Strong self-permanent inequalities on five-row bipartite supports

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Luminary Oddities

Abstract

For every complex positive semidefinite matrix whose off-diagonal support is bipartite with one part of size at most five, the permanent of its entrywise squared modulus is at most its permanent times the product of its diagonal entries. The other support part has arbitrary finite size, and complex cycle phases are unrestricted. A row-dependent Carlen–Lieb–Loss estimate retains fourth-moment information in the higher minor sectors. A rational tangent bound and a low-row/high-row split produce two budgets separately affine in those moments. Their continuum positivity is proved by an exact certificate of 2,097,152 tensor Bernstein coefficients, with its full reconstruction specified in the article. A four-row structural core and a large-entry transfer cover the complementary domain. For positive diagonal entries, equality occurs exactly for disjoint rank-one two-vertex correlation blocks and isolated vertices; zero diagonals give automatic vanishing equality separately. The proof also yields an explicit positive gap in the capped high-row region. A separate energy criterion gives the ordinary self-Chollet inequality when the smaller support part has size at most twelve.

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1 Introduction and main results

For a Hermitian positive semidefinite matrix $B = (b_{ij})$, consider the comparison

$$\operatorname{per}(B \circ \overline{B}) \le \mathrm{d}(B) \operatorname{per} B, \qquad \mathrm{d}(B) = \prod_i b_{ii}. \tag{1}$$

Here $\circ$ is the entrywise product, so $B \circ \overline{B} = |B|^2$. This is a strong self-conjugate permanent inequality: permanent dominance $\operatorname{per} B \ge \mathrm{d}(B)$ then gives the ordinary self-Chollet bound $\operatorname{per} |B|^2 \le \operatorname{per}(B)^2$. The strong comparison is a classical target [1]; its structural domains matter because stronger permanent comparisons fail in general [4].

The question here is how a zero pattern can preserve this comparison while allowing arbitrary complex phases. A nonnegative permanent expansion can be estimated term by term, whereas complex cycles introduce cancellation in $\operatorname{per} B$. The main result proves (1) when the off-diagonal support is bipartite with one part of size at most five. The other part has arbitrary finite size. A fixed structural width therefore permits the stronger comparison in unbounded matrix order. The proof retains row-moment dependence in the higher minor sectors instead of paying them by a common cap. Its row-domain split is made before minimizing the moment variables, so separate affinity remains valid.

1.1 Main results in normalized form

A support graph specifies allowed off-diagonal entries; allowed edges may have weight zero. Bipartite support means that, after one simultaneous permutation of rows and columns, the within-part off-diagonal entries vanish. For a finite complex $p \times q$ rectangle X, write

$$A(X) = \begin{pmatrix} I_p & X \\ X^* & I_q \end{pmatrix}, \qquad \mathcal{P}(H) = \operatorname{per} H, \qquad \mathcal{Q}(H) = \operatorname{per} |H|^2,$$
$$P = \mathcal{P}(A(X)), \qquad Q = \mathcal{Q}(A(X)).$$

The condition $A(X) \succeq 0$ is equivalent to $\|X\|_{\mathrm{op}} \leq 1$. This normalization is explained immediately below the main results.

Theorem 1.1 (All-column five-row strong comparison)**.** *For every finite q, every complex $5 \times q$ operator contraction satisfies $Q \leq P$. Consequently every Hermitian positive semidefinite matrix B supported in $K_{5,q}$ satisfies*

$$\operatorname{per} |B|^2 \leq \mathrm{d}(B) \operatorname{per} B.$$

Arbitrary phases, missing edges, zero rows or columns and singular matrices are included. In particular every bipartite support through total order eleven satisfies this strong comparison.

Theorem 1.2 (Equality and a capped-domain gap)**.** *For a complex operator contraction X with at most five rows, $P = Q$ if and only if its nonzero entries have modulus one and occupy distinct rows and columns. For positive-diagonal B in Theorem 1.1, this means its normalized matrix is a direct sum of isolated vertices and rank-one two-vertex correlation blocks. A zero diagonal instead gives vanishing equality automatically. If X has five rows, all squared entries are at most $7/10$, and all row squared norms are at least $1/2$, then*

$$P - Q \geq c_H := \frac{2104602658688797}{6480000000000000} > 0.$$

This is a sufficient region-specific gap, not a dimension-uniform stability estimate or an optimal gap.

The proof is computer-assisted on a scalar domain independent of q. It includes all analytic reductions, exact degree and case counts, rational coefficient reconstruction, and standalone integer code in Appendix A. The four-row theorem below is an analytically proved support-induction input, not a bound on the final column count.

Theorem 1.3 (Four-row structural input)**.** *For every finite q, every complex $4 \times q$ operator contraction satisfies $Q \leq P$.*

1.2 Diagonal normalization and the matrix interpretation

For positive-diagonal positive semidefinite B with bipartite support, write $B = EA(X)E$, where $E_{ii} = \sqrt{b_{ii}}$. The within-part blocks are identities and their Schur complement gives $\|X\|_{\mathrm{op}} \leq 1$. The squared magnitudes of X therefore have row and column sums at most one, with $\sum_{a,j} |x_{aj}|^2 \leq \min(p, q)$. Permanent scaling gives

$$\operatorname{per} B = \mathrm{d}(B) P, \qquad \operatorname{per} |B|^2 = \mathrm{d}(B)^2 Q.$$

Consequently the normalized strong assertion is $\operatorname{per} |B|^2 \leq \mathrm{d}(B) \operatorname{per} B$, and the ordinary assertion scales identically on both sides. A zero diagonal of a positive semidefinite matrix forces a zero row and column; then both permanents are zero. Zero-column rectangles, when used as compressed cores, have $P = Q = 1$. All singular and missing-edge cases are therefore included.

1.3 Prior work and the complex-phase distinction

Chollet's conjecture concerns $\operatorname{per}(B \circ C) \le \operatorname{per}(B)\operatorname{per}(C)$ for positive semidefinite factors [3]. Li, Zhao and Luo report the ordinary comparison for arbitrary complex positive semidefinite matrices through total order six [6]. The present strong comparison concerns a specified bipartite support width while allowing unbounded total order. The strong self-comparison is the condition in Beasley's formulation [1, Lemma 2.2]. Beasley also treats entrywise-nonnegative positive semidefinite factors [1, Theorem 3.2]. Pant and Singh prove a strong comparison for real symmetric Z-matrices with nonnegative diagonal and bipartite support [8, Theorem 3.1]. Their sign hypotheses make the nonzero permutation terms nonnegative. Sa-nguansin and Rodtes study specified correlation-matrix regions [9]. Drury's counterexample [4] shows the importance of retaining the structural hypotheses in stronger permanent comparisons.

Complex bipartite supports contain phase patterns outside those nonnegative or sign-restricted classes. For example,

$$X_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is unitary, so $A(X_0)$ is positive semidefinite. Its four-cycle phase product is $-1/4$, invariant under diagonal unitary changes of phase. Such a change cannot make all four cross entries nonnegative or all nonpositive. Padding two zero rows places this example in the four-row class. For this example the minor expansion gives $P = 3$ and $Q = 9/4$. Thus phase cancellation is present even in a small member of the target class. Multiplying X_0 by any $0 < \varepsilon < 1$ gives a positive-definite $A(\varepsilon X_0)$. More generally, strict contractions give full-rank matrices of order $p + q$: the structural width here is distinct from a rank restriction.

The three-row core, dimension-independent large-entry transfer and all-column four-row argument are proved fully below. The five-row extension uses the same collision budget with a new row-dependent higher-sector payment and an exact two-domain certificate. No separate bounded-column theorem is needed as a premise. The matching equality pattern has sign-compatible antecedents in [8]; the classification here permits arbitrary complex phases on every finite five-row support.

The proof uses classical results at their points of application: Carlen–Lieb–Loss for square permanent norms [2], Lieb for positive semidefinite blocks [7], and Edmonds for the matching polytope [5]. Exact rectangular Bernstein verification appears in earlier work on Chollet inequalities [10]. Here it verifies the particular coupled minor budget after its analytic and symmetry reductions.

Table 1: The structural increment in the strong comparison.

Source	Domain	Contribution relevant here
Beasley [1]	Nonnegative or gaugeable positive semidefinite factors	Strong comparison from a sign-compatible class
Pant–Singh [8]	Real symmetric Z-matrices with bipartite support	Strong comparison using nonnegative permutation products
Theorem 1.3	Complex $K_{4,q}$, every finite q	Analytic structural input
Theorem 1.1	Complex $K_{5,q}$, every finite q	Row-dependent moment budget and exact scalar certificate

1.4 How the column restriction is removed

The normalized permanent is a sum of squared complex minor permanents. Its entrywise-modulus counterpart has nonnegative minor terms. These two expansions allow the proof to retain the overlap terms that pay for complex cancellation. For the four-row input, when all squared entries are at most 7/10, the relevant overlap graph has only four row vertices. A matching-polytope estimate controls it independently of q. The remaining row-moment budget is affine in each fourth moment. Deleting zero-endpoint partner pairs first, then using separate concavity, reduces that budget to a fixed rational corner table. The order of deletion is essential. If an entry exceeds the cap, Lieb's block inequality transfers the comparison to a three-row core. These two branches cover every contraction.

The five-row extension replaces the common higher-sector caps by row-dependent bounds. It uses no finite-column lower moment endpoint. A low-row/high-row split preserves separate affinity in the moments; the resulting endpoint polynomials have coordinate degree at most three. A separate ordinary-energy criterion remains a supporting consequence.

Shared estimates, the three-row core and the large-entry transfer precede the analytic four-row input. The all-five-row certificate and equality proof follow. Support and ordinary-energy consequences complete the main text.

2 Minor expansions and basic comparisons

For the proof, write $W = (w_{aj}) = |X|^2$, $r_a = \sum_j w_{aj}$, $c_j = \sum_a w_{aj}$, $R_a = \sum_j w_{aj}^2$ and $S = \sum_a r_a$. The star assumptions are

$$r_a \le 1, \qquad c_j \le 1. \tag{2}$$

They follow from operator contraction. Several estimates below use these scalar assumptions alone. Let e_k denote the elementary symmetric polynomial, with $e_0 = 1$.

Define

$$F_k = \sum_{|I|=|J|=k} |\operatorname{per} X[I,J]|^2, \qquad Q_k = \sum_{|I|=|J|=k} \operatorname{per}(W[I,J])^2, \quad W = (w_{aj}).$$

Empty minors have permanent one; unavailable orders contribute zero. Direct permutation expansion of the identity blocks gives

$$P = \sum_k F_k, \qquad Q = \sum_k Q_k, \qquad F_0 = Q_0 = 1, \quad F_1 = S. \tag{3}$$

Indeed a permutation moves equally many row and column vertices across the bipartition, and the two cross-permutations contribute conjugate permanents. Thus every F_k is nonnegative, even without positive semidefiniteness of $A(X)$.

For fixed row set I, nonnegativity and distinct-column assignments imply

$$\sum_{|J|=|I|} \operatorname{per} W[I,J] \le \prod_{a\in I} r_a.$$

Squaring the sum bounds the sum of squares, so

$$Q \le \prod_a (1 + r_a^2). \tag{4}$$

Set $G = XX^*$, $O = \sum_{a<b} |G_{ab}|^2$, and $D_{ab} = \sum_j w_{aj} w_{bj}$. Expanding the two-row minors gives

$$F_2 = e_2(r) + O - 2 \sum_{a<b} D_{ab}. \tag{5}$$

Since $2 \sum D_{ab} = \sum_j c_j^2 - \sum_a R_a \le S - \sum_a R_a$, (3) yields

$$P \ge 1 + S, \qquad P \ge 1 + e_2(r), \qquad P \ge 1 + \max\{S, e_2(r)\}. \tag{6}$$

No restriction on the number of rows was used.

3 Shared overlap, norm and collision estimates

Let $D_{ab} = \sum_j w_{aj} w_{bj}$, $E_{ab} = \sum_j w_{aj}^2 w_{bj}^2$, $\delta_a = r_a - R_a$, $G = XX^*$ and $K = X^*X$. Write $C = \sum_j c_j(1 - c_j)$ and $O = \sum_{a<b} |G_{ab}|^2$. The exact second-minor identity for Q and the trace identity are

$$Q_2 = e_2(R) + \sum_{a<b} D_{ab}^2 - 2 \sum_{a<b} E_{ab}, \tag{7}$$

$$C + O = \frac{1}{2} \sum_a r_a(1 - r_a) + \frac{1}{2} \sum_j c_j(1 - c_j) + \sum_{j<l} |K_{jl}|^2. \tag{8}$$

Thus, for any row number under stars,

$$P - Q \ge \frac{1}{2} \sum_a r_a(1 - r_a) + e_2(r) - \sum_{k=2}^{p} Q_k. \tag{9}$$

Indeed $F_1 - Q_1 = C + 2 \sum D_{ab}$, so (5) cancels the latter overlap term; all remaining F_k are nonnegative.

Lemma 3.1 (Matching bound)**.** *Let $z_{ij} \ge 0$ be edge weights of a finite loopless graph, with weighted degrees $h_i = \sum_j z_{ij} \le \delta_i$. Then*

$$\sum_{ij \in E} z_{ij}^2 \le \max_{M \text{ a matching}} \sum_{ij \in M} \delta_i \delta_j.$$

Proof. Discard zero-degree vertices and set $B_{ij} = z_{ij}/\sqrt{h_i h_j}$, $B_{ii} = 0$. Weighted $2ab \le a^2 + b^2$ implies $|v^T B v| \le \sum_i v_i^2$, so B is a real symmetric contraction. Set $y_{ij} = B_{ij}^2$. Then $\sum_j y_{ij} = (B^2)_{ii} \le 1$.

For an odd vertex set of size m, its principal compression T is a symmetric contraction of trace zero. Its eigenvalues lie in $[-1, 1]$ and sum to zero. The convex function $\sum \lambda_i^2$ on this compact polytope has a maximum at an extreme point, where at most one coordinate is interior. Since m is odd, the other $m - 1$ endpoint coordinates sum to an even integer and the remaining coordinate must be zero. Hence $\operatorname{Tr} T^2 \le m - 1$ and

$$\sum_{ij \in E(S)} y_{ij} \le (m - 1)/2.$$

These are exactly the degree and odd-set inequalities of Edmonds's matching-polytope theorem [5]. Thus y is a convex combination of matching incidence vectors, and

$$\sum z_{ij}^2 = \sum h_i h_j y_{ij} \le \max_M \sum_{ij \in M} h_i h_j \le \max_M \sum_{ij \in M} \delta_i \delta_j.$$

No division is made at a zero degree. The lemma is an application of the classical matching-polytope theorem, with the required spectral odd-set verification supplied explicitly. □

Column stars give $\sum_{b\neq a} D_{ab} \le \delta_a$. Lemma 3.1 and (7) therefore imply

$$Q_2 \le e_2(R) + M(\delta), \tag{10}$$

where M is the maximum weighted matching sum on the row vertices. For four rows this is the maximum over three perfect pairings; for five rows over fifteen two-edge matchings. Nonnegative weights allow shorter matchings to be extended.

The square Carlen–Lieb–Loss inequality [2, Theorem 1.1] gives

$$|\operatorname{per} M_k| \le \frac{k!}{k^{k/2}} \prod_{a=1}^{k} \|(M_k)_{a,*}\|_2. \tag{11}$$

Under stars and the cap $w_{aj} \le 7/10$, concentration of each nonnegative row gives

$$0 \le R_a \le f(r_a) \le r_a, \qquad f(t) = \begin{cases} t^2, & 0 \le t \le 7/10, \\ 49/100 + (t-7/10)^2, & 7/10 \le t \le 1. \end{cases} \tag{12}$$

At most one filled cap and one remainder are needed since $r_a \le 1 < 2(7/10)$. In particular $R_a \le 29/50$. Applying (11) to W minors and using their nonnegative distinct-column sum gives

$$Q_3 \le g_3 e_3(r), \quad Q_4 \le g_4 e_4(r), \quad Q_5 \le g_5 e_5(r), \quad (g_3, g_4, g_5) = \left(\frac{511}{1000}, \frac{2523}{5000}, \frac{11}{20}\right). \tag{13}$$

The bounds for individual minors are $k!(29/50)^{k/2}/k^{k/2}$. The fourth cap is exact; the squared safety margins of the third and fifth caps are $583/600000$ and $992213/19531250000$. Unavailable orders vanish.

Lemma 3.2 (Three-row collision identity)**.** *For any three selected rows, write $r_a = G_{aa}$, $g_{ab} = G_{ab}$ and let c denote the remaining row in a pair a, b. Set $T = \sum_j w_{0j} w_{1j} w_{2j}$ and $H_{ab} = \sum_j w_{cj} x_{aj} \overline{x_{bj}}$. The contribution F_3 of that row triple satisfies*

$$F_3 = \operatorname{per} G - 2 \sum_{a<b} r_c D_{ab} - 4\Re \sum_{a<b} \overline{g_{ab}} H_{ab} + 12T. \tag{14}$$

Proof. Expand the Gram permanent by column multisets with multiplicity factorials. Distinct columns contribute F_3. A double column j and different column k contribute $2|\sum_a x_{ak} \prod_{b\neq a} x_{bj}|^2$; a triple column contributes $6\prod_a w_{aj}$. Extending the double-column sum to $k = j$ adds $18\prod_a w_{aj}$. Hence

$$F_3 = \operatorname{per} G - 2 \sum_{j,k} \left| \sum_a x_{ak} \prod_{b\neq a} x_{bj} \right|^2 + 12T.$$

The diagonal part of the extended square sums to $\sum r_c D_{ab}$ and the cross part to $2\Re \sum \overline{g_{ab}} H_{ab}$, proving (14). The identity is polynomial and needs no operator bound. □

For an operator contraction $0 \le G \le I$, so $|g_{ab}| \le 1/2$. For $u, v, z \in [0, 1/2]$, AM–GM gives $2uvz \le (u^2 + v^2 + z^2)/3$. Therefore

$$2\Re(g_{ab} g_{bc} g_{ca}) \ge -\frac{1}{3}(|g_{ab}|^2 + |g_{bc}|^2 + |g_{ca}|^2). \tag{15}$$

4 The three-row structural core

The following complete argument supplies the support-induction core.

Proposition 4.1. *Every complex $3 \times q$ operator contraction satisfies $Q \le P$.*

Proof. Retain the notation of Lemma 3.2. Put $Z_k = \sum_{|I|=|J|=k} \operatorname{per} W[I,J]$. Every nonnegative square minor has permanent at most its product of row sums, hence at most one. Thus

$$E = \sum_{k=2}^{3} \sum_{|I|=|J|=k} \operatorname{per} W[I,J](1 - \operatorname{per} W[I,J]) \ge 0.$$

Distinct-column inclusion–exclusion gives

$$Z_2 = e_2(r) - \sum D_{ab}, \qquad Z_3 = r_0 r_1 r_2 - \sum r_c D_{ab} + 2T.$$

Using (5), (14) and $S - \sum R_a = C + 2\sum D_{ab}$, direct substitution yields

$$\begin{aligned} P - Q = E + C + \sum_{a<b} (1 - r_c) D_{ab} + \sum_{a<b} (1 + r_c) |g_{ab}|^2 \\ + 2\Re(g_{01} g_{12} g_{20}) - 4\Re \sum_{a<b} \overline{g_{ab}} H_{ab} + 10T. \end{aligned}$$

Weighted Cauchy gives $|H_{ab}|^2 \le r_c T$. Put $a_c = 2/3 + r_c$. By (15) and completion of the square,

$$a_c |g_{ab}|^2 - 4\Re(\overline{g_{ab}} H_{ab}) = a_c |g_{ab} - 2H_{ab}/a_c|^2 - 4|H_{ab}|^2/a_c.$$

Since $4r_c/a_c \le 12/5$, summing subtracts at most $36T/5$ from $10T$. Consequently

$$P - Q \ge E + C + \sum (1 - r_c) D_{ab} + \sum a_c |g_{ab} - 2H_{ab}/a_c|^2 + \frac{14}{5} T \ge 0.$$

The strictly positive a_c causes no division at a zero row. Zero columns and missing orders are included. □

5 A dimension-independent large-entry transfer

The following border argument states its core hypothesis explicitly, so its successive support steps have no circular dependency.

Lemma 5.1. *Let $p \ge 2$, and let X be any complex $p \times q$ operator contraction. Suppose its core D, obtained by deleting one row and column, satisfies $\mathcal{Q}(A(D)) \le \mathcal{P}(A(D))$. If the deleted entry has squared magnitude $t \ge 7/10$, then*

$$P - Q \ge \frac{7}{2000}(1 - t)\mathcal{Q}(A(D)) \ge 0.$$

The non-strict conclusion also holds when $t \ge (5 - \sqrt{13})/2$.

Proof. Place the entry first, and write $X = \left(\begin{smallmatrix} a & b \\ c & D \end{smallmatrix}\right)$, $t = |a|^2$, $s = 1 - t$, $u_j = |b_j|^2$, $v_i = |c_i|^2$, $U_b = \sum u_j$, $V_b = \sum v_i$. Contraction implies $U_b, V_b \le s$ and D is contractive. For $q = 1$, the support is a star and $P - Q = \sum_a w_a(1 - w_a) \ge t(1 - t)$, while $\mathcal{Q}(A(D)) = 1$. Hence assume $q \ge 2$.

Partition the actual positive semidefinite $A(X)$ into the two endpoints of a and the complementary $A(D)$. Lieb's block permanent inequality [7] and the core hypothesis give

$$P \geq (1+t)\mathcal{P}(A(D)) \geq (1+t)\mathcal{Q}(A(D)). \tag{16}$$

Put $W = |D|^2$. For fixed t, W, v, every minor permanent of the nonnegative rectangle $\left(\begin{smallmatrix} t & u \\ v & W \end{smallmatrix}\right)$ is affine in u. Its square is convex; hence their sum is convex in u, and similarly in v at fixed u. Successive maximization over the fixed-total simplices bounds Q by the largest value with $u = U_b e_j$, $v = V_b e_i$. These auxiliary concentrated rectangles need not be contractions; only the original matrix was used in (16).

Let $d_{I,J} = \operatorname{per} W[I,J]$, over equal-cardinality core subsets, including the empty pair. Then $\|d\|^2 = \mathcal{Q}(A(D))$. Use the Hilbert space with orthonormal basis $e_{I,J}$ indexed by all core row subsets I and core column subsets J, allowing unequal cardinalities. Embed d by setting $d_{I,J} = 0$ when $|I| \neq |J|$. Define

$$C_j e_{I,J} = \begin{cases} e_{I,J\cup\{j\}}, & j \notin J, \\ 0, & j \in J, \end{cases} \qquad R_i e_{I,J} = \begin{cases} e_{I\cup\{i\},J}, & i \notin I, \\ 0, & i \in I. \end{cases}$$

Each map takes distinct unoccupied basis states to distinct occupied states, so it is a partial isometry. The maps commute; $L = R_i C_j$ adds both indices. Direct expansion of all concentrated bordered minors gives

$$\begin{aligned} Q_{\mathrm{conc}} = (1+t^2)\|d\|^2 + U_b^2\|C_j d\|^2 + V_b^2\|R_i d\|^2 \\ + U_b^2 V_b^2\|Ld\|^2 + 2tU_bV_b\Re\langle d, Ld\rangle. \end{aligned} \tag{17}$$

Indeed a minor containing both border indices has permanent $td_{I,J} + U_bV_b d_{I\setminus\{i\},J\setminus\{j\}}$ when both selected core indices are present. Single-border minors give the two single-creation norms.

Group coordinates according to occupation of i, j. For fixed remaining subsets, the extra quadratic form in (17) has the 00/11 block

$$B(U_b, V_b) = \begin{pmatrix} U_b^2 + V_b^2 + U_b^2V_b^2 & tU_bV_b \\ tU_bV_b & 0 \end{pmatrix};$$

the other occupation types have diagonal U_b^2 or V_b^2. Missing states give compressions. The largest eigenvalue

$$\lambda(U_b, V_b) = \frac{A + \sqrt{A^2 + 4t^2U_b^2V_b^2}}{2}, \qquad A = U_b^2 + V_b^2 + U_b^2V_b^2,$$

dominates those diagonal values and increases in both nonnegative variables. Consequently

$$Q \leq [1 + t^2 + \lambda(s,s)]\mathcal{Q}(A(D)).$$

Set $M = tsI - B(s,s)$. Direct calculation yields

$$\det M = ts^2(1 - 3s - s^2), \qquad \operatorname{Tr} M = s(2 - 4s - s^3) \leq 2s.$$

For $0 < s \leq 3/10$, the lower-right entry ts is positive and $1 - 3s - s^2 \geq 1/100$, so M is positive definite. Its least eigenvalue is at least determinant divided by trace, hence

$$ts - \lambda(s,s) \geq \frac{ts^2/100}{2s} \geq \frac{7}{2000}s.$$

Combining this with (16) proves the quantitative result. For $s = 0$ the border terms vanish directly. The non-strict extension follows from $1 - 3s - s^2 \geq 0$, namely $t \geq (5 - \sqrt{13})/2$, with the same diagonal test; no division at zero slack is required. □

6 All finite-column four-row supports

Proposition 6.1. *Every complex $4 \times q$ rectangle satisfying stars and $w_{aj} \le 7/10$ satisfies $Q \le P$, without assuming operator contraction.*

For each perfect pairing π of the four rows define

$$B_\pi(r,R) = \frac{1}{2}\sum_a r_a(1-r_a) + e_2(r) - e_2(R) - \sum_{\{a,b\}\in\pi}(r_a - R_a)(r_b - R_b) - g_3 e_3(r) - g_4 e_4(r). \quad (18)$$

Equations (9), (10) and (13) give $P - Q \ge \min_\pi B_\pi$. We prove every such polynomial nonnegative on $0 \le r_a \le 1$, $0 \le R_a \le f(r_a)$.

Paired deletion and separate concavity

Fix π, r. The scalar (18) is separately affine in R, so its minimum occurs at endpoints $R_i = 0$ or $R_i = f(r_i)$. Mark rows according to the chosen endpoint; the two choices may coincide when $r_i = 0$. Hypothetical zero endpoints need not be realizable matrix rows. We first delete zero-marked partners together, then any remaining zero-marked row. A deleted row is set to $r_i = R_i = 0 = f(0)$.

Suppose two zero-marked partners have row sums t, u, and the other rows have sums v, w. The exact difference between the scalar before and after deleting the pair is

$$\begin{aligned}&\frac{1}{2}\big[t(1-t) + u(1-u)\big] + (t+u)(v+w)\\ &\quad - g_3\big[tu(v+w) + (t+u)vw\big] - g_4 tuvw. \end{aligned} \quad (19)$$

The tu term of $e_2(r)$ cancels the paired δ term because both R endpoints are zero. Since $2tu \le t+u$ and $2vw \le v+w$, (19) is at least

$$\frac{1}{2}\big[t(1-t)+u(1-u)\big] + (t+u)(v+w)(1-g_3-g_4/4) = \frac{1}{2}\big[t(1-t)+u(1-u)\big] + \frac{7257}{20000}(t+u)(v+w) \ge 0. \quad (20)$$

Thus all zero-marked partner pairs may be removed, regardless of the other endpoints.

Any remaining zero-marked row now has an f-marked partner. Write its row sum as t, its partner's as p, and the other two as u, v. Here $R_p = f(p)$. Deleting t changes the scalar by $t(1-t)/2 + tL(p,u,v)$, where

$$L = f(p) + (u+v) - g_3[p(u+v) + uv] - g_4 puv \ge f(p) + (u+v)\big[1 - g_3/2 - p(g_3 + g_4/2)\big].$$

For $0 \le p \le 1/2$, the bracket is at least $1 - g_3 - g_4/4 = 7257/20000 > 0$. For $1/2 \le p \le 1$, it is at least $1 - 3g_3/2 - g_4/2 = -47/2500$, while $f(p) \ge 1/4$ because f is increasing and continuous. Since $u + v \le 2$, this case gives $L \ge 1/4 - 94/2500 = 531/2500 > 0$. Every remaining zero-marked row can therefore be deleted safely. This argument does not assume that arbitrary single-row deletion is valid before partner pairs are removed.

It remains to prove $J_\pi(r) := B_\pi(r, f(r)) \ge 0$. Hold the other rows fixed and vary $t = r_i$, paired with p. The coefficient of $f(t)$ is

$$L = \delta_p - \sum_{a\ne i} f(r_a) \le p - 2f(p) \le \frac{1}{8}.$$

Indeed on the lower interval $p - 2p^2 = 1/8 - 2(p - 1/4)^2$, and on the upper interval $p - 2f(p) = -2(p - 19/20)^2 - 31/200$. Apart from $Lf(t)$, coordinate dependence is linear minus $t^2/2$, since $e_3(r), e_4(r)$ are separately linear. On each of the two intervals,

$$J''_\pi(t) = -1 + 2L \le -3/4.$$

Separate concavity on the piecewise boxes reduces the minimum to $r_a \in \{0, 7/10, 1\}$. Table 2 prints the minimum over the three pairings for each count pattern. These fifteen nonnegative rational values cover all 81 labelled corners and 243 pairing evaluations. Thus $J_\pi \ge 0$, then the deletion steps and affinity prove every $B_\pi \ge 0$ on the enlarged box. This proves Proposition 6.1 for every finite q, including $q = 1, 2$ where higher minor orders vanish.

Table 2: Exact corner minima of $B_\pi(r, f(r))$; n_0, n_7, n_1 count coordinates $0, 7/10, 1$.

n_0	n_7	n_1	$\min_\pi B_\pi$
0	0	4	5401/5000
0	1	3	7133/6250
0	2	2	551383/500000
0	3	1	1350349/1250000
0	4	0	50447677/50000000
1	0	3	6517/5000
1	1	2	10661/10000
1	2	1	95291/100000
1	3	0	845327/1000000
2	0	2	609/1250
2	1	1	2163/5000
2	2	0	2079/5000
3	0	1	0
3	1	0	21/200
4	0	0	0

Four-row assertion of Theorem 1.3. In the capped domain Proposition 6.1 applies. Otherwise Lemma 5.1 applies with its three-row core hypothesis supplied by Proposition 4.1. This proves all $4 \times q$ contractions, including the common cap boundary. A smaller number of rows is included by padding zero rows; identity padding preserves both permanents. □

7 A shared five-row minor budget

In this section there are five rows. The star assumptions imply that W itself is an operator contraction. In fact, weighted Cauchy–Schwarz gives

$$\sum_a \left| \sum_j w_{aj} v_j \right|^2 \le \sum_a r_a \sum_j w_{aj} |v_j|^2 \le \sum_j c_j |v_j|^2 \le \sum_j |v_j|^2.$$

Thus $H = WW^T$ is a positive semidefinite contraction, with diagonal R_a and off-diagonal D_{ab}. Set $U = \sum_a R_a$. Its trace inequality gives

$$\sum_{a<b} D_{ab}^2 \le \frac{1}{2} \left(U - \sum_a R_a^2 \right). \tag{21}$$

Combining (7) with (21) and $\sum R_a^2 \ge U^2/5$ yields

$$Q_2 \le \frac{3}{10}U^2 + \frac{1}{2}U. \tag{22}$$

Lemma 7.1 (Five-row third-sector budget)**.** *If X is an operator contraction, then for every $0 \le \theta \le 1$,*

$$P - Q \ge \frac{1-\theta}{2}\sum_a r_a(1-r_a) + e_2(r) + \theta e_3(r) - \sum_{k=2}^{5} Q_k - 2\theta \sum_{a<b} A_{ab}D_{ab}, \tag{23}$$

where $A_{ab} = \sum_{c\notin\{a,b\}} r_c$.

Proof. Write $g_{ab} = G_{ab}$, $T_{abc} = \sum_j w_{aj}w_{bj}w_{cj}$, $T = \sum_{a<b<c} T_{abc}$, and

$$H_{ab} = \sum_j \left(\sum_{c\notin\{a,b\}} w_{cj} \right) x_{aj}\overline{x_{bj}}.$$

The three-row collision identity, summed over row triples, is

$$\begin{aligned} F_3 = e_3(r) &+ \sum_{a<b} A_{ab}|g_{ab}|^2 + 2\Re \sum_{a<b<c} g_{ab}g_{bc}g_{ca} \\ &- 2\sum_{a<b} A_{ab}D_{ab} - 4\Re \sum_{a<b} \overline{g_{ab}}H_{ab} + 12T. \end{aligned} \tag{24}$$

Identity (24) follows by summing Lemma 3.2 over row triples.

Since $0 \le G \le I$, every $|g_{ab}| \le 1/2$. For $x, y, z \in [0, 1/2]$, AM–GM gives $2xyz \le (x^2+y^2+z^2)/3$. Thus each cubic Gram cycle is at least minus one third of its three pair squares. Each pair belongs to three triples, so their total contribution is at least $-O$.

Weighted Cauchy–Schwarz gives

$$|H_{ab}|^2 \le A_{ab} \sum_{c\notin\{a,b\}} T_{abc}.$$

For $A_{ab} > 0$, completion of the square yields

$$A_{ab}|g_{ab}|^2 - 4\Re(\overline{g_{ab}}H_{ab}) \ge -4|H_{ab}|^2/A_{ab} \ge -4\sum_c T_{abc}.$$

If $A_{ab} = 0$, the corresponding H, T vanish, so the same conclusion holds directly. Summing consumes at most $12T$. Combining with (5) now gives

$$F_2 + \theta F_3 \ge e_2(r) + \theta e_3(r) + (1-\theta)O - 2\sum D_{ab} - 2\theta \sum A_{ab}D_{ab}.$$

All unretained F_k are nonnegative. Finally $C + (1-\theta)O \ge (1-\theta)(C+O)$, and the shared trace identity proves (23). Zero rows and missing minor orders are included. □

8 All finite-column five-row supports

We prove Theorem 1.1. The large-entry lemma, using the already-proved all-column four-row input, covers any squared entry at least 7/10. It remains to consider the capped domain

$$0 \le r_i \le 1, \qquad 0 \le R_i \le f(r_i), \qquad i = 0, \dots, 4.$$

No finite-column lower moment bound is imposed. Allowing $R_i = 0$ at positive r_i enlarges the domain and will require care for equality.

8.1 Row-dependent higher-sector bounds

For a fixed k-row set I, CLL gives for every k-column set J

$$\operatorname{per} W[I,J] \le c_k \prod_{i\in I} \sqrt{R_i}, \qquad c_k = \frac{k!}{k^{k/2}}.$$

Multiplying by the same nonnegative permanent and summing over J, using the distinct-column bound $\sum_J \operatorname{per} W[I,J] \le \prod_{i\in I} r_i$, yields

$$Q_k \le c_k e_k(r_i\sqrt{R_i}), \qquad k = 3,4,5. \tag{25}$$

This retains the row moments and holds for every finite number of columns; missing minor orders vanish. The tangent inequality $(\sqrt{R} - 3/4)^2 \ge 0$ gives $\sqrt{R} \le 3/8 + 2R/3$. Put

$$z_i = r_i(3/8 + 2R_i/3), \qquad H(r,R) = \frac{231}{200}e_3(z) + \frac{3}{2}e_4(z) + \frac{537}{250}e_5(z).$$

All factors are nonnegative, and the rational upper constants are valid:

$$(231/200)^2 - 4/3 = 83/120000 > 0, \qquad c_4 = 3/2, \qquad (537/250)^2 - 576/125 = 369/62500 > 0.$$

Thus

$$Q_3 + Q_4 + Q_5 \le H(r,R). \tag{26}$$

The tangent bound restores separate affinity in R while retaining moment information lost by a common higher-sector allowance.

8.2 Select the row domain before minimizing moments

For a two-edge matching π on five row vertices write $M_\pi(\delta) = \sum_{\{i,j\}\in\pi} \delta_i\delta_j$. Equation (10) gives $Q_2 \le e_2(R) + \max_\pi M_\pi(\delta)$.

If some $r_i \le 1/2$, take $\theta = 0$ in (23). Equations (10) and (26) give $P - Q \ge \min_\pi B_{\mathrm{L},\pi}$, where

$$B_{\mathrm{L},\pi} = \tfrac{1}{2}\sum_i r_i(1-r_i) + e_2(r) - e_2(R) - M_\pi(\delta) - H(r,R). \tag{27}$$

If all $r_i \ge 1/2$, the potential $y_i = S/2 - r_i$ is nonnegative: the other four row sums total at least two and $r_i \le 1$. Since $y_i + y_j = A_{ij}$ and overlap degrees are at most δ_i,

$$\sum_{i<j} A_{ij}D_{ij} \le \sum_i \delta_i(S/2 - r_i).$$

Taking $\theta = 3/5$ in (23) gives $P - Q \ge \min_\pi B_{\mathrm{H},\pi}$ with

$$\begin{aligned} B_{\mathrm{H},\pi} = {} & \tfrac{1}{5}\sum_i r_i(1-r_i) + e_2(r) + \tfrac{3}{5}e_3(r) - e_2(R) - M_\pi(\delta) - H(r,R) \\ & - \tfrac{6}{5}\sum_i \delta_i(S/2 - r_i). \end{aligned} \tag{28}$$

The low and high regions cover the row cube, with their shared boundaries included. The signed potential is used only where it is actually nonnegative; it is not extended to a dominating-row region.

Each selected fixed-matching budget is separately affine in every R_i: products in elementary symmetric polynomials use distinct indices, matching edges are disjoint, z_i is affine in R_i, and the dual term is linear in δ_i. Hence its minimum over the moment box occurs at independent endpoints $R_i = 0$ or $f(r_i)$. This step does not minimize a maximum of affine functions: the row-domain choice has already selected one individual budget.

8.3 The exact finite scalar cover

Fix $\pi_0 = \{\{0,1\},\{2,3\}\}$, with row four unmatched. Every matching is a relabeling of this one. In the low region choose one row with $r_i \le 1/2$. If matched, a matching-preserving permutation sends it to row zero; if unmatched, it becomes row four. Thus there are two representative domains, each with its special row in $[0,1/2]$ and the other four in $[0,1]$. Split the other intervals at $7/10$: sixteen row boxes and thirty-two moment choices give 512 polynomials in each group. In the high region, the cube $[1/2,1]^5$ is invariant under every row permutation. Splitting each coordinate into $[1/2,7/10]$ and $[7/10,1]$, with all moment choices, gives 1024 polynomials. These cases cover every matching and every allowed moment profile.

After endpoint substitution, each coordinate degree is at most three: $f(r_i)$ is quadratic on a row interval, z_i is cubic, and the own-coordinate factor in $\delta_i(S/2 - r_i)$ is cubic. All other products use distinct coordinates. On each row box, mapped affinely to $t \in [0,1]^5$, write the exact tensor cubic Bernstein expansion

$$B(t) = \sum_{\alpha \in \{0,1,2,3\}^5} \beta_\alpha \prod_{i=0}^{4} \binom{3}{\alpha_i} t_i^{\alpha_i} (1-t_i)^{3-\alpha_i}. \tag{29}$$

The basis is nonnegative and sums to one. Appendix A gives a complete integer formula and standalone reconstruction for every coefficient. Its exact results are

Group	Polynomials	Coefficients	Zeros	Minimum positive
Low, special 0	512	524288	168	1/20
Low, special 4	512	524288	176	1/20
High	1024	1048576	0	c_H
Total	2048	2097152	344	1/20

Every coefficient is nonnegative. Thus every selected endpoint budget is nonnegative on its entire closed row box. Separate affinity extends the result to the full moment box, and the row-domain and matching cover give $P \ge Q$ for every capped five-row contraction. If an entry is larger than the cap, Lemma 5.1 invokes an all-column four-row core supplied by Theorem 1.3. This covers all finite q, including zero cores and the shared cap boundary, and completes Theorem 1.1. The diagonal scaling already given transfers it to all stated PSD bipartite matrices.

9 Strictness and equality

We prove Theorem 1.2. Every high-region coefficient in (29) is at least c_H. The partition of unity therefore gives $B_{\mathrm{H},\pi} \ge c_H$ at every moment endpoint; affine interpolation and matching symmetry preserve the same gap for every actual high-row capped contraction.

For low-row strictness it suffices first to examine the all-upper moment endpoint $R_i = f(r_i)$ of (27). Every row-box vertex has its special row in $\{0,1/2\}$ and the other four in $\{0,7/10,1\}$. Direct rational substitution gives the following minima by the number k of positive row sums, with 162 vertices per group. Each group has respectively $1,9,32,56,48,16$ vertices for $k = 0,\dots,5$.

k	Special row 0	Special row 4
0	0	0
1	0	0
2	3/8	2079/5000
3	12725891826529/14400000000000	12725891826529/14400000000000
4	1071853731000479/864000000000000	18705983227/17280000000
5	360934627338073/518400000000000	323920867338073/518400000000000

This table is specified by a small direct enumeration: fix the special coordinate to zero or $1/2$, substitute all 3^4 values of the remaining coordinates, form f, z, δ by their displayed rational formulas, and evaluate (27) at π_0 and $R = f(r)$. All 324 values are covered by this rule. In particular every such vertex with at least two active rows is strictly positive.

Take any point of a low row box with at least two positive coordinates. For each positive coordinate choose a positive interval endpoint with strictly positive endpoint Bernstein factor at that point; for a zero coordinate choose the zero endpoint. At interval boundaries choose the endpoint at which the coordinate lies. This produces a vertex retaining at least two active rows. Its vertex Bernstein coefficient is its actual polynomial value, strictly positive by the table, and its product basis weight is strictly positive. All other coefficients are nonnegative. Thus the all-upper moment endpoint polynomial is strictly positive throughout this part of the box. No concavity of the new cubic endpoint polynomial is asserted or needed.

For an actual finite rectangle, $r_i > 0$ implies $R_i > 0$ and $f(r_i) > 0$. The multiaffine interpolation of the budget in the moment box assigns the all-upper active-row endpoint weight

$$\prod_{i:r_i>0} R_i/f(r_i) > 0.$$

Zero rows have coincident endpoints and are omitted, so there is no zero division. Every other endpoint is nonnegative. Therefore each fixed-matching actual low budget is strictly positive with at least two active rows; the minimum over fifteen matchings is still positive. If exactly one row is active, the minor identity gives directly $P - Q = \sum_j w_j(1 - w_j) \geq (3/10)r > 0$ under the cap. The zero rectangle has $P = Q = 1$. Hence every nonzero capped rectangle is strict.

If a squared entry t lies in $[7/10, 1)$, Lemma 5.1 gives $P - Q \geq (7/2000)(1 - t)Q(A(D)) > 0$, because $Q(A(D)) \geq 1$. For $t = 1$, the row and column star constraints isolate that entry. Its two-vertex block contributes two to each permanent, so $P = 2P_{\text{core}}$ and $Q = 2Q_{\text{core}}$. Remove such unit entries successively. Equality persists exactly when the remaining core has equality. A nonzero final core would fall into a strict capped or strict nonunit-heavy case. Thus equality leaves a zero core, namely a partial unit-modulus matching. Conversely these matchings give $P = Q = 2^s$ for s nonzero entries.

Positive-diagonal scaling preserves this classification and makes unit cross magnitude equivalent to $|b_{ij}|^2 = b_{ii}b_{jj}$. If a diagonal entry of B vanishes, both full permanents vanish and equality is automatic, separately from the matching classification. This proves all parts of Theorem 1.2.

10 Support consequences

A bipartition on at most eleven vertices has smaller part at most five, so Theorem 1.1 applies after zero padding or interchanging the parts. The ordinary criterion in Theorem 11.1 instead covers smaller part at most twelve, and hence every bipartite support through total order twenty-five. These are different strengths of conclusion.

For two positive semidefinite matrices of the same order, if each satisfies a proved ordinary self-bound, permutation Cauchy–Schwarz gives

$$|\operatorname{per}(B \circ C)|^2 \le \operatorname{per}|B|^2 \operatorname{per}|C|^2 \le \operatorname{per}(B)^2 \operatorname{per}(C)^2.$$

Their bipartitions need not coincide. A strong class also has its ordinary self-bound by permanent dominance $\operatorname{per} B \ge \mathrm{d}(B)$, or by the normalized minor expansion.

11 A dimension-free ordinary energy criterion

The strong theorems above use the size of the smaller support part. A separate minor-budget argument yields the ordinary self-comparison from a dimension-free bound on total squared energy. Its hypotheses are the star conditions (2); operator contraction implies these conditions and also bounds S by the size of the smaller part.

Theorem 11.1 (Ordinary energy criterion)**.** *For every finite complex rectangle satisfying* (2)*,*

$$S \le \sigma := 12 + \frac{201 + 16\sqrt{537}}{988} \quad \Longrightarrow \quad Q \le P^2.$$

In particular every positive semidefinite $K_{p,q}$ support with $\min(p,q) \le 12$ *satisfies the ordinary self-comparison. This includes every bipartite support through order* 25*.*

Proof of Theorem 11.1. Write $S = k + t$, with k a nonnegative integer and $0 \le t \le 1$. At an integer either adjacent representation can be used. For $0 \le x \le 1$,

$$\frac{d^2}{dx^2} \log(1 + x^2) = \frac{2(1 - x^2)}{(1 + x^2)^2} \ge 0.$$

At fixed sum, shifting two interior coordinates toward an endpoint therefore does not decrease $\sum \log(1 + r_a^2)$. The same operation maximizes $\sum r_a^2$ and hence minimizes $e_2(r) = (S^2 - \sum r_a^2)/2$. Iterating gives k ones, one t and zeros. Consequently

$$Q \le 2^k(1 + t^2), \qquad e_2(r) \ge \binom{k}{2} + kt, \qquad P \ge L_k(t) := 1 + \max\{k + t, \tbinom{k}{2} + kt\}.$$

For $k = 0, 1, 2$, $L_k = 1 + k + t$; for $k \ge 3$, $L_k = 1 + \binom{k}{2} + kt$. Table 3 gives the exact quadratic $L_k(t)^2 - 2^k(1+t^2)$. For negative quadratic coefficients the minimum is at an endpoint. For $k = 0, 3, 4$ the coefficients show nonnegativity directly. All listed minima are nonnegative, proving the claim through $S = 12$.

For $k = 12$ the polynomial is $393 + 1608t - 3952t^2$. Its positive root is $(201 + 16\sqrt{537})/988 \in (0, 1)$, since its discriminant is $128^2 \cdot 537$. It is nonnegative through that root, proving the closed endpoint $S \le \sigma$. This is the endpoint of the displayed coarse comparisons. □

Declarations

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Table 3: Coefficients of $L_k(t)^2 - 2^k(1+t^2)$ and its minimum on $[0,1]$.

k	constant	linear	quadratic	minimum
0	0	2	0	0
1	2	4	−1	2
2	5	6	−3	5
3	8	24	1	8
4	33	56	0	33
5	89	110	−7	89
6	192	192	−28	192
7	356	308	−79	356
8	585	464	−192	585
9	857	666	−431	857
10	1092	920	−924	1088
11	1088	1232	−1927	393

Data availability. All mathematical reductions and the exact coefficient reconstruction algorithm are contained in this article.

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A Exact reconstruction of the all-column certificate

All finite data below reconstruct the scalar proof, without an input matrix or a bound on its column count. At one-coordinate Bernstein nodes $t_i = m_i/3$, $m_i \in \{0,1,2,3\}$, for a row interval $[\ell_i/10, h_i/10]$ set

$$r_i = n_i/30, \qquad n_i = 3\ell_i + (h_i - \ell_i)m_i.$$

The endpoints ℓ_i, h_i are among $0, 5, 7, 10$. Put $R_i = N_i/900$, with $N_i = 0$ at its zero endpoint, and at its upper moment endpoint

$$N_i = \begin{cases} n_i^2, & h_i \le 7, \\ 441 + (n_i - 21)^2, & (\ell_i, h_i) = (7, 10). \end{cases}$$

Define $d_i = 30n_i - N_i$, $s = \sum_i n_i$, $Z = 162000$ and $u_i = n_i(2025 + 4N_i)$, so $\delta_i = d_i/900$ and $z_i = u_i/Z$. Let

$$D = \operatorname{lcm}(4500, 1800, 900, 45000, 810000, 200Z^3, 2Z^4, 250Z^5).$$

The exact low-budget nodal numerator is

$$\begin{aligned} DB_{\mathrm{L},\pi_0} = {} & \frac{D}{1800}\sum_i n_i(30 - n_i) + \frac{D}{900}e_2(n) - \frac{D}{810000}\left[e_2(N) + d_0d_1 + d_2d_3\right] \\ & - \frac{231D}{200Z^3}e_3(u) - \frac{3D}{2Z^4}e_4(u) - \frac{537D}{250Z^5}e_5(u). \end{aligned} \tag{30}$$

For the high budget replace the first coefficient by $D/4500$ and add

$$\frac{D}{45000}e_3(n) - \frac{D}{45000}\sum_i d_i(s-2n_i).$$

All scalar coefficients here are integers. The one-coordinate cubic Bernstein evaluation matrix is $B_{27}/27$, where

$$B_{27} = \begin{pmatrix} 27 & 0 & 0 & 0 \\ 8 & 12 & 6 & 1 \\ 1 & 6 & 12 & 8 \\ 0 & 0 & 0 & 27 \end{pmatrix}, \qquad M_6 = \begin{pmatrix} 6 & 0 & 0 & 0 \\ -5 & 18 & -9 & 2 \\ 2 & -9 & 18 & -5 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

The exact identity $B_{27}M_6 = 162I$ gives the inverse $M_6/6$. Apply M_6 successively on the five tensor axes to the nodal numerators. The outputs are the Bernstein numerators at common denominator

$$D6^5 = 216905884017408000000000000000000.$$

The degree bound in the main text makes the interpolation an identity, not a numerical fit.

The following complete standard-library program implements these formulas. It explicitly rejects a negative coefficient or incomplete domain count. Its case order is low special row zero, low special row four, then high; within groups it uses lexicographic row-region bits and moment-endpoint bits. Tensor indices are lexicographic with the final coordinate fastest. No floating arithmetic enters any sign or coverage decision; reduced fractions in the printed minima are exact.

```
"""New all-column five-row certificate; integer arithmetic, no dependencies."""
from itertools import product
from fractions import Fraction
from math import lcm
import json,time
Z=162000
DEN=lcm(4500,1800,900,45000,810000,200*Z**3,2*Z**4,250*Z**5)
M=((6,0,0,0),(-5,18,-9,2),(2,-9,18,-5),(0,0,0,6))
GRID=list(product(range(4),repeat=5))
def elementary(x,k):
 e=[1]+[0]*k
 for a in x:
  for j in range(k,0,-1):e[j]+=a*e[j-1]
 return e[k]
def nodes(regions,ends,high):
 vals=[]
 for point in GRID:
  n=[3*lo+(hi-lo)*t for (lo,hi),t in zip(regions,point)]
  N=[(x*x if hi<=7 else 441+(x-21)**2) if end else 0
                 for x,(_,hi),end in zip(n,regions,ends)]
  d=[30*x-y for x,y in zip(n,N)]
  z=[x*(2025+4*y) for x,y in zip(n,N)]
  row=sum(x*(30-x) for x in n)
  v=(DEN//(4500 if high else 1800))*row
  v+=(DEN//900)*elementary(n,2)
  if high:v+=(DEN//45000)*elementary(n,3)
  v-=(DEN//810000)*(elementary(N,2)+d[0]*d[1]+d[2]*d[3])
  if high:v-=(DEN//45000)*sum(x*(sum(n)-2*y) for x,y in zip(d,n))
  v-=231*(DEN//(200*Z**3))*elementary(z,3)
  v-=3*(DEN//(2*Z**4))*elementary(z,4)
  v-=537*(DEN//(250*Z**5))*elementary(z,5)
  vals.append(v)
```

```
 return vals

def coefficients(vals):
 a=list(vals)
 for axis in range(5):
  stride=4**(4-axis)
  for start in range(0,1024,4*stride):
   for off in range(stride):
    ids=[start+off+j*stride for j in range(4)]
    old=[a[i] for i in ids]
    for j in range(4):a[ids[j]]=sum(M[j][k]*old[k] for k in range(4))
 return a

def cases():
 for special in (0,4):
  for bits in product((0,1),repeat=4):
   it=iter(bits);reg=[]
   for i in range(5):reg.append(
    (0,5) if i==special else ((0,7) if next(it)==0 else (7,10)))
   for ends in product((0,1),repeat=5):yield 'low'+str(special),bits,ends,reg,False
 for bits in product((0,1),repeat=5):
  reg=[(5,7) if x==0 else (7,10) for x in bits]
  for ends in product((0,1),repeat=5):yield 'high',bits,ends,reg,True

def run():
 groups={};gmin=None;location=None;t=time.time()
 for name,bits,ends,reg,high in cases():
  c=coefficients(nodes(reg,ends,high))
  if min(c)<0:raise AssertionError(('negative',name,bits,ends,min(c)))
  st=groups.setdefault(name,dict(
   polynomials=0,coefficients=0,zero=0,min_positive=None))
  st['polynomials']+=1;st['coefficients']+=1024;st['zero']+=c.count(0)
  pos=min((x for x in c if x>0),default=None)
  if pos is not None and (st['min_positive'] is None
                          or pos<st['min_positive']):st['min_positive']=pos
  if pos is not None and (gmin is None
                          or pos<gmin):gmin=pos;location=(name,bits,ends,c.index(pos))
  if sum(x['polynomials'] for x in groups.values())%512==0:print(
   'progress',groups.keys(),'seconds',round(time.time()-t,2),flush=True)
 counts={k:v['polynomials'] for k,v in groups.items()}
 expected={'low0':512,'low4':512,'high':1024}
 if counts!=expected:raise AssertionError(('incomplete coverage',counts,expected))
 for st in groups.values():st['min_positive']=str(Fraction(
  st['min_positive'],DEN*6**5))
 out=dict(status='PASS_NEW_EXACT_ALL_COLUMN_CERTIFICATE',
          groups=groups,polynomials=2048,coefficients=2097152,
          denominator=DEN*6**5,
          min_positive=str(Fraction(gmin,DEN*6**5)),
          min_positive_location=location)
 print(json.dumps(out,indent=2),flush=True)
 return out
if __name__=='__main__':run()
```

The resulting 344 zero coefficients are retained. In particular the proof does not replace nonnegativity by an unverified strictness claim; the separate vertex argument supplies equality classification.

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