Stable stars, rank-one replacement and negative edge correlation in random forests

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Luminary Oddities

Abstract

Pick a random spanning forest of a finite graph with probability proportional to the product of its positive edge weights. A conjecture of Kahn and of Grimmett and Winkler, in its weighted form, asserts that any two edges are nonpositively correlated. We prove this for every pair of edges sharing a vertex, on every finite graph and at all positive weights: the generating polynomial of the edges at any vertex is real stable, so every star is strongly Rayleigh and every vertex degree is a sum of independent Bernoulli variables. Two incident edges are strictly negatively correlated exactly when some cycle contains both, with an explicit lower bound. For disjoint edges we develop an exact replacement principle: a weighted background attached through activities of product form $s_h r_v$ can be replaced by finitely many isolated vertices whose weights are the negated reciprocal roots of a real-rooted one-hub polynomial, preserving the joint law of all retained edges. This yields strict negative correlation of two disjoint edges whose four endpoints form a rank-one clique over an arbitrary weighted background, and hence strict negative correlation of every pair of edges of K_n with vertex-factor weights $w_{ij} = t_i t_j$, including every common activity. We also prove nonpositive pairwise correlation at all positive weights on every graph with a vertex cover of size at most three, on rank-one attachments whose replacement graph has at most seven vertices, and on graphs assembled by 2-sums from such pieces; these last results use Erickson's finite theorem for graphs on at most seven vertices.

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1 Introduction

Let G be a finite graph with positive edge weights w_e. A *forest* is a set of edges containing no cycle; isolated vertices and the empty set are allowed. The weighted random forest, also called the arboreal gas, chooses a forest F with probability

$$\mathbb{P}_G(F) = \frac{w(F)}{Z_G}, \qquad w(F) = \prod_{e \in F} w_e, \qquad Z_G = \sum_F w(F). \tag{1}$$

There are no root choices and no factors depending on component sizes. Write $X_e = \mathbf{1}_{\{e \in F\}}$. A conjecture circulated by Kahn [16] and stated by Grimmett and Winkler [12] (see also Pemantle's survey [18]) asserts that $\mathrm{Cov}(X_e, X_f) \le 0$ for any two distinct edges of any finite graph under the uniform forest measure. The weighted form asks for the same inequality at all positive weights; equivalently, the forest polynomial satisfies the *Rayleigh inequality*

$$Z_e Z_f - Z Z_{ef} \ge 0,$$

where subscripts denote derivatives in w_e, w_f. A graph with this property for all pairs and all positive weights is called *I-Rayleigh*. Cocks [7] and Erickson [10] showed that the uniform statement for all graphs implies the weighted one. The corresponding statement for spanning trees holds at all weights, through Kirchhoff's determinant and the transfer-current theorem; for forests the conjecture has been established on the graph classes listed in Table 1, by a variety of methods.

Edges at a vertex. Our first result settles the conjecture, at all positive weights and on every finite graph, for every pair of edges that share a vertex. This extends Huang's all-weight theorem for pairs of edges that series and parallel reductions identify or separate [15, Corollary 2.7] to every pair of incident edges. In fact we prove more: at each vertex v, the generating polynomial

$$S_v(z) = \sum_F w(F) \prod_{vi \in F} z_i$$

of the incident edges is real stable (Theorem 2.2). Consequently the joint law of the edges at v is strongly Rayleigh in the sense of Borcea, Brändén and Liggett [3]; it is negatively associated, stays so under conditioning, and the degree of v is distributed as a sum of independent Bernoulli variables. The proof identifies a classical component-marked forest polynomial with a stable determinantal differential operator applied to a signed matching polynomial. Earlier results for general incident pairs hold at large activity: Huang [15] proves incident negative correlation on every finite connected graph for sufficiently large common activity, and Çiçeksiz and Ravichandran [8] prove negative correlation at sufficiently large activity whenever one of the two leading coefficients of the Rayleigh difference is nonzero, together with a complete-graph random-cluster theorem for common edge weights bounded below.

We then determine exactly when the inequality for incident edges is strict: precisely when some cycle contains both edges, with an explicit lower bound in terms of any path joining their far endpoints (Theorem 3.1).

Disjoint edges and rank-one replacement. For disjoint edges we introduce an exact replacement principle (Theorem 4.2). Suppose a weighted background graph H meets the rest of the graph only through edges of product form $w_{hv} = s_h r_v$, where h runs over retained vertices and v over background vertices. Then H can be replaced by finitely many isolated vertices, one for each v with $r_v > 0$, joined to h with activities $s_h \lambda_i$; the λ_i are the negated reciprocal roots of the background's one-hub polynomial, and they are positive by the star theorem. The replacement preserves the joint law of all retained edges and all retained degrees, for an arbitrary retained graph. The proof is an exact elimination in the Grassmann representation of unrooted forests due to Caracciolo, Jacobsen, Saleur, Sokal and Sportiello [5].

The replacement leads to a strict inequality for two disjoint edges ab, cd whose endpoints carry scales t_a, t_b, t_c, t_d, with cross edges $\gamma t_h t_j$ and legs $t_h r_v$ into an arbitrary weighted background (Theorem 5.3). It includes an explicit lower bound on the Rayleigh difference. As a consequence every two edges of K_n with vertex-factor weights $w_{ij} = t_i t_j$ are strictly negatively correlated (Theorem 6.1). Constant vertex factors give every common activity. For the uniform measure on K_n this was proved by Gupta [13]; for every common activity it was proved independently by Shmalo [23] through a Riccati barrier and in the first edition of our earlier manuscript [19] through Hermite counts. The argument here is short and covers inhomogeneous vertex factors.

Small vertex covers, finite hosts and 2-sums. Section 7 proves nonpositive correlation of all pairs at all positive weights on every graph with a vertex cover of size at most three (Theorem 7.1),

for instance on $K_{3,n}$ with arbitrary edges added inside the part of size three. The proof compresses the graph onto its three cover vertices, proves one of the two disjoint cases by an elementary inequality, and reduces the other to a single pair of disjoint edges of $K_{3,3}$, where it invokes Erickson's computer-assisted theorem that every graph on at most seven vertices is I-Rayleigh [10]. The same theorem combined with rank-one replacement gives nonpositive correlation whenever the replacement graph has at most seven vertices (Corollary 7.3); for example, four vertices with arbitrary mutual weights attached in product form to an arbitrary background through at most three attachment vertices. The theorem of Wagner and of Cocks that 2-sums preserve the Rayleigh property [27, 7] then extends nonpositive correlation to every graph whose 3-connected components have at most seven vertices or a vertex cover of size at most three (Theorem 7.4).

Related work. Table 1 summarizes the results most closely related to ours. Semple and Welsh [22] proved negative correlation on series-parallel graphs, and Erickson [9] gave a weighted proof by sums of squares. Stark [24] proved negative correlation for uniform forests of K_n for all sufficiently large n. Cao and Fu [4] proved strict negative correlation for uniform forests of every $K_{m,n}$, $m, n \geq 2$; the companion paper [21] treats $K_{m,n}$ at every common activity. Bado [1] treated uniform forests on wheels. Zhao and Zhou [29] proved that every matroid of rank or corank at most three is Z-Rayleigh; letting $q \to 0$ with $y_e = qw_e$ turns their inequality into the weighted forest inequality for graphs whose cycle matroid has rank or corank at most three; for graphs this case is also contained in Theorem 7.4, since every 3-connected component of such a graph is K_4. Tang and Zhang proved negative correlation for several uniform spanning-subgraph models of large complete graphs, including forests with a prescribed number of components [25], and quantitative correlation bounds for random-cluster measures with cluster weight at most one, uniform in the remaining activities [26].

Organization. Section 2 proves star stability, and Section 3 the strict incident theorem. Section 4 proves rank-one replacement. Section 5 treats two hubs, split pairs and the endpoint clique, and Section 6 deduces the vertex-factor complete-graph theorem. Section 7 treats vertex covers of size three, finite rank-one hosts and 2-sums.

2 Optional component marks and stable stars

A real polynomial is *real stable* if it does not vanish when all its variables have strictly positive imaginary parts. Limits, real specialization and positive affine substitution preserve stability, with the identically zero polynomial allowed as a limit; these facts follow from the multivariate Hurwitz theorem [28].

Theorem 2.1 (Stable optional component marks)**.** *Let H be a finite simple graph on $\{1, \dots, d\}$ with nonnegative edge weights $C_{ij} = C_{ji}$, $C_{ii} = 0$. The polynomial*

$$\Phi_C(b) = \sum_{A \text{ forest of } H} \prod_{e \in A} C_e \prod_{B \in \mathcal{C}(A)} \Big(1 + \sum_{i \in B} b_i\Big) \tag{2}$$

is real stable. Here $\mathcal{C}(A)$ is the set of components of A, singletons included.

Expanding the product, each component is either unmarked, with contribution one, or carries exactly one marked vertex i, with contribution b_i. This component-marked forest identity is the graph specialization of Caracciolo, Sokal and Sportiello [6, Theorem 5.4], with $t_i = 1 + b_i$ and edge parameter one, since a tree has one more vertex than edge. We reconstruct it in a form suited to stability.

Graphs or pairs	Weights	Reference
Series-parallel graphs	uniform; all positive	Semple–Welsh [22]; Erickson [9]
All graphs on at most 7 vertices	all positive	Erickson [10] (computer-assisted)
2-sums of Rayleigh matroids	all positive	Wagner [27]; Cocks [7]
Pairs with a bridge, or identified by series–parallel reduction	all positive	Huang [15, Cor. 2.7]
Incident pairs, connected graphs	large common activity	Huang [15]
Pairs with a nonzero leading Rayleigh coefficient	large activity	Çiçeksiz–Ravichandran [8]
Incident pairs, all graphs	**all positive**	**Theorem 2.2**
K_n, large n	uniform	Stark [24]
K_n, every n	uniform	Gupta [13]
K_n, every n	every common activity	Shmalo [23]; [19]
K_n, every n	**vertex factors t_it_j**	**Theorem 6.1**
$K_{m,n}$	uniform	Cao–Fu [4]
$K_{m,n}$	every common activity	companion paper [21]
Wheels	uniform	Bado [1]
Rank or corank at most 3	all positive	Zhao–Zhou [29]
Vertex cover at most 3	**all positive**	**Theorem 7.1**
2-sums of the above pieces	**all positive**	**Theorem 7.4**

Table 1: Negative correlation for random forests: results related to this paper. Bold rows are proved here.

2.1 A coefficient identity

For $d = 0$ set $\Phi_C = 1$. Otherwise introduce formal rooted tree series

$$T_i = x_i \exp\Big(\sum_j C_{ij} T_j\Big), \qquad \mathcal{U} = \sum_i T_i - \frac{1}{2}\sum_{i,j} C_{ij} T_i T_j.$$

They are exponential generating series for labeled trees whose vertices carry colors from $\{1, \dots, d\}$, rooted at a vertex of color i in T_i. The squarefree coefficient $[x_1 \cdots x_d]$ uses each color exactly once and so describes the weighted host itself. Each unrooted tree is counted once per vertex in $\sum_i T_i$ and once per edge in the quadratic term; since a tree has one more vertex than edge, $\mathcal{U}$ counts it once. The set decomposition into components gives

$$\Phi_C(b) = [x_1 \cdots x_d] \exp\Big(\mathcal{U} + \sum_i b_i T_i\Big), \tag{3}$$

since each tree is either unmarked or rooted at its marked vertex. All coefficient factorials equal one in this squarefree extraction.

The formal substitution $x_i = t_i \exp(-\sum_j C_{ij} t_j)$ has identity linear part. Its logarithmic Jacobian is

$$D_C(t) = \det(I - C \operatorname{diag} t),$$

because the (i, j) entry of $\partial \log x_i / \partial \log t_j$ is $\delta_{ij} - C_{ij} t_j$. In the formal residue each x_i^{-1} contributes $\exp(\sum_j C_{ij} t_j)$. The multivariate Lagrange–Good formula [11] therefore gives

$$\Phi_C(b) = [t_1 \cdots t_d]\, D_C(t) \exp\Big(\sum_i \alpha_i t_i - \sum_{i<j} C_{ij} t_i t_j\Big), \qquad \alpha_i = 1 + \sum_j C_{ij} + b_i. \tag{4}$$

This is a formal identity. Define the signed matching polynomial

$$Q_C(\alpha) = \sum_{M \text{ matching of } H} (-1)^{|M|} \prod_{ij \in M} C_{ij} \prod_{i \notin V(M)} \alpha_i.$$

It is the squarefree coefficient of the exponential in (4), because the selected quadratic terms must use disjoint pairs. Multiplication by t_i before extraction is differentiation in α_i, and D_C is multiaffine. Hence

$$P_C(\alpha) := D_C(\partial_\alpha) Q_C(\alpha), \qquad \Phi_C(b) = P_C(1 + \deg_C + b), \qquad (\deg_C)_i = \sum_j C_{ij}. \tag{5}$$

2.2 Stability

The weighted multivariate matching stability of Heilmann and Lieb [14] has the following induction in this normalization. For an induced vertex set J and $v \in J$, deleting v or its matching partner gives

$$\frac{Q_J}{Q_{J-v}} = \alpha_v - \sum_{u \in J-v} \frac{C_{vu}}{Q_{J-v}/Q_{J-v-u}}.$$

Let all α_i lie in the upper half-plane. Inductively every ratio in a denominator lies there, so its reciprocal has negative imaginary part; since $C_{vu} \geq 0$, the displayed ratio has strictly positive imaginary part. Starting from $Q_\varnothing = 1$, every Q_J is nonzero, so Q_C is real stable.

The polynomial D_C is real stable. If C is invertible, $D_C(t) = \det(C) \det(C^{-1} - \operatorname{diag} t)$; for upper-half-plane t the second matrix has negative definite imaginary part, and a nonzero kernel vector ξ would give $\operatorname{Im} \xi^*(C^{-1} - \operatorname{diag} t)\xi < 0$, a contradiction. For singular C approximate by the invertible real symmetric matrices $C + \epsilon I$ and pass to the limit; the limit is nonzero since $D_C(0) = 1$.

We use the multiaffine Lieb–Sokal lemma [28, Lemma 3.2]: if $g(\alpha) + yf(\alpha)$ is stable and $\deg_{\alpha_i} f \leq 1$, then $g - \partial_i f$ is stable or zero. For completeness, here is the proof. The polynomials f and g are stable or zero, as a leading coefficient and a specialization. If $f = 0$ or $\partial_i f = 0$ there is nothing to prove. Otherwise $yf(\alpha_i - 1/y, \alpha_{\neq i}) = yf(\alpha) - \partial_i f(\alpha)$ is stable, because $-1/y$ lies in the upper half-plane. For upper-half-plane α, nonvanishing of $g + yf$ for all upper-half-plane y gives $\operatorname{Im}(g/f) \geq 0$, and the shifted polynomial gives $\operatorname{Im}(-\partial_i f/f) \geq 0$. Adding, $g - \partial_i f + yf$ is stable, and $y = 0$ proves the lemma.

Since D_C has real coefficients, $D_C(-y)$ is stable, and $D_C(-y)Q_C(\alpha)$ is stable in the disjoint variable sets y, α. Applying the lemma successively, replacing y_i by $-\partial_{\alpha_i}$, keeps every intermediate polynomial multiaffine in each remaining variable, so all degree hypotheses hold. The result is $D_C(\partial_\alpha) Q_C = P_C$, stable or zero; its coefficient of $\alpha_1 \cdots \alpha_d$ is one, so it is stable. The same conclusion follows from the Borcea–Brändén symbol criterion [2, Corollary 4.1], with symbol $D_C(-w)$. The real translation in (5) proves Theorem 2.1.

2.3 The star theorem

Theorem 2.2 (Stable stars)**.** *For every vertex v of every finite simple graph with positive edge weights, the polynomial $S_v(z) = \sum_F w(F) \prod_{vi \in F} z_i$ is real stable. The normalized law of the edges at v is strongly Rayleigh, hence negatively associated, and every two edges at v satisfy* $\operatorname{Cov}(X_{vi}, X_{vj}) \leq 0$.

Proof. Delete v and fix a forest A of $G - v$. Each tree of A either stays unattached to v or receives exactly one edge from v; two edges into the same tree would close a cycle, and every single choice is legal. Hence

$$S_v(z) = \Phi_C(b), \qquad b_i = w_{vi} z_i, \tag{6}$$

where C is the weighted adjacency matrix of $G - v$ and $w_{vi} = 0$ for nonneighbors. Positive scaling and real specialization preserve stability or give zero, and $S_v(\mathbf{1}) = Z_G > 0$. A multiaffine real stable probability generating polynomial defines a strongly Rayleigh law, which is negatively associated under conditioning and external fields [3, Theorem 4.9]. In particular $(\partial_i S_v)(\partial_j S_v) - S_v \partial_{ij} S_v \geq 0$ at $z = \mathbf{1}$, which is the covariance assertion. □

Conversely, attaching a new vertex to every vertex of H with weight one turns its star polynomial into $\Phi_C(b)$; thus Theorems 2.1 and 2.2 are equivalent formulations. The constant one in each component factor of (2) distinguishes Φ_C from the rooted-forest polynomial, whose component factors $\sum_{i \in B} b_i$ weight forests by products of component sizes [3, Section 3.4].

Corollary 2.3 (Degree laws)**.** *For every vertex v with d_v neighbors in a finite simple graph with positive weights, the degree polynomial $S_v(z, \dots, z)$ has only strictly negative real zeros, and*

$$\frac{S_v(z, \dots, z)}{S_v(1, \dots, 1)} = \prod_{j=1}^{d_v} (1 - p_j + p_j z), \qquad 0 < p_j < 1.$$

Thus the degree of v is distributed as a sum of d_v independent Bernoulli variables.

Proof. Diagonal specialization of a real stable polynomial with real coefficients is real-rooted. Every set of edges at v is a forest, so all coefficients from degree 0 to d_v are positive, and all zeros are strictly negative. □

Corollary 2.4 (Conditioning)**.** *Condition on disjoint edge sets A, B being present and absent, respectively, with positive probability. The conditional law of the remaining edges at any fixed vertex is strongly Rayleigh. More generally the same holds for the remaining boundary edges of one connected component of A.*

Proof. The set A is a forest. Delete B and contract A. Forests satisfying the conditioning correspond, up to the constant factor $\prod_{e \in A} w_e$, to forests of the resulting multigraph with loops removed. A forest uses at most one edge of each parallel class c; replace the class by one edge of weight $W_c = \sum_{e \in c} w_e$, and substitute

$$y_c = \frac{\sum_{e \in c \text{ tracked}} w_e z_e + \sum_{e \in c \text{ untracked}} w_e}{W_c}$$

in the simple-graph star polynomial. Then $W_c y_c$ is the original marked parallel sum. Tracked classes give upper-half-plane affine images and untracked classes specialize to $y_c = 1$, so the conditional marked polynomial is stable; its value at all marks one is positive. The boundary of one component of A becomes a star after contraction. □

3 Strict incident correlation

Theorem 3.1 (Incident strictness)**.** *Let $e = vi$ and $f = vj$ be distinct edges of a finite simple graph G with positive weights. Then $\operatorname{Cov}(X_e, X_f) < 0$ if and only if some cycle contains both e and f; otherwise X_e, X_f are independent. Quantitatively, put $H = G - v$ with weighted adjacency C, let $b_k = w_{vk}$, with $b_k = 0$ for nonneighbors, and let π be any path from i to j in H with vertex set W. Put $b'_k = b_k + \sum_{u \in W} C_{ku}$ for $k \notin W$. Then*

$$-\operatorname{Cov}(X_e, X_f) \;\geq\; b_i b_j \Big(\prod_{uu' \in E(\pi)} C_{uu'} \Big) \frac{\Phi_{H-W}(b')^2}{\Phi_H(b)^2} > 0,$$

where Φ_{H-W} is the polynomial (2) of the induced weighted graph on the vertices outside W, equal to one when W is everything.

Proof. For independent variables α define

$$E_C(\alpha) = \det(\operatorname{diag}(\alpha) - C), \qquad T_{uu'} = 1 - C_{uu'}\partial_u\partial_{u'}, \qquad P_C(\alpha) = \prod_{uu' \in E(H)} T_{uu'}\, E_C(\alpha).$$

The operators commute. Expanding the determinant gives $E_C(\alpha) = D_C(\partial_\alpha)\prod_k \alpha_k$ with D_C as in Section 2, and the signed matching polynomial is $Q_C = \prod_{uu'} T_{uu'} \prod_k \alpha_k$, by multiaffinity. Commuting the operators shows that this P_C coincides with $D_C(\partial_\alpha)Q_C$ from (5), so $P_C(1 + \deg_C + b) = \Phi_C(b)$.

The determinant E_C is real stable, because for upper-half-plane α the matrix $\operatorname{diag}(\alpha) - C$ has positive definite imaginary part. Each $T_{uu'}$ maps multiaffine real stable polynomials to real stable polynomials or zero: multiply by the stable polynomial $1 - C_{uu'}y_u y_{u'}$ in new variables and apply the Lieb–Sokal lemma of Section 2 twice, replacing $y_u \mapsto -\partial_u$ and $y_{u'} \mapsto -\partial_{u'}$. The coefficient of $\prod_k \alpha_k$ stays one, so no intermediate polynomial vanishes. In particular, for any edge uu' of H, the polynomial obtained by omitting the factor $T_{uu'}$ from the product defining P_C is again multiaffine and real stable.

For a multiaffine polynomial q write $q_i = \partial_i q$ and $\Delta_{ij}(q) = q_i q_j - q q_{ij}$. If q is real stable, then $\Delta_{ij}(q) \geq 0$ at every real point: specializing the other variables to real values keeps stability or gives zero, and a real stable multiaffine polynomial in two variables has nonnegative Δ on $\mathbb{R}^2$ [3, 28]. Multiaffinity gives the identities

$$\begin{aligned} \Delta_{ij}(T_{ij}q) &= \Delta_{ij}(q) + C_{ij}q_{ij}^2, \\ \Delta_{ij}(T_{ik}q) &= \Delta_{ij}(q) + C_{ik}\Delta_{jk}(q_i) \qquad (i, j, k \text{ distinct}). \end{aligned}$$

For the second, $(T_{ik}q)_i = q_i$, $(T_{ik}q)_j = q_j - C_{ik}q_{ijk}$ and $(T_{ik}q)_{ij} = q_{ij}$; multiplying out gives the formula.

Let q be P_C with the factor T_{ik} omitted, where ik is an edge of H. Then q is real stable, as shown above, and so is q_i. Since ∂_i annihilates every operator term involving i on multiaffine polynomials and $\partial_i E_C$ is the principal minor deleting i, we have $q_i = P_{C[V\setminus\{i\}]}$, where $C[U]$ denotes the restriction to U, and similarly $(P_C)_{ij} = P_{C[V\setminus\{i,j\}]}$. Now let $\pi = (i = v_0, v_1, \ldots, v_\ell = j)$. For $\ell = 1$, the first identity with $q = P_C$ minus the factor T_{ij} gives $\Delta_{ij}(P_C) \geq C_{ij}P^2_{C[V\setminus\{i,j\}]}$. For $\ell \geq 2$, the second identity with $k = v_1$ gives $\Delta_{ij}(P_C) \geq C_{iv_1}\Delta_{v_1 j}(P_{C[V\setminus\{i\}]})$, and induction on the remaining path yields, for every real α,

$$\Delta_{ij}(P_C)(\alpha) \geq \Big(\prod_{h=0}^{\ell-1} C_{v_h v_{h+1}}\Big) P_{C[V\setminus W]}(\alpha_{V\setminus W})^2.$$

At $\alpha = 1 + \deg_C + b$ the coordinates outside W equal $1 + \deg_{H-W} + b'$, so the right side involves $\Phi_{H-W}(b')$, which is positive: every forest term is nonnegative and the empty forest contributes $\prod_{k\notin W}(1 + b'_k) > 0$. By (6) and the chain rule, $-\operatorname{Cov}(X_e, X_f) = b_i b_j \Delta_{ij}(\Phi_C)(b)/\Phi_C(b)^2$, which gives the bound.

If i and j lie in different components of H, then Φ_C factors over the components and $\Delta_{ij}(\Phi_C) = 0$; binary variables with zero covariance are independent. Finally, i and j are joined by a path in $G - v$ exactly when some cycle of G contains vi and vj. □

4 Rank-one replacement

Throughout this section parallel edges are allowed, as labeled alternatives of which a forest selects at most one per parallel class; merging a class into one edge with the summed activity leaves all partition functions unchanged.

4.1 Statement

Fix a background graph H on a vertex set V with nonnegative activities and a profile $r = (r_v)_{v \in V} \geq 0$. Adjoin one new vertex, the *hub*, with edges of activities qr_v to each $v \in V$, and let $f(q)$ be the partition function of the result. Put

$$Z_0 = Z_H, \qquad P(q) = f(q)/Z_0, \qquad R = \sum_v r_v, \qquad m = |\{v : r_v > 0\}|. \tag{7}$$

Corollary 4.1 (One-hub polynomial)**.** *There are $\lambda_1, \ldots, \lambda_m > 0$ with*

$$P(q) = \prod_{i=1}^{m} (1 + \lambda_i q), \qquad \sum_i \lambda_i = R. \tag{8}$$

Proof. We may take H simple by merging parallel classes. For a fixed forest J of H the hub sends at most one edge into each component of J, so

$$f(q) = \sum_{J \in \mathcal{F}(H)} w(J) P_J(q), \qquad P_J(q) = \prod_{B \in \mathcal{C}(J)} (1 + q a_B), \qquad a_B = \sum_{v \in B} r_v. \tag{9}$$

This is Φ_C of Theorem 2.1 after the real specialization $b_v = 0$ for $r_v = 0$ and the substitution $b_v = qr_v$ otherwise, so f is real-rooted. Its coefficients are nonnegative with constant term $Z_0 > 0$, so all roots are negative. No forest uses more than m hub edges, and the empty background forest allows all m, so f has degree m. Finally $\sum_B a_B = R$ for every J, so $P'(0) = R$. □

We call the λ_i the *site weights* of (H, r). Let Q be any graph on a vertex set U disjoint from V, with edge activities b_e, and let $s = (s_h)_{h \in U} \geq 0$ be scales. Let $G(H, Q; r, s)$ be the graph obtained from $H \cup Q$ by adding the cross edges

$$w_{hv} = s_h r_v \qquad (h \in U,\ v \in V). \tag{10}$$

Let $G_*(Q; \lambda, s)$ be obtained from Q by adding m new vertices, the *sites*, with no edges among themselves, and edges of activity $s_h \lambda_i$ from each $h \in U$ to site i.

Theorem 4.2 (Rank-one replacement)**.** *For every graph Q,*

$$Z_{G(H,Q;r,s)} = Z_0 \, Z_{G_*(Q;\lambda,s)} \tag{11}$$

as polynomials in all activities b_e and scales s_h. Consequently the two forest laws give the same joint distribution to all edge indicators of Q together with the numbers of cross edges at the vertices of Q, and hence also to the total degrees of the vertices of Q.

4.2 The Grassmann representation

Use anticommuting generators $\bar\psi_v, \psi_v$ and set $\tau_v = \bar\psi_v\psi_v$. The Berezin integral is the coefficient functional normalized by $\int \prod_v \tau_v = 1$. Even elements commute, and every series below is a finite polynomial in the generators. For a loopless graph G put

$$\mathcal{A}_G = \sum_v \tau_v + \sum_{uv\in E(G)} w_{uv}(\tau_u + \tau_v - \bar\psi_u\psi_v - \bar\psi_v\psi_u - \tau_u\tau_v),$$

with parallel alternatives summed.

Lemma 4.3 (Unrooted forests [5])**.** $Z_G = \int \exp(\mathcal{A}_G)$.

Proof. This is the case $t = 1$, $u = -1$ of [5, Eq. (13)]; we verify the normalization. Split the action as $\bar\psi L\psi + \sum_v \tau_v - \sum_{uv} w_{uv}\tau_u\tau_v$, where L is the weighted Laplacian. The last two sums are even and commute, so their exponential is $\prod_v(1+\tau_v)\prod_{uv}(1-w_{uv}\tau_u\tau_v)$, and nilpotency leaves vertex-disjoint marker families. For a vertex set A, $\int \prod_{v\in A}\tau_v\, e^{\bar\psi L\psi} = \det L[V\setminus A]$, which by the principal-minor matrix-tree theorem counts forests with exactly one root from A in each tree. Joining each marked edge to the two rooted trees at its endpoints, each component becomes a tree marked by one vertex, with coefficient $+1$, or by one edge, with coefficient -1; deleting the marked edge recovers the two rooted trees, so all choices are counted with the correct multiplicity. The total multiplier of a tree T is $|V(T)| - |E(T)| = 1$, which gives (1), singletons included. □

4.3 Eliminating isolated sites

Fix Q, scales s_h, and masses $a_1, \dots, a_c \ge 0$ with $\sum_j a_j = R$. Join each $h \in U$ to site j with activity $s_h a_j$. Abbreviate

$$S = \sum_h s_h, \quad T = \sum_h s_h\tau_h, \quad A = S - T, \quad B = \sum_h s_h\bar\psi_h, \quad C = \sum_h s_h\psi_h, \quad D = BC. \tag{12}$$

Then $B^2 = C^2 = D^2 = 0$. The even factor carried by U is $E_{Q,R} = \exp[\mathcal{A}_Q + RT]$. At a site of mass a with generators $\bar\eta, \eta$ the remaining action is $(1 + aS - aT)\bar\eta\eta - aB\eta - a\bar\eta C$, and the coefficient of $\bar\eta\eta$ in its exponential is

$$1 + aA - a^2D, \tag{13}$$

because $(-aB\eta)(-a\bar\eta C) = -a^2BC\bar\eta\eta$ and terms containing $\bar\eta\eta$ together with a source vanish. Let $p(q) = \prod_j(1 + a_jq)$. Since $D^2 = 0$,

$$\prod_j(1 + a_jA - a_j^2D) = p(A) - D\sum_j a_j^2\prod_{\ell\ne j}(1 + a_\ell A) = p(A) - D\,(\mathcal{H}_R p)(A), \tag{14}$$

where $(\mathcal{H}_R p)(q) = (Rp(q) - p'(q))/q$ is a polynomial because $Rp(q) - p'(q) = q\sum_j a_j^2\prod_{\ell\ne j}(1 + a_\ell q)$. On the linear space of polynomials with $p'(0) = Rp(0)$ define

$$\mathcal{L}_{Q,s,R}[p] = \int_U E_{Q,R}\{p(A) - D\,(\mathcal{H}_R p)(A)\}. \tag{15}$$

It is linear in p, and by Lemma 4.3 and (14), integrating the sites first, $\mathcal{L}_{Q,s,R}[p]$ is the partition function of Q joined to the sites of masses a_j. The factor $E_{Q,R}$ depends on the masses only through R.

4.4 Averaging over the background

Fix a forest J of H. The forests F of $G = G(H, Q; r, s)$ with $F \cap E(H) = J$ are the sets $J \cup X$ with X disjoint from $E(H)$, and $J \cup X$ is a forest of G exactly when X is a forest of the contraction G/J with parallel edges kept [17, Section 3.1]. Contract each component B of J to one site of mass a_B; the edges from h into B become a parallel class of total activity $s_h a_B$. No edge of Q is contracted, so no loops arise, and the unrooted law carries no vertex or component factors. Hence the conditional partition function is the isolated-site partition just computed, with $p = P_J$. As every J has total mass R,

$$\frac{Z_{G(H,Q;r,s)}}{Z_0} = \mathbb{E}_J \, \mathcal{L}_{Q,s,R}[P_J] = \mathcal{L}_{Q,s,R}[P], \tag{16}$$

by (9) and linearity.

Proof of Theorem 4.2. By Corollary 4.1, $P = \prod_i (1 + \lambda_i q)$ with $\sum_i \lambda_i = R$. The isolated-site computation with masses $\lambda_1, \dots, \lambda_m$ has the same functional, so (16) equals $Z_{G_*(Q;\lambda,s)}$. For markers y_e on the edges of Q and z_h on cross degrees, replace b_e by $b_e y_e$ and s_h by $s_h z_h$ in (11); the site weights depend only on H, r. Dividing by the value at $y = z = 1$, which is positive, gives equality of the joint probability generating functions. The total degree of h is its cross degree plus its number of incident edges of Q. □

4.5 Equivalence of backgrounds and an example

Say that two background-profile pairs have the same *response* if, for every Q and s, their normalized partition functions Z_G/Z_0 agree as polynomials in the activities of Q and the scales.

Corollary 4.4. *Two pairs have the same response if and only if their polynomials P coincide. In that case all joint laws of edges of Q and cross degrees agree.*

Proof. Equal P gives equal $R = P'(0)$, so (16) gives sufficiency. For necessity take Q to be one vertex: its normalized partition function is $P(s)$. □

For example, let H be one edge of activity $w > 0$ and $r = (1, 1)$. Then $Z_0 = 1 + w$, $P(q) = 1 + 2q + q^2/(1 + w)$, and the site weights are $\lambda_\pm = 1 \pm \sqrt{w/(1 + w)} > 0$. For every Q, replacing this background by two isolated sites with activities $s_h \lambda_\pm$ and multiplying by $1 + w$ reproduces the partition function exactly.

5 Two hubs, split pairs and the endpoint clique

5.1 Two hubs

Let Q consist of two isolated vertices with scales $s, t > 0$, and put $q = s + t$, $p = st$. By Theorem 4.2 the replacement graph is $K_{2,m}$. A forest of $K_{2,m}$ joins at most one site to both hubs, so the two-hub partition function $F(s, t)$ satisfies

$$\frac{F(s,t)}{Z_0} = P(q) + p\,\frac{RP(q) - P'(q)}{q} = P(q)\{1 + p\mathsf{B}(q)\}, \qquad \mathsf{B}(q) = \sum_i \frac{\lambda_i^2}{1 + \lambda_i q}. \tag{17}$$

Corollary 5.1 (Two-hub degrees)**.** *The numbers D_1, D_2 of cross edges at the two hubs satisfy* $\mathrm{Cov}(D_1, D_2) < 0$ *when $m \geq 2$, and are independent when $m \leq 1$.*

Proof. Put $\mathsf{C}(q) = \sum_i \lambda_i^2/(1+\lambda_i q)^2$, so that $\mathsf{B} + q\mathsf{B}' = \mathsf{C}$ and $(\log P)'' = -\mathsf{C}$. Writing $K = 1 + p\mathsf{B}$, one has $K_s = t\mathsf{B} + p\mathsf{B}'$, $K_t = s\mathsf{B} + p\mathsf{B}'$, $K_{st} = \mathsf{C} + p\mathsf{B}''$, and substitution into $\partial_s\partial_t \log F = -\mathsf{C} + (KK_{st} - K_sK_t)/K^2$ gives

$$\partial_s\partial_t \log F = \frac{p(\mathsf{B}'' - 2\mathsf{BC}) + p^2(\mathsf{BB}'' - (\mathsf{B}')^2 - \mathsf{B}^2\mathsf{C})}{(1+p\mathsf{B})^2}. \tag{18}$$

For $m \geq 1$ write $b_i = \lambda_i^2/(1+\lambda_i q)$ and $z_i = \lambda_i/(1+\lambda_i q)$. Then $\mathsf{B} = \sum b_i$, $\mathsf{C} = \sum z_i^2$, $\mathsf{B}' = -\sum b_i z_i$ and $\mathsf{B}'' = 2\sum b_i z_i^2$, so $2\mathsf{BC} - \mathsf{B}'' = 2\sum_{i\neq j} b_i z_j^2$, which is strictly positive for $m \geq 2$. With $\rho_i = b_i/\mathsf{B}$, $\mu = \sum_i \rho_i z_i$ and $M = \max_i z_i$, we have $\mathbb{E}_\rho z^2 \leq M\mu$ and

$$\frac{\mathsf{BB}'' - (\mathsf{B}')^2}{\mathsf{B}^2} = 2\mathbb{E}_\rho z^2 - \mu^2 \leq 2M\mu - \mu^2 \leq M^2 \leq \mathsf{C}.$$

Both coefficients in (18) are therefore nonpositive, the first strictly for $m \geq 2$, and $\mathrm{Cov}(D_1, D_2) = st\,\partial_s\partial_t \log F$. For $m = 1$, $F = Z_0(1+\lambda_1 s)(1+\lambda_1 t)$, and for $m = 0$, $F = Z_0$. □

5.2 Two split pairs

Attach two disjoint pairs u_ℓ, v_ℓ, $\ell = 1, 2$, to H, with no edges between the pairs. Pair ℓ has an internal edge of activity $\theta_\ell > 0$ and edges $x_\ell r_v$ from u_ℓ and $y_\ell r_v$ from v_ℓ to each $v \in V$, with $x_\ell, y_\ell > 0$. Set $s_\ell = x_\ell + y_\ell$ and $k_\ell = x_\ell y_\ell / s_\ell$.

For one pair and a fixed background forest with component masses a_B: if the internal edge is present, no component attaches to both endpoints; if it is absent, at most one component attaches to both. The partition function of H with one pair is therefore

$$(1+\theta)f(s) + xy\sum_J w(J)\sum_B a_B^2 \prod_{B'\neq B}(1 + s a_{B'}) = (1 + \theta + kR)f(s) - kf'(s),$$

where a_B^2 accounts for both endpoints attaching to the same component, at equal or different vertices. This derivation uses H only as a background of total mass R. Apply it to pair 1, with background H plus pair 2 and the profile extended by zero; then apply it to pair 2, with background H plus a hub of scale s_1 and the profile extended by zero, differentiating in s_1 where needed. With $F(s_1, s_2)$ the two-hub function of the previous subsection and Z the split partition function,

$$Z = A_1A_2F - k_1A_2F_1 - k_2A_1F_2 + k_1k_2F_{12}, \qquad A_\ell = 1 + \theta_\ell + k_\ell R,$$

where $F_\ell = \partial_{s_\ell} F$ and A_ℓ, k_ℓ are held fixed. Differentiating in θ_1, θ_2 gives $Z_{\theta_1}Z_{\theta_2} - ZZ_{\theta_1\theta_2} = k_1k_2(F_1F_2 - FF_{12})$, so

$$\mathrm{Cov}(X_{u_1v_1}, X_{u_2v_2}) = \frac{\theta_1\theta_2k_1k_2F^2}{s_1s_2Z^2}\,\mathrm{Cov}(D_1, D_2). \tag{19}$$

By Corollary 5.1 the two internal edges are strictly negatively correlated when the profile has at least two positive coordinates, and independent otherwise. For instance, in $K_n \vee 2K_2$ with any common activity $\beta > 0$ the two edges of $2K_2$ are strictly negatively correlated for every $n \geq 2$.

Remark 5.2. In the last example with unit weights, the unit-current potentials across the two internal edges are supported on disjoint pairs, so the transfer current between them and the next electrical sum of squares in [8, Theorem 1.5] both vanish. The strict sign at every finite activity comes from (19).

5.3 The endpoint clique

Let H be a finite graph on V with positive activities, $V = \varnothing$ allowed, and let $r_v \geq 0$. Add four vertices a, b, c, d with scales $t_a, t_b, t_c, t_d > 0$ and edges hv of activity $t_h r_v$ for $h \in \{a, b, c, d\}$, $v \in V$. Fix $\gamma > 0$, give ac, ad, bc, bd the activities $\gamma t_h t_j$, and give $e = ab$ and $f = cd$ arbitrary activities $W_e, W_f > 0$. Let Z be the partition function and $\Delta = Z_e Z_f - Z Z_{ef}$, with derivatives in W_e, W_f. Write

$$s_1 = t_a + t_b, \quad s_2 = t_c + t_d, \quad q_0 = s_1 + s_2, \quad p = s_1 s_2.$$

Let ρ_i be the site weights of (H, r) from Corollary 4.1 and $\mathsf{B}_0 = \sum_i \rho_i^2/(1 + \rho_i q_0)$, an empty sum when $r = 0$. Let F_* be the partition function of H together with two vertices of scales s_1, s_2, edges $s_\ell r_v$ to V, and one edge of activity $\gamma s_1 s_2$ between them.

Theorem 5.3 (Endpoint clique)**.** *In this setting*

$$\Delta \;\geq\; \frac{2\gamma^2 t_a t_b t_c t_d \, F_*^2}{[1 + p(\mathsf{B}_0 + \gamma)]^2} > 0, \tag{20}$$

and $F_* = Z_{ef}$ is independent of W_e, W_f. Consequently $\mathrm{Cov}(X_e, X_f) = -W_e W_f \Delta / Z^2 < 0$ *for all $W_e, W_f > 0$.*

A weak-site approximation

Fix labeled activities $b_e, b_f > 0$ on two internal edges ab and cd, and for $m \geq 2$ adjoin m weak sites, with no edges among themselves or to H, each joined to a, b, c, d with activities $t_h \epsilon_m$, where $\epsilon_m = \sqrt{\gamma/m}$. The background is now the disjoint union of H and the weak sites, with profile r on H and ϵ_m on each weak site, so the split-pair formula (19) applies for every m.

Retain Q_0, consisting of H, its edges to a, b, c, d, and the two internal edges; give a, b, c, d the scales t_h and the vertices of H scale zero. In (12) then $S = q_0$. With $E_{Q_0} = \exp(\mathcal{A}_{Q_0})$, formula (14) with $m\epsilon_m^2 = \gamma$ gives

$$Z_m = \int E_{Q_0} e^{m\epsilon_m T} (1 + \epsilon_m A)^m \Big(1 - \frac{\gamma D}{1 + \epsilon_m A}\Big), \tag{21}$$

where the inverse is the finite nilpotent inverse with scalar part $1 + \epsilon_m S > 0$. Normalize by $c_m = (1 + \epsilon_m S)^m$ and put $\alpha_m = \epsilon_m/(1 + \epsilon_m S)$. In the finite algebra of Q_0, with $N = |V(Q_0)|$ so that $T^{N+1} = 0$,

$$m\epsilon_m T + m \log(1 - \alpha_m T) = \frac{\gamma S}{1 + \epsilon_m S} T - \sum_{j=2}^{N} \frac{m\alpha_m^j}{j} T^j,$$

with $m\alpha_m^2 \to \gamma$ and $m\alpha_m^j \to 0$ for $j \geq 3$. The last factor in (21) tends to $1 - \gamma D = e^{-\gamma D}$. Hence

$$\frac{Z_m}{c_m} \longrightarrow Z_* := \int E_{Q_0} \exp\{\gamma(ST - T^2/2 - D)\}, \tag{22}$$

and

$$\gamma(ST - T^2/2 - D) = \sum_{h<j} \gamma t_h t_j (\tau_h + \tau_j - \bar\psi_h \psi_j - \bar\psi_j \psi_h - \tau_h \tau_j),$$

since the diagonal terms $t_h^2 \tau_h$ cancel between ST and D and $\tau_h^2 = 0$; zero scales on H create no new edges at H. By Lemma 4.3, Z_* is the partition function of the endpoint clique with the extra labeled edges b_e, b_f parallel to ab, cd. The convergence holds coefficientwise in markers on b_e, b_f, since c_m does not involve them, so the joint law and covariance of the two labeled edges converge.

A uniform gap

The weak-site background has one-hub polynomial $f_m(q)/f_m(0) = (f_0(q)/Z_H)(1+\epsilon_m q)^m$, with site weights the ρ_i and m copies of ϵ_m. At q_0,

$$\mathsf{B}_m = \mathsf{B}_0 + \frac{m\epsilon_m^2}{1+\epsilon_m q_0} \le \mathsf{B}_0 + \gamma,$$

and keeping only pairs of distinct weak sites in the proof of Corollary 5.1,

$$2\mathsf{B}_m\mathsf{C}_m - \mathsf{B}''_m \ge g_m := \frac{2m(m-1)\epsilon_m^4}{(1+\epsilon_m q_0)^3} \longrightarrow 2\gamma^2.$$

As the second coefficient in (18) is nonpositive, the merged hub degrees satisfy

$$\mathrm{Cov}_m(D_1, D_2) \le -\frac{p^2 g_m}{[1+p(\mathsf{B}_0+\gamma)]^2}. \tag{23}$$

Let F_m be the twice-merged and Z_m the split partition function, and $\kappa_1 = t_a t_b/s_1$, $\kappa_2 = t_c t_d/s_2$. By (19),

$$\mathrm{Cov}_m(\mathbf{1}_{b_e}, \mathbf{1}_{b_f}) = K_m \,\mathrm{Cov}_m(D_1, D_2), \qquad K_m = \frac{b_e b_f \kappa_1 \kappa_2 F_m^2}{p Z_m^2} > 0.$$

Merging endpoints adds their scales, so the total scale is still S and the same c_m normalizes both partition functions. By (22) applied to the merged graph, $F_m/c_m \to F_* > 0$, the four cross alternatives merging into the activity $\gamma s_1 s_2$; also $Z_m/c_m \to Z_* > 0$. Hence $K_m \to K_* = b_e b_f \kappa_1 \kappa_2 F_*^2/(pZ_*^2)$ and

$$\mathrm{Cov}_*(\mathbf{1}_{b_e}, \mathbf{1}_{b_f}) \le -\frac{2p^2\gamma^2 K_*}{[1+p(\mathsf{B}_0+\gamma)]^2} < 0. \tag{24}$$

Parallel thinning

Merge each parallel class at e, f, with activities $W'_e = \gamma t_a t_b + b_e$ and $W'_f = \gamma t_c t_d + b_f$. Given the merged forest, the labeled alternative in a selected class is chosen with probability b_e/W'_e, respectively b_f/W'_f, independently, because acyclicity depends only on which classes are selected. Hence

$$\mathrm{Cov}_*(\mathbf{1}_{b_e}, \mathbf{1}_{b_f}) = \frac{b_e}{W'_e}\frac{b_f}{W'_f}\,\mathrm{Cov}(X_e, X_f) = -\frac{b_e b_f \Delta}{Z_*^2}.$$

Substituting this and K_* into (24) and using $\kappa_1\kappa_2 p = t_a t_b t_c t_d$ gives (20) at W'_e, W'_f. For fixed remaining activities, $Z = A_0 + W_e A_e + W_f A_f + W_e W_f A_{ef}$ and $\Delta = A_e A_f - A_0 A_{ef}$, which does not involve W_e, W_f; so the bound holds at all $W_e, W_f > 0$. Contracting both targets identifies $A_{ef} = F_*$: the edges at merged endpoints add to $s_\ell r_v$, and the four cross edges add to $\gamma s_1 s_2$. This proves Theorem 5.3.

6 Complete graphs with vertex factors

Theorem 6.1 (Vertex-factor complete graphs)**.** *Let $n \ge 3$ and $t_1, \dots, t_n > 0$. In the forest law on K_n with activities $w_{ij} = t_i t_j$, every two distinct edges are strictly negatively correlated. For a disjoint pair ab, cd the strict inequality, with the bound* (20) *for $\gamma = 1$, persists when the activities of ab and cd are arbitrary positive numbers and the edges among the other $n-4$ vertices carry arbitrary nonnegative activities.*

Proof. Two incident edges vi, vj lie on the triangle vij, so Theorem 3.1 gives strict negative correlation. For a disjoint pair ab, cd, apply Theorem 5.3 with H the graph on the other $n-4$ vertices, possibly empty, with its given activities and zero activities omitted, and with $r_v = t_v$ and $\gamma = 1$. The edges from $\{a, b, c, d\}$ to H then have activities $t_h t_v$, the four cross edges $t_h t_j$, and the targets arbitrary positive activities. □

With constant factors $t_i = \beta^{1/2}$ the theorem gives strict negative correlation on K_n at every common activity $\beta > 0$. At $\beta = 1$ this is Gupta's theorem [13]; for every β it was obtained by Shmalo [23] and in [19] by different methods. Here the disjoint case is a direct specialization of the endpoint-clique bound, which also gives the explicit estimate (20).

7 Small vertex covers, finite hosts and 2-sums

7.1 Graphs with a vertex cover of size three

A vertex cover is a set of vertices meeting every edge. We use one external finite theorem, due to Erickson [10, Section 5.3, Theorem 5.3.1 and Corollary 5.3.2]: every simple graph on at most seven vertices is I-Rayleigh. Erickson proves it by a finite deletion–contraction and sum-of-squares computation. The proof below applies it to one pair of disjoint edges of $K_{3,3}$.

Theorem 7.1 (Vertex cover three)**.** *Let G be a finite simple graph with a vertex cover of size at most three. For all positive edge weights, every two distinct edges of G satisfy* $\mathrm{Cov}(X_e, X_f) \le 0$. *In particular this holds on every $K_{3,n}$, also after adding any edges inside its part of size three.*

Terminal states. Let $T = \{1, 2, 3\}$ be a vertex cover, padding by isolated vertices if necessary; isolated vertices do not change the forest law. Vertices outside T are pairwise nonadjacent. A graph B meeting the rest of G only in T will be called a *background*. Each forest of B induces one of the five partitions 1|2|3, 12|3, 13|2, 23|1, 123 of T. Dividing the total weights of the forests inducing each partition by the positive total weight $w_\varnothing(B)$ of those inducing 1|2|3 gives the *state* $(1, h, j, k, C)$ of B.

Lemma 7.2 (State constraints)**.** *Let B consist of edges inside T and of vertices outside T with their edges to T. Its state satisfies $C = E + d$ with $E = hj + hk + jk$ and $d \ge 0$, and*

$$d^2 + dE \le hjk. \tag{25}$$

In particular $d > 0$ forces $h, j, k > 0$ and $d \le \min\{hj/(h+j),\, hk/(h+k),\, jk/(j+k)\}$.

Proof. Decompose B into columns: each edge inside T, and each outside vertex with its edges to T. A column of an outside vertex with weights x, y, z to $1, 2, 3$, zeros allowed, has unnormalized weights $A = 1 + x + y + z$, xy, xz, yz, xyz for the five partitions, hence pair weights $h_i = xy/A$, $j_i = xz/A$, $k_i = yz/A$ and $xyz/A = h_i j_i + h_i k_i + j_i k_i + d_i$ with $d_i = xyz/A^2 \ge 0$. An edge inside T of weight c has pair weight c of its type and $d_i = 0$. When columns are combined, two pair mergers of the same type close a cycle, as do three pair mergers or any merger after a connected column; all other combinations are forests. Summing, with $h = \sum h_i$, $j = \sum j_i$, $k = \sum k_i$ and $d = \sum d_i$, gives $C = E + d$.

Mark the terminals in the polynomial (2) of B and set all other marks to zero. Components avoiding T contribute one, so after division by $w_\varnothing(B)$ the stable polynomial of Theorem 2.1 becomes

$$\begin{aligned} P(z) = {} & (1+z_1)(1+z_2)(1+z_3) + h(1+z_1+z_2)(1+z_3) + j(1+z_1+z_3)(1+z_2) \\ & + k(1+z_2+z_3)(1+z_1) + C(1+z_1+z_2+z_3). \end{aligned}$$

With $Z = 1 + h + j + k + C$ and $z_3 = t$ real, its Rayleigh difference in z_1, z_2 at $z_1 = z_2 = 0$ is the real quadratic

$$[Z + (1 + h + k)t][Z + (1 + h + j)t] - Z(1 + t)(1 + j + k + t),$$

which is nonnegative for all real t by stability. Its discriminant equals $4Z(d^2+dE-hjk)$ and is therefore nonpositive (when the leading coefficient vanishes, so does the linear one). This proves (25). If $d > 0$ then $hjk \geq d^2 > 0$; and $dh(j + k) \leq dE \leq hjk$ gives $d \leq jk/(j + k)$, the other two bounds following by symmetry. □

For each forest F_R of the remaining edges, acyclicity of $F_R \cup F_B$ with a background forest F_B depends only on the partition of T induced by F_B: contract the trees of F_B and test F_R in the quotient. Hence the full partition function, marked on any edges outside B, equals $w_\varnothing(B)$ times a linear functional of the state, and its Rayleigh difference is $w_\varnothing(B)^2$ times the Rayleigh difference computed from the state.

Proof of Theorem 7.1. Pairs sharing a vertex are covered by Theorem 2.2. Two disjoint edges cannot both lie inside T, so up to relabeling either (a) $e = 12$ and $f = 3x$ with $x \notin T$, or (b) $e = 1x$ and $f = 2y$ with distinct $x, y \notin T$. In each case let B consist of all edges except e, f and the other edges at x (and at y); it is a background of the form in Lemma 7.2. Write Δ for the Rayleigh difference computed from the state.

Case (a). Let x have weights x_1, x_2 to the terminals $1, 2$. Expanding the four marked partition functions over the five states gives

$$\Delta = (jx_2 - kx_1)^2 + jx_2(1 + j + x_2) + kx_1(1 + k + x_1) - d(x_1 + x_2)(1 + x_1 + x_2).$$

If $d = 0$ this is nonnegative. If $d > 0$, put $c = jk/(j + k)$, so that $d \leq c \leq \min(j, k)$ by Lemma 7.2. Then $c(x_1 + x_2) \leq jx_2 + kx_1$, and by the Cauchy–Schwarz inequality $c(x_1 + x_2)^2 \leq (kx_1^2 + jx_2^2)\, c\,(1/k + 1/j) = kx_1^2 + jx_2^2$. Hence $\Delta \geq (jx_2 - kx_1)^2 \geq 0$.

Case (b). Let x have weights a_2, a_3 to $2, 3$ and y weights b_1, b_3 to $1, 3$, and put $A = 1 + a_2 + a_3$, $B = 1 + b_1 + b_3$, $\alpha = a_2a_3/A$, $\beta = b_1b_3/B$. Composing a core edge of weight c on 12 with a state replaces (h, j, k) by $(h + c, j, k)$ and keeps d, so the state partition function of a background with core edges added is $\varphi(H, J, K) + d$ with

$$\varphi(H, J, K) = 1 + H + J + K + HJ + HK + JK.$$

With e and f contracted, x merges with 1 and y with 2, so the other edges at x, y become core edges 12 (weights a_2 and b_1), 13 (a_3) and 23 (b_3). With e deleted, x is a vertex adjacent only to $2, 3$, equivalent to the factor A times a core edge 23 of weight α; similarly for f and y. Therefore

$$\Delta = AB\big[(\varphi_1 + d)(\varphi_2 + d) - (\varphi_3 + d)(\varphi_4 + d)\big] = AB\big[\varphi_1\varphi_2 - \varphi_3\varphi_4 - dM\big],$$

where $\varphi_1 = \varphi(h + a_2, j + a_3 + \beta, k)$, $\varphi_2 = \varphi(h + b_1, j, k + b_3 + \alpha)$, $\varphi_3 = \varphi(h + a_2 + b_1, j + a_3, k + b_3)$, $\varphi_4 = \varphi(h, j + \beta, k + \alpha)$, and

$$M = \varphi_3 + \varphi_4 - \varphi_1 - \varphi_2 = a_2(b_3 - \beta) + b_1(a_3 - \alpha) + a_3b_3 + \alpha\beta \ \geq 0,$$

since $\beta \leq b_3$ and $\alpha \leq a_3$. Thus Δ is nonincreasing in d. For $h, j, k > 0$ let $d_{\max} = \frac{1}{2}(\sqrt{E^2 + 4hjk} - E)$ be the largest value allowed by (25). The state $(h, j, k, d_{\max})$ is the state of a single outside vertex z with weights $\zeta_1 = hj/d_{\max}$, $\zeta_2 = hk/d_{\max}$, $\zeta_3 = jk/d_{\max}$ to $1, 2, 3$: indeed equality in (25) gives $1 + \zeta_1 + \zeta_2 + \zeta_3 = hjk/d_{\max}^2$, whence $\zeta_1\zeta_2/(1 + \zeta_1 + \zeta_2 + \zeta_3) = h$, and similarly

for j, k and $d_{\max} = \zeta_1\zeta_2\zeta_3/(1+\zeta_1+\zeta_2+\zeta_3)^2$. For this background the whole graph is $K_{3,3}$ on $\{1,2,3\} \cup \{x,y,z\}$, and its Rayleigh difference for the disjoint edges $1x, 2y$ is nonnegative by Erickson's theorem. Hence $\Delta(h,j,k,d) \geq \Delta(h,j,k,d_{\max}) \geq 0$ whenever $h,j,k > 0$. If one of h,j,k vanishes then $d = 0$ by Lemma 7.2, and since $d_{\max}$ extends continuously by zero to the boundary of the orthant, $\Delta(h,j,k,0) \geq 0$ follows by continuity.

In both cases the actual Rayleigh difference is $w_\varnothing(B)^2\Delta \geq 0$. □

7.2 Finite rank-one hosts

Corollary 7.3 (Finite hosts)**.** *Let Q be a finite simple graph on k vertices with positive activities, attached to a background (H,r) with m positive profile coordinates through cross edges $s_h r_v$ as in* (10)*. If $k+m \leq 7$, or if Q has a vertex cover of size at most three containing every vertex with $s_h > 0$, then any two distinct edges of Q are nonpositively correlated in the forest law of $G(H,Q;r,s)$.*

Proof. By Theorem 4.2 the joint law of the edges of Q equals their law in $G_*(Q;\lambda,s)$, a simple graph on $k+m$ vertices with positive activities once zero activities are omitted. Every site is adjacent only to vertices with $s_h > 0$, so a vertex cover of Q containing these vertices covers G_*. Erickson's theorem in the first case and Theorem 7.1 in the second give the conclusion. □

For example, four vertices a,b,c,d with arbitrary positive activities on all six edges among them, attached with edges $t_h r_v$ to an arbitrary weighted background whose profile has at most three positive coordinates, have all pairs among these six edges nonpositively correlated. The endpoint-clique theorem treats arbitrary profiles with product-form cross edges among a,b,c,d, and this corollary treats arbitrary cross edges with at most three attachment vertices.

7.3 Two-sums

Recall that the cycle matroid of a 2-connected graph is obtained by iterated 2-sums from the cycle matroids of its 3-connected components, which are polygons, bonds and simple 3-connected graphs, each including its virtual edges [17, Section 8.3]. Forests of a graph are the independent sets of its cycle matroid, so the I-Rayleigh property is a property of the matroid.

Theorem 7.4 (Two-sum closure)**.** *Let G be a finite graph, loops and parallel edges allowed. Suppose each simple 3-connected graph in the decomposition of each 2-connected block of G, including its virtual edges, has at most seven vertices or a vertex cover of size at most three. Then for all positive weights every two distinct edges of G satisfy* $\operatorname{Cov}(X_e, X_f) \leq 0$.

Proof. A loop never lies in a forest, and its indicator has zero covariance with every edge. A cycle lies inside one block, so the forest law is the product of the forest laws of the blocks, and edges in different blocks are independent. A block with one edge has nothing to check. A polygon with weights w_i has $Z = \prod_i(1+w_i) - \prod_i w_i$, and a direct computation gives $Z_eZ_f - ZZ_{ef} = \prod_{i\neq e,f}(1+w_i)\prod_{i\neq e,f} w_i \geq 0$; a bond has $Z = 1+\sum_i w_i$ and $Z_eZ_f - ZZ_{ef} = 1$. The 3-connected components are I-Rayleigh by Erickson's theorem or Theorem 7.1. Wagner [27, Theorem 5.8(b)] and, independently, Cocks [7] proved that 2-sums of matroids preserve the Rayleigh property for independent sets. By induction along the decomposition, the cycle matroid of each block is I-Rayleigh, which is the claim. □

Since series-parallel graphs are exactly the graphs whose 2-connected blocks decompose into polygons and bonds alone, Theorem 7.4 contains the series-parallel theorem of Semple–Welsh and Erickson [22, 9].

Data availability. All arguments are contained in this article and the cited works. The script `code/check_identities.py` accompanying this article verifies in exact symbolic arithmetic the finite identities of Section 7 (the state composition, the Rayleigh quadratic of Lemma 7.2 and its discriminant, the case (a) formula, the case (b) formula and slope, and the identification of the extreme state with $K_{3,3}$), the worked replacement example of Section 4 and the two-hub formula (17) on $K_{2,3}$.

Use of AI tools. OpenAI ChatGPT/Codex and Anthropic Claude assisted with literature discovery, proof exploration, drafting, editing and exact verification code. Separate AI-assisted internal checks were performed. The author takes full responsibility for the content.

Prior versions. Sections 2–3 and the vertex-cover theorem revise material first posted in the Luminary Research Archive [19]; Sections 4–6 revise a second Archive manuscript [20].

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