Strong permanent contraction on four-by-seven bipartite supports

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Abstract

We prove that every complex positive semidefinite matrix A supported in $K_{4,7}$ satisfies $\operatorname{per}(|A|^2) \le \operatorname{per}(A)\prod_i a_{ii}$. The normalized off-diagonal block is a four-by-seven operator contraction. The new argument retains third-order minor contributions and controls their coupled phase cancellations through the row Gram matrix. A split according to the smallest squared row norm separates an analytic corner reduction from a balanced-row polynomial inequality. The latter is verified by an exact rational tensor Bernstein certificate on 256 boxes. An established all-column large-entry theorem covers the complementary region. The computer-assisted result includes arbitrary phases, missing edges and singular endpoints.

1 Statement and context

For Hermitian A, put $\mathcal{P}(A) = \operatorname{per}(A)$, $\mathcal{Q}(A) = \operatorname{per}(|A|^2)$ and $\mathrm{d}(A) = \prod_i a_{ii}$, with absolute value and square entrywise. A support graph specifies allowed off-diagonal positions; allowed entries may vanish.

Theorem 1 (Computer-assisted)**.** *Let X be any complex 4×7 matrix with $\|X\|_{\mathrm{op}} \le 1$ and set*

$$A(X) = \begin{pmatrix} I_4 & X \\ X^* & I_7 \end{pmatrix}.$$

Then $\mathcal{Q}(A(X)) \le \mathcal{P}(A(X))$. Equivalently, every Hermitian positive semidefinite matrix B supported in $K_{4,7}$ satisfies

$$\mathcal{Q}(B) \le \mathrm{d}(B)\mathcal{P}(B). \tag{1}$$

All missing-edge and singular cases are included.

The strong self-inequality is a classical formulation. Beasley [1, Lemma 2.2] relates its universal correlation-matrix version to a maximum-form Hadamard-product proposal. His order-three theorem and entrywise nonnegative first-factor theorem do not exhaust arbitrary complex bipartite supports with cycles. For example, a padded 2×2 Hadamard contraction has a negative four-cycle product, invariant under diagonal unitary conjugation; entrywise nonnegative matrices have nonnegative cycle products. Unitary changes inside a bipartition do not preserve the permanent.

Chollet's conjecture is $\operatorname{per}(A \circ B) \le \operatorname{per}(A)\operatorname{per}(B)$ for positive semidefinite inputs [2]. Its self-case follows from (1), but the strong bound fails for unrestricted positive semidefinite matrices [3]. Structural graph methods are developed by Pant and Singh [5]; their bipartite Z-matrix theorem has real sign hypotheses. The available abstract of Sa-nguansin and Rodtes [6] describes correlation-matrix annuli and order-five ridge results.

The author's preceding paper [9] proves the four-row theorem through six columns. The seven-column constants invalidate a direct reuse of its scalar estimate. Here a retained third-order mechanism repairs that loss on a balanced-row region. We use the classical rectangular inequality of

Carlen–Lieb–Loss [4], an elementary overlap estimate, and exact polynomial positivity. The large-entry region is covered by the following published input [9, Lemma 4]: for every finite q, a complex $4 \times q$ contraction with a squared entry $t \geq 7/10$ satisfies

$$\mathcal{P}(A(X)) - \mathcal{Q}(A(X)) \geq \frac{7}{2000}(1-t)\mathcal{Q}(A(D)) \geq 0, \tag{2}$$

where D is obtained by removing that entry's row and column. That proof uses Lieb's block inequality and the published three-row theorem [7]. We reproduce the entire new small-entry argument below.

Exact rational rectangular Bernstein certificates have already been used in Chollet research: Zeng, Liu and Ratnavelu [8, Section 6, Lemma 4] use them for a structured real order-five family and supply an independent verifier. Their target is a self-Chollet ratio with denominator $\mathcal{P}^2$, rather than the complex order-eleven strong-support bound here. The basis method is established; the new content here is the retained third-minor budget, row-domain split and structural application.

2 Minor expansions and overlap constraints

Rows are indexed by $0, 1, 2, 3$. Set $W = |X|^2$, $w_{aj} = |x_{aj}|^2$ and

$$r_a = \sum_j w_{aj}, \quad c_j = \sum_a w_{aj}, \quad R_a = \sum_j w_{aj}^2, \quad \delta_a = r_a - R_a, \quad G = XX^*, \quad K = X^*X.$$

Contraction gives $0 \leq r_a, c_j \leq 1$ and $0 \preceq G \preceq I$. Write e_k for elementary symmetric polynomials in four variables and $r^2 = (r_a^2)_a$. For equal-size row and column subsets define

$$F_k = \sum_{|I|=|J|=k} |\operatorname{per} X[I,J]|^2, \qquad Q_k = \sum_{|I|=|J|=k} \operatorname{per} W[I,J]^2.$$

The empty permanent is one. Crossing permutations between the two parts choose equally many vertices on each side, and their two bijections are independent. Thus

$$\mathcal{P}(A(X)) = \sum_{k=0}^{4} F_k, \qquad \mathcal{Q}(A(X)) = \sum_{k=0}^{4} Q_k. \tag{3}$$

Let $D_{ab} = \sum_j w_{aj} w_{bj}$ and $E_{ab} = \sum_j w_{aj}^2 w_{bj}^2$. Expansion of two-by-two permanents gives

$$F_2 = e_2(r) + \sum_{a<b} |G_{ab}|^2 - 2\sum_{a<b} D_{ab}, \tag{4}$$

$$Q_2 = e_2(R) + \sum_{a<b} D_{ab}^2 - 2\sum_{a<b} E_{ab}. \tag{5}$$

Put $C = \sum_j c_j(1-c_j)$ and $O = \sum_{a<b} |G_{ab}|^2$. The trace identity $\operatorname{Tr} G^2 = \operatorname{Tr} K^2$ yields

$$C + O = \frac{1}{2}\sum_a r_a(1-r_a) + \frac{1}{2}\sum_j c_j(1-c_j) + \sum_{j<l} |K_{jl}|^2 \geq \frac{1}{2}\sum_a r_a(1-r_a). \tag{6}$$

Using $F_3, F_4 \geq 0$ in (3), with $F_1 - Q_1 = \sum w_{aj}(1-w_{aj}) = C + 2\sum D_{ab}$, gives

$$\mathcal{P} - \mathcal{Q} \geq \frac{1}{2}\sum_a r_a(1-r_a) + e_2(r) - Q_2 - Q_3 - Q_4. \tag{7}$$

Lemma 2. *For six nonnegative weights z_{ab} on pairs of four vertices, whose degrees $h_a = \sum_{b\neq a} z_{ab}$ satisfy $h_a \le \delta_a$, one has*

$$\sum_{a<b} z_{ab}^2 \le M(\delta) := \max_{\pi} \sum_{\{a,b\}\in\pi} \delta_a \delta_b,$$

where π ranges over the three perfect pairings.

Proof. Relabel so $a = z_{12}$ is a largest edge. Write $f = z_{34}$, $b = z_{13}$, $c = z_{14}$, $d = z_{23}$ and $e = z_{24}$. Then

$$h_1h_2 + h_3h_4 = a^2 + f^2 + (a+f)(b+c+d+e) + (b+c)(d+e) + (b+d)(c+e).$$

Since $a(b+c+d+e) \ge b^2+c^2+d^2+e^2$, this is at least the sum of all six squared weights. Now use $h_1h_2 + h_3h_4 \le \delta_1\delta_2 + \delta_3\delta_4 \le M(\delta)$. □

Column sums imply

$$\sum_{b\neq a} D_{ab} = \sum_j w_{aj}(c_j - w_{aj}) \le \delta_a.$$

Seven-column Cauchy–Schwarz gives $E_{ab} \ge D_{ab}^2/7$. Hence (5) and Lemma 2 yield

$$Q_2 \le e_2(R) + \frac{5}{7} M(\delta). \tag{8}$$

The maximizing pairing may depend on the actual overlaps; below we prove each fixed-pairing budget nonnegative.

The rectangular CLL inequality [4, Theorem 3.1] on selected rows of the seven-column W gives

$$Q_3 \le \frac{180}{49} e_3(R), \qquad Q_4 \le \frac{2880}{343} e_4(R). \tag{9}$$

The squared constants are $\binom{7}{k}(k!)^2/7^k$. Independently,

$$Q_k \le e_k(r^2). \tag{10}$$

Indeed for fixed I, the nonnegative sum $\sum_J \operatorname{per} W[I,J]$ permits only distinct column assignments and is at most $\prod_{a\in I} r_a$. Its sum of squared summands is at most its square. Sum over I. Any convex mixture of (9) and (10) is therefore valid.

Assume henceforth every $w_{aj} \le 7/10$. Convex concentration at fixed row sum gives

$$0 \le R_a \le f(r_a) \le r_a, \qquad f(t) = \begin{cases} t^2, & 0 \le t \le 7/10, \\ 49/100 + (t-7/10)^2, & 7/10 \le t \le 1. \end{cases} \tag{11}$$

There is at most one filled cap and one remainder because $r_a \le 1 < 2(7/10)$. Scalar endpoints of this enlarged box need not be realizable rows.

3 The small-row region

Lemma 3. *If every squared entry is at most $7/10$ and some $r_i \le 13/20$, then $\mathcal{Q}(A(X)) \le \mathcal{P}(A(X))$.*

Proof. Use mixture weight 4/5 on (9) and 1/5 on (10). By (7) and (8), $\mathcal{P} - \mathcal{Q} \geq \min_\pi B_\pi$, where

$$B_\pi(r,R) = \frac{1}{2}\sum_a r_a(1-r_a) + e_2(r) - e_2(R) - \frac{5}{7}\sum_{\{a,b\}\in\pi}\delta_a\delta_b \\ - \frac{144}{49}e_3(R) - \frac{2304}{343}e_4(R) - \frac{1}{5}\big[e_3(r^2) + e_4(r^2)\big]. \tag{12}$$

Fix π, r. Separate affinity in R reduces the minimum to $R_i = 0$ or $f(r_i)$. If $R_i = 0$ is inactive, replace $r_i = t$ by zero. With partner p, $S = \sum_{a\neq i} r_a$ and $H = e_2(r^2_{\neq i}) + e_3(r^2_{\neq i})$, the exact difference between the budget before and after deletion is

$$\frac{1}{2}t(1-t) + tS - \frac{5}{7}t\delta_p - \frac{1}{5}t^2H.$$

For three variables in $[0,1]$, $e_2(u^2) \leq \sum u$ and $e_3(u^2) \leq (\sum u)/3$, by $u^2v^2 \leq uv \leq (u+v)/2$ and $u^2v^2z^2 \leq uvz \leq (u+v+z)/3$. Thus $H \leq 4S/3$ and $\delta_p \leq S$. The difference is at least

$$\frac{1}{2}t(1-t) + \left(\frac{2}{7} - \frac{4}{15}\right)tS = \frac{1}{2}t(1-t) + \frac{2}{105}tS \geq 0.$$

Delete all inactive rows. Existence of a row at most 13/20 is preserved, and $f(0) = 0$.

Substitute $R = f(r)$ and hold other coordinates fixed. The coefficient of $f(t)$ is

$$L = -\sum_{a\neq i} f(r_a) + \frac{5}{7}\delta_p - \frac{144}{49}e_2(f(r_{\neq i})) - \frac{2304}{343}e_3(f(r_{\neq i})) \leq \frac{5}{7}r_p - \frac{12}{7}f(r_p) \leq \frac{25}{336}.$$

The last quadratic has maximum 25/336 at 5/24 on the lower interval; its upper-interval maximum is $-2231/8400$. Apart from $Lf(t)$, the remaining coordinate dependence is a linear function minus $t^2/2 + t^2H/5$. Therefore on each interval the second derivative is at most $-1 + 2(25/336) = -143/168$. No concavity across the breakpoint is required.

Choose a distinguished small row in $[0, 13/20]$, and split the other row intervals at 7/10. Separate concavity reduces to endpoints: the distinguished row is 0 or 13/20, and the others are $0, 7/10, 1$. A sufficient larger set consists of all count patterns in $\{0, 13/20, 7/10, 1\}^4$ containing a zero or 13/20. There are $35 - 5 = 30$ patterns up to coordinate permutation. Evaluate (12) at $R = f(r)$ and take the minimum over the three pairings. All are nonnegative; the smallest positive value is 21/200 (the all-zero pattern and the pattern with one unit coordinate give zero). The accompanying `verify_k47_certificate.py` reconstructs these rational values from (12), including all 720 labelled corner/pairing cases. This is a finite exact evaluation following the continuum reduction. It proves $B_\pi \geq 0$ and the lemma. □

4 Retaining third-order contributions

For a row triple I put $T_I = \sum_j \prod_{a\in I} w_{aj}$. For each pair a, b, let c, d be the other two rows and set

$$z_{ab} = r_c + r_d, \qquad H_{ab} = \sum_j (w_{cj} + w_{dj})x_{aj}\overline{x_{bj}}.$$

Lemma 4. *For every four-row operator contraction,*

$$F_2 + F_3 \geq e_2(r) + e_3(r) - 2\sum_{a<b}(1 + z_{ab})D_{ab} + \frac{1}{3}O. \tag{13}$$

Proof. We first give the exact third-order collision identity:

$$F_3 = \sum_{|I|=3} \operatorname{per} G[I] - 2\sum_{a<b} z_{ab} D_{ab} - 4\Re \sum_{a<b} \overline{G_{ab}} H_{ab} + 12 \sum_{|I|=3} T_I. \tag{14}$$

For a fixed row triple, expand the permanent of its row Gram matrix by column assignments and group equal columns. Distinct columns contribute its squared-minor sum. For columns j, j, k with $j \neq k$, the repeated-column permanent is $2\sum_a x_{ak} \prod_{b\neq a} x_{bj}$, so division by 2! contributes twice the squared modulus of that sum. Triple columns contribute $6T_I$. Extending the double-column sum to $k = j$ adds $18T_I$, hence its subtraction leaves $+12T_I$. Expanding the extended square gives the terms $-2\sum r_c D_{ab}$ and $-4\Re\sum \overline{G_{ab}} H_{ab}$ on that triple. Summing the four triples proves (14), with the stated conjugations.

Because $0 \preceq G \preceq I$, $\|G - I/2\| \leq 1/2$ and $|G_{ab}| \leq 1/2$ for $a \neq b$. For $u, v, z \in [0, 1/2]$,

$$u^2 + v^2 + z^2 \geq 3(uvz)^{2/3} \geq 6uvz.$$

Thus each cubic Gram term in a triple permanent is at least minus one third of the sum of its three off-diagonal squares. Each pair belongs to two triples. Adding (4) to (14), the remaining coefficient of $|G_{ab}|^2$ is at least $1/3 + z_{ab}$. Weighted Cauchy–Schwarz gives

$$|H_{ab}|^2 \leq z_{ab}(T_{abc} + T_{abd}).$$

Reserve $1/3$ of the pair square and complete the rest:

$$z_{ab}|G_{ab}|^2 - 4\Re(\overline{G_{ab}} H_{ab}) \geq -4(T_{abc} + T_{abd}).$$

If $z_{ab} = 0$, all corresponding terms vanish and the same bound holds directly. Every triple is charged by three pairs, consuming exactly its $12T_I$ contribution. This proves (13) without choosing phases independently. □

Equations (3), (13) imply

$$\mathcal{P} - \mathcal{Q} \geq C + O/3 + e_2(r) + e_3(r) - Q_2 - Q_3 - Q_4 - 2\sum_{a<b} D_{ab} z_{ab}.$$

Since $C \geq 0$, (6) gives $C + O/3 \geq (C + O)/3 \geq \sum r_a(1 - r_a)/6$. Mix this lower bound with (7), giving weight $3/4$ to the new bound. The result has row-defect coefficient $1/4$ and $e_3(r)$ coefficient $3/4$.

5 Balanced rows and exact polynomial positivity

Assume all $r_a \geq 13/20$. Put $S = \sum r_a$ and $y_a = S/2 - r_a$. These potentials are nonnegative: the other three rows sum to at least $39/20 > 1 \geq r_a$. Moreover $y_a + y_b = z_{ab}$, so

$$\sum_{a<b} D_{ab} z_{ab} = \sum_a y_a \sum_{b\neq a} D_{ab} \leq \sum_a \delta_a y_a. \tag{15}$$

This allowance is used only on the balanced region where its coefficients are nonnegative.

Use the higher-minor mixture with weight 3/4 on CLL and 1/4 on (10). The mixed lower bound, (8) and (15) show that it suffices to prove, for every pairing,

$$\mathcal{R}_\pi(r,R) = \frac{1}{4}\sum_a r_a(1-r_a) + e_2(r) + \frac{3}{4}e_3(r) - e_2(R) - \frac{5}{7}\sum_{\{a,b\}\in\pi}\delta_a\delta_b \\ - \frac{135}{49}e_3(R) - \frac{2160}{343}e_4(R) - \frac{1}{4}\big[e_3(r^2)+e_4(r^2)\big] - \frac{3}{2}\sum_a \delta_a(S/2-r_a) \geq 0. \tag{16}$$

For fixed r, this polynomial is separately affine in R, so it is enough to substitute $R_a = \epsilon_a f(r_a)$ for all 16 activity vectors $\epsilon \in \{0,1\}^4$. Split each row interval into $[13/20, 7/10]$ and $[7/10, 1]$, giving 16 interval boxes. Fix $\pi = \{01, 23\}$; coordinate permutations give the other pairings on the same full domain.

We describe the exact certificate, including its basis conversion. For $u \in \{0,1\}^4$, set $(l_i, h_i) = (13/20, 7/10)$ if $u_i = 0$, and $(l_i, h_i) = (7/10, 1)$ otherwise. Substitute

$$r_i = l_i + (h_i - l_i)x_i, \qquad R_i = \epsilon_i \begin{cases} r_i^2, & u_i = 0, \\ r_i^2 - (7/5)r_i + 49/50, & u_i = 1. \end{cases}$$

Equation (16) becomes a rational polynomial $p(x) = \sum_a c_a x^a$ of coordinate degree at most $n_i = 2 + \epsilon_i \leq 3$. Its tensor Bernstein coefficients are

$$b_k = \sum_{a\leq k} c_a \prod_{i=0}^{3} \frac{\binom{k_i}{a_i}}{\binom{n_i}{a_i}}, \qquad p(x) = \sum_{0\leq k\leq n} b_k \prod_{i=0}^{3} \binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i}. \tag{17}$$

The basis functions are nonnegative and sum to one on $[0,1]^4$. Thus nonnegative coefficients prove the full closed-box inequality.

Exact rational evaluation of (17) for all 256 activity/interval boxes gives the per-activity minima in Table 1. In particular every coefficient is at least $707119/400000000 > 0$. The variable-degree representation has $16(3+4)^4 = 38416$ coefficients. Elevating every coordinate to degree three gives $256 \cdot 4^4 = 65536$ coefficients and the same global minimum. Degree elevation changes the representation, not the domain or polynomial.

The accompanying standalone `verify_k47_certificate.py` reconstructs (16) by direct rational budget evaluations and tensor interpolation, checks coefficient positivity on every box, and checks reconstruction at rational points away from the interpolation grid. It also verifies the small-row evaluations of (12). There is no optimizer premise or floating tolerance. Together with (17), this finite exact certificate proves (16). Separate affinity extends the result from every R endpoint to its whole box, proving the balanced-row case.

6 Exhaustion, normalization and scope

Proof of Theorem 1. If a squared entry is at least 7/10, apply the published input (2). Otherwise all entries are capped. If some $r_i \leq 13/20$, Lemma 3 applies; otherwise all rows lie in the balanced region just proved. Each bound includes its boundary, so the cases exhaust the domain.

For positive-diagonal B supported in $K_{4,7}$, diagonal normalization gives $B = EA(X)E$, where $E_{ii} = \sqrt{b_{ii}}$. The diagonal blocks become identities and their Schur complement gives $\|X\|_{\mathrm{op}} \leq 1$. Permanent scaling gives

$$\mathcal{P}(B) = \mathrm{d}(B)\mathcal{P}(A(X)), \qquad \mathcal{Q}(B) = \mathrm{d}(B)^2\mathcal{Q}(A(X)),$$

Table 1: Minimum Bernstein coefficient over all interval boxes for each activity vector, with pairing 01/23.

$\epsilon_0\epsilon_1\epsilon_2\epsilon_3$	Minimum coefficient
0000	9/28
0001	697100024153/716800000000
0010	697100024153/716800000000
0011	913636344153/716800000000
0100	697100024153/716800000000
0101	143575934879/102400000000
0110	143575934879/102400000000
0111	6612711345071/5017600000000
1000	697100024153/716800000000
1001	143575934879/102400000000
1010	143575934879/102400000000
1011	6612711345071/5017600000000
1100	913636344153/716800000000
1101	6612711345071/5017600000000
1110	6612711345071/5017600000000
1111	707119/400000000

which proves (1). If a diagonal vanishes, positive semidefiniteness forces its row and column to vanish, so both permanents are zero. Missing edges and singular cases require no inverse of a singular row Gram matrix. □

Padding X by zero rows or columns adds isolated unit diagonal vertices. Thus the theorem includes smaller four-row rectangles through seven columns.

There is a paired consequence when *both* positive semidefinite factors belong to proved strong classes. Permutation Cauchy–Schwarz gives $|\operatorname{per}(A \circ B)|^2 \le \mathcal{Q}(A)\mathcal{Q}(B)$. In correlation normalization, strong contraction implies $\mathcal{P}(A) \ge \mathcal{Q}(A) \ge 1$ and likewise for B. Hence $\operatorname{per}(A \circ B) \le \sqrt{\mathcal{P}(A)\mathcal{P}(B)} \le \mathcal{P}(A)\mathcal{P}(B)$. Diagonal scaling and zero-row cases extend this assertion.

Reproducibility and acknowledgment

The theorem combines analytic continuum reductions with a finite rational positivity certificate. The supplied verifier uses exact arithmetic to establish the finite coefficient and corner calculations. AI tools assisted mathematical research, drafting and verification. The author is responsible for the statements, proofs, attribution and final text.

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