Strong permanent inequalities for complex positive semidefinite cycle and single-chord supports

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Abstract

For a complex Hermitian positive semidefinite matrix A, write $\mathcal{Q}(A) = \operatorname{per}(A \circ \overline{A})$ and $\mathfrak{d}(A) = \prod_i a_{ii}$. We prove $\mathcal{Q}(A) \le \mathfrak{d}(A) \operatorname{per}(A)$ whenever the off-diagonal support of A is contained in a cycle of length at least four, or in a theta graph with endpoint-path lengths $1, b, c$, where $b, c \ge 3$ and $b + c \ge 7$. The entries may have arbitrary phases and unequal magnitudes, and singular matrices are included. For a single cycle, the loss between two weighted matching partition functions pays the worst possible directed-cycle contribution. For a single-chord cycle, a simultaneous matching-loss allocation pays all three overlapping cycles. We give the complex form of the known one-point strong-closure argument and obtain hereditary inequalities for graphs assembled from these blocks. These are structural results.

1 Introduction and statements

For a square matrix $A = (a_{ij})$ of order n, its permanent is

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

We use the empty permanent convention $\operatorname{per}(\varnothing) = 1$. Throughout, $A \succeq 0$ means that A is complex Hermitian positive semidefinite. Write

$$\mathcal{P}(A) = \operatorname{per}(A), \qquad \mathcal{Q}(A) = \operatorname{per}(A \circ \overline{A}), \qquad \mathfrak{d}(A) = \prod_i a_{ii},$$

where $\circ$ denotes entrywise multiplication. A correlation matrix is a positive semidefinite matrix with diagonal entries all equal to one.

Chollet's conjecture asks whether

$$\operatorname{per}(A \circ B) \le \operatorname{per}(A) \operatorname{per}(B) \qquad (A, B \succeq 0). \tag{1}$$

Its self-conjugate form is $\mathcal{Q}(A) \le \mathcal{P}(A)^2$ [1]. The stronger inequality considered here is

$$\mathcal{Q}(A) \le \mathfrak{d}(A)\mathcal{P}(A). \tag{2}$$

This is the self-conjugate specialization of the Bapat–Sunder inequality. It is false for general positive semidefinite matrices: Drury's complex correlation matrix of order seven has $\mathcal{P}(A) = 45$

and $\mathcal{Q}(A) = 6185/128 > 45$ [2]. The support hypotheses below are therefore essential to our argument.

The support graph of a Hermitian matrix has its indices as vertices, with an edge $\{i, j\}$ when $i \neq j$ and $a_{ij} \neq 0$. Saying that the support is contained in a graph allows some of its edges to have zero matrix entries. A theta graph consists of three internally vertex-disjoint paths between two endpoints; their lengths count edges.

Theorem 1.1. *Inequality* (2) *holds for every complex Hermitian positive semidefinite matrix whose support is contained in either of the following graphs:*

1. *a cycle C_n, with $n \geq 4$;*
2. *a theta graph with path lengths $1, b, c$, where $b, c \geq 3$ and $b + c \geq 7$.*

The result includes arbitrary complex phases, unequal edge magnitudes, singular matrices, and missing edges.

The second class is a cycle of length $b + c$ with one chord whose two endpoint paths along the outer cycle have lengths at least three. The hypotheses exclude the theta graph with path lengths $1, 3, 3$ from this theorem.

Theorem 1.2. *Let G be a graph each of whose blocks is an edge, a cycle of length at least four, or a theta graph from Theorem 1.1. If $A \succeq 0$ has support contained in G, then every principal submatrix D satisfies*

$$\mathcal{Q}(D) \leq \mathfrak{d}(D)\mathcal{P}(D).$$

Isolated vertices and disconnected graphs are allowed.

Here blocks are maximal two-connected subgraphs, with bridges regarded as edge blocks. Theorem 1.2 uses the established one-point strong-closure mechanism of Pant and Singh [3, Theorem 4.3 and Appendix A]. We spell out its complex Hermitian algebra and the positive semidefinite diagonal allocation needed to apply it to arbitrary matrices of the prescribed support.

Pant and Singh prove strong inequalities for graph-Laplacian cycles and cacti, as well as structural one-point closure [3, Theorems 5.3–5.4]. Our cycle statement allows arbitrary complex edge phases and unequal weights subject to positive semidefiniteness. The single-chord statement treats three cycles sharing endpoints and paths, where cut-vertex closure alone does not apply. The combinatorial expansions are standard cycle-cover expansions, in the Sachs framework, including its complex gain-graph form [4, Theorem 3.2]; the substantive estimate here is an allocation of nonnegative matching losses.

2 Normalization and matching loss

If a positive semidefinite matrix has a zero diagonal entry, its corresponding row and column vanish. Both sides of (2) are then zero. Otherwise let $D = \operatorname{diag}(\sqrt{a_{11}}, \ldots, \sqrt{a_{nn}})$ and $C = D^{-1}AD^{-1}$. Then C is a correlation matrix with the same support, and

$$\mathcal{P}(A) = \mathfrak{d}(A)\mathcal{P}(C), \qquad \mathcal{Q}(A) = \mathfrak{d}(A)^2\mathcal{Q}(C). \tag{3}$$

It therefore suffices to prove $\mathcal{Q}(C) \leq \mathcal{P}(C)$ in correlation normalization. In this normalization $|c_{ij}| \leq 1$ by the two-by-two principal minors.

For a graph H with edge weights $0 \le w_e \le 1$, let

$$Z_H(w) = \sum_{M \text{ a matching of } H} \prod_{e \in M} w_e.$$

The empty matching contributes one. Put

$$L_H(w) = Z_H(w) - Z_H(w^2) = \sum_M m_M(1 - m_M), \qquad m_M = \prod_{e \in M} w_e.$$

Every summand is nonnegative. This elementary observation permits us to discard some matching losses or distribute them among different cycle costs.

Lemma 2.1. *Suppose a correlation matrix is supported on a triangle-free graph. If edges e, f meet at a vertex, then $w_e + w_f \le 1$, where $w_e = |a_e|^2$. For any simple cycle C of length $\ell \ge 4$, put $h_C = (\prod_{e \in C} w_e)^{1/2}$. Then*

$$\sum_{e \in C} w_e(1 - w_e) \ge \ell h_C^{4/\ell}, \tag{4}$$

$$h_C \le 2^{-\ell/2}. \tag{5}$$

Proof. The three vertices supporting e, f induce a path. Its principal determinant is $1 - w_e - w_f$, which is nonnegative. Number the edges of C cyclically. Therefore

$$\sum_i w_i(1 - w_i) \ge \sum_i w_i w_{i+1} \ge \ell \left(\prod_i w_i w_{i+1} \right)^{1/\ell} = \ell h_C^{4/\ell},$$

by AM–GM. Summing $w_i + w_{i+1} \le 1$ gives $\sum_i w_i \le \ell/2$. A second application of AM–GM proves (5). Zero weights cause no exception. □

The lemma uses necessary local positive semidefinite constraints only. In the support classes below these constraints suffice for the matching-loss estimates, without any separate assumption about the compatibility of cycle phases.

3 A single cycle

Proof of Theorem 1.1, part (1). Normalize the diagonal to one and write $w_i = |a_{i,i+1}|^2$, with indices cyclic modulo n. Let H be the directed product of the entries around the full cycle, $r = \operatorname{Re} H$, and $h = |H| = (\prod_i w_i)^{1/2}$. A nonzero permutation term consists either of fixed vertices and disjoint edge transpositions, or of one of the two orientations of the full cycle. Thus

$$\mathcal{P}(A) = Z_{C_n}(w) + 2r, \qquad \mathcal{Q}(A) = Z_{C_n}(w^2) + 2h^2. \tag{6}$$

These identities include $n = 4$ and zero edge entries. Since $r \ge -h$, retaining singleton matching losses and using Lemma 2.1 yields

$$\mathcal{P}(A) - \mathcal{Q}(A) \ge n h^{4/n} - 2h - 2h^2.$$

For $n \ge 4$ and $0 \le h \le 1$, one has $h^{4/n} \ge h$. Hence the right-hand side is at least $h(n - 2 - 2h) \ge 0$. This proves the normalized strong inequality, and (3) proves its general form. □

Corollary 3.1. *For a correlation matrix as in part (1), let* $S = \sum_i |a_{i,i+1}|^2$ *and* $\gamma_n = 1 - 2/n$. *Then*

$$\mathcal{P}(A)^2 - \mathcal{Q}(A) \geq \gamma_n S + \gamma_n^2 S^2.$$

Proof. Since $Z(w) \geq 1 + S$ and $S \geq nh^{2/n} \geq nh$, equation (6) gives $\mathcal{P}(A) \geq 1 + \gamma_n S$. The proved inequality $\mathcal{Q}(A) \leq \mathcal{P}(A)$ now gives the result. □

For a full nonzero cycle support the strong inequality is strict: adjacent constraints give $h < 1$, so $h(n - 2 - 2h) > 0$. Missing-edge cases can have equality, for example a direct sum of normalized rank-one two-by-two blocks. The constants in Corollary 3.1 are sufficient bounds.

4 Single-chord cycles

Fix path lengths $1, b, c$ as in Theorem 1.1. Interchanging b, c, assume $3 \leq b \leq c$, so $c \geq 4$. The graph has exactly three simple long cycles, denoted C_1, C_2, C_3, with lengths

$$\ell_1 = b + 1 \geq 4, \qquad \ell_2 = c + 1 \geq 5, \qquad \ell_3 = b + c \geq 7.$$

The last is the outer cycle and contains all vertices. Every edge belongs to exactly two of these cycles.

For each C_j, let $Z_j(w)$ be the matching partition function on the induced graph outside its vertices; in particular $Z_3(w) = 1$. Let H_j be its directed entry product, $r_j = \operatorname{Re} H_j$, and $h_j = |H_j|$. All cycles share both endpoints, so a permutation cover cannot contain two long cycles. Consequently the exact expansion is

$$\mathcal{P}(A) = Z_G(w) + 2\sum_{j=1}^{3} r_j Z_j(w), \tag{7}$$

$$\mathcal{Q}(A) = Z_G(w^2) + 2\sum_{j=1}^{3} h_j^2 Z_j(w^2). \tag{8}$$

In particular, $Z_j(w^2) \leq Z_j(w)$ and $r_j \geq -h_j$ imply

$$\mathcal{P}(A) - \mathcal{Q}(A) \geq L_G(w) - \sum_{j=1}^{3} E_j, \qquad E_j = 2h_j(1 + h_j)Z_j(w). \tag{9}$$

This bound permits the worst sign for every cycle separately. It therefore covers all jointly realizable phases without replacing the common matrix by independent phase data.

Lemma 4.1. *Set*

$$R_j = \left(\sum_{e \in C_j} w_e(1 - w_e) \right) Z_j(w).$$

Then

$$E_1 \leq \tfrac{5}{8} R_1, \qquad E_2 \leq \tfrac{3}{8} R_2, \qquad E_3 \leq \tfrac{1}{8} R_3.$$

Proof. For a cycle of length $\ell \geq 4$ and $h > 0$, Lemma 2.1 bounds the ratio E/R by

$$\frac{2}{\ell} h^{1-4/\ell}(1 + h) \leq \frac{2}{\ell} 2^{-(\ell-4)/2}(1 + 2^{-\ell/2}).$$

If $\ell \geq 4$, this is at most $(2/4)(5/4) = 5/8$. If $\ell \geq 5$, it is at most

$$\frac{2}{5}\frac{1}{\sqrt{2}}\frac{5}{4} = \frac{1}{2\sqrt{2}} < \frac{3}{8}.$$

If $\ell \geq 7$, it is at most

$$\frac{2}{7}\frac{1}{2\sqrt{2}}\frac{9}{8} = \frac{9}{56\sqrt{2}} < \frac{1}{8}.$$

The strict comparisons follow respectively from $16 < 18$ and $81 < 98$. When $h = 0$, the corresponding cost is zero, so no division is needed. □

Lemma 4.2. *The reservoirs satisfy*

$$\tfrac{5}{8}R_1 + \tfrac{3}{8}R_2 + \tfrac{1}{8}R_3 \leq L_G(w).$$

Proof. A summand of R_j is specified by an edge e of C_j and a matching T on its complementary vertices. Their union $M = \{e\} \cup T$ is a matching of G, and the summand is

$$w_e(1 - w_e) \prod_{f \in T} w_f = m_M(1 - w_e) \leq m_M(1 - m_M),$$

since $m_M \leq w_e$. For a fixed j, a matching M occurs at most once: it must have exactly one edge on C_j, and all its other edges lie entirely outside that cycle.

If M has at least two edges, it cannot occur in R_3, whose complementary graph is empty. Its total allocation coefficient is therefore at most $5/8 + 3/8 = 1$. If M is a singleton, it can be charged only by the two cycles containing its edge. The possible coefficient sums are 1, 3/4, and 1/2. The empty matching is never charged. Thus the total allocated weight of each matching is at most its own loss $m_M(1 - m_M)$. Summing proves the lemma. □

Proof of Theorem 1.1, part (2). Lemmas 4.1 and 4.2 show that $\sum_j E_j \leq L_G(w)$. Equation (9) gives $\mathcal{Q}(A) \leq \mathcal{P}(A)$ for correlation matrices. Scaling proves (2). All estimates allow zero edge weights, so no missing-edge or singular limit is omitted. □

The two bounds for lengths at least five and seven are strict when their cycle products are nonzero. In particular the strong inequality is strict for full nonzero support. The argument allocates multi-edge matching losses as well as singleton losses; replacing L_G by only its singleton contribution is not justified.

5 Hereditary strong closure

We give the closure argument explicitly, crediting its source [3, Proposition 1, Theorem 4.3 and Appendix A]. The change from real symmetric to complex Hermitian matrices introduces conjugation in $\mathcal{Q}$, but no additional sign assumption in the algebra.

Lemma 5.1. *Suppose*

$$A = \begin{pmatrix} B_1 & 0 & x_1 \\ 0 & B_2 & x_2 \\ x_1^* & x_2^* & d \end{pmatrix} \succeq 0.$$

There exist $t_1, t_2 \geq 0$ with $t_1 + t_2 = d$ such that

$$A_i = \begin{pmatrix} B_i & x_i \\ x_i^* & t_i \end{pmatrix} \succeq 0 \quad (i = 1, 2).$$

Proof. Positivity forces x_i to lie in range(B_i): a component along $\ker(B_i)$ would make a suitable quadratic form negative. Completing squares on the ranges gives

$$\rho_i = x_i^* B_i^+ x_i \geq 0, \qquad \rho_1 + \rho_2 \leq d,$$

where B_i^+ denotes the Moore–Penrose inverse. Choose $t_1 = \rho_1 + d - \rho_1 - \rho_2$ and $t_2 = \rho_2$. Completing the same squares proves positivity of A_i. This retains all Schur slack and includes singular B_i. □

Lemma 5.2. *With the notation of Lemma 5.1, suppose* (2) *holds for A_1, A_2, B_1, B_2. It then holds for A.*

Proof. Put $a_i = \mathcal{P}(B_i)$, $c_i = \mathcal{Q}(B_i)$, $p_i = \mathcal{P}(A_i)$, $q_i = \mathcal{Q}(A_i)$, and $D_i = \mathfrak{d}(B_i)$. A permutation cycle through the common last vertex cannot visit both sides. Expanding it, including the fixed-vertex contribution $(t_1 + t_2)^2$ in $\mathcal{Q}$, gives

$$\mathcal{P}(A) = a_2 p_1 + a_1 p_2, \qquad \mathcal{Q}(A) = c_2 q_1 + c_1 q_2 + 2 c_1 c_2 t_1 t_2.$$

The strong inputs give $c_i \leq D_i a_i$ and $q_i \leq t_i D_i p_i$. Lieb's positive semidefinite block permanent inequality gives $p_i \geq t_i a_i$ [5]. All quantities are nonnegative. Hence

$$\begin{aligned} \mathcal{Q}(A) &\leq D_1 D_2 (a_2 p_1 t_1 + a_1 p_2 t_2 + 2 a_1 a_2 t_1 t_2) \\ &\leq D_1 D_2 (t_1 + t_2)(a_2 p_1 + a_1 p_2) = \mathfrak{d}(A) \mathcal{P}(A). \end{aligned}$$

The second inequality has difference

$$D_1 D_2 \{ t_2 a_2 (p_1 - t_1 a_1) + t_1 a_1 (p_2 - t_2 a_2) \} \geq 0.$$

No division by diagonal entries or invertibility of B_i is used. □

Proof of Theorem 1.2. For an edge block, normalization gives $\mathcal{P} = 1 + w$ and $\mathcal{Q} = 1 + w^2$, with $0 \leq w \leq 1$. Single vertices are immediate. Cycle and theta blocks satisfy Theorem 1.1. Every proper induced subgraph of a cycle is a forest. Every proper induced subgraph of a theta block here contains at most one long cycle: any two distinct long cycles together use all three endpoint paths and hence all vertices. Its remaining support is a triangle-free cactus, whose blocks are edges or cycles of length at least four. Thus the collection of support graphs in the theorem is closed under taking induced subgraphs.

Proceed by induction on matrix order in this collection. Disconnected matrices split into direct sums, for which both permanents and diagonal products multiply. A connected host graph with more than one block has a cut vertex. Split the components of its deletion into two nonempty groups and apply Lemma 5.1. The resulting A_i and B_i have strictly smaller order, and their host graphs remain in the collection; only a diagonal has been allocated. Induction supplies all four strong inputs, and Lemma 5.2 proves the inequality for A. Applying the established assertion to every principal submatrix proves the hereditary statement. □

6 Paired consequences and limits

For complex matrices of the same order, the permanent Cauchy–Schwarz inequality is

$$|\operatorname{per}(A \circ B)|^2 \leq \mathcal{Q}(A) \mathcal{Q}(B).$$

It follows by applying Cauchy–Schwarz to the permutation-indexed products of entries. If *both* positive semidefinite matrices belong to the proved structural classes (their support graphs need not coincide), then

$$\operatorname{per}(A \circ B) \le \sqrt{\mathfrak{d}(A)\mathfrak{d}(B)\mathcal{P}(A)\mathcal{P}(B)} \le \mathcal{P}(A)\mathcal{P}(B).$$

The last step uses the classical inequality $\mathcal{P}(A) \ge \mathfrak{d}(A)$ and its counterpart for B, consequences of Lieb's block inequality. In correlation normalization the first bound reads

$$\operatorname{per}(A \circ B) \le \sqrt{\mathcal{P}(A)\mathcal{P}(B)}.$$

These statements require a proved self-inequality for each factor; they do not give Bapat–Sunder domination with an arbitrary second input.

The matching-loss proof does not settle general theta graphs, the excluded path-length configuration $1, 3, 3$, graphs with further chords, or arbitrary irreducible matrices of order seven. In particular a structural theorem on rank-six seven-vertex examples is distinct from a proof for every singular matrix of that order. The strong inequality is known to fail outside restricted classes, while the unrestricted Chollet conjecture remains beyond the conclusions established here.

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References

[1] J. Chollet, Is there a permanental analogue to Oppenheim's inequality?, *American Mathematical Monthly* **89** (1982), 57–58.

[2] S. W. Drury, A counterexample to a question of Bapat and Sunder, *Electronic Journal of Linear Algebra* **31** (2016), 69–70. `https://journals.uwyo.edu/index.php/ela/article/view/1631`.

[3] P. Pant and R. Singh, Structural Classes for Chollet's Permanent Conjecture, arXiv:2604.24192v2 (2026), `https://arxiv.org/abs/2604.24192v2`.

[4] R. Mehatari, M. R. Kannan and A. Samanta, On the adjacency matrix of a complex unit gain graph, *Linear and Multilinear Algebra* **70** (2022), no. 9, 1798–1813. `https://doi.org/10.1080/03081087.2020.1776672`.

[5] E. H. Lieb, Proofs of some conjectures on permanents, *Journal of Mathematics and Mechanics* **16** (1966), 127–134.