Strong permanent inequalities for complex two- and three-chord seven-cycle supports

Scott Schubr
Luminary Oddities

Abstract

We prove $\mathrm{per}(|A|^2) \le \mathrm{per}(A)$ for complex positive semidefinite correlation matrices supported on a seven-cycle with chords $03, 15, 26$. More precisely, the difference dominates one fiftieth of the matching-partition loss. The proof combines elementary cycle estimates with an explicit nonnegative integer polynomial identity, supplied with a standard-library verification program. The two-chord subgraph with chords $03, 15$ inherits the same bound. For the crossing placement $03, 14$ we give a separate analytic proof with constant $1/12$. All results include missing edges and singular matrices. The normalized arguments require only vertex-star magnitude bounds, and we describe equality and diagonal scaling.

1 Statements and context

For a Hermitian matrix set $\mathcal{P}(A) = \mathrm{per}(A)$ and $\mathcal{Q}(A) = \mathrm{per}(|A|^2)$, where the square and absolute value are entrywise. A correlation matrix is positive semidefinite with diagonal entries one. Let G be the graph on $0, \dots, 6$ with ordered edges

$$01, 12, 23, 34, 45, 56, 06, 03, 15, 26. \tag{1}$$

Let G_- omit 26, and let $G_\times$ have the seven cycle edges and chords $03, 14$. Support contained in a graph permits zero entries on any of its edges. For edge weights $w_e = |a_e|^2$ and a matching M, put

$$m_M = \prod_{e \in M} w_e, \quad Z_H(w) = \sum_{M \text{ matching of } H} m_M, \quad L_H(w) = Z_H(w) - Z_H(w^2).$$

The empty product is one. Thus $L_H = \sum_M m_M(1 - m_M)$ when $0 \le w_e \le 1$.

Theorem 1. *For every complex correlation matrix with support contained in G or G_-,*

$$\mathcal{P}(A) - \mathcal{Q}(A) \ge \frac{1}{50} L_H(w) \ge 0. \tag{2}$$

For support contained in $G_\times$, the constant $1/50$ can be replaced by $1/12$. These conclusions also hold for any Hermitian matrix with diagonal one, the indicated support, and $\sum_{e \ni v} w_e \le 1$ at every vertex, without a full positive semidefiniteness assumption.

Chollet's conjecture asks for a permanent analogue of Oppenheim's inequality for positive semidefinite matrices [1]. The self-inequality $\mathcal{Q}(A) \le \mathcal{P}(A)$ is stronger than its normalized self-case $\mathcal{Q}(A) \le \mathcal{P}(A)^2$, and is false in unrestricted dimension [2]. Structural graph methods for the conjecture are developed by Pant and Singh [3]. This paper follows the author's cycle and single-chord support results [4]; the present contribution treats overlapping cycles created by two and three

chords, with the three-chord theorem subsuming one two-chord placement. Our mechanism is a matching-loss allocation on overlapping cycles. The cycle expansion below is of the classical Sachs type; the complex gain-graph version is discussed in [6]. We give the complete expansion needed here directly.

2 Cycle costs and elementary reservoirs

All three graphs in Theorem 1 are triangle-free. A vertex and its neighbors induce a star. A correlation-PSD principal submatrix on that star has form $\left(\begin{smallmatrix} 1 & z^* \\ z & I \end{smallmatrix}\right)$; its Schur complement gives

$$s_v := 1 - \sum_{e \ni v} w_e \geq 0. \tag{3}$$

We henceforth use only these inequalities and $w_e \geq 0$. In particular $w_e \leq 1$ and adjacent cycle weights satisfy $w_i + w_{i+1} \leq 1$.

None of the graphs has two vertex-disjoint cycles: each cycle has length at least four and there are seven vertices. For an unoriented simple cycle C, choose either orientation and let H_C be the product of corresponding matrix entries. Put $h_C = |H_C| = (\prod_{e \in C} w_e)^{1/2}$, and let $\bar{C}$ denote the induced graph outside its vertices. A permutation consists of fixed points, transpositions, and at most one longer cycle. The two orientations of that cycle contribute conjugate products. Consequently

$$\mathcal{P}(A) = Z_H(w) + 2 \sum_C \Re(H_C) Z_{\bar{C}}(w), \tag{4}$$

$$\mathcal{Q}(A) = Z_H(w^2) + 2 \sum_C h_C^2 Z_{\bar{C}}(w^2). \tag{5}$$

Since $Z_{\bar{C}}(w^2) \leq Z_{\bar{C}}(w)$, these exact formulas imply

$$\mathcal{P}(A) - \mathcal{Q}(A) \geq L_H - \sum_C E_C, \qquad E_C = 2h_C(1 + h_C) Z_{\bar{C}}(w). \tag{6}$$

This bound allows independently unfavorable cycle signs, so requires no assumption on the compatibility of cycle phases.

Order the edges of a cycle cyclically as e_i, with indices modulo ℓ. Multiplication of $w_i w_{i+1} \leq 1/4$ gives

$$h_C \leq 2^{-\ell/2}. \tag{7}$$

Define the angle reservoir

$$A_C = \Big(\sum_i w_i w_{i+1} \Big) Z_{\bar{C}}(w).$$

For $\ell \geq 5$, select exactly the ℓ distinct distance-two matchings $I_i = \{e_i, e_{i+2}\}$. For $\ell = 4$, select the two opposite-edge matchings, each once. Define

$$J_C = \sum_{I \text{ selected}} m_I(1 - m_I), \qquad B_C = J_C Z_{\bar{C}}(w).$$

In particular J_C does not mean all size-two matchings when $\ell \geq 6$.

Lemma 2. *For $4 \le \ell \le 7$, $E_C \le \alpha_\ell A_C$ and $E_C \le \beta_\ell B_C$, where*

ℓ	4	5	6	7
α_ℓ	5/8	1/3	3/16	1/8
β_ℓ	5/3	4/9	1/4	3/20

Proof. AM–GM gives $\sum_i w_i w_{i+1} \ge \ell h_C^{4/\ell}$. The angle ratio is at most $(2/\ell)h^{1-4/\ell}(1+h)$, increasing on the interval (7). For lengths $4, 5, 6$ its endpoint values are $5/8$, $\sqrt{2}/5 + 1/20 < 1/3$, and $3/16$. For length seven the endpoint is at most $9/(56\sqrt{2}) < 1/8$, using $h \le 1/8$ in the factor $1 + h$.

For the pair estimate with $\ell \ge 5$, put $m_i = w_i w_{i+2}$, $q_i = \sqrt{m_i m_{i+1}}$, and $t_i = m_i + m_{i+1}$. Cauchy–Schwarz gives

$$\sqrt{m_i} + \sqrt{m_{i+1}} \le \sqrt{(w_i + w_{i+1})(w_{i+2} + w_{i+3})} \le 1.$$

Hence $q_i \le 1/4$ and $2q_i \le t_i \le 1 - 2q_i$. The identity

$$m_i(1 - m_i) + m_{i+1}(1 - m_{i+1}) - 2q_i(1 - q_i) = (t_i - 2q_i)(1 - t_i - 2q_i) \ge 0 \tag{8}$$

sums, counting each selected loss twice, to

$$J_C \ge \sum_i q_i(1 - q_i) \ge \frac{3}{4} \sum_i q_i \ge \frac{3\ell}{4} h_C^{4/\ell}.$$

The ratio E_C/B_C is at most $(8/(3\ell))h^{1-4/\ell}(1 + h)$. Its endpoint values for lengths $5, 6, 7$ are $(4\sqrt{2} + 1)/15 < 4/9$, $1/4$, and $(8\sqrt{2} + 1)/84 < 3/20$. The strict radical comparisons follow by squaring, respectively $288 < 289$ and $800 < 841$.

For length four let p, q be the opposite-pair weights. Then $\sqrt{p} + \sqrt{q} \le 1$ and $\sqrt{pq} = h_C$. Identity (8) gives $J_C \ge 2h_C(1 - h_C)$. Thus the ratio is at most $(1 + h_C)/(1 - h_C) \le 5/3$, since $h_C \le 1/4$. All zero-product cases follow directly or by continuity. □

3 The three-chord allocation and its integer certificate

Sort cycles by length and then lexicographically by their increasing edge-index lists in (1). There are five four-cycles, four five-cycles, two six-cycles and four seven-cycles. The four-cycle lists and their weights x_C are

$(0, 1, 2, 7)$	2187/8750
$(0, 1, 6, 9)$	51/125
$(0, 5, 6, 8)$	4341/17500
$(1, 5, 8, 9)$	4341/17500
$(2, 6, 7, 9)$	2187/8750

Assign $x_C = 27/80$ to every five-cycle and $x_C = 1$ to every six- or seven-cycle. Lemma 2 gives the convex estimate

$$E_C \le (1 - x_C)\alpha_{|C|} A_C + x_C \beta_{|C|} B_C. \tag{9}$$

A term in B_C selects an internal pair I and a matching T entirely outside C. Their union is a graph matching M, and

$$m_I(1 - m_I)m_T \le m_M(1 - m_M). \tag{10}$$

For fixed C, the matching M determines I, T uniquely. Call M eligible for C precisely when its edges on C form exactly one selected pair, and all remaining edges have both endpoints outside C. Set

$$c_M = \sum_{C:\, M \text{ eligible}} 3600 x_C \beta_{|C|}, \qquad A_{\mathrm{mix}} = \sum_C 3600(1 - x_C)\alpha_{|C|} A_C. \tag{11}$$

There is no singleton pair charge.

Lemma 3. *For this graph and these rational coefficients,*

$$F := \sum_{1 \le |M| \le 2} (3528 - c_M) m_M (1 - m_M) - A_{\mathrm{mix}} \ge 0, \tag{12}$$

$$c_M \le 3528 \qquad (|M| = 3). \tag{13}$$

Proof. Nonnegative monomial products of defining linear slacks are a standard polynomial-positivity framework [5]. Here we use a particular explicit identity, not an existence theorem for strictly positive polynomials: the verified identity applies directly to all closed faces of the star domain. The accompanying file `certificate.json` specifies the following exact integer polynomial identity:

$$350F = \sum_{j=1}^{323} N_j w^{a_j} \prod_{v \in V_j} s_v. \tag{14}$$

Every N_j is a positive integer, every exponent in a_j is a nonnegative integer, and V_j is a multiset of at most two vertices. Each summand has total degree at most four. The right side is nonnegative by (3).

For precision, the certificate contains the ordered graph edges, cycles represented by edge-index lists, the rational mixture weights, scale 3600, budget 3528, integer multiplier 350, and the 323 terms. In a term, `monomial` is the ten-entry exponent vector a_j, `slacks` lists the vertices of V_j with multiplicity, and `integer_coefficient` is N_j. Thus (14), together with (11)–(12), defines the asserted polynomial without any optimization software.

The accompanying `verify_certificate.py` reconstructs all simple cycles and matchings from the graph, the selected distance-two pairs, and all charges in (11). It expands both sides of (14) over exact rational numbers and requires their coefficient difference to be identically zero. It also checks nonnegative integer coefficients, the allowed slack factors, the rational mixtures, and (13). There are 53 matchings, with counts $1, 10, 26, 16$ in sizes $0, 1, 2, 3$; the maximum size-three charge is exactly 3528. The target has 141 nonzero monomials. This is an explicit finite polynomial proof; solver status, approximate coefficients and numerical tolerances are not its hypotheses. The complete coefficient list is supplied electronically to avoid replacing an inspectable integer identity by a long table of integers. □

Proof of Theorem 1 for G and G_-. Equations (9) and (10) give

$$3600 \sum_C E_C \le A_{\mathrm{mix}} + \sum_M c_M m_M (1 - m_M).$$

Apply (12) to matchings of sizes one and two, and (13) to size three. The right side is at most $3528 L_G$. Equation (6) proves $\mathcal{P} - \mathcal{Q} \ge L_G/50$. Setting the weight of edge 26 to zero gives the same theorem for G_-. Every estimate is valid on the star domain, including all zero weights. □

4 An analytic sharper bound for the crossing placement

We give a certificate-free proof for $G_\times$. Its core B on vertices $0,1,2,3,4$ is $K_{2,3}$ with hubs $1,3$ and leaves $0,2,4$. Put $u_j = w_{1j}$ and $v_j = w_{3j}$, and $h_{jk} = \sqrt{u_j u_k v_j v_k}$. The hub-star bounds give $\sum_j u_j \le 1$ and $\sum_j v_j \le 1$. Therefore the singleton core loss is at least

$$2\sum_{j<k}(u_j u_k + v_j v_k) \ge 4\sum_{j<k} h_{jk}.$$

For each leaf pair the opposite matching weights are $p = u_j v_k$ and $q = u_k v_j$. Cauchy–Schwarz gives $\sqrt{p} + \sqrt{q} \le 1$, so (8) supplies loss at least $2h_{jk}(1 - h_{jk})$. Each core size-two matching belongs to exactly one leaf pair and the core has no larger matching. Since $h_{jk} \le 1/4$,

$$L_B \ge \sum_{j<k}(6h_{jk} - 2h_{jk}^2) \ge \frac{11}{2}\sum_{j<k} h_{jk}. \tag{15}$$

The same argument on a single four-cycle gives $L_C \ge (11/2)h_C$.

Put $t = w_{06}$, $q = w_{56}$ and $r = w_{45}$. There are three four-cycles C_{02}, C_{04}, C_{24} indexed by leaf pairs, two five-cycles D_1, D_3 using the indicated hub and the external path $0,6,5,4$, and two spanning seven-cycles. These exhaust the cycles, since a cycle either stays in the core or uses the entire external path and one of the four simple core paths from 0 to 4. The complementary matching factors of the three four-cycles are $1 + q + r, 1 + q, 1 + q + t$. Define

$$R_B = (1 + q)L_B + rL_{C_{02}} + tL_{C_{24}}.$$

Equation (15) and $2h(1 + h) \le (5/2)h$ show

$$\sum_{C \text{ four-cycle}} E_C \le \frac{5}{11} R_B. \tag{16}$$

Each term of R_B charges a full matching at most once: its external-edge pattern is respectively empty, 56, 45, or 06. These patterns are mutually exclusive. Adding one external edge of weight z to a core matching of weight m charges $zm(1 - m) \le zm(1 - zm)$.

For a cycle put

$$R_C = \Big(\sum_{e \in C} w_e(1 - w_e)\Big) Z_{\bar{C}}(w).$$

The adjacent inequalities yield

$$\sum_{e \in C} w_e(1 - w_e) \ge \sum_i w_i w_{i+1} \ge |C| h_C^{4/|C|}.$$

The endpoint calculations in Lemma 2 therefore give $E_C \le R_C/3$ for five-cycles and $E_C \le R_C/8$ for seven-cycles. A term of R_C selects one cycle edge e and a complementary matching T. It charges $M = \{e\} \cup T$ by $m_M(1 - w_e) \le m_M(1 - m_M)$. For fixed C, the selection is unique.

It remains to count total coefficients. An external singleton edge $45, 56, 06$ has no nonzero cluster charge and lies in both five- and both seven-cycles, giving $2/3 + 1/4 = 11/12$. An outer core singleton $01, 03, 14, 34$ receives at most $5/11 + 1/3 + 1/8 = 241/264 < 11/12$. A middle singleton $12, 23$ receives at most $5/11 + 1/4 = 31/44 < 11/12$.

No matching of size at least two is charged by a spanning seven-cycle reservoir. A size-two matching cannot be charged by both five-cycle reservoirs: their complementary edges are respectively 23 and 12, which meet. Its coefficient is at most $5/11 + 1/3 = 26/33 < 11/12$. A matching

of size at least three receives only the cluster coefficient 5/11, since each five-cycle complement has only one edge. The empty matching has zero loss. Hence

$$\frac{5}{11}R_B + \frac{1}{3}(R_{D_1} + R_{D_3}) + \frac{1}{8}\sum_{C \text{ seven-cycle}} R_C \le \frac{11}{12}L_{G_\times}.$$

Together with (16) and (6), this proves the crossing assertion of Theorem 1.

5 Equality, scaling and the scope of the result

Let δ be the constant 1/50 or 1/12 applicable to the support. Strong equality $\mathcal{P}(A) = \mathcal{Q}(A)$ in correlation normalization holds exactly when all nonzero edge weights are one and their edges form a matching. Indeed the singleton terms of L_H force every w_e to be zero or one, and the star bounds forbid two incident ones. Conversely such a matrix is a direct sum of isolated vertices and 2-by-2 blocks with unit-modulus off-diagonal entries. Both permanents are two on each block, independently of phase.

There is also an explicit ordinary self-Chollet deficit. Put $S = \sum_e w_e$, $T = \sum_e w_e^2$, and $U = S - T$. Since $L_H \ge U$, $\mathcal{Q} \ge 1 + T$ and $\mathcal{P} \ge \mathcal{Q} \ge 1$,

$$\mathcal{P}(A)^2 - \mathcal{Q}(A) = \mathcal{P}(A)(\mathcal{P}(A) - 1) + (\mathcal{P}(A) - \mathcal{Q}(A)) \ge \delta S + (1 - \delta)T + T^2. \tag{17}$$

Ordinary equality consequently requires $A = I$.

For a positive semidefinite matrix with positive diagonal, let $d = \prod_i a_{ii}$ and let C be its diagonal normalization. Then $\mathcal{P}(A) = d\mathcal{P}(C)$ and $\mathcal{Q}(A) = d^2\mathcal{Q}(C)$, so

$$d\mathcal{P}(A) - \mathcal{Q}(A) \ge \delta d^2 L_H(|C|^2).$$

If a diagonal entry is zero, positivity forces its row and column to vanish, and both permanents are zero. No invertibility or limiting argument is needed for this case.

The theorem is a statement about the specified supports and their subgraphs. In particular a strong self-inequality on these supports does not imply domination with an arbitrary second positive semidefinite factor. The finite certificate exploits the fixed graph and should not be interpreted as a general graph theorem.

Reproducibility and acknowledgment

The electronic supplement contains the exact integer certificate and a verification program using only the Python standard library. It verifies algebraic identity and finite graph enumeration; the analytic reduction and inequalities are proved above. AI tools assisted with drafting and verification. The author is responsible for the statements, proofs, attribution and final text.

References

[1] J. Chollet, Is there a permanental analogue to Oppenheim's inequality?, *American Mathematical Monthly* **89** (1982), 57–58.

[2] S. W. Drury, A counterexample to a question of Bapat and Sunder, *Electronic Journal of Linear Algebra* **31** (2016), 69–70. `https://journals.uwyo.edu/index.php/ela/article/view/1631`.

[3] P. Pant and R. Singh, Structural Classes for Chollet's Permanent Conjecture, arXiv:2604.24192v2 (2026). https://arxiv.org/abs/2604.24192v2.

[4] S. Schubr, Strong permanent inequalities for complex positive semidefinite cycle and single-chord supports, Luminary Research Archive, first edition (2026). Luminary Research Archive, edition 1.

[5] D. Handelman, Representing polynomials by positive linear functions on compact convex polyhedra, *Pacific Journal of Mathematics* **132** (1988), no. 1, 35–62. https://doi.org/10.2140/pjm.1988.132.35.

[6] R. Mehatari, M. R. Kannan and A. Samanta, On the adjacency matrix of a complex unit gain graph, *Linear and Multilinear Algebra* **70** (2022), no. 9, 1798–1813. https://doi.org/10.1080/03081087.2020.1776672.