Strong and energy-bounded permanent inequalities on bipartite supports

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Luminary Oddities

Abstract

We prove strong permanent contraction for every complex positive semidefinite matrix supported in $K_{4,q}$, with any finite q, and in $K_{5,5}$. The four-row proof is analytic; the five-by-five proof combines a retained third-minor budget with finite-column moment bounds and an exact tensor Bernstein certificate comprising $3,145,728$ rational coefficients. Independently, a dimension-free energy criterion proves the ordinary self-Chollet inequality whenever the off-diagonal rectangle has squared row and column sums at most one and total squared energy at most $12+(201+16\sqrt{537})/988$, including all positive semidefinite $K_{12,q}$ supports. Strong five-row estimates for arbitrary finite column number are also obtained near row-isometries and under dispersed fourth moments. The structural core proofs, large-entry transfer and analytic reductions are given in full, with classical permanent norm, block and matching-polytope theorems credited. Arbitrary complex phases, missing edges and singular endpoints are included.

1 Principal results and relation to prior work

For a finite complex $p \times q$ rectangle X, put

$$A(X) = \begin{pmatrix} I_p & X \\ X^* & I_q \end{pmatrix}, \qquad P = \operatorname{per} A(X), \qquad Q = \operatorname{per} |A(X)|^2.$$

Absolute values and squares are entrywise. Write $W = (w_{aj}) = |X|^2$, $r_a = \sum_j w_{aj}$, $c_j = \sum_a w_{aj}$, $R_a = \sum_j w_{aj}^2$ and $S = \sum_a r_a$. For Hermitian B let $\mathrm{d}(B) = \prod_i b_{ii}$. A support graph specifies allowed off-diagonal entries; allowed edges may have weight zero. The ordinary self-comparison is $\operatorname{per} |B|^2 \le \operatorname{per}(B)^2$, whereas the stronger comparison is $\operatorname{per} |B|^2 \le \mathrm{d}(B) \operatorname{per}(B)$.

Theorem 1 (Strong bipartite supports)**.** *For every finite q, every complex $4 \times q$ operator contraction satisfies $Q \le P$. Every complex 5×5 operator contraction also satisfies $Q \le P$. Consequently every Hermitian positive semidefinite matrix supported in $K_{4,q}$ or $K_{5,5}$ satisfies*

$$\operatorname{per} |B|^2 \le \mathrm{d}(B) \operatorname{per} B.$$

This includes every bipartite support through total order 10*.*

The $K_{5,5}$ conclusion is computer-assisted. Its complete finite verification protocol is given in Section 8, with the standalone exact checker `code/verify_bipartite_certificate.py`. The $K_{4,q}$ conclusion is analytic, using a fifteen-value rational table.

The *star assumptions* are

$$r_a \le 1, \qquad c_j \le 1. \tag{1}$$

They follow from operator contraction but are weaker; $A(X)$ need not be positive semidefinite under stars alone.

Theorem 2 (Ordinary energy criterion)**.** *For every finite complex rectangle satisfying* (1)*,*

$$S \le \sigma := 12 + \frac{201 + 16\sqrt{537}}{988} \quad \Longrightarrow \quad Q \le P^2.$$

In particular every positive semidefinite $K_{p,q}$ support with $\min(p,q) \le 12$ satisfies the ordinary self-comparison. This includes every bipartite support through order 25*.*

Theorem 3 (Unbounded-column five-row regions)**.** *Let X be a complex $5 \times q$ operator contraction, with any finite q. If $S \ge 249/50$, then $Q \le P$. Under the additional cap $\max w_{aj} \le 7/10$, its gap is at least $739/75000$. If all five row norms equal one, the capped gap is at least $451/3000$; without the cap the non-strict comparison still holds. Thus all five-row row-isometries, including all complex five-by-five unitaries, satisfy the strong comparison.*

The uniform positive margins apply only to the capped branch; permutation rectangles can have equality. A second unbounded-column result is useful beyond this energy region.

Theorem 4 (Dispersed fourth moments)**.** *For every complex $5 \times q$ rectangle satisfying* (1) *and $R_a \le 1/2$,*

$$P - Q \ge \frac{1}{2}\sum_a r_a(1 - r_a) + e_2(r) - \frac{5603}{600}.$$

In particular $r_a \ge 97/100$ for all a implies $P - Q \ge 53/750$.

Here and below e_k is the elementary symmetric polynomial and $e_0 = 1$. Theorem 4 needs no operator bound. Its operator-contraction subdomain is not contained in Theorem 3: a Fourier unitary scaled to squared row norms $97/100$ has $S = 97/20$ and $R_a = (97/100)^2/5$.

Chollet's conjecture concerns $\operatorname{per}(B \circ C) \le \operatorname{per}(B)\operatorname{per}(C)$ for positive semidefinite factors [2]. The stronger self-comparison has classical antecedents, including Beasley's formulation [1, Lemma 2.2], but fails for general positive semidefinite matrices [3]. Pant and Singh develop structural methods [7]; their bipartite theorem has real nonpositive off-diagonal sign hypotheses. Sa-nguansin and Rodtes study correlation-matrix regions [8], including annuli and an order-five ridge.

The author's published three-row and bounded-column four-row results [9, 10, 11, 12] are predecessors. This paper gives one proof chain, including the three-row core, the all-column four-row argument and the finite five-by-five certificate. Classical tools are used explicitly: Carlen–Lieb–Loss for square permanent norms [4], Lieb for positive semidefinite blocks [5], and Edmonds for the matching polytope [6]. Exact rectangular Bernstein verification already appears in work on Chollet inequalities [13]. The additional content is the coupled minor budgets and their support and moment domains.

2 Minor expansions and bounded energy

Define

$$F_k = \sum_{|I|=|J|=k} |\operatorname{per} X[I,J]|^2, \qquad Q_k = \sum_{|I|=|J|=k} \operatorname{per}(W[I,J])^2, \quad W = (w_{aj}).$$

Empty minors have permanent one; unavailable orders contribute zero. Direct permutation expansion of the identity blocks gives

$$P = \sum_k F_k, \qquad Q = \sum_k Q_k, \qquad F_0 = Q_0 = 1, \quad F_1 = S. \tag{2}$$

Indeed a permutation moves equally many row and column vertices across the bipartition, and the two cross-permutations contribute conjugate permanents. Thus every F_k is nonnegative, even without positive semidefiniteness of $A(X)$.

For fixed row set I, nonnegativity and distinct-column assignments imply

$$\sum_{|J|=|I|} \operatorname{per} W[I,J] \le \prod_{a\in I} r_a.$$

Squaring the sum bounds the sum of squares, so

$$Q \le \prod_a (1+r_a^2). \tag{3}$$

Set $G = XX^*$, $O = \sum_{a<b} |G_{ab}|^2$, and $D_{ab} = \sum_j w_{aj}w_{bj}$. Expanding the two-row minors gives

$$F_2 = e_2(r) + O - 2\sum_{a<b} D_{ab}. \tag{4}$$

Since $2\sum D_{ab} = \sum_j c_j^2 - \sum_a R_a \le S - \sum_a R_a$, (2) yields

$$P \ge 1+S, \qquad P \ge 1+e_2(r), \qquad P \ge 1+\max\{S, e_2(r)\}. \tag{5}$$

No restriction on the number of rows was used.

Proof of Theorem 2. Write $S = k+t$, with k a nonnegative integer and $0 \le t \le 1$. At an integer either adjacent representation can be used. For $0 \le x \le 1$,

$$\frac{d^2}{dx^2}\log(1+x^2) = \frac{2(1-x^2)}{(1+x^2)^2} \ge 0.$$

At fixed sum, shifting two interior coordinates toward an endpoint therefore does not decrease $\sum \log(1+r_a^2)$. The same operation maximizes $\sum r_a^2$ and hence minimizes $e_2(r) = (S^2 - \sum r_a^2)/2$. Iterating gives k ones, one t and zeros. Consequently

$$Q \le 2^k(1+t^2), \qquad e_2(r) \ge \binom{k}{2} + kt, \qquad P \ge L_k(t) := 1+\max\{k+t, \tbinom{k}{2}+kt\}.$$

For $k = 0,1,2$, $L_k = 1+k+t$; for $k \ge 3$, $L_k = 1+\binom{k}{2}+kt$. Table 1 gives the exact quadratic $L_k(t)^2 - 2^k(1+t^2)$. For negative quadratic coefficients the minimum is at an endpoint. For $k = 0,3,4$ the coefficients show nonnegativity directly. All listed minima are nonnegative, proving the claim through $S = 12$. For $k = 12$ the polynomial is $393 + 1608t - 3952t^2$. Its positive root is $(201+16\sqrt{537})/988 \in (0,1)$, since its discriminant is $128^2 \cdot 537$. It is nonnegative through that root, proving the closed endpoint $S \le \sigma$. This is the endpoint of the displayed coarse comparisons. □

3 Shared overlap, norm and collision estimates

Let $D_{ab} = \sum_j w_{aj}w_{bj}$, $E_{ab} = \sum_j w_{aj}^2 w_{bj}^2$, $\delta_a = r_a - R_a$, $G = XX^*$ and $K = X^*X$. Write $C = \sum_j c_j(1-c_j)$ and $O = \sum_{a<b}|G_{ab}|^2$. The exact second-minor identity for Q and the trace identity are

$$Q_2 = e_2(R) + \sum_{a<b} D_{ab}^2 - 2\sum_{a<b} E_{ab}, \tag{6}$$

$$C + O = \frac{1}{2}\sum_a r_a(1-r_a) + \frac{1}{2}\sum_j c_j(1-c_j) + \sum_{j<l} |K_{jl}|^2. \tag{7}$$

Table 1: Coefficients of $L_k(t)^2 - 2^k(1+t^2)$ and its minimum on $[0,1]$.

k	constant	linear	quadratic	minimum
0	0	2	0	0
1	2	4	−1	2
2	5	6	−3	5
3	8	24	1	8
4	33	56	0	33
5	89	110	−7	89
6	192	192	−28	192
7	356	308	−79	356
8	585	464	−192	585
9	857	666	−431	857
10	1092	920	−924	1088
11	1088	1232	−1927	393

Thus, for any row number under stars,

$$P - Q \geq \frac{1}{2}\sum_a r_a(1-r_a) + e_2(r) - \sum_{k=2}^{p} Q_k. \tag{8}$$

Indeed $F_1 - Q_1 = C + 2\sum D_{ab}$, so (4) cancels the latter overlap term; all remaining F_k are nonnegative.

Lemma 5 (Matching bound)**.** *Let $z_{ij} \geq 0$ be edge weights of a finite loopless graph, with weighted degrees $h_i = \sum_j z_{ij} \leq \delta_i$. Then*

$$\sum_{ij\in E} z_{ij}^2 \leq \max_{M \text{ a matching}} \sum_{ij\in M} \delta_i\delta_j.$$

Proof. Discard zero-degree vertices and set $B_{ij} = z_{ij}/\sqrt{h_i h_j}$, $B_{ii} = 0$. Weighted $2ab \leq a^2 + b^2$ implies $|v^T B v| \leq \sum_i v_i^2$, so B is a real symmetric contraction. Set $y_{ij} = B_{ij}^2$. Then $\sum_j y_{ij} = (B^2)_{ii} \leq 1$.

For an odd vertex set of size m, its principal compression T is a symmetric contraction of trace zero. Its eigenvalues lie in $[-1,1]$ and sum to zero. The convex function $\sum \lambda_i^2$ on this compact polytope has a maximum at an extreme point, where at most one coordinate is interior. Since m is odd, the other $m-1$ endpoint coordinates sum to an even integer and the remaining coordinate must be zero. Hence $\operatorname{Tr} T^2 \leq m-1$ and

$$\sum_{ij\in E(S)} y_{ij} \leq (m-1)/2.$$

These are exactly the degree and odd-set inequalities of Edmonds's matching-polytope theorem [6]. Thus y is a convex combination of matching incidence vectors, and

$$\sum z_{ij}^2 = \sum h_i h_j y_{ij} \leq \max_M \sum_{ij\in M} h_i h_j \leq \max_M \sum_{ij\in M} \delta_i\delta_j.$$

No division is made at a zero degree. The lemma is an application of the classical matching-polytope theorem, with the required spectral odd-set verification supplied explicitly. □

Column stars give $\sum_{b\neq a} D_{ab} \le \delta_a$. Lemma 5 and (6) therefore imply

$$Q_2 \le e_2(R) + M(\delta), \tag{9}$$

where M is the maximum weighted matching sum on the row vertices. For four rows this is the maximum over three perfect pairings; for five rows over fifteen two-edge matchings. Nonnegative weights allow shorter matchings to be extended.

The square Carlen–Lieb–Loss inequality [4, Theorem 1.1] gives

$$|\operatorname{per} M_k| \le \frac{k!}{k^{k/2}} \prod_{a=1}^{k} \|(M_k)_{a,*}\|_2. \tag{10}$$

Under stars and the cap $w_{aj} \le 7/10$, concentration of each nonnegative row gives

$$0 \le R_a \le f(r_a) \le r_a, \qquad f(t) = \begin{cases} t^2, & 0 \le t \le 7/10, \\ 49/100 + (t-7/10)^2, & 7/10 \le t \le 1. \end{cases} \tag{11}$$

At most one filled cap and one remainder are needed since $r_a \le 1 < 2(7/10)$. In particular $R_a \le 29/50$. Applying (10) to W minors and using their nonnegative distinct-column sum gives

$$Q_3 \le g_3 e_3(r), \quad Q_4 \le g_4 e_4(r), \quad Q_5 \le g_5 e_5(r), \quad (g_3, g_4, g_5) = \left(\frac{511}{1000}, \frac{2523}{5000}, \frac{11}{20}\right). \tag{12}$$

The bounds for individual minors are $k!(29/50)^{k/2}/k^{k/2}$. The fourth cap is exact; the squared safety margins of the third and fifth caps are 583/600000 and 992213/19531250000. Unavailable orders vanish.

Lemma 6 (Three-row collision identity)**.** *For any three selected rows, write $r_a = G_{aa}$, $g_{ab} = G_{ab}$ and let c denote the remaining row in a pair a, b. Set $T = \sum_j w_{0j} w_{1j} w_{2j}$ and $H_{ab} = \sum_j w_{cj} x_{aj} \overline{x_{bj}}$. The contribution F_3 of that row triple satisfies*

$$F_3 = \operatorname{per} G - 2 \sum_{a<b} r_c D_{ab} - 4\Re \sum_{a<b} \overline{g_{ab}} H_{ab} + 12T. \tag{13}$$

Proof. Expand the Gram permanent by column multisets with multiplicity factorials. Distinct columns contribute F_3. A double column j and different column k contribute $2|\sum_a x_{ak} \prod_{b\neq a} x_{bj}|^2$; a triple column contributes $6\prod_a w_{aj}$. Extending the double-column sum to $k = j$ adds $18\prod_a w_{aj}$. Hence

$$F_3 = \operatorname{per} G - 2 \sum_{j,k} \left| \sum_a x_{ak} \prod_{b\neq a} x_{bj} \right|^2 + 12T.$$

The diagonal part of the extended square sums to $\sum r_c D_{ab}$ and the cross part to $2\Re \sum \overline{g_{ab}} H_{ab}$, proving (13). The identity is polynomial and needs no operator bound. □

For an operator contraction $0 \le G \le I$, so $|g_{ab}| \le 1/2$. For $u, v, z \in [0, 1/2]$, AM–GM gives $2uvz \le (u^2 + v^2 + z^2)/3$. Therefore

$$2\Re(g_{ab} g_{bc} g_{ca}) \ge -\frac{1}{3}(|g_{ab}|^2 + |g_{bc}|^2 + |g_{ca}|^2). \tag{14}$$

4 The three-row structural core

We include the published three-row argument [9] to make the support induction explicit.

Proposition 7. *Every complex $3 \times q$ operator contraction satisfies $Q \le P$.*

Proof. Retain the notation of Lemma 6. Put $Z_k = \sum_{|I|=|J|=k} \operatorname{per} W[I,J]$. Every nonnegative square minor has permanent at most its product of row sums, hence at most one. Thus

$$E = \sum_{k=2}^{3} \sum_{|I|=|J|=k} \operatorname{per} W[I,J](1 - \operatorname{per} W[I,J]) \ge 0.$$

Distinct-column inclusion–exclusion gives

$$Z_2 = e_2(r) - \sum D_{ab}, \qquad Z_3 = r_0 r_1 r_2 - \sum r_c D_{ab} + 2T.$$

Using (4), (13) and $S - \sum R_a = C + 2\sum D_{ab}$, direct substitution yields

$$\begin{aligned} P - Q = E + C + \sum_{a<b}(1 - r_c)D_{ab} + \sum_{a<b}(1 + r_c)|g_{ab}|^2 \\ + 2\Re(g_{01}g_{12}g_{20}) - 4\Re\sum_{a<b}\overline{g_{ab}}H_{ab} + 10T. \end{aligned}$$

Weighted Cauchy gives $|H_{ab}|^2 \le r_c T$. Put $a_c = 2/3 + r_c$. By (14) and completion of the square,

$$a_c|g_{ab}|^2 - 4\Re(\overline{g_{ab}}H_{ab}) = a_c|g_{ab} - 2H_{ab}/a_c|^2 - 4|H_{ab}|^2/a_c.$$

Since $4r_c/a_c \le 12/5$, summing subtracts at most $36T/5$ from $10T$. Consequently

$$P - Q \ge E + C + \sum(1 - r_c)D_{ab} + \sum a_c|g_{ab} - 2H_{ab}/a_c|^2 + \frac{14}{5}T \ge 0.$$

The strictly positive a_c causes no division at a zero row. Zero columns and missing orders are included. □

5 A dimension-independent large-entry transfer

The following reproduces the established border mechanism of [10, Lemma 4]; its stated core hypothesis allows two successive support steps without circularity.

Lemma 8. *Let $p \ge 2$, and let X be any complex $p \times q$ operator contraction. Suppose its core D, obtained by deleting one row and column, satisfies $\mathcal{Q}(A(D)) \le \mathcal{P}(A(D))$. If the deleted entry has squared magnitude $t \ge 7/10$, then*

$$P - Q \ge \frac{7}{2000}(1 - t)\mathcal{Q}(A(D)) \ge 0.$$

The non-strict conclusion also holds when $t \ge (5 - \sqrt{13})/2$.

Proof. Place the entry first, and write $X = \left(\begin{smallmatrix} a & b \\ c & D \end{smallmatrix}\right)$, $t = |a|^2$, $s = 1 - t$, $u_j = |b_j|^2$, $v_i = |c_i|^2$, $U_b = \sum u_j$, $V_b = \sum v_i$. Contraction implies $U_b, V_b \le s$ and D is contractive. For $q = 1$, the support is a star and $P - Q = \sum_a w_a(1 - w_a) \ge t(1 - t)$, while $\mathcal{Q}(A(D)) = 1$. Hence assume $q \ge 2$.

Partition the actual positive semidefinite $A(X)$ into the two endpoints of a and the complementary $A(D)$. Lieb's block permanent inequality [5] and the core hypothesis give

$$P \ge (1 + t)\mathcal{P}(A(D)) \ge (1 + t)\mathcal{Q}(A(D)). \tag{15}$$

Put $W = |D|^2$. For fixed t, W, v, every minor permanent of the nonnegative rectangle $\left(\begin{smallmatrix} t & u \\ v & W \end{smallmatrix}\right)$ is affine in u. Its square is convex; hence their sum is convex in u, and similarly in v at fixed u. Successive maximization over the fixed-total simplices bounds Q by the largest value with $u = U_b e_j$, $v = V_b e_i$. These auxiliary concentrated rectangles need not be contractions; only the original matrix was used in (15).

Let $d_{I,J} = \operatorname{per} W[I, J]$, over equal-cardinality core subsets, including the empty pair. Then $\|d\|^2 = \mathcal{Q}(A(D))$. Let C_j and R_i add the indicated column or row index to a subset when absent and vanish when already occupied; their individual targets may have unequal cardinalities. They are partial isometries. Let $L = R_i C_j$, which adds both indices. Direct expansion of all concentrated bordered minors gives

$$\begin{aligned} Q_{\mathrm{conc}} = (1 + t^2)\|d\|^2 + U_b^2\|C_j d\|^2 + V_b^2\|R_i d\|^2 \\ + U_b^2 V_b^2\|Ld\|^2 + 2tU_b V_b \Re\langle d, Ld\rangle. \end{aligned} \tag{16}$$

Indeed a minor containing both border indices has permanent $td_{I,J} + U_b V_b d_{I\setminus\{i\},J\setminus\{j\}}$ when both selected core indices are present. Single-border minors give the two single-creation norms.

Group coordinates according to occupation of i, j. For fixed remaining subsets, the extra quadratic form in (16) has the 00/11 block

$$B(U_b, V_b) = \begin{pmatrix} U_b^2 + V_b^2 + U_b^2 V_b^2 & tU_b V_b \\ tU_b V_b & 0 \end{pmatrix};$$

the other occupation types have diagonal U_b^2 or V_b^2. Missing states give compressions. The largest eigenvalue

$$\lambda(U_b, V_b) = \frac{A + \sqrt{A^2 + 4t^2 U_b^2 V_b^2}}{2}, \qquad A = U_b^2 + V_b^2 + U_b^2 V_b^2,$$

dominates those diagonal values and increases in both nonnegative variables. Consequently

$$Q \le [1 + t^2 + \lambda(s, s)]\mathcal{Q}(A(D)).$$

Set $M = tsI - B(s, s)$. Direct calculation yields

$$\det M = ts^2(1 - 3s - s^2), \qquad \operatorname{Tr} M = s(2 - 4s - s^3) \le 2s.$$

For $0 < s \le 3/10$, the lower-right entry ts is positive and $1 - 3s - s^2 \ge 1/100$, so M is positive definite. Its least eigenvalue is at least determinant divided by trace, hence

$$ts - \lambda(s, s) \ge \frac{ts^2/100}{2s} \ge \frac{7}{2000}s.$$

Combining this with (15) proves the quantitative result. For $s = 0$ the border terms vanish directly. The non-strict extension follows from $1 - 3s - s^2 \ge 0$, namely $t \ge (5 - \sqrt{13})/2$, with the same diagonal test; no division at zero slack is required. □

6 All finite-column four-row supports

Proposition 9. *Every complex $4 \times q$ rectangle satisfying stars and $w_{aj} \le 7/10$ satisfies $Q \le P$, without assuming operator contraction.*

For each perfect pairing π of the four rows define

$$B_\pi(r,R) = \frac{1}{2}\sum_a r_a(1-r_a) + e_2(r) - e_2(R) - \sum_{\{a,b\}\in\pi}(r_a - R_a)(r_b - R_b) - g_3 e_3(r) - g_4 e_4(r). \quad (17)$$

Equations (8), (9) and (12) give $P - Q \ge \min_\pi B_\pi$. We prove every such polynomial nonnegative on $0 \le r_a \le 1$, $0 \le R_a \le f(r_a)$.

Paired deletion and separate concavity

Fix π, r. The scalar (17) is separately affine in R, so its minimum occurs at endpoints $R_i = 0$ or $R_i = f(r_i)$. Mark rows according to the chosen endpoint; the two choices may coincide when $r_i = 0$. Hypothetical zero endpoints need not be realizable matrix rows. We first delete zero-marked partners together, then any remaining zero-marked row. A deleted row is set to $r_i = R_i = 0 = f(0)$.

Suppose two zero-marked partners have row sums t, u, and the other rows have sums v, w. The exact difference between the scalar before and after deleting the pair is

$$\begin{aligned}&\frac{1}{2}\big[t(1-t) + u(1-u)\big] + (t+u)(v+w)\\ &\quad - g_3\big[tu(v+w) + (t+u)vw\big] - g_4 tuvw. \end{aligned} \quad (18)$$

The tu term of $e_2(r)$ cancels the paired δ term because both R endpoints are zero. Since $2tu \le t+u$ and $2vw \le v+w$, (18) is at least

$$\frac{1}{2}\big[t(1-t)+u(1-u)\big] + (t+u)(v+w)(1-g_3-g_4/4) = \frac{1}{2}\big[t(1-t)+u(1-u)\big] + \frac{7257}{20000}(t+u)(v+w) \ge 0. \quad (19)$$

Thus all zero-marked partner pairs may be removed, regardless of the other endpoints.

Any remaining zero-marked row now has an f-marked partner. Write its row sum as t, its partner's as p, and the other two as u, v. Here $R_p = f(p)$. Deleting t changes the scalar by $t(1-t)/2 + tL(p,u,v)$, where

$$L = f(p) + (u+v) - g_3[p(u+v) + uv] - g_4 puv \ge f(p) + (u+v)\big[1 - g_3/2 - p(g_3 + g_4/2)\big].$$

For $0 \le p \le 1/2$, the bracket is at least $1 - g_3 - g_4/4 = 7257/20000 > 0$. For $1/2 \le p \le 1$, it is at least $1 - 3g_3/2 - g_4/2 = -47/2500$, while $f(p) \ge 1/4$ because f is increasing and continuous. Since $u + v \le 2$, this case gives $L \ge 1/4 - 94/2500 = 531/2500 > 0$. Every remaining zero-marked row can therefore be deleted safely. This argument does not assume that arbitrary single-row deletion is valid before partner pairs are removed.

It remains to prove $J_\pi(r) := B_\pi(r, f(r)) \ge 0$. Hold the other rows fixed and vary $t = r_i$, paired with p. The coefficient of $f(t)$ is

$$L = \delta_p - \sum_{a\ne i} f(r_a) \le p - 2f(p) \le \frac{1}{8}.$$

Indeed on the lower interval $p - 2p^2 = 1/8 - 2(p - 1/4)^2$, and on the upper interval $p - 2f(p) = -2(p - 19/20)^2 - 31/200$. Apart from $Lf(t)$, coordinate dependence is linear minus $t^2/2$, since $e_3(r), e_4(r)$ are separately linear. On each of the two intervals,

$$J''_\pi(t) = -1 + 2L \le -3/4.$$

Separate concavity on the piecewise boxes reduces the minimum to $r_a \in \{0, 7/10, 1\}$. Table 2 prints the minimum over the three pairings for each count pattern. These fifteen nonnegative rational values cover all 81 labelled corners and 243 pairing evaluations. Thus $J_\pi \ge 0$, then the deletion steps and affinity prove every $B_\pi \ge 0$ on the enlarged box. This proves Proposition 9 for every finite q, including $q = 1, 2$ where higher minor orders vanish.

Table 2: Exact corner minima of $B_\pi(r, f(r))$; n_0, n_7, n_1 count coordinates $0, 7/10, 1$.

n_0	n_7	n_1	$\min_\pi B_\pi$
0	0	4	5401/5000
0	1	3	7133/6250
0	2	2	551383/500000
0	3	1	1350349/1250000
0	4	0	50447677/50000000
1	0	3	6517/5000
1	1	2	10661/10000
1	2	1	95291/100000
1	3	0	845327/1000000
2	0	2	609/1250
2	1	1	2163/5000
2	2	0	2079/5000
3	0	1	0
3	1	0	21/200
4	0	0	0

Four-row assertion of Theorem 1. In the capped domain Proposition 9 applies. Otherwise Lemma 8 applies with its three-row core hypothesis supplied by Proposition 7. This proves all $4 \times q$ contractions, including the common cap boundary. A smaller number of rows is included by padding zero rows; identity padding preserves both permanents. □

7 Five-row coupling and unbounded-column regions

In this section there are five rows. The star assumptions imply that W itself is an operator contraction. In fact, weighted Cauchy–Schwarz gives

$$\sum_a \left| \sum_j w_{aj} v_j \right|^2 \le \sum_a r_a \sum_j w_{aj} |v_j|^2 \le \sum_j c_j |v_j|^2 \le \sum_j |v_j|^2.$$

Thus $H = WW^T$ is a positive semidefinite contraction, with diagonal R_a and off-diagonal D_{ab}. Set $U = \sum_a R_a$. Its trace inequality gives

$$\sum_{a<b} D_{ab}^2 \le \frac{1}{2}\left(U - \sum_a R_a^2\right). \tag{20}$$

Combining (6) with (20) and $\sum R_a^2 \ge U^2/5$ yields

$$Q_2 \le \frac{3}{10}U^2 + \frac{1}{2}U. \tag{21}$$

Lemma 10 (Five-row third-sector budget)**.** *If X is an operator contraction, then for every $0 \le \theta \le 1$,*

$$P - Q \ge \frac{1-\theta}{2}\sum_a r_a(1-r_a) + e_2(r) + \theta e_3(r) - \sum_{k=2}^{5} Q_k - 2\theta \sum_{a<b} A_{ab}D_{ab}, \tag{22}$$

where $A_{ab} = \sum_{c\notin\{a,b\}} r_c$.

Proof. Write $g_{ab} = G_{ab}$, $T_{abc} = \sum_j w_{aj}w_{bj}w_{cj}$, $T = \sum_{a<b<c} T_{abc}$, and

$$H_{ab} = \sum_j \left(\sum_{c\notin\{a,b\}} w_{cj}\right) x_{aj}\overline{x_{bj}}.$$

The three-row collision identity, summed over row triples, is

$$\begin{aligned} F_3 = e_3(r) &+ \sum_{a<b} A_{ab}|g_{ab}|^2 + 2\Re \sum_{a<b<c} g_{ab}g_{bc}g_{ca} \\ &- 2\sum_{a<b} A_{ab}D_{ab} - 4\Re\sum_{a<b} \overline{g_{ab}}H_{ab} + 12T. \end{aligned} \tag{23}$$

Identity (23) follows by summing Lemma 6 over row triples.

Since $0 \le G \le I$, every $|g_{ab}| \le 1/2$. For $x, y, z \in [0, 1/2]$, AM–GM gives $2xyz \le (x^2+y^2+z^2)/3$. Thus each cubic Gram cycle is at least minus one third of its three pair squares. Each pair belongs to three triples, so their total contribution is at least $-O$.

Weighted Cauchy–Schwarz gives

$$|H_{ab}|^2 \le A_{ab} \sum_{c\notin\{a,b\}} T_{abc}.$$

For $A_{ab} > 0$, completion of the square yields

$$A_{ab}|g_{ab}|^2 - 4\Re(\overline{g_{ab}}H_{ab}) \ge -4|H_{ab}|^2/A_{ab} \ge -4\sum_c T_{abc}.$$

If $A_{ab} = 0$, the corresponding H, T vanish, so the same conclusion holds directly. Summing consumes at most $12T$. Combining with (4) now gives

$$F_2 + \theta F_3 \ge e_2(r) + \theta e_3(r) + (1-\theta)O - 2\sum D_{ab} - 2\theta\sum A_{ab}D_{ab}.$$

All unretained F_k are nonnegative. Finally $C + (1-\theta)O \ge (1-\theta)(C+O)$, and the shared trace identity proves (22). Zero rows and missing minor orders are included. □

For five capped rows, (12) gives the shared higher-sector allowance

$$Q_3 + Q_4 + Q_5 \le 10g_3 + 5g_4 + g_5 = \frac{8183}{1000}. \tag{24}$$

Proposition 11. *For a five-row operator contraction with squared entry cap* $7/10$*, unit row norms imply* $P - Q \ge 451/3000$*. More generally,* $d := \sum_a (1 - r_a) \le 1/50$ *implies* $P - Q \ge 739/75000$*.*

Proof. The product inequality $\prod_{i\in I}(1 - d_i) \ge 1 - \sum_{i\in I} d_i$, for $0 \le d_i \le 1$, implies

$$e_2(r) \ge 10 - 4d, \qquad e_3(r) \ge 10 - 6d.$$

Also $A_{ab} \le 3$ and $2\sum D_{ab} = \sum_j c_j^2 - U \le 5 - U$. Taking the better of $\theta = 0, 1$ in Lemma 10, and using (21) and (24), gives

$$P - Q \ge 10 - 4d + \max\{0, 3U - 5 - 6d\} - \frac{3}{10}U^2 - \frac{1}{2}U - \frac{8183}{1000}. \tag{25}$$

The cap ensures $0 \le U \le 29/10$. Put $b = 5/3+2d$. For $U \le b$, the quadratic allowance $3U^2/10+U/2$ increases, so (25) is at least

$$\frac{451}{3000} - 7d - \frac{6}{5}d^2.$$

For $b \le U \le 29/10$, the lower budget is the concave quadratic

$$5 - 10d + \frac{5}{2}U - \frac{3}{10}U^2 - \frac{8183}{1000}.$$

Its value at b is the previous expression and its value at $29/10$ is $193/125 - 10d$. For $0 \le d \le 1/50$ we have $b < 29/10$, the first expression is at least $739/75000$, and the second endpoint is at least $168/125$. This proves the deficit claim. At $d = 0$ the same endpoints are $451/3000$ and $193/125$, proving the unit-row margin. □

The large-entry lemma in Section 5 removes the entry cap from the non-strict conclusion. If all $r_a = 1$ and $G \le I$, then $I - G$ is positive semidefinite with zero diagonal, hence zero; thus $XX^* = I_5$. For a square matrix this says precisely that X is unitary. Proposition 11 and that lemma prove Theorem 3.

Proof of Theorem 4. The overlap degrees satisfy

$$\sum_{b\ne a} D_{ab} = \sum_j w_{aj}(c_j - w_{aj}) \le r_a - R_a.$$

Equation (9) implies

$$Q_2 \le e_2(R) + M(r - R) \le e_2(R) + M(1 - R),$$

where $M(v)$ is the maximum sum of the two disjoint pair products on five vertices. Nonnegative weights allow shorter matchings to be extended; there are $5\cdot 3 = 15$ two-edge matchings. For a fixed such matching the expression is separately affine in $R \in [0, 1/2]^5$. At a vertex let k be the number of half-valued coordinates, $s \in \{0, 1\}$ indicate whether the unmatched coordinate is half-valued, and ℓ count matching pairs both half-valued. Its value is

$$\frac{k(k-1)}{8} + 2 - \frac{k - s}{2} + \frac{\ell}{4}.$$

The maxima for $k = 0, 1, 2, 3, 4, 5$ are respectively $2, 2, 7/4, 2, 9/4, 3$. Thus $Q_2 \le 3$. This is also a direct explanation of the $15 \cdot 32 = 480$ labelled corner evaluations.

Under $R_a \le 1/2$, (10) gives individual W-minor bounds $1/\sqrt{6} < 49/120$, $3/8$, and $6/(5\sqrt{10}) < 19/50$ for orders $3, 4, 5$. The squared rational margins for the two strict roundings are $1/14400$ and $1/2500$. Summing minor permanents before squaring gives

$$Q_3 + Q_4 + Q_5 \le 10\frac{49}{120} + 5\frac{3}{8} + \frac{19}{50} = \frac{3803}{600}.$$

Equation (8) proves the stated result. If each $r_a \ge 97/100$, then $e_2(r) \ge 10(97/100)^2$, giving $53/750$ after discarding the nonnegative row-deficit term. At unit row norms the displayed bound is $397/600$. □

8 The five-by-five finite-moment certificate

We now prove the remaining strong assertion of Theorem 1. Throughout this section X is a 5×5 operator contraction. Lemma 8 and the four-row assertion already proved cover every squared entry at least $7/10$. We may therefore assume the cap $w_{aj} \le 7/10$.

8.1 A polynomial lower budget with finite-column moments

The number of columns now matters. Cauchy–Schwarz and (11) give the necessary moment bounds

$$\frac{r_i^2}{5} \le R_i \le f(r_i), \qquad 0 \le r_i \le 1. \tag{26}$$

The following argument proves positivity on this entire enlarged moment box. It does not require every profile in the box to be realizable by a matrix.

Five-column Cauchy–Schwarz also gives $E_{ab} \ge D_{ab}^2/5$. Hence (6) and Lemma 5 imply

$$Q_2 \le e_2(R) + \frac{3}{5}M(\delta), \qquad \delta_i = r_i - R_i, \tag{27}$$

where M is the maximum of the fifteen two-edge matching sums. There is exactly one full fifth-order minor. Squaring (10) directly on that minor yields

$$Q_5 \le \frac{576}{125}e_5(R). \tag{28}$$

Unlike the constant fifth-sector allowance in (12), this retains its full row-moment dependence and uses exactly five columns.

Let $y_i = \max\{0, S/2 - r_i\}$. Since $y_a + y_b \ge S - r_a - r_b = A_{ab}$ and the overlap degrees are at most δ_i,

$$\sum_{a<b} A_{ab}D_{ab} \le \sum_i \delta_i y_i.$$

Take $\theta = 1/2$ in (22), and use (12), (27) and (28). This proves $P - Q \ge B(r, R)$, where

$$\begin{aligned} B = {} & \frac{1}{4}\sum_i r_i(1 - r_i) + e_2(r) - \frac{11}{1000}e_3(r) - \frac{2523}{5000}e_4(r) \\ & - e_2(R) - \frac{3}{5}M(\delta) - \frac{576}{125}e_5(R) - \sum_i \delta_i \max\{0, S/2 - r_i\}. \end{aligned} \tag{29}$$

It remains to certify $B \geq 0$ throughout (26).

For a fixed matching π define $B_{\pi,-1}$ by replacing M in (29) with its fixed matching sum and replacing the last maximum by the signed expression $S/2 - r_i$. For $i = 0, \ldots, 4$ set

$$B_{\pi,i} = B_{\pi,-1} - \delta_i(r_i - S/2). \tag{30}$$

At most one row has $r_i > S/2$. Since $\delta_i \geq 0$, the nondominant corrections in (30) are nonpositive. It follows exactly that

$$B(r, R) = \min_{\pi,\, b\in\{-1,0,1,2,3,4\}} B_{\pi,b}(r, R). \tag{31}$$

If a row dominates, its correction removes precisely the negative signed-dual entry; if none dominates, the signed branch is the minimum. This identity includes balanced/dominant boundaries, without pretending that a signed potential is feasible on a dominant-row region.

Each $B_{\pi,b}$ is separately affine in R. In particular, matching edges are disjoint and $e_5(R)$ is multiaffine. At each fixed r its minimum over (26) is therefore attained by independently choosing $R_i = r_i^2/5$ or $R_i = f(r_i)$. No separate concavity after endpoint substitution is assumed.

Every matching can be relabelled to $\pi_0 = \{\{0, 1\}, \{2, 3\}\}$, with vertex 4 unmatched. Its stabilizer is transitive on the four matched vertices and fixes vertex 4. As all ordered row boxes and endpoint choices will be enumerated, it suffices to check π_0 with branches $b = -1, 0, 4$. This accounts for all fifteen matchings and all six branches in (31).

8.2 Exact Bernstein verification protocol

Split each row interval at $7/10$. For $\epsilon \in \{0, 1\}^5$, use $[0, 7/10]$ when $\epsilon_i = 0$, and $[7/10, 1]$ otherwise. On each of these 32 closed boxes, substitute any of the 32 moment endpoint choices into $B_{\pi_0,b}$ for $b = -1, 0, 4$. There are 3072 resulting polynomials. Each has coordinate degree at most three: R_i has degree two, and only its multiplication by its own row in the signed or corrected dual term can raise that degree to three. The fifth-moment product has degree two in each coordinate.

Map each row interval affinely to $t_i \in [0, 1]$. The tensor Bernstein representation of coordinate degree three is

$$p(t) = \sum_{\alpha\in\{0,1,2,3\}^5} \beta_\alpha \prod_{i=0}^{4} \binom{3}{\alpha_i} t_i^{\alpha_i}(1 - t_i)^{3-\alpha_i}. \tag{32}$$

Its basis functions are nonnegative and sum to one. Thus nonnegative coefficients prove positivity on the entire closed box, including zero endpoints.

Here the coefficients are recovered by exact interpolation, not floating evaluation. In one coordinate use nodes $0, 1/3, 2/3, 1$. Their Bernstein evaluation matrix is $B_{27}/27$, where

$$B_{27} = \begin{pmatrix} 27 & 0 & 0 & 0 \\ 8 & 12 & 6 & 1 \\ 1 & 6 & 12 & 8 \\ 0 & 0 & 0 & 27 \end{pmatrix}, \qquad M_6 = \begin{pmatrix} 6 & 0 & 0 & 0 \\ -5 & 18 & -9 & 2 \\ 2 & -9 & 18 & -5 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

Since $B_{27}M_6 = 162I$, its inverse is $M_6/6$. Applying M_6 successively along the five tensor axes to the nodal values therefore produces $6^5\beta_\alpha$. The known degree bound makes this interpolation an identity.

For a completely specified integer calculation, write $r_i = n_i/30$ at a node. If its row interval is lower, $n_i = 7m_i$; on an upper interval, $n_i = 21 + 3m_i$, where $m_i \in \{0, 1, 2, 3\}$. Write $R_i = N_i/4500$:

$$N_i = \begin{cases} n_i^2, & \text{lower moment endpoint,} \\ 5n_i^2, & \text{upper endpoint, lower row interval,} \\ 5[441 + (n_i - 21)^2], & \text{upper endpoint, upper row interval.} \end{cases}$$

Put $d_i = 150n_i - N_i$, $s = \sum_i n_i$, and $L = 125 \cdot 4500^5$. The exact nodal numerator $LB_{\pi_0,-1}$ is

$$\begin{aligned} &\frac{L}{3600}\sum_i n_i(30 - n_i) + \frac{L}{900}e_2(n) - \frac{11L}{1000 \cdot 30^3}e_3(n) - \frac{2523L}{5000 \cdot 30^4}e_4(n) \\ &\quad - \frac{L}{4500^2}e_2(N) - \frac{3L}{5 \cdot 4500^2}(d_0d_1 + d_2d_3) \\ &\quad - \frac{576L}{125 \cdot 4500^5}\prod_i N_i - \frac{L}{270000}\sum_i d_i(s - 2n_i). \end{aligned} \tag{33}$$

For branch $b = 0$ or 4, subtract $Ld_b(2n_b - s)/270000$. All displayed scalar coefficients are integers. Applying M_6 on the five axes gives coefficient numerators with common denominator

$$L6^5 = 17936133750000000000000000.$$

The standalone supplement `code/verify_bipartite_certificate.py` reconstructs this formula with arbitrary-size integer arithmetic, applies these five exact basis transforms, and rejects any negative coefficient or incomplete case count. The result is summarized in Table 3. The minimum strictly positive coefficient over all cases is exactly $1/40$; 160 coefficients are zero. Zero coefficients are retained and cause no strict-positivity assumption. The program uses the lexicographic order

Table 3: Exact coefficient verification for the three symmetry-representative branches. Each polynomial has 1024 coefficients.

branch	polynomials	coefficients	zero coefficients
-1	1024	1048576	32
0	1024	1048576	64
4	1024	1048576	64
all	3072	3145728	160

of region bits, moment endpoint bits, branches $(-1, 0, 4)$ and tensor coefficient indices. It prints the canonical coefficient digest and checks the complete enumeration. No external data, numerical optimizer or input matrix is needed. The supplement also checks the small rational tables used elsewhere in this paper.

Every coefficient is nonnegative, so (32) proves every endpoint polynomial nonnegative. Separate affinity extends this to all moments in (26); (31) proves $B \geq 0$, and (29) gives $P \geq Q$ in the capped domain. The large-entry transfer, with the already established four-row core, covers the complement and the shared boundary. This proves the 5×5 assertion of Theorem 1.

9 Normalization, support consequences and limitations

For positive-diagonal positive semidefinite B with bipartite support, write $B = EA(X)E$, where $E_{ii} = \sqrt{b_{ii}}$. The within-part blocks are identities and their Schur complement gives $\|X\|_{\mathrm{op}} \leq 1$.

Thus stars hold and $S \le \min(p, q)$. Permanent scaling gives

$$\operatorname{per} B = \mathrm{d}(B)P, \qquad \operatorname{per}|B|^2 = \mathrm{d}(B)^2 Q.$$

Consequently the normalized strong assertion is $\operatorname{per}|B|^2 \le \mathrm{d}(B)\operatorname{per} B$, and the ordinary assertion scales identically on both sides. A zero diagonal of a positive semidefinite matrix forces a zero row and column; then both permanents are zero. Zero-column rectangles, when used as compressed cores, have $P = Q = 1$. All singular and missing-edge cases are therefore included.

For at most ten vertices a bipartition either has a part of size at most four or has both parts of size five. Padding zero rows or columns, or interchanging parts, invokes Theorem 1. For at most twenty-five vertices a bipartition has smaller part at most twelve, so Theorem 2 applies. These are different strengths of conclusion.

For two positive semidefinite matrices of the same order, if *each* satisfies a proved ordinary self-bound, permutation Cauchy–Schwarz gives

$$|\operatorname{per}(B \circ C)|^2 \le \operatorname{per}|B|^2 \operatorname{per}|C|^2 \le \operatorname{per}(B)^2 \operatorname{per}(C)^2.$$

Their bipartitions need not coincide. This is a both-factor assertion, not a theorem with an arbitrary second factor. A strong class also satisfies its ordinary self-bound, by positive semidefinite permanent dominance $\operatorname{per}(B) \ge \mathrm{d}(B)$ (or the block-minor expansion after normalization).

The finite $K_{5,5}$ certificate uses the moment lower endpoint $r_i^2/5$ and the unique full fifth minor. The unrestricted five-row family is covered only on the regions in Theorems 3 and 4. The ordinary endpoint σ is sharp only for the stated coarse-bound comparison.

Reproducibility and acknowledgment

The all-four-row, energy and auxiliary five-row arguments are analytic, with their small rational tables printed or explicitly described. The full five-by-five result is computer-assisted through the finite exact Bernstein calculation in Section 8. Run `python -B` followed by `code/verify_bipartite_certificate.py`; the same explicit acceptance checks remain active under Python's optimized mode. The standalone checker uses only the standard library, requires no external certificate file and writes no files. Its finite calculations verify the stated coefficients and arithmetic, not the classical input theorems or the continuum reductions by themselves. AI tools assisted mathematical research, drafting and verification. The author is responsible for the statements, proofs, attribution and final text.

References

[1] L. B. Beasley, An inequality on permanents of Hadamard products, *Bulletin of the Korean Mathematical Society* **37** (2000), no. 3, 633–639.

[2] J. Chollet, Is there a permanental analogue to Oppenheim's inequality?, *American Mathematical Monthly* **89** (1982), 57–58.

[3] S. W. Drury, A counterexample to a question of Bapat and Sunder, *Electronic Journal of Linear Algebra* **31** (2016), 69–70. `https://journals.uwyo.edu/index.php/ela/article/view/1631`.

[4] E. Carlen, E. H. Lieb and M. Loss, An inequality of Hadamard type for permanents, *Methods and Applications of Analysis* **13** (2006), no. 1. `https://doi.org/10.4310/MAA.2006.v13.n1.a1`. Square form: arXiv:math/0508096v1, Theorem 1.1, `https://arxiv.org/html/math/0508096v1`.

[5] E. H. Lieb, Proofs of some conjectures on permanents, *Journal of Mathematics and Mechanics* **16** (1966), 127–134.

[6] J. Edmonds, Maximum matching and a polyhedron with 0,1-vertices, *Journal of Research of the National Bureau of Standards, Section B* **69B** (1965), 125–130. https://doi.org/10.6028/jres.069B.013.

[7] P. Pant and R. Singh, Structural Classes for Chollet's Permanent Conjecture, arXiv:2604.24192v2 (2026). https://arxiv.org/abs/2604.24192v2.

[8] S. Sa-nguansin and K. Rodtes, Permanents of correlation matrices and the Chollet permanental conjecture, *Linear and Multilinear Algebra* **73** (2025), no. 18, 4084–4096. https://doi.org/10.1080/03081087.2025.2588584.

[9] S. Schubr, Strong permanent inequalities for three-row bipartite and triangle-free supports, Luminary Research Archive, first edition (2026). Archive page.

[10] S. Schubr, Strong permanent contraction on four-row bipartite supports through six columns, Luminary Research Archive, first edition (2026). Archive page.

[11] S. Schubr, Strong permanent contraction on four-by-seven bipartite supports, Luminary Research Archive, first edition (2026). Archive page.

[12] S. Schubr, Row-dependent bounds for strong permanent contraction on four-by-eight supports, Luminary Research Archive, first edition (2026). Archive page.

[13] Z. Zeng, H. Liu and K. Ratnavelu, Sharp Self-Chollet Inequalities in Order Five: Rank Two and Structured Rank Three, Preprints.org (2026), version 1, posted 18 August 2026, Section 6. https://www.preprints.org/manuscript/202608.1240/v1.