20

For $\ell \ge 7$ also $\rho_\ell < 1/8$.

Proof. AM–GM gives $\sum_i w_i w_{i+1} \ge \ell h^{4/\ell}$, so $E_Y/A_Y \le \frac{2}{\ell} h^{1-4/\ell}(1+h)$, which increases in h; by (9.2) it is at most ρ_ℓ. The ratio $\rho_{\ell+1}/\rho_\ell = \frac{\ell}{\ell+1} 2^{-1/2} \frac{1+2^{-(\ell+1)/2}}{1+2^{-\ell/2}}$ is below one. The values are $\rho_4 = 5/8$, $\rho_5 = \sqrt{2}/5 + 1/20 < 1/3$ (as $288 < 289$), $\rho_6 = 3/16$ and $\rho_7 = 1/(7\sqrt{2}) + 1/112 < 1/8$ (as $256 < 338$).

For pairs with $\ell \ge 5$ put $m_i = w_i w_{i+2}$ and $q_i = \sqrt{m_i m_{i+1}}$. Cauchy–Schwarz and the adjacent bounds give $\sqrt{m_i} + \sqrt{m_{i+1}} \le \sqrt{(w_i + w_{i+1})(w_{i+2} + w_{i+3})} \le 1$, hence $q_i \le 1/4$. Summing Lemma 9.1 over i, each selected loss counted twice, gives $J_Y \ge \sum_i q_i(1 - q_i) \ge \frac{3}{4} \sum_i q_i \ge \frac{3\ell}{4} h^{4/\ell}$, because $\prod_i q_i = \prod_i m_i = h^4$. The ratio E_Y/B_Y is at most $\frac{4}{3}\rho_\ell$ at the cap, namely $(4\sqrt{2}+1)/15 < 4/9$ (as $288 < 289$), $1/4$, and $(8\sqrt{2}+1)/84 < 3/20$ (as $800 < 841$). For $\ell = 4$ the opposite pairs p, q satisfy $\sqrt{p} + \sqrt{q} \le 1$ and $\sqrt{pq} = h$, so $J_Y \ge 2h(1-h)$ and $E_Y/B_Y \le (1+h)/(1-h) \le 5/3$. Zero products need no division. □

Lemma 9.3 (Charging)**.** *A term of R_Y is indexed by an edge $e \in Y$ and a matching T of $\bar{Y}$; a term of B_Y by a selected pair I and such a T. In either case $M = \{e\} \cup T$, respectively $M = I \cup T$, is a matching of H, the term is at most $m_M(1 - m_M)$, and for fixed Y the matching M determines its index uniquely.*

Proof. $w_e(1 - w_e) m_T = m_M(1 - w_e) \le m_M(1 - m_M)$ and $m_I(1 - m_I) m_T \le m_M(1 - m_M)$, since $m_M \le \min(w_e, m_I)$. The edges of M on Y are exactly e or I. □

A matching M is *eligible* for Y (pair reservoir) when its edges on Y form one selected pair and its other edges have both endpoints outside Y; for the singleton reservoir, when exactly one edge lies on Y and the others lie outside Y.

10 Cycles and single-chord cycles

Proof of Theorem 1.3 for cycles. Normalize to a correlation matrix. Lemma 2.1 applies with a single cycle $Y = C_n$, for which $Z_{\bar{Y}} = 1$, and Lemma 9.2 gives $E_Y \le \frac{5}{8} R_Y$. Since R_Y is the singleton part of L_{C_n}, $\mathcal{P} - \mathcal{Q} \ge L_{C_n} - \frac{5}{8} R_Y \ge \frac{3}{8} L_{C_n} \ge 0$. Scaling (2.1) gives the general case. □

The gap shows that on a cycle, strong equality holds exactly when the nonzero normalized edge weights equal one and form a matching: $L = 0$ forces $w_e \in \{0, 1\}$, and (9.1) separates the ones.

Proof of Theorem 1.3 for single-chord cycles. The graph $\theta(1, 3, 3)$ is a six-cycle with a chord joining vertices at distance three, hence of opposite parity: it is bipartite with parts of size three, and Theorem 1.1 (indeed Proposition 4.1) applies. Otherwise assume $3 \le b \le c$ and $b + c \ge 7$. The three cycles have lengths $\ell_1 = b + 1 \ge 4$, $\ell_2 = c + 1 \ge 5$ and $\ell_3 = b + c \ge 7$; any two share both endpoints of the chord, so Lemma 2.1 applies, and every edge lies on exactly two of them. By Lemma 9.2, $E_1 \le \frac{5}{8} R_1$, $E_2 \le \frac{3}{8} R_2$ and $E_3 \le \frac{1}{8} R_3$. By Lemma 9.3, a matching with at least two edges is charged only by Y_1, Y_2 (the complement of Y_3 is empty), with total coefficient at most $\frac{5}{8} + \frac{3}{8} = 1$; a singleton is charged by the two cycles through its edge, with total 1, $\frac{3}{4}$ or $\frac{1}{2}$; the empty matching has zero loss. Hence $\sum_j E_j \le L$, and (2.6) gives $\mathcal{Q} \le \mathcal{P}$. □

11 Chorded seven-cycles

The graphs $G, G_-, G_\times$ are triangle-free on seven vertices, so no two cycles are disjoint and Section 9 applies. For G order the edges as

$$01, 12, 23, 34, 45, 56, 06, 03, 15, 26, \tag{11.1}$$

and sort cycles by length, then lexicographically by edge-index lists. There are five four-cycles, four five-cycles, two six-cycles and four seven-cycles. Give the four-cycles $(0,1,2,7)$, $(0,1,6,9)$, $(0,5,6,8)$, $(1,5,8,9)$, $(2,6,7,9)$ the weights $x_Y = 2187/8750$, $51/125$, $4341/17500$, $4341/17500$, $2187/8750$, every five-cycle $x_Y = 27/80$, and every longer cycle $x_Y = 1$. Lemma 9.2 gives the convex estimate

$$E_Y \le (1 - x_Y)\alpha_{|Y|}A_Y + x_Y\beta_{|Y|}B_Y. \tag{11.2}$$

Put

$$c_M = \sum_{Y:\, M \text{ eligible}} 3600 x_Y \beta_{|Y|}, \qquad A_{\mathrm{mix}} = \sum_Y 3600(1 - x_Y)\alpha_{|Y|}A_Y.$$

Lemma 11.1 (Integer certificate)**.** *For G and these weights,*

$$F := \sum_{1 \le |M| \le 2} (3528 - c_M) m_M (1 - m_M) - A_{\mathrm{mix}} \ge 0 \quad \text{on (9.1)}, \qquad c_M \le 3528 \ (|M| = 3).$$

Proof. The supplement file `chorded_seven_cycle_certificate.json` lists an exact identity

$$350F = \sum_{j=1}^{323} N_j w^{a_j} \prod_{v \in V_j} s_v \tag{11.3}$$

with positive integers N_j, nonnegative integer exponent vectors a_j and multisets V_j of at most two vertices, each term of total degree at most four. Its right side is nonnegative on (9.1). The verifier reconstructs all 15 cycles and 53 matchings (sizes $0, 1, 2, 3$ counted $1, 10, 26, 16$), the selected pairs and the charges, expands both sides over the rationals and finds a zero residual; the maximal size-three charge is exactly 3528. Products of slacks are a classical positivity framework [8]; here the explicit identity itself is the proof, valid on all closed faces. □

Proof of Theorem 1.4 for G and G_-. By (11.2) and Lemma 9.3, $3600 \sum_Y E_Y \le A_{\mathrm{mix}} + \sum_M c_M m_M(1 - m_M) \le 3528 L_G$, using Lemma 11.1 for sizes one and two and the charge bound for size three. Lemma 2.1 gives $\mathcal{P} - \mathcal{Q} \ge \frac{72}{3600} L_G = L_G/50$. Setting $w_{26} = 0$ gives G_-. □

Lemma 11.2 ($K_{2,3}$ core)**.** *Let $B \subseteq H$ be a $K_{2,3}$ with hubs x, y and leaves $j = 1, 2, 3$, put $u_j = w_{xj}$, $v_j = w_{yj}$ and $h_{jk} = \sqrt{u_j u_k v_j v_k}$, the cycle product of the four-cycle $xjyk$. Under* (9.1)*,*

$$L_B \ge \tfrac{11}{2} \sum_{j<k} h_{jk}, \qquad L_{Y_{jk}} \ge \tfrac{11}{2} h_{jk}, \qquad E_{Y_{jk}} \le \tfrac{5}{2} h_{jk} Z_{\bar{Y}_{jk}}(w).$$

Proof. The hub stars give $\sum_j u_j, \sum_j v_j \le 1$, so the singleton loss is at least $2\sum_{j<k}(u_j u_k + v_j v_k) \ge 4 \sum h_{jk}$. A size-two core matching uses both hubs and one leaf pair, with opposite weights $p = u_j v_k$, $q = u_k v_j$, $\sqrt{p} + \sqrt{q} \le 1$ and $\sqrt{pq} = h_{jk}$; Lemma 9.1 gives loss at least $2h_{jk}(1 - h_{jk})$. The core has no larger matching and $h_{jk} \le 1/4$, so $L_B \ge \sum(6h_{jk} - 2h_{jk}^2) \ge \frac{11}{2} \sum h_{jk}$; the same count applies to one four-cycle. Finally $2h(1 + h) \le \frac{5}{2} h$. □

Proof of Theorem 1.4 for $G_\times$. The core on $0,1,2,3,4$ is $K_{2,3}$ with hubs $1,3$ and leaves $0,2,4$. Put $t = w_{06}$, $q = w_{56}$, $r = w_{45}$. A cycle either stays in the core or uses the whole external path $0,6,5,4$ with one of the four core paths from 0 to 4; thus there are three four-cycles Y_{02}, Y_{04}, Y_{24}, two five-cycles D_1, D_3 through one hub, and two seven-cycles. The complement factors of the four-cycles are $1+q+r$, $1+q$, $1+q+t$. With $R_B = (1+q)L_B + rL_{Y_{02}} + tL_{Y_{24}}$, Lemma 11.2 gives $\sum_{\text{four-cycles}} E_Y \le \frac{5}{11} R_B$. Each term of R_B charges a matching at most once, with external pattern empty, 56, 45 or 06; adding an external edge z to a core matching m costs $zm(1-m) \le zm(1-zm)$. Lemma 9.2 gives $E_Y \le \frac{1}{3} R_Y$ for five-cycles and $E_Y \le \frac{1}{8} R_Y$ for seven-cycles.

Total coefficients: an external singleton $45, 56, 06$ has no cluster charge and lies on both five- and both seven-cycles, giving $\frac{2}{3} + \frac{1}{4} = \frac{11}{12}$; an outer core singleton $01, 03, 14, 34$ receives at most $\frac{5}{11} + \frac{1}{3} + \frac{1}{8} = \frac{241}{264}$; a middle singleton $12, 23$ at most $\frac{5}{11} + \frac{1}{4} = \frac{31}{44}$. Seven-cycles charge only singletons; a size-two matching is charged by at most one five-cycle, because their complementary edges 23 and 12 meet, giving at most $\frac{5}{11} + \frac{1}{3} = \frac{26}{33}$; larger matchings receive only $\frac{5}{11}$. All are at most $\frac{11}{12}$, so $\sum_Y E_Y \le \frac{11}{12} L$ and $\mathcal{P} - \mathcal{Q} \ge L/12$. □

Equality and scaling in Theorem 1.4. The singleton terms of L_H vanish only if every $w_e \in \{0,1\}$, and (9.1) forbids two incident ones; conversely a direct sum of isolated vertices and unit-modulus 2×2 blocks has $\mathcal{P} = \mathcal{Q} = 2^s$. The scaled bound follows from (2.1). □

12 Triangle-free supports through order seven

Let H_t have edges $05, 06, 15, 16, 25, 26, 34, 36, 45$ (a pentagon blow-up with one part tripled) and let H_n have edges

$$05, 06, 15, 16, 24, 26, 34, 36, 45 \tag{12.1}$$

(two nonadjacent parts doubled).

Lemma 12.1. *Under* (9.1)*, every unit-diagonal Hermitian C supported in H_t satisfies $\mathcal{P}(C) - \mathcal{Q}(C) \ge L_{H_t}/700$.*

Proof. The core is $K_{2,3}$ with hubs $5,6$ and leaves $0,1,2$; the external path is $5,4,3,6$. The cycles are three four-cycles, each with complement the edge 34 of weight z, and three five-cycles, each with complement two isolated leaves. Lemma 11.2 gives $\sum_{\text{four}} E_Y \le \frac{5}{2}(1+z)\sum h_{jk} \le \frac{5}{11} R_B$ with $R_B = (1+z)L_B$, which charges each matching at most once. For five-cycles $E_Y \le \rho_5 R_Y$ and $\rho_5 = \sqrt{2}/5 + 1/20 < 233/700$ (as $\sqrt{2} < 99/70$). An external singleton lies on all three five-cycles and receives $699/700$; a core singleton at most $\frac{5}{11} + \frac{233}{700} = \frac{6063}{7700}$; larger matchings only $\frac{5}{11}$. Hence $\sum E_Y \le \frac{699}{700} L_{H_t}$. □

Lemma 12.2. *Under* (9.1)*, every unit-diagonal Hermitian C supported in H_n satisfies $\mathcal{P}(C) - \mathcal{Q}(C) \ge L_{H_n}/50$.*

Proof. With the edge order (12.1), the cycles are the four-cycles $(0,1,2,3)$, $(4,5,6,7)$ and the five-cycles $(0,1,4,5,8)$, $(0,1,6,7,8)$, $(2,3,4,5,8)$, $(2,3,6,7,8)$. Take $x_Y = 147/250$ on four-cycles and $x_Y = 1$ on five-cycles in (11.2), and define c_M and A_{mix} as in Section 11. The supplement file `nonadjacent_certificate.json` lists an exact identity

$$\sum_{1 \le |M| \le 2} (3528 - c_M) m_M (1 - m_M) - A_{\text{mix}} = \sum_{j=1}^{119} N_j w^{a_j} \prod_{v \in V_j} s_v$$

of the same form, with positive integers N_j; its verifier reconstructs all 38 matchings and checks $c_M \le 3528$ for size three. As in the proof for G, $3600 \sum E_Y \le 3528 L_{H_n}$. □

Lemma 12.3 (Classification)**.** *Up to isomorphism the maximal triangle-free graphs on seven vertices are $K_{1,6}$, $K_{2,5}$, $K_{3,4}$ and the pentagon blow-ups with part sizes $(2,2,1,1,1)$, $(2,1,2,1,1)$ and $(3,1,1,1,1)$ in cyclic order, represented by G, H_n and H_t.*

Proof. Every labelled triangle-free graph on $\{0,\dots,n\}$ arises exactly once from one on $\{0,\dots,n-1\}$ by adding vertex n adjacent to an independent set. The supplement program `classify_trianglefree.py` enumerates all 133501 labelled triangle-free graphs on seven vertices, keeps the 1743 in which every nonedge has a common neighbour (the maximal ones), and matches each to exactly one of six embedded representatives by exhaustive degree-preserving isomorphism tests, with multiplicities $7, 21, 35, 630, 630, 420$. The script `laplacian_strong_exact.py` checks that the embedded $(2,2,1,1,1)$ representative is isomorphic to G. □

Proof of Theorem 1.5. Normalize, pad with isolated unit vertices to order seven, and extend the support to a maximal triangle-free host, new edges carrying zero entries. Theorem 1.1 covers the three bipartite hosts, Theorem 1.4 covers G, and Lemmas 12.2 and 12.1 cover H_n and H_t. Relabelling preserves both permanents. □

A maximal triangle-free extension of a nonbipartite support keeps an odd cycle and is one of the three blow-ups. Hence every nonbipartite triangle-free correlation matrix of order at most seven satisfies $\mathcal{P} - \mathcal{Q} \geq L_H/700$.

13 The hereditary block class

Lemma 13.1. *Suppose $A = \begin{pmatrix} B_1 & 0 & x_1 \\ 0 & B_2 & x_2 \\ x_1^* & x_2^* & d \end{pmatrix} \succeq 0$. There are $t_1, t_2 \geq 0$ with $t_1 + t_2 = d$ and $A_i = \begin{pmatrix} B_i & x_i \\ x_i^* & t_i \end{pmatrix} \succeq 0$.*

Proof. Positivity places x_i in the range of B_i, and with the Moore–Penrose inverse, $\rho_i = x_i^* B_i^+ x_i$ satisfies $\rho_1 + \rho_2 \leq d$. Take $t_1 = d - \rho_2$ and $t_2 = \rho_2$; completing squares shows $A_i \succeq 0$. □

Lemma 13.2 (Complex one-point closure)**.** *In Lemma 13.1, if* (1.1) *holds for A_1, A_2, B_1, B_2, then it holds for A.*

Proof. This is the mechanism of Pant and Singh [12, Theorem 4 and Appendix A] in complex form. Put $a_i = \mathcal{P}(B_i)$, $c_i = \mathcal{Q}(B_i)$, $p_i = \mathcal{P}(A_i)$, $q_i = \mathcal{Q}(A_i)$ and $D_i = \mathrm{d}(B_i)$. A permutation cycle through the last vertex stays on one side, and the fixed point contributes $(t_1 + t_2)^2$ to $\mathcal{Q}$, so

$$\mathcal{P}(A) = a_2 p_1 + a_1 p_2, \qquad \mathcal{Q}(A) = c_2 q_1 + c_1 q_2 + 2 c_1 c_2 t_1 t_2.$$

The hypotheses give $c_i \leq D_i a_i$ and $q_i \leq t_i D_i p_i$, and Lieb's inequality gives $p_i \geq t_i a_i$ [10]. Therefore

$$\mathcal{Q}(A) \leq D_1 D_2 (a_2 p_1 t_1 + a_1 p_2 t_2 + 2 a_1 a_2 t_1 t_2) \leq D_1 D_2 (t_1 + t_2)(a_2 p_1 + a_1 p_2) = \mathrm{d}(A)\mathcal{P}(A),$$

the last difference being $D_1 D_2 \{t_2 a_2 (p_1 - t_1 a_1) + t_1 a_1 (p_2 - t_2 a_2)\} \geq 0$. □

Lemma 13.3. *Let $\mathcal{G}$ be the class of graphs all of whose blocks lie in $\mathcal{B}$. Then $\mathcal{G}$ is closed under induced subgraphs.*

Proof. A block of an induced subgraph $G[S]$ lies in a single block B of G and is a block of $B[S\cap V(B)]$; blocks are induced subgraphs. It therefore suffices that every block of an induced subgraph of a member of $\mathcal{B}$ lies in $\mathcal{B}$. For bipartite graphs of width five and triangle-free graphs on at most seven vertices this is immediate. A proper induced subgraph of C_n is a linear forest, with blocks K_2. In $\theta(1,b,c)$ with $b,c\geq 3$ every vertex lies on at least two of the three cycles, so a proper induced subgraph contains at most one cycle; its blocks are edges and at most one cycle of length at least four. □

Proof of Theorem 1.6. Every member of $\mathcal{B}$ satisfies (1.1) for all supported $A\succeq 0$, by Theorems 1.1, 1.3 and 1.5. Induct on the order of A, over host graphs in $\mathcal{G}$. Direct sums multiply both sides. A connected host with one block lies in $\mathcal{B}$. A connected host with several blocks has a cut vertex; split the components of its deletion into two nonempty groups and apply Lemma 13.1. The matrices A_i, B_i have smaller order and host graphs in $\mathcal{G}$ by Lemma 13.3, so Lemma 13.2 applies. A principal submatrix has support in an induced subgraph, again in $\mathcal{G}$. □

14 Graph Laplacians

Proof of Corollary 1.7. If $B\in\mathcal{B}$, the Laplacian L_B is positive semidefinite with support B, so Theorem 1.6 shows that L_B is hereditarily strong: every principal submatrix satisfies (1.1). For an arbitrary bipartite block this is [12, Corollary 3], and for a cycle C_n with $n\geq 3$, the triangle included, it is [12, Theorem 7]. A connected graph is obtained from its blocks by successive vertex coalescence, and the Laplacian of a coalescence is the one-point sum $L_{G_1}\star L_{G_2}$, which adds the two diagonal entries at the shared vertex [12, Section 5.1]. The one-point sum of hereditarily strong real positive semidefinite matrices is hereditarily strong [12, Theorem 4]. Components multiply, and an isolated vertex makes both sides zero. □

Proof of Theorem 1.8. The supplement script `laplacian_strong_exact.py` forms L_G for each of the $2^{\binom{n}{2}}$ labelled graphs with $2\leq n\leq 7$ and computes $\operatorname{per} L_G$ and $\operatorname{per}(L_G\circ L_G)$ by Ryser's formula in 64-bit integer arithmetic, which is exact since every intermediate value is below $42^7\cdot 128<2^{63}$. It compares the two sides of (1.2) exactly and finds no violation among the 1, 4, 41, 768, 27449 and 1887284 graphs without isolated vertices for $n=2,\dots,7$. It records equality exactly on the perfect matchings and a maximal ratio of $13/16$ for $n=3,5,7$, attained by K_3 plus a perfect matching of the remaining vertices. The independent script `check_laplacian_strong.py` reproduces the same counts and maxima. □

For K_3 one has $\operatorname{per} L=12$, $\prod\deg=8$ and $\operatorname{per}(L\circ L)=78$, the ratio $13/16$. Random samples $G(n,p)$ with $n=8,9,10$ in `check_laplacian_strong.py` reach a largest ratio of about 0.72.

15 Ordinary consequences

For complex matrices of the same order, Cauchy–Schwarz over permutations gives $|\operatorname{per}(A\circ B)|^2\leq \mathcal{Q}(A)\mathcal{Q}(B)$. Hence, if $A,B\succeq 0$ both belong to the classes of Theorems 1.1–1.6, with possibly different supports,

$$|\operatorname{per}(A\circ B)|\leq\sqrt{\mathrm{d}(A)\,\mathrm{d}(B)\mathcal{P}(A)\mathcal{P}(B)}\leq\mathcal{P}(A)\mathcal{P}(B),$$

which is Chollet's inequality for the pair. The ordinary self-case also follows from a dimension-free energy bound.

Proposition 15.1 (Energy criterion)**.** *For every finite complex rectangle satisfying* (2.2)*,*

$$S \le \sigma := 12 + \frac{201 + 16\sqrt{537}}{988} \quad \Longrightarrow \quad Q \le P^2.$$

In particular every $A \succeq 0$ with bipartite support of width twelve, hence every bipartite support on at most 25 *vertices, satisfies $\mathcal{Q}(A) \le \mathcal{P}(A)^2$.*

Proof. Write $S = k + t$ with $k \in \mathbb{Z}_{\ge 0}$ and $t \in [0, 1]$. Since $\log(1 + x^2)$ is convex on $[0, 1]$, shifting two interior coordinates toward an endpoint at fixed sum does not decrease $\sum \log(1 + r_a^2)$ and does not increase $e_2(r)$. Iterating gives k ones, one t and zeros, so by $Q \le \prod_a (1 + r_a^2)$ and the bound $P \ge 1 + \max\{S, e_2(r)\}$,

$$Q \le 2^k(1 + t^2), \qquad P \ge L_k(t) = 1 + \max\{k + t, \tbinom{k}{2} + kt\}.$$

Table 3 lists $L_k(t)^2 - 2^k(1 + t^2)$ and its minimum on $[0, 1]$ for $k \le 11$; all minima are nonnegative. For $k = 12$ the polynomial is $393 + 1608t - 3952t^2$, with discriminant $128^2 \cdot 537$ and positive root $(201 + 16\sqrt{537})/988 \in (0, 1)$. Operator contraction gives $S \le \min(p, q)$. □

Table 3: Coefficients of $L_k(t)^2 - 2^k(1 + t^2)$ and its minimum on $[0, 1]$.

k	const.	lin.	quad.	min	k	const.	lin.	quad.	min
0	0	2	0	0	6	192	192	−28	192
1	2	4	−1	2	7	356	308	−79	356
2	5	6	−3	5	8	585	464	−192	585
3	8	24	1	8	9	857	666	−431	857
4	33	56	0	33	10	1092	920	−924	1088
5	89	110	−7	89	11	1088	1232	−1927	393

16 Supplement and reproduction

The supplement accompanies this article and uses Python 3 with the standard library, with `numpy` for the Laplacian and adversarial scripts and `scipy` for one search. Each file is run as `python -I file [input]`, except `check_strong_small_orders.py`, which runs with `-E -P` so that an installed `scipy` is found; and `run_all.sh` runs all of them. The verifiers are in the folder `verifiers`, their inputs in `certificates`, and the independent reconstructions in `reproduction`.

- `verify_fiverow_certificate.py`: the program of Appendix A; it reports 2048 polynomials, 2097152 nonnegative coefficients and the table of Section 7.
- `verify_chorded_seven_cycle_certificate.py`, reading `chorded_seven_cycle_certificate.json`: Lemma 11.1.
- `verify_nonadjacent_certificate.py`, reading `nonadjacent_certificate.json`: Lemma 12.2.
- `classify_trianglefree.py`: Lemma 12.3.
- `check_fiverow_certificate.py` and `check_fiverow_strictness.py`: independent reconstructions, by monomial expansion and tensor Bernstein transform, of Table 1, the five-row certificate and Table 2.

- `laplacian_strong_exact.py` and `check_laplacian_strong.py`: Theorem 1.8 and the graph facts used in Sections 10–12.
- `check_seven_cycle_chords.py`, `check_strong_permanent_proofs.py`, `check_strong_permanent_supports.py`, `check_strong_small_orders.py` and `check_drury_support.py`: numerical cross-checks of the identities (4.2), the gaps of Theorem 1.4 and Lemmas 12.1–12.2, adversarial maximization of $\mathcal{Q}/(\mathrm{d}\,\mathcal{P})$ on the proved supports (all maxima below one), and a recomputation of Drury's matrix ($\mathcal{P} = 45$, $\mathcal{Q} = 6185/128$, support K_7 minus an edge, 30 triangles), which the same search recovers at 1.072 on full support of order seven.

The outputs of every run are included in the supplement directory `outputs`.

Prior versions

This article merges and supersedes the following seven records of the Luminary Research Archive by the author.

1. *Strong self-permanent inequalities on five-row bipartite supports*, edition 5, `https://research.luminaryoddities.org/strong-energy-bounded-permanent-inequalities-bipartite-supports`.
2. *Strong permanent inequalities on three chorded seven-cycle supports*, edition 2, `https://research.luminaryoddities.org/strong-permanent-inequalities-complex-two-three-chord-seven-cycle-supports`.
3. *Strong permanent inequalities for three-row bipartite supports and triangle-free supports through order seven*, edition 2, `https://research.luminaryoddities.org/strong-permanent-inequalities-three-row-bipartite-triangle-free-supports`.
4. *Strong permanent inequalities for cycles and a class of single-chord supports*, edition 2, `https://research.luminaryoddities.org/strong-permanent-inequalities-complex-positive-semidefinite-cycle-single-chord-supports`.
5. *Strong permanent contraction on four-by-seven bipartite supports*, edition 2, `https://research.luminaryoddities.org/strong-permanent-contraction-four-by-seven-bipartite-supports`.
6. *Strong permanent contraction on four-row bipartite supports through six columns*, edition 2, `https://research.luminaryoddities.org/strong-permanent-contraction-four-row-bipartite-supports-through-six-columns`.
7. *Row-dependent bounds for strong permanent contraction on four-by-eight supports*, edition 2, `https://research.luminaryoddities.org/row-dependent-bounds-strong-permanent-contraction-four-by-eight-supports`.

The $K_{4,6}$, $K_{4,7}$ and $K_{4,8}$ theorems of the last three records, including their computer-assisted certificates, are special cases of Theorem 6.2, which the present article proves analytically for every number of columns.

Declarations

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Data availability. All reductions are in the article; the certificates, verifiers, reproduction scripts and their outputs form the supplement.

A Exact reconstruction of the five-row certificate

At Bernstein nodes $t_i = m_i/3$, $m_i \in \{0,1,2,3\}$, on a row interval $[\ell_i/10, h_i/10]$ with $\ell_i, h_i \in \{0,5,7,10\}$, set $r_i = n_i/30$ with $n_i = 3\ell_i + (h_i - \ell_i)m_i$, and $R_i = N_i/900$ with $N_i = 0$ at the zero endpoint and, at the upper endpoint, $N_i = n_i^2$ if $h_i \le 7$ and $N_i = 441 + (n_i - 21)^2$ if $(\ell_i, h_i) = (7, 10)$. Put $d_i = 30n_i - N_i$, $s = \sum_i n_i$, $Z = 162000$ and $u_i = n_i(2025 + 4N_i)$, so that $\delta_i = d_i/900$ and $z_i = u_i/Z$. With $D = \operatorname{lcm}(4500, 1800, 900, 45000, 810000, 200Z^3, 2Z^4, 250Z^5)$, the low-budget nodal numerator is

$$\begin{aligned} DB_{\mathrm{L},\pi_0} = {} & \frac{D}{1800}\sum_i n_i(30 - n_i) + \frac{D}{900}e_2(n) - \frac{D}{810000}\left[e_2(N) + d_0d_1 + d_2d_3\right] \\ & - \frac{231D}{200Z^3}e_3(u) - \frac{3D}{2Z^4}e_4(u) - \frac{537D}{250Z^5}e_5(u), \end{aligned}$$

and for the high budget the first coefficient becomes $D/4500$ and the terms $\frac{D}{45000}e_3(n) - \frac{D}{45000}\sum_i d_i(s - 2n_i)$ are added. All scalar coefficients are integers. The one-coordinate cubic Bernstein evaluation matrix is $B_{27}/27$, and M_6 below satisfies $B_{27}M_6 = 162I$:

$$B_{27} = \begin{pmatrix} 27 & 0 & 0 & 0 \\ 8 & 12 & 6 & 1 \\ 1 & 6 & 12 & 8 \\ 0 & 0 & 0 & 27 \end{pmatrix}, \qquad M_6 = \begin{pmatrix} 6 & 0 & 0 & 0 \\ -5 & 18 & -9 & 2 \\ 2 & -9 & 18 & -5 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

Applying M_6 along the five tensor axes to the nodal numerators gives the Bernstein numerators over the common denominator $D \cdot 6^5 = 216905884017408000000000000000000$. The degree bound makes the interpolation an identity. The following standard-library program implements these formulas, rejects a negative coefficient or an incomplete case count, and uses no floating-point arithmetic in any decision. Its case order is low special row 0, low special row 4, then high, with lexicographic region and moment bits and the last tensor coordinate fastest.

```
"""New all-column five-row certificate; integer arithmetic, no dependencies."""
from itertools import product
from fractions import Fraction
from math import lcm
import json,time
Z=162000
DEN=lcm(4500,1800,900,45000,810000,200*Z**3,2*Z**4,250*Z**5)
M=((6,0,0,0),(-5,18,-9,2),(2,-9,18,-5),(0,0,0,6))
GRID=list(product(range(4),repeat=5))
def elementary(x,k):
 e=[1]+[0]*k
 for a in x:
  for j in range(k,0,-1):e[j]+=a*e[j-1]
 return e[k]
def nodes(regions,ends,high):
 vals=[]
 for point in GRID:
  n=[3*lo+(hi-lo)*t for (lo,hi),t in zip(regions,point)]
  N=[(x*x if hi<=7 else 441+(x-21)**2) if end else 0
```

```
                for x,(_,hi),end in zip(n,regions,ends)]
  d=[30*x-y for x,y in zip(n,N)]
  z=[x*(2025+4*y) for x,y in zip(n,N)]
  row=sum(x*(30-x) for x in n)
  v=(DEN//(4500 if high else 1800))*row
  v+=(DEN//900)*elementary(n,2)
  if high:v+=(DEN//45000)*elementary(n,3)
  v-=(DEN//810000)*(elementary(N,2)+d[0]*d[1]+d[2]*d[3])
  if high:v-=(DEN//45000)*sum(x*(sum(n)-2*y) for x,y in zip(d,n))
  v-=231*(DEN//(200*Z**3))*elementary(z,3)
  v-=3*(DEN//(2*Z**4))*elementary(z,4)
  v-=537*(DEN//(250*Z**5))*elementary(z,5)
  vals.append(v)
 return vals

def coefficients(vals):
 a=list(vals)
 for axis in range(5):
  stride=4**(4-axis)
  for start in range(0,1024,4*stride):
   for off in range(stride):
    ids=[start+off+j*stride for j in range(4)]
    old=[a[i] for i in ids]
    for j in range(4):a[ids[j]]=sum(M[j][k]*old[k] for k in range(4))
 return a

def cases():
 for special in (0,4):
  for bits in product((0,1),repeat=4):
   it=iter(bits);reg=[]
   for i in range(5):reg.append(
    (0,5) if i==special else ((0,7) if next(it)==0 else (7,10)))
   for ends in product((0,1),repeat=5):yield 'low'+str(special),bits,ends,reg,False
 for bits in product((0,1),repeat=5):
  reg=[(5,7) if x==0 else (7,10) for x in bits]
  for ends in product((0,1),repeat=5):yield 'high',bits,ends,reg,True

def run():
 groups={};gmin=None;location=None;t=time.time()
 for name,bits,ends,reg,high in cases():
  c=coefficients(nodes(reg,ends,high))
  if min(c)<0:raise AssertionError(('negative',name,bits,ends,min(c)))
  st=groups.setdefault(name,dict(
   polynomials=0,coefficients=0,zero=0,min_positive=None))
  st['polynomials']+=1;st['coefficients']+=1024;st['zero']+=c.count(0)
  pos=min((x for x in c if x>0),default=None)
  if pos is not None and (st['min_positive'] is None
                          or pos<st['min_positive']):st['min_positive']=pos
  if pos is not None and (gmin is None
                          or pos<gmin):gmin=pos;location=(name,bits,ends,c.index(pos))
  if sum(x['polynomials'] for x in groups.values())%512==0:print(
   'progress',groups.keys(),'seconds',round(time.time()-t,2),flush=True)
 counts={k:v['polynomials'] for k,v in groups.items()}
 expected={'low0':512,'low4':512,'high':1024}
 if counts!=expected:raise AssertionError(('incomplete coverage',counts,expected))
 for st in groups.values():st['min_positive']=str(Fraction(
  st['min_positive'],DEN*6**5))
 out=dict(status='PASS_NEW_EXACT_ALL_COLUMN_CERTIFICATE',
          groups=groups,polynomials=2048,coefficients=2097152,
          denominator=DEN*6**5,
          min_positive=str(Fraction(gmin,DEN*6**5)),
```

```
          min_positive_location=location)
 print(json.dumps(out,indent=2),flush=True)
 return out
if __name__=='__main__':run()
```

References

[1] R. B. Bapat and V. S. Sunder, On majorization and Schur products, *Linear Algebra Appl.* **72** (1985), 107–117. https://doi.org/10.1016/0024-3795(85)90147-8.

[2] R. B. Bapat and V. S. Sunder, An extremal property of the permanent and the determinant, *Linear Algebra Appl.* **76** (1986), 153–163. https://doi.org/10.1016/0024-3795(86)90220-X.

[3] L. B. Beasley, An inequality on permanents of Hadamard products, *Bull. Korean Math. Soc.* **37** (2000), no. 3, 633–639.

[4] E. Carlen, E. H. Lieb and M. Loss, An inequality of Hadamard type for permanents, *Methods Appl. Anal.* **13** (2006), no. 1, 1–18. https://doi.org/10.4310/MAA.2006.v13.n1.a1.

[5] J. Chollet, Is there a permanental analogue to Oppenheim's inequality?, *Amer. Math. Monthly* **89** (1982), no. 1, 57–58.

[6] S. W. Drury, A counterexample to a question of Bapat and Sunder, *Electron. J. Linear Algebra* **31** (2016), 69–70. https://doi.org/10.13001/1081-3810.3178.

[7] J. Edmonds, Maximum matching and a polyhedron with 0,1-vertices, *J. Res. Nat. Bur. Standards Sect. B* **69B** (1965), 125–130. https://doi.org/10.6028/jres.069B.013.

[8] D. Handelman, Representing polynomials by positive linear functions on compact convex polyhedra, *Pacific J. Math.* **132** (1988), no. 1, 35–62. https://doi.org/10.2140/pjm.1988.132.35.

[9] Q. Li, Q. Zhao and W. Luo, Chollet's permanent conjecture through order six via border induction, Preprints.org (2026), posted 31 August 2026. https://doi.org/10.20944/preprints202608.2196.v1.

[10] E. H. Lieb, Proofs of some conjectures on permanents, *J. Math. Mech.* **16** (1966), 127–134. https://doi.org/10.1512/iumj.1967.16.16008.

[11] R. Mehatari, M. R. Kannan and A. Samanta, On the adjacency matrix of a complex unit gain graph, *Linear Multilinear Algebra* **70** (2022), no. 9, 1798–1813. https://doi.org/10.1080/03081087.2020.1776672.

[12] P. Pant and R. Singh, Structural classes for Chollet's permanent conjecture, arXiv:2604.24192v2 (2026). https://arxiv.org/abs/2604.24192v2.

[13] L. Pioge, K. K. Pietrasz, B. Seron, L. Novo and N. J. Cerf, A logical implication between two conjectures on matrix permanents, *Linear Algebra Appl.* **725** (2025), 309–318. https://doi.org/10.1016/j.laa.2025.07.011.

[14] K. Rodtes, Chollet's permanent conjecture for 4×4 matrices, *Linear Multilinear Algebra* **72** (2024), no. 16, 2633–2638. https://doi.org/10.1080/03081087.2023.2279150.

[15] S. Sa-nguansin and K. Rodtes, Permanents of correlation matrices and the Chollet permanental conjecture, *Linear Multilinear Algebra* **73** (2025), no. 18, 4084–4096. https://doi.org/10.1080/03081087.2025.2588584.

[16] V. S. Shchesnovich, The permanent-on-top conjecture is false, *Linear Algebra Appl.* **490** (2016), 196–201. https://doi.org/10.1016/j.laa.2015.10.034.

[17] F. Zhang, An update on a few permanent conjectures, *Spec. Matrices* **4** (2016), 305–316. https://doi.org/10.1515/spma-2016-0030.