Strong permanent contraction on four-row bipartite supports through six columns

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Abstract

We prove that every complex positive semidefinite matrix A supported in $K_{4,6}$ satisfies $\mathrm{per}(|A|^2) \le \mathrm{per}(A)\prod_i a_{ii}$. The normalized off-diagonal block is a four-by-six operator contraction. For squared entries at most $7/10$, the proof combines a four-vertex overlap estimate with a fixed mixture of rectangular permanent bounds. Separate affinity, deletion of inactive rows and piecewise separate concavity reduce the remaining inequality to fifteen rational corner values. A convex-border argument covers larger entries for every number of columns, using Lieb's block inequality and the published three-row theorem. Arbitrary phases, missing edges and singular endpoints are included. The four-by-four and four-by-five conclusions follow by padding.

1 Statement and relation to prior work

Write $\mathcal{P}(A) = \mathrm{per}(A)$, $\mathcal{Q}(A) = \mathrm{per}(|A|^2)$ and $\mathrm{d}(A) = \prod_i a_{ii}$, with absolute value and square entrywise. A support graph specifies allowed off-diagonal entries; allowed entries may vanish.

Theorem 1. *For every complex 4×6 matrix X with $\|X\|_{\mathrm{op}} \le 1$, let*

$$A(X) = \begin{pmatrix} I_4 & X \\ X^* & I_6 \end{pmatrix}.$$

Then $\mathcal{Q}(A(X)) \le \mathcal{P}(A(X))$. Equivalently, every Hermitian positive semidefinite matrix B supported in $K_{4,6}$ satisfies

$$\mathcal{Q}(B) \le \mathrm{d}(B)\mathcal{P}(B). \tag{1}$$

The strong self-inequality is a classical formulation. Beasley [1, Lemma 2.2] relates its universal correlation-matrix version to a maximum-form Hadamard-product proposal. His Theorem 3.1 treats matrix order three; Theorem 3.2 treats an entrywise nonnegative real positive semidefinite first factor, and its corollaries include tridiagonal Hermitian supports. These important classes do not exhaust arbitrary complex bipartite supports with cycles: a padded 2×2 Hadamard contraction has a negative four-cycle product, invariant under diagonal unitary conjugation, whereas entrywise nonnegative matrices have nonnegative cycle products. Unitary changes within a part do not preserve the permanent.

Chollet's conjecture concerns $\mathrm{per}(A \circ B) \le \mathrm{per}(A)\,\mathrm{per}(B)$ for positive semidefinite inputs [2]. The strong self-bound (1) is more restrictive than its self-case and fails for general positive semidefinite matrices [3]. Pant and Singh develop structural graph methods [6]; their bipartite Z-matrix theorem has real sign hypotheses. The available abstract of Sa-nguansin and Rodtes [7] describes correlation-matrix annuli and order-five ridge results.

The present argument consolidates the four-row structural development through six columns. Its elementary graph estimate, convex mixing and scalar reductions are used together with the

classical inequalities of Carlen–Lieb–Loss and Lieb [4, 5]. The only earlier structural theorem used without reproduction is the published three-row result [8]:

$$\mathcal{Q}(A(D)) \le \mathcal{P}(A(D)) \quad \text{for every complex } 3 \times q \text{ contraction } D. \tag{2}$$

Here and below $A(D)$ denotes the block matrix with identity diagonal blocks of the appropriate sizes. The large-entry proof is reproduced completely.

2 Minor expansions and the four-vertex overlap budget

Set $W = |X|^2$, $w_{aj} = |x_{aj}|^2$, and

$$r_a = \sum_j w_{aj}, \quad c_j = \sum_a w_{aj}, \quad R_a = \sum_j w_{aj}^2, \quad \delta_a = r_a - R_a, \quad G = XX^*, \quad K = X^*X.$$

Contraction implies $r_a, c_j \le 1$. Write e_k for the elementary symmetric polynomial in four variables and $r^2 = (r_a^2)_a$. For equal-size row and column subsets define

$$F_k = \sum_{|I|=|J|=k} |\operatorname{per} X[I,J]|^2, \qquad Q_k = \sum_{|I|=|J|=k} \operatorname{per} W[I,J]^2.$$

The empty permanent is one. A permutation of $A(X)$ chooses equally many vertices of its two parts to cross; the two crossing bijections are independent. Consequently

$$\mathcal{P}(A(X)) = \sum_{k=0}^{4} F_k, \qquad \mathcal{Q}(A(X)) = \sum_{k=0}^{4} Q_k. \tag{3}$$

Put $D_{ab} = \sum_j w_{aj} w_{bj}$ and $E_{ab} = \sum_j w_{aj}^2 w_{bj}^2$. Expanding two-by-two permanents gives

$$F_2 = e_2(r) + \sum_{a<b} |G_{ab}|^2 - 2\sum_{a<b} D_{ab}, \qquad Q_2 = e_2(R) + \sum_{a<b} D_{ab}^2 - 2\sum_{a<b} E_{ab}.$$

Since $F_3, F_4 \ge 0$, (3) implies

$$\mathcal{P} - \mathcal{Q} \ge T + e_2(r) - Q_2 - Q_3 - Q_4, \quad T = \sum_j c_j(1-c_j) + \sum_{a<b} |G_{ab}|^2. \tag{4}$$

The identity $\operatorname{Tr} G^2 = \operatorname{Tr} K^2$ yields

$$T = \frac{1}{2}\sum_a r_a(1-r_a) + \frac{1}{2}\sum_j c_j(1-c_j) + \sum_{j<l} |K_{jl}|^2 \ge \frac{1}{2}\sum_a r_a(1-r_a). \tag{5}$$

Lemma 2. *For nonnegative weights z_{ab} on the six pairs of four vertices with weighted degrees $h_a \le \delta_a$, one has*

$$\sum_{a<b} z_{ab}^2 \le M(\delta) := \max_{\pi} \sum_{\{a,b\}\in\pi} \delta_a \delta_b,$$

where π ranges over the three perfect pairings.

Proof. Relabel so $a = z_{12}$ is a largest edge. Write $f = z_{34}$, $b = z_{13}$, $c = z_{14}$, $d = z_{23}$, $e = z_{24}$, and $S = b + c + d + e$. Direct expansion gives

$$h_1h_2 + h_3h_4 = a^2 + f^2 + (a + f)S + (b + c)(d + e) + (b + d)(c + e).$$

Since $aS \geq b^2 + c^2 + d^2 + e^2$, the right side is at least the sum of all six squared weights. Finally $h_1h_2 + h_3h_4 \leq \delta_1\delta_2 + \delta_3\delta_4 \leq M(\delta)$. Zeros and ties cause no difficulty. □

Column sums and six-column Cauchy–Schwarz give

$$\sum_{b\neq a} D_{ab} = \sum_j w_{aj}(c_j - w_{aj}) \leq \delta_a, \qquad E_{ab} \geq D_{ab}^2/6.$$

Thus Lemma 2 implies

$$Q_2 \leq e_2(R) + \frac{2}{3}M(\delta). \tag{6}$$

The maximum is essential: its maximizing pairing may depend on the actual overlaps.

3 Higher minor bounds and a scalar reduction

The rectangular squared-minor inequality of Carlen–Lieb–Loss [4, Theorem 3.1], applied to each selected set of rows of the six-column W, gives

$$Q_3 \leq \frac{10}{3}e_3(R), \qquad Q_4 \leq \frac{20}{3}e_4(R).$$

The constants are $\binom{6}{k}(k!)^2/6^k$. There is also the elementary bound $Q_k \leq e_k(r^2)$. Indeed, for fixed rows I, the nonnegative sum $\sum_J \operatorname{per} W[I, J]$ permits distinct column assignments only and is at most $\prod_{a\in I} r_a$. The sum of its squared summands is at most its square. Summing over I proves the bound.

Use the fixed convex weights 9/10 and 1/10 for these two bounds:

$$Q_3 \leq 3e_3(R) + \frac{1}{10}e_3(r^2), \qquad Q_4 \leq 6e_4(R) + \frac{1}{10}e_4(r^2). \tag{7}$$

Equations (4)–(7) show $\mathcal{P} - \mathcal{Q} \geq \min_\pi B_\pi(r, R)$, where

$$\begin{aligned} B_\pi(r, R) = \frac{1}{2}\sum_a r_a(1 - r_a) + e_2(r) - e_2(R) - \frac{2}{3}\sum_{\{a,b\}\in\pi}(r_a - R_a)(r_b - R_b) \\ - 3e_3(R) - 6e_4(R) - \frac{1}{10}\big[e_3(r^2) + e_4(r^2)\big]. \end{aligned} \tag{8}$$

We prove every fixed-pairing budget nonnegative.

Lemma 3. *If $r_a, c_j \leq 1$ and all $w_{aj} \leq 7/10$, then $\mathcal{P}(A(X)) \geq \mathcal{Q}(A(X))$.*

Proof. Convex concentration of a row at fixed sum gives $0 \leq R_a \leq f(r_a) \leq r_a$, where

$$f(t) = \begin{cases} t^2, & 0 \leq t \leq 7/10, \\ 49/100 + (t - 7/10)^2, & 7/10 \leq t \leq 1. \end{cases}$$

There can be at most one filled cap and one remainder since $1 < 2(7/10)$. Fix π and r. The polynomial B_π is separately affine in R_a, so its minimum on the enlarged box $0 \le R_a \le f(r_a)$ is attained with every R_a equal to zero or $f(r_a)$. These scalar vertices need not be realizable rows.

At such a vertex call a row inactive if $R_i = 0$. Replace $r_i = t$ by zero, leaving $R_i = 0$. With partner p, $S = \sum_{a \neq i} r_a$, and $H = e_2(r^2_{\neq i}) + e_3(r^2_{\neq i})$, the exact difference in (8) is

$$\frac{1}{2}t(1-t) + tS - \frac{2}{3}t\delta_p - \frac{1}{10}t^2 H.$$

For three variables in $[0,1]$, $e_2(u^2) \le \sum u$ and $e_3(u^2) \le (\sum u)/3$: use $u^2v^2 \le uv \le (u+v)/2$ and $u^2v^2z^2 \le uvz \le (u+v+z)/3$. Hence $H \le 4S/3$ and $\delta_p \le S$. The difference is at least

$$\frac{1}{2}t(1-t) + \frac{1}{3}tS - \frac{2}{15}t^2 S \ge \frac{1}{2}t(1-t) + \frac{1}{5}tS \ge 0.$$

Delete inactive rows successively. Since $f(0) = 0$, it remains to prove $J_\pi(r) := B_\pi(r, f(r)) \ge 0$.

Hold the other coordinates fixed. Its dependence on $t = r_i$, paired with p, is a constant plus

$$t/2 - t^2/2 + t\sum_{a \neq i} r_a - \frac{2}{3}t\delta_p + Lf(t) - \frac{1}{10}t^2 H,$$

where

$$L = -\sum_{a \neq i} f(r_a) + \frac{2}{3}\delta_p - 3e_2(f(r_{\neq i})) - 6e_3(f(r_{\neq i})).$$

Discarding only nonpositive terms gives $L \le (2/3)r_p - (5/3)f(r_p) \le 1/15$. On the lower interval this last expression is $1/15 - (5/3)(r_p - 1/5)^2$; on the upper it is $-17/60 - (5/3)(r_p - 9/10)^2$. Therefore on each interval

$$\frac{\partial^2 J_\pi}{\partial t^2} = -1 + 2L - H/5 \le -13/15.$$

Separate concavity on each of the sixteen piecewise boxes reduces the minimum to $r_a \in \{0, 7/10, 1\}$. We apply this to each fixed pairing, not to their minimum. Table 1 records the minimum over all three pairings for each count pattern, and hence covers all 81 labelled corners and 243 pairing evaluations. All entries are nonnegative. Endpoint minimization, inactive deletion and separate affinity now prove $B_\pi \ge 0$, and thus the lemma. □

4 A large-entry estimate by convex border concentration

This section applies to a complex $4 \times q$ operator contraction, for any finite q; here $A(X) = \left(\begin{smallmatrix} I_4 & X \\ X^* & I_q \end{smallmatrix}\right)$, and $A(D)$ denotes the analogous block-identity matrix.

Lemma 4. *If a squared entry t of X satisfies $t \ge 7/10$, then $\mathcal{P}(A(X)) \ge \mathcal{Q}(A(X))$. More precisely, after placing that entry first and writing*

$$X = \begin{pmatrix} a & b \\ c & D \end{pmatrix}, \qquad t = |a|^2,$$

one has

$$\mathcal{P}(A(X)) - \mathcal{Q}(A(X)) \ge \frac{7}{2000}(1-t)\mathcal{Q}(A(D)). \tag{9}$$

Table 1: Exact worst-pairing values of J_π; n_0, n_7, n_1 count coordinates $0, 7/10, 1$.

n_0	n_7	n_1	$\min_\pi J_\pi$
4	0	0	0
3	1	0	21/200
2	2	0	861/2000
1	3	0	6705881/10000000
0	4	0	50099539/1000000000
3	0	1	0
2	1	1	231/500
1	2	1	197729/250000
0	3	1	2816317/25000000
2	0	2	273/500
1	1	2	234773/250000
0	2	2	1003051/6250000
1	0	3	148483/125000
0	1	3	713471/3125000
0	0	4	706457/3125000

Proof. Write $s = 1-t$, $u_j = |b_j|^2$, $v_i = |c_i|^2$, $U = \sum_j u_j$, and $V = \sum_i v_i$. Contraction gives $U, V \le s$, and D is a contraction. If $q = 1$, the support is a star and $P - Q = \sum_a w_a(1 - w_a) \ge t(1 - t)$. Since $\mathcal{Q}(A(D)) = 1$, this already proves (9). Assume henceforth $q \ge 2$.

Partition the actual positive semidefinite matrix $A(X)$ into the two endpoints of a and the complementary principal block $A(D)$. Lieb's two-block permanent inequality [5] and (2) give

$$\mathcal{P}(A(X)) \ge (1 + t)\mathcal{P}(A(D)) \ge (1 + t)\mathcal{Q}(A(D)). \tag{10}$$

The three-row input applies to the $3 \times (q - 1)$ compression D, including singular cases.

Put $W = |D|^2$. For equal-size core subsets I, J, including the empty ones, let $d_{I,J} = \mathrm{per}(W[I, J])$. Then the Euclidean squared norm of this minor vector is $\|d\|^2 = \mathcal{Q}(A(D))$.

Fix t, W, v and consider $\mathcal{Q}(A(X))$ purely as a polynomial in the nonnegative border vector u. Each minor permanent of $\left(\begin{smallmatrix} t & u \\ v & W \end{smallmatrix}\right)$ is affine in u, since it uses its border row at most once. Its square is convex. The sum of those squares is therefore convex in u, and likewise in v with u fixed. Successive convexity on the fixed-total simplices gives

$$Q(t, u, v, W) \le \max_{i,j} Q(t, Ue_j, Ve_i, W). \tag{11}$$

Here Q denotes the squared-minor functional of the nonnegative rectangle. Zero totals follow by the remaining convexity step or continuity. The concentrated borders need not be contractions: they are auxiliary inputs for an upper bound only. Positive semidefiniteness was used on the original matrix in (10), not on these fictitious inputs.

Fix i, j. Let C_j add column index j to a subset if absent, and give zero if it is already present; let R_i similarly add row i. These are partial-isometry maps between subset-index spaces. Their targets individually may have unequal row and column cardinalities. Their product $L = R_iC_j$ acts on the equal-cardinality space by adding both indices when both are absent, and vanishes otherwise. Expanding the concentrated bordered minors gives exactly

$$\begin{aligned} Q(t, Ue_j, Ve_i, W) = (1 + t^2)\|d\|^2 + U^2\|C_jd\|^2 + V^2\|R_id\|^2 \\ + U^2V^2\|Ld\|^2 + 2tUV\Re\langle d, Ld\rangle. \end{aligned} \tag{12}$$

Indeed, a minor containing both border indices has permanent $td_{I,J}+UVd_{I\setminus\{i\},J\setminus\{j\}}$ when $i \in I, j \in J$. The single-border minors yield the two single-creation norms. All minor orders are retained.

Group input coordinates according to whether they contain i and j. For fixed remaining subsets, L connects occupancy 00 to 11 with unit weight. The extra quadratic form beyond $(1+t^2)\|d\|^2$ in (12) has the block

$$B(U,V) = \begin{pmatrix} A & tUV \\ tUV & 0 \end{pmatrix}, \qquad A = U^2 + V^2 + U^2V^2.$$

The other occupancy types have diagonal values U^2 or V^2. Missing states take compressions and cannot raise the largest eigenvalue. That eigenvalue is

$$\lambda(U,V) = \frac{A + \sqrt{A^2 + 4t^2U^2V^2}}{2},$$

which dominates U^2, V^2 and increases with $U, V \geq 0$. Using $U, V \leq s$ in (11)–(12) yields

$$\mathcal{Q}(A(X)) \leq [1 + t^2 + \lambda(s,s)]\mathcal{Q}(A(D)). \tag{13}$$

Let $M = tsI - B(s,s)$. Since $t = 1-s$, its lower-right diagonal is ts and its determinant is exactly

$$\det M = ts^2(1 - 3s - s^2). \tag{14}$$

For $0 < s \leq 3/10$, this determinant and the lower-right diagonal are positive, because $1 - 3s - s^2 \geq 1/100$. Thus M is positive definite and $\lambda(s,s) \leq ts$. At $s = 0$ both matrices vanish directly. Equations (10) and (13) prove the non-strict inequality.

For the quantitative bound, $\operatorname{Tr} M = s(2 - 4s - s^3) \leq 2s$. A positive two-by-two matrix has least eigenvalue at least its determinant divided by its trace. Hence, for $s > 0$,

$$ts - \lambda(s,s) \geq \frac{ts^2/100}{2s} \geq \frac{7}{2000}s.$$

Substitution proves (9); $s = 0$ was handled separately. No inverse or division at a zero-slack endpoint is used. □

The same determinant argument gives non-strict contraction whenever $t \geq (5-\sqrt{13})/2$, slightly below $7/10$. The rational threshold is sufficient for the domain union and the explicit remainder above.

5 Full domain, scaling and consequences

Proof of Theorem 1. A contraction has squared row and column sums at most one. If every squared entry is at most $7/10$, Lemma 3 applies; otherwise Lemma 4 applies. Both closed estimates include the common boundary.

For positive diagonal B supported in $K_{4,6}$, normalize as $B = EA(X)E$, where $E_{ii} = \sqrt{b_{ii}}$. The diagonal blocks become identities and the Schur complement gives $\|X\|_{\mathrm{op}} \leq 1$. Permanent scaling gives $\mathcal{P}(B) = \mathrm{d}(B)\mathcal{P}(A(X))$ and $\mathcal{Q}(B) = \mathrm{d}(B)^2\mathcal{Q}(A(X))$, proving (1). If a diagonal entry vanishes, positivity forces its row and column to vanish, and both permanents are zero. Thus singular endpoints and missing edges are included. □

Corollary 5. *The strong inequality* (1) *holds on every $K_{p,q}$ with $p \leq 4$, $q \leq 6$, and on every bipartite positive semidefinite support of order at most nine.*

Proof. Zero-row or zero-column padding of X adds isolated unit diagonal vertices, preserving both normalized permanents. This includes $K_{4,4}$ and $K_{4,5}$. For a bipartition of total size at most nine, either its smaller part has size at most three and (2) applies, or its two parts have sizes at most four and five, respectively. □

There is also a paired consequence when *both* factors belong to proved strong classes. Permutation Cauchy–Schwarz gives

$$|\operatorname{per}(A \circ B)|^2 \le \mathcal{Q}(A)\mathcal{Q}(B).$$

For normalized positive semidefinite inputs, the strong bounds and the identity term imply $\mathcal{P}(A) \ge \mathcal{Q}(A) \ge 1$, and similarly for B. Hence

$$\operatorname{per}(A \circ B) \le \sqrt{\mathcal{P}(A)\mathcal{P}(B)} \le \mathcal{P}(A)\mathcal{P}(B).$$

Diagonal scaling and the zero-row cases extend this to unnormalized inputs. This does not cover an arbitrary second factor.

The six-column estimate is essential in the small-entry argument. The heavy-entry lemma holds for every finite number of columns.

Reproducibility and acknowledgment

The continuum reductions are analytic. The fifteen entries in Table 1 are rational evaluations of (8); an accompanying exact-arithmetic checker can replay all labelled corners and pairings without numerical optimization. Such arithmetic verifies the finite calculation rather than replacing the affinity, deletion and concavity arguments. AI tools assisted mathematical research, drafting and verification. The author is responsible for the statements, proofs, attribution and final text.

References

[1] L. B. Beasley, An inequality on permanents of Hadamard products, *Bulletin of the Korean Mathematical Society* **37** (2000), no. 3, 633–639.

[2] J. Chollet, Is there a permanental analogue to Oppenheim's inequality?, *American Mathematical Monthly* **89** (1982), 57–58.

[3] S. W. Drury, A counterexample to a question of Bapat and Sunder, *Electronic Journal of Linear Algebra* **31** (2016), 69–70. https://journals.uwyo.edu/index.php/ela/article/view/1631.

[4] E. Carlen, E. H. Lieb and M. Loss, An inequality of Hadamard type for permanents, *Methods and Applications of Analysis* **13** (2006), no. 1. https://doi.org/10.4310/MAA.2006.v13.n1.a1. Rectangular form: arXiv:math/0508096v1, Theorem 3.1, https://arxiv.org/html/math/0508096v1.

[5] E. H. Lieb, Proofs of some conjectures on permanents, *Journal of Mathematics and Mechanics* **16** (1966), 127–134.

[6] P. Pant and R. Singh, Structural Classes for Chollet's Permanent Conjecture, arXiv:2604.24192v2 (2026). https://arxiv.org/abs/2604.24192.

[7] S. Sa-nguansin and K. Rodtes, Permanents of correlation matrices and the Chollet permanental conjecture, *Linear and Multilinear Algebra* **73** (2025), no. 18, 4084–4096. https://doi.org/10.1080/03081087.2025.2588584.

[8] S. Schubr, Strong permanent inequalities for three-row bipartite and triangle-free supports, Luminary Research Archive, first edition (2026). First-edition archive page.