Spectral escape for interacting and coalescing signed potential forms

Scott Schubr
Luminary Oddities

October 25, 2026

Abstract

We determine how the spectrum of a concentrating signed potential form splits between escaping eigenvalues and a bounded remainder as its centers coalesce. Zero profile integrals alone do not rule out escape, and relative leading-order estimates need not determine logarithmic shifts to bounded error. For two unequal asymmetric zero-mean profiles separated by more than their width, we obtain logarithmic formulas with bounded additive remainders on both escaping scales throughout the mesoscopic collision regime. Exact primitive correlations and Green geometry, together with a cancellation in a quadratic spectral pencil, control mixed trace terms that can exceed the slower scale. For a noncancelling cluster at bounded separation-to-width ratios and zero background, we identify the operator remaining after removal of the actual escaping pair. Aggregate inverse-square tails determine its essential and absolutely continuous channels; vanishing aggregate tails give trace-norm convergence. Fixed-threshold eigenvalue counts and signed spectral masses quantify how a bounded bulk balances the escaping pair's logarithmic trace drift. At exact cancellation, a primitive estimate controls the complete operator for measurable profiles. An anchored cusp family exhibits regularity-dependent transition scales and a logarithmic endpoint, with a vanishing rank-two approximation error along every joint path. Identical even profiles supply explicit scalar matching through all collision ratios.

2020 Mathematics Subject Classification: 47A55, 34L20, 47B50.
Keywords: signed potential forms, spectral concentration, Dirichlet Green matrix, coalescing centers, eigenvalue asymptotics, Hardy inequality.

1 Introduction

1.1 What escapes when signed profiles collide?

Let $I = (a, b)$ be a bounded interval and equip $H = H_0^1(I)$ with derivative energy. For a real potential V_ε, define the compact selfadjoint energy operator C_ε by

$$e(C_\varepsilon u, z) = q_\varepsilon(u, z), \qquad e(u, z) = \int_a^b u'\overline{z'}\,\mathrm{d}x, \qquad q_\varepsilon(u, z) = \tfrac{1}{2}\int_a^b V_\varepsilon u\overline{z}\,\mathrm{d}x. \tag{1}$$

Its eigenvalues are the extremal signed potential energies relative to $\int |u'|^2$. The corresponding equation is $-u'' = V_\varepsilon u/(2\lambda)$ for a nonzero energy eigenvalue λ. We consider finitely many concentrating profiles,

$$V_\varepsilon(x) = \sum_{j=1}^{m} \varepsilon^{-2} v_j\big((x - r_{j,\varepsilon})/\varepsilon\big) + W_\varepsilon(x), \tag{2}$$

where ε is their width, the centers may approach one another, and W_ε is real and uniformly bounded whenever it is included. The zero-mean profiles satisfy

$$\int_{\mathbb{R}} v_j = 0, \qquad |v_j(t)| \le C_j/(1+t^2), \qquad P_j(t) = \int_{-\infty}^{t} v_j(s)\,\mathrm{d}s.$$

The results that resolve logarithmic terms specify independent left and right inverse-square coefficients. Each fixed-ε potential is bounded, so compactness follows from the embedding $H_0^1(I) \hookrightarrow L^2(I)$.

The rational profile

$$v_0(t) = \frac{2(t^2-1)}{(1+t^2)^2}, \qquad P_0(t) = \frac{-2t}{1+t^2}$$

illustrates why signed concentration is different from a mass-only approximation. Its negative core and positive wings have total integral zero, while $\int P_0^2 = 2\pi$. A concentrating copy nevertheless produces one diverging eigenvalue of each sign. Opposite coincident copies cancel identically; displacing them restores a nonzero primitive. The relevant questions are therefore both quantitative and structural: what determines the escaping scales during collision, what bounded operator survives escape, and how does profile regularity alter a degenerate cancellation?

The interval geometry enters through the Dirichlet Green kernel

$$G(r,s) = \frac{(\min(r,s)-a)(b-\max(r,s))}{b-a}. \tag{3}$$

For marked centers the matrix $\mathbf{G} = (G(r_i,r_j))$ is the energy Gram matrix of their point-evaluation representatives. For example, centers $-1/2, 1/2$ in $(-1,1)$ give $\mathbf{G} = \left(\begin{smallmatrix} 3/8 & 1/8 \\ 1/8 & 3/8 \end{smallmatrix}\right)$, whose two eigenvalues are $1/2$ and $1/4$. Even before collision, replacing this matrix by its diagonal loses the leading interaction. As the gap shrinks, one Green direction degenerates; an error small relative to the fast scale can then conceal the slower scale or its logarithmic drift.

1.2 The precision law and the surviving bulk

The first main result resolves two unequal asymmetric profiles at $r \pm h/2$, with $h \to 0$ and $h/\varepsilon \to \infty$. Theorem 2.2 gives exactly two escaping eigenvalues of each sign, with a uniformly bounded complementary spectrum, and

$$\lambda_f^{\pm} = \pm a_\varepsilon - p_\varepsilon + O(1), \qquad \lambda_s^{\pm} = \pm b_\varepsilon - t_\varepsilon + O(1), \qquad a_\varepsilon \asymp \varepsilon^{-1/2}, \quad b_\varepsilon \asymp \sqrt{h/\varepsilon}.$$

The point is the additive precision on both scales. A relative leading estimate for each pair permits errors that grow with its scale; it does not by itself determine the logarithmic drift to bounded error. The coefficients above retain exact first-primitive correlations and Green geometry; the shifts are explicit two-sided tail increments. A cancellation in a quadratic spectral pencil controls mixed trace entries that can exceed the slow scale, yielding bounded errors under $h \to 0$ and $h/\varepsilon \to \infty$ alone.

A bounded complementary spectrum can contain increasing numbers of nonvanishing eigenvalues. For bounded collision ratios and a nonzero aggregate primitive, Theorem 3.2 identifies the actual spectrally deflated operator after an exact anchor/trace energy transport. With zero background its strong limit is

$$[(I - P_w)K_{\boldsymbol{c}}(I - P_w)] \oplus 0,$$

where w' is the normalized aggregate primitive and $K_{\boldsymbol{c}}$ is the two-half-line energy operator of the aggregate profile. The aggregate tails determine its essential interval and absolutely continuous

channels. If both tails vanish, convergence holds in trace norm. Theorem 3.4 also counts the finite-ε spectrum at fixed thresholds, allowing bounded backgrounds and cancellation. The resulting signed mass law and Proposition 3.6 explain how the bounded part balances the escaping pair's logarithmic trace shift.

At an exact cancellation configuration, Proposition 4.2 controls the whole operator by the aggregate primitive, under measurable profile hypotheses. Theorem 4.3 resolves a degenerate direction where the first primitive displacement variation vanishes: for the anchored primitive $\chi(t)|t|^{\beta}$, the same second-order stencil has a continuum of critical scales, a logarithmic endpoint at $\beta = 3/2$, and a full $O(\delta^2)$ operator error. The relative pair laws hold on every joint path of width and offset approaching zero. Section 5 provides scalar matching through all collision ratios for identical even profiles, with the full parity estimates in Appendix A.

1.3 Mechanism, predecessors and organization

The common reduction separates functions vanishing at the centers from their Green-tent trace complement. We call this trace-zero subspace the reserve. Hardy's inequality bounds the entire reserve. First-primitive energies determine its coupling to the traces; critical tails generate logarithmic trace terms. Spectral deflation then removes the actual escaping directions, and logarithmic coordinates identify the remaining inverse-square channels. Near cancellation the same integration-by-parts estimate transfers primitive regularity into an error bound for the original energy operator.

The author's research preprint *Collision and cancellation spectra for concentrated energy forms* (Luminary Research Archive, edition 2, 2026; https://research.luminaryoddities.com/collision-cancellation-spectra-concentrated-energy-forms/v/2) already supplies the collision-uniform bounded-error matrix model, compatible trace factorization and smooth multipole thresholds. We recall the needed model with its proof in Section 2. The present precision law retains the exact quadratures while extracting bounded-remainder logarithmic terms on two collapsing scales. The cancellation section recovers the inherited smooth dipole scale under measurable hypotheses and treats the additional anchored regularity boundary. An earlier version of this work, *Spectral escape and mean crossover for concentrating signed potential forms* (same Archive, edition 4, 2026; https://research.luminaryoddities.com/spectral-transitions-concentrated-zero-mean-energy-forms/v/4), develops the fixed-separated-center, mean-crossover and mode-energy laws. Here the exposition concentrates on collision, the bounded channel and cancellation.

Primitive-energy effects in one-dimensional weak binding are classical: Simon [11] and Blankenbecler, Goldberger and Simon [1] supply essential antecedents. In our normalization the single-center rescaling of the energy eigenvalue equation combines reciprocal coupling with an expanding Dirichlet interval. Golovaty [6] studies nonanalytic width-dependent threshold families. The anchored stencil estimates here retain a uniform vanishing error for the full bounded-interval energy operator. Golovaty and Hryniv [7] analyze escaping negative Schrödinger eigenvalues despite norm-resolvent convergence, and Kielty [9] treats singular limits of fixed monotone indefinite-form families. These comparisons distinguish the operator, parameter regime and quantitative conclusion.

Coalescing regularizations of singular interactions are developed by Exner, Neidhardt and Zagrebnov [2], Golovaty [5] and Zolotaryuk [12]. Fassari, Gadella, Nieto and Rinaldi [3] give a coalescence limit for a nonlocal double-δ' well. The forms studied here are signed multiplication forms measured in derivative energy. For the bounded part, the Hardy–Mellin, trace-class and inverse-square counting arguments are classical; their precise roles are identified where used. The collision laws retain the primitive overlap, the finite Green geometry and the full bounded spectral remainder.

Section 2 proves the unequal-profile precision law and the bounded-ratio pair formula. Section 3 identifies the deflated operator, its spectral channels, counts and signed masses. Section 4 treats cancellation and regularity. Section 5 gives the identical-even scalar matching and examples. Appendix A supplies its uniform parity estimates.

2 Unequal asymmetric collisions

For unequal profiles the primitive overlaps and the trace need to be kept together as centers merge. We first recall the uniform finite matrix model and compatible trace factorization from the author's research preprint *Collision and cancellation spectra for concentrated energy forms*, cited in Section 1.3. Their bounded-error conclusion supplies the all-ratio description. We then refine its resolved two-center regime to scalar logarithmic expansions with bounded remainders.

2.1 The uniform primitive matrix model

Fix a finite number $m \geq 1$ of real measurable profiles v_j with zero integral and $|v_j(t)| \leq b_j/(1+t^2)$. Put $P_j(t) = \int_{-\infty}^{t} v_j$ and $c_j = \sup_t |tP_j(t)| < \infty$. On a fixed interval $I = (A, B)$, choose arbitrary distinct interior centers $r_1 < \cdots < r_m$, allowed to depend on ε. Let W_ε be real with a common bound W_*, and consider the energy operator with form $q_\varepsilon(u, z) = \frac{1}{2}\int_I V_\varepsilon u\overline{z}\,\mathrm{d}x$, where

$$V_\varepsilon(x) = \sum_j \varepsilon^{-2} v_j((x - r_j)/\varepsilon) + W_\varepsilon(x).$$

Let H_* be the simultaneous value-zero space at these centers, called the reserve, and let $e_1, \ldots, e_m$ be the canonical real energy-orthonormal basis of its Green-tent complement. Its evaluation matrix is $\mathbf{E} = \mathbf{G}^{1/2}$, $E_{ji} = e_i(r_j)$. Let Π subtract the mean on every marked cell and set

$$K_j(x) = \varepsilon^{-1} P_j((x - r_j)/\varepsilon), \qquad M_{ij} = \int_I (\Pi K_i)(\Pi K_j)\,\mathrm{d}x,$$
$$\mathbf{S} = \tfrac{1}{2}(\mathbf{EME})^{1/2}, \qquad A_{il} = q_\varepsilon(e_i, e_l), \qquad \mathbf{F} = \begin{pmatrix} 0 & \mathbf{S} \\ \mathbf{S} & \mathbf{A} \end{pmatrix}. \tag{4}$$

Theorem 2.1 (Uniform primitive matrix model)**.** *There is an isometry $\mathcal{J}_\varepsilon : \mathbb{C}^{2m} \to H_0^1(I)$ such that*

$$\|C_\varepsilon - \mathcal{J}_\varepsilon \mathbf{F} \mathcal{J}_\varepsilon^*\| \leq 2\sum_j b_j + 2\sum_j c_j + \frac{W_*(B-A)^2}{\pi^2}.$$

The bound is independent of every center gap, boundary distance and primitive rank. At most m eigenvalues of either sign lie outside a fixed bounded interval, and the zero-completed signed variational sequences of the original operator and the finite matrix model differ by at most this bound. Here the positive list is ordered downwards and the negative list from most negative towards zero, with missing entries filled by zero. Coincident centers are handled by summing their profiles and marking only distinct locations, with m replaced by the number of those locations in the finite model.

Proof. Hardy at each center and Poincaré give, on the reserve,

$$\|B\|_\varepsilon \leq 2\sum_j b_j + \frac{W_*(B-A)^2}{2\pi^2}.$$

For an arbitrary energy function the one-sided value-difference Hardy inequality is

$$\|(u-u(r_j))/(x-r_j)\|_2 \le 2\,\|u'\|_2 .$$

Integration by parts gives, for $z \in H_*$,

$$q_\varepsilon(u,z) = -\tfrac{1}{2}\sum_j \int K_j(u'\overline{z} + u\overline{z'})\,\mathrm{d}x + \tfrac{1}{2}\int W_\varepsilon u\overline{z}\,\mathrm{d}x.$$

Replace u in its second primitive term by $u(r_j)$ plus its value difference. Each of the two remaining primitive pairings is bounded by $2c_j\,\|u'\|_2\,\|z'\|_2$ before the factor $1/2$. Thus the actual trace-to-reserve coupling differs from $\mathbf{T}_0 = -\mathbf{R}\mathbf{E}/2$ by at most $2\sum_j c_j + W_*(B-A)^2/(2\pi^2)$, where the reserve vector R_j has derivative $R_j' = \Pi K_j$. The derivative map of H_* onto cell-mean-zero functions is an isometry: integrate on each cell, using its zero mean to sew zero endpoint values. Hence $\mathbf{T}_0^*\mathbf{T}_0 = \mathbf{S}^2$. Extend the polar partial isometry across its kernel into the infinite-dimensional reserve to obtain an isometry $\mathbf{U}$ with $\mathbf{T}_0 = \mathbf{U}\mathbf{S}$. Combining this isometry with the trace basis yields $\mathcal{J}_\varepsilon$. The residual block has the bounded reserve and coupling error and zero trace block, so its norm is at most their summed bounds. The codimension-m reserve gives the signed count by minmax; compact selfadjoint norm perturbation gives the variational comparison. □

The same integration by parts also gives the compatible trace identity needed below. Set $B_{ji} = \int_I K_j e_i'\,\mathrm{d}x$ and $\mathbf{C} = \mathbf{E}\mathbf{B}/2$. Then

$$\mathbf{A} = -\mathbf{C} - \mathbf{C}^{\mathsf{T}} + O(1) \tag{5}$$

uniformly in the moving geometry. Indeed replace each undifferentiated spline value in the trace integral by its value at r_j plus its difference. The nodal terms give the displayed matrices; the two remaining terms are bounded in operator norm by the same value-difference Hardy estimate. A bounded background adds a bounded trace operator.

Cancellation is retained in the receiver. For example, opposite profiles at a common center sum to zero, leaving only the bounded background. The finite matrix determines which candidates escape when its primitive rank changes.

2.2 An unequal asymmetric mesoscopic logarithmic refinement

Fix $A < r < B$, two fixed real measurable profiles v_1, v_2, each nonzero on a set of positive measure and of zero integral, $|v_j(t)| \le C/(1+t^2)$, and two-sided tails $v_j(t) = \alpha_j^{\pm}/t^2 + O(|t|^{-2-\eta_j^{\pm}})$ with all $\eta_j^{\pm} > 0$. Let $r_1 = r - h/2$, $r_2 = r + h/2$, $h \to 0$, and $c = h/\varepsilon \to \infty$. The centers therefore remain in a fixed compact interior region. Write $P_j(t) = \int_{-\infty}^t v_j$, $d_j = \int_{\mathbb{R}} P_j^2 > 0$, $K_j(x) = \varepsilon^{-1}P_j((x-r_j)/\varepsilon)$, $\ell = B - A$, and

$$N_{ij} = \int_A^B K_i K_j\,\mathrm{d}x, \qquad w_j = \ell^{-1/2}\int_A^B K_j\,\mathrm{d}x, \qquad \mathbf{E} = \mathbf{G}_h^{1/2}.$$

Here $\mathbf{G}_h$ is the Dirichlet Green matrix at r_1, r_2. Let e_1, e_2 be the canonical energy-orthonormal trace basis and set

$$B_{jl} = \int_A^B K_j e_l'\,\mathrm{d}x, \qquad \mathbf{C} = \mathbf{E}\mathbf{B}/2, \qquad \mathbf{H} = \mathbf{E}(\mathbf{N} - \mathbf{w}\mathbf{w}^{\mathsf{T}})\mathbf{E}/4.$$

These retain the unprojected overlaps and the global constant component. Denote the eigenvalues of $\mathbf{H}$ by $a^2 > b^2 > 0$ and choose real unit eigenvectors $\mathbf{x}_f, \mathbf{x}_s$. Set $t_i = \sum_l (\mathbf{x}_i)_l e_l$, $y_{ij} = t_i(r_j)$ and $p = \mathbf{x}_f^{\mathsf{T}}\mathbf{C}\mathbf{x}_f$, $t = \mathbf{x}_s^{\mathsf{T}}\mathbf{C}\mathbf{x}_s$.

Theorem 2.2 (Two-term unequal mesoscopic refinement)**.** *For sufficiently small ε, the full energy-Riesz operator has exactly two escaping eigenvalues of each sign, with bounded remaining spectrum. They satisfy*

$$\lambda_f^{\pm} = \pm a - p + O(1), \qquad \lambda_s^{\pm} = \pm b - t + O(1).$$

The same conclusions hold after adding any uniformly bounded real background. The bounded errors hold under $h \to 0$ and $h/\varepsilon \to \infty$ alone. Moreover $a \asymp \varepsilon^{-1/2}$, $b \asymp \sqrt{h/\varepsilon}$. The corrections can be replaced to bounded error by explicit tail-log increments: put

$$L_j(u) = -\alpha_j^{\operatorname{sgn} u} \log(1 + |u|), \quad L_j(0) = 0, \qquad r_0 = A, \quad r_3 = B,$$

then for $i = f, s$,

$$\mathbf{x}_i^{\mathsf{T}} \mathbf{C} \mathbf{x}_i = \frac{1}{2} \sum_{j=1}^{2} y_{ij} \sum_{q=0}^{2} t_i'|_{(r_q, r_{q+1})} \left[L_j\left(\frac{r_{q+1} - r_j}{\varepsilon} \right) - L_j\left(\frac{r_q - r_j}{\varepsilon} \right) \right] + O(1).$$

Thus all geometry-dependent logarithmic terms, including ones that cannot be replaced uniformly by constant facing-tail weights, are retained.

The earlier resolved law determines the leading Green scales to relative error. Here the logarithmic shifts are resolved to bounded error even when the mixed trace coupling is larger than the slow scale. Keeping $\mathbf{H}$ and $\mathbf{C}$ compatible is essential to that refinement.

Proof. Use Theorem 2.1 and the compatible trace identity (5). The cell-constant space has the $L^2(I)$-orthonormal basis $e_1', e_2', \ell^{-1/2}$. Hence its primitive Gram is $\mathbf{B}\mathbf{B}^{\mathsf{T}} + \mathbf{w}\mathbf{w}^{\mathsf{T}}$ exactly. The principal coupling square and compatible trace are therefore

$$\mathbf{S}^2 = \mathbf{H} - \mathbf{C}\mathbf{C}^{\mathsf{T}}, \qquad \mathbf{A}_{\mathrm{pr}} = -\mathbf{C} - \mathbf{C}^{\mathsf{T}}.$$

The full operator differs by uniformly bounded norm from the finite block $[0, \mathbf{S}; \mathbf{S}, -\mathbf{C} - \mathbf{C}^{\mathsf{T}}]$, extended by zero. These estimates bound every reserve, coupling and trace remainder.

Write $L = \log(2/\varepsilon)$ and $l = \log(2 + c)$. The envelope convolution and fixed-boundary tail estimates yield

$$\varepsilon \mathbf{N} = \operatorname{diag}(d_1, d_2) + O(l/c + \varepsilon), \qquad |\mathbf{w}| = O(L),$$

Indeed the whole-line diagonal is d_j/ε and the omitted boundary tails are $O(1)$ because the centers stay in a compact interior region. For the mixed entry, splitting at 0 and c gives

$$\int_{\mathbb{R}} \frac{\mathrm{d}z}{(1 + |z|)(1 + |z - c|)} = 2 \log(1 + c) \left(\frac{1}{c} + \frac{1}{c + 2} \right).$$

The two exterior pieces each equal $\log(1 + c)/c$; partial fractions give the middle piece $2 \log(1 + c)/(c + 2)$. This gives the stated mixed bound. Each global mean is $O(L)$ by integrating the primitive envelope. Thus $\varepsilon(\mathbf{N} - \mathbf{w}\mathbf{w}^{\mathsf{T}}) = \operatorname{diag}(d_j) + o(1)$ is uniformly positive. The explicit Green kernel gives $\det \mathbf{G}_h = (r_1 - A)(B - r_2)h/\ell$ and $\operatorname{tr} \mathbf{G}_h \to 2(r - A)(B - r)/\ell > 0$. Its eigenvalues therefore have orders 1 and h. Green congruence before taking eigenvalues gives $a \asymp \varepsilon^{-1/2}$ and $b \asymp \sqrt{c}$, with $b/a \asymp \sqrt{h} \to 0$.

Here are the geometry bounds needed below. The fast nodal vector $\mathbf{y}_f = \mathbf{E}\mathbf{x}_f$ is bounded and its two entries differ by $O(h)$. Indeed $\mathbf{G}_h(\mathbf{N} - \mathbf{w}\mathbf{w}^{\mathsf{T}})\mathbf{y}_f = 4a^2 \mathbf{y}_f$, the two rows of $\mathbf{G}_h$ differ by $O(h)$, the normalized primitive matrix is bounded, and εa^2 is bounded below. Hence all three slopes of t_f are $O(1)$. Its common nodal part is bounded away from zero: unit energy and an $O(h)$

middle difference leave a nonzero amount of outer energy. Energy orthogonality to t_f, together with $|t_s(r_2) - t_s(r_1)| \le \sqrt{h}$, then gives $\mathbf{y}_s = O(\sqrt{h})$; its outer slopes are $O(\sqrt{h})$ and its middle slope is $O(h^{-1/2})$. For clarity the orthogonality uses the exact nodal stiffness form: its outer terms force the common nodal part of $\mathbf{y}_s$ to be $O(\sqrt{h})$, while the middle term is $O(h)O(\sqrt{h})/h = O(\sqrt{h})$.

The primitive integrals on the middle cell are $O(l)$; those on the outer cells are $O(L)$. In the $\mathbf{H}$ eigenbasis write

$$\mathbf{C} = \begin{pmatrix} p & q \\ u & t \end{pmatrix}.$$

The preceding nodal/slope bounds imply

$$p = O(L), \quad q = O(l/\sqrt{h} + \sqrt{h}L), \quad u = O(\sqrt{h}L), \quad t = O(l + hL).$$

In particular $p/a \to 0$, $t/b \to 0$, and $\|\mathbf{C}\|/a \to 0$. Since $J_j(u) = \int_0^u P_j = L_j(u) + O(1)$ on each half-line, replacing the exact primitive cell integrals by the displayed L_j increments changes either diagonal by $O(1)$: products $y_{fj}t'_f$ are bounded, and products $y_{sj}t'_s$ are bounded even on the middle cell. This proves the tail-log formula.

It remains to control the two scales without an invalid absolute trace perturbation. Eliminating the first two coordinates in the finite block eigenvalue equation gives the characteristic polynomial

$$P(\lambda) = \det[(\lambda\mathbf{I} + \mathbf{C})(\lambda\mathbf{I} + \mathbf{C}^{\mathsf{T}}) - \mathbf{H}].$$

This polynomial identity extends through $\lambda = 0$. Direct expansion gives

$$\begin{aligned} P(\lambda) = {} & [(\lambda + p)^2 - a^2][(\lambda + t)^2 - b^2] - 2qu(\lambda + p)(\lambda + t) \\ & + q^2u^2 - a^2u^2 - b^2q^2. \end{aligned}$$

The cancellation of an apparent a^2q^2 term is essential.

Near $\lambda = \pm a - p$, divide the perturbing terms by the base root slope, comparable with a^3. Their root displacement is bounded by

$$E_f = C_0 \left(\frac{|qu| + u^2}{a} + \frac{b^2q^2 + q^2u^2}{a^3} \right) = o(1).$$

Near $\lambda = \pm b - t$, the other base factor is comparable with $-a^2$ even when $p \gg b$. Dividing by the root slope comparable with a^2b gives

$$E_s = C_0 \left(\frac{|qu|(b + L)}{a^2} + \frac{u^2}{b} + \frac{bq^2}{a^2} + \frac{q^2u^2}{a^2b} \right) = o(1).$$

Indeed these estimates are bounded by a fixed multiple of $l^2/\sqrt{c} + \sqrt{\varepsilon}L^4$, which tends to zero on every stated path. The bounds persist uniformly in unit neighborhoods of the four base roots. At the two endpoints of a sufficiently large fixed multiple of each error radius (with an additional positive vanishing radius if an error is zero), the base polynomial dominates the error and changes sign. Thus the intermediate value theorem gives a root near each base root. The neighborhoods are disjoint because $a/b \to \infty$ and $b \to \infty$; a quartic has no further roots. Consequently the finite block roots are $\pm a - p + o(1)$ and $\pm b - t + o(1)$.

Restoring the uniformly bounded full-operator remainder yields the claimed expansions. Each of the four branches diverges with its stated sign. The simultaneous two-trace Hardy compression permits at most two eigenvalues per sign outside a fixed interval, so these are exactly the escaping branches and all other spectrum is bounded. □

2.3 An asymmetric cluster at bounded separation-to-width ratios

Let $I = (A, B)$ and fix $r \in (A, B)$. Fix a finite $m \geq 1$. Let $v_1, \ldots, v_m$ be real measurable profiles such that, for $1 \leq j \leq m$,

$$\int_{\mathbb{R}} v_j = 0, \qquad |v_j(t)| \leq C_j(1+t^2)^{-1}, \qquad v_j(t) = \frac{\alpha_j^{\pm}}{t^2} + O(|t|^{-2-\eta_j^{\pm}}) \quad (t \to \pm\infty), \quad \eta_j^{\pm} > 0.$$

The profiles may be asymmetric. Put $P_j(t) = \int_{-\infty}^{t} v_j(u)\,\mathrm{d}u$, $\alpha^{\pm} = \sum_{j=1}^{m} \alpha_j^{\pm}$, and, for $\mathbf{c} = (c_1, \ldots, c_m)$,

$$V_{\mathbf{c}}(t) = \sum_{j=1}^{m} v_j(t - c_j), \quad F_{\mathbf{c}}(t) = \sum_{j=1}^{m} P_j(t - c_j), \quad D_{\mathbf{c}} = \int_{\mathbb{R}} |F_{\mathbf{c}}(t)|^2\,\mathrm{d}t.$$

Theorem 2.3 (A bounded-ratio asymmetric cluster)**.** *Let $\mathcal{C} \subset \mathbb{R}^m$ be compact and suppose $\inf_{\mathbf{c}\in\mathcal{C}} D_{\mathbf{c}} = d_0 > 0$. Let $W_{\varepsilon,\mathbf{c}}$ be real measurable with a common $L^\infty(I)$ bound. On $H_0^1(I)$ with derivative energy let $C_{\varepsilon,\mathbf{c}}$ represent the form*

$$q_{\varepsilon,\mathbf{c}}(u, z) = \frac{1}{2}\int_A^B \left[\varepsilon^{-2} V_{\mathbf{c}}\left(\frac{x-r}{\varepsilon}\right) + W_{\varepsilon,\mathbf{c}}(x)\right] u\overline{z}\,\mathrm{d}x.$$

Put $\tau = (r-A)(B-r)/(B-A)$. There exist M and $\varepsilon_0 > 0$, independent of $\mathbf{c} \in \mathcal{C}$, such that for $0 < \varepsilon < \varepsilon_0$ exactly one eigenvalue lies above M and exactly one below $-M$. Both are simple, the entire remaining spectrum lies in $[-M, M]$, and uniformly in $\mathbf{c}$,

$$\lambda_{\varepsilon,\mathbf{c}}^{\pm} = \pm\frac{\sqrt{\tau D_{\mathbf{c}}}}{2\sqrt{\varepsilon}} - \frac{\alpha^+(r-A) + \alpha^-(B-r)}{2(B-A)}\log(1/\varepsilon) + O(1).$$

In particular $\mathbf{c}$ may depend arbitrarily on ε inside $\mathcal{C}$.

Proof. Write $\eta = \min(1, \min_{1\leq j\leq m}\eta_j^+, \min_{1\leq j\leq m}\eta_j^-) > 0$. Boundedness of the offsets gives, uniformly in $\mathbf{c} \in \mathcal{C}$,

$$|V_{\mathbf{c}}(t)| \leq C(1+t^2)^{-1}, \quad \int V_{\mathbf{c}} = 0, \quad V_{\mathbf{c}}(t) = \alpha^{\pm}/t^2 + O(|t|^{-2-\eta}) \quad (t \to \pm\infty).$$

Thus $F_{\mathbf{c}} = O((1+|t|)^{-1})$ and $F_{\mathbf{c}}(t) = -\alpha^{\pm}/t + O(|t|^{-1-\eta})$ on the corresponding tails. On bounded sets it is uniformly bounded; its value at zero need not vanish. For $J_{\mathbf{c}}(t) = \int_0^t F_{\mathbf{c}}(u)\,\mathrm{d}u$,

$$J_{\mathbf{c}}(t) = -\alpha^{\pm}\log(1+|t|) + O(1) \quad (t \to \pm\infty),$$

with uniform errors. Translation continuity in L^2 makes $D_{\mathbf{c}}$ continuous, and its upper bound is uniform as well.

Let $H_* = \{u : u(r) = 0\}$ and let $e_r = G_r/\sqrt{\tau}$ be its energy-unit trace complement. The common envelope gives $|\varepsilon^{-2}V_{\mathbf{c}}((x-r)/\varepsilon)| \leq C/(x-r)^2$ away from r. Two one-sided Hardy inequalities and Poincare therefore bound the compression $R_{\varepsilon,\mathbf{c}}$ to H_* uniformly. In the decomposition $H_* \oplus \mathbb{C}e_r$ write

$$C_{\varepsilon,\mathbf{c}} = \begin{pmatrix} R_{\varepsilon,\mathbf{c}} & T_{\varepsilon,\mathbf{c}} \\ T_{\varepsilon,\mathbf{c}}^* & a_{\varepsilon,\mathbf{c}} \end{pmatrix}.$$

Set $b = T_{\varepsilon,\mathbf{c}}1$. On each of the two fixed cells, write $e_r(x) = \sqrt{\tau} + k(x-r)$. An exact derivative primitive for the profile contribution to b is

$$-\frac{\sqrt{\tau}}{2\varepsilon}F_{\mathbf{c}}(t) - \frac{k}{2}tF_{\mathbf{c}}(t) + \frac{k}{2}J_{\mathbf{c}}(t), \qquad t = (x-r)/\varepsilon.$$

Subtract its cell mean to impose both Dirichlet values. The bounded W contribution has a uniformly bounded derivative primitive. On each cell subtract the appropriate constant logarithmic part of J before centering; it cancels under centering. The remaining correction is bounded by $C[1+|\log(|x-r|+\varepsilon)|]$, with uniformly bounded L^2 norm and mean. Its cross integral with $\varepsilon^{-1}F_{\mathbf{c}}$ is $O(\log^2(1/\varepsilon))$: inside $|x-r|\le\varepsilon$ use $|F_{\mathbf{c}}|\le C$, and outside use $C/|x-r|$. The means of the leading primitive are $O(\log(1/\varepsilon))$, so centering changes its Gram by $O(\log^2(1/\varepsilon))$. Finally the omitted squared primitive tails have mass $O(\varepsilon)$ uniformly. Consequently

$$\|T_{\varepsilon,\mathbf{c}}\|^2=\frac{\tau D_{\mathbf{c}}}{4\varepsilon}+O(\log^2(1/\varepsilon)).$$

The lower bound d_0 yields

$$\|T_{\varepsilon,\mathbf{c}}\|=\frac{\sqrt{\tau D_{\mathbf{c}}}}{2\sqrt{\varepsilon}}+O(\sqrt{\varepsilon}\log^2(1/\varepsilon)).$$

For the trace, integration by parts is exact:

$$a_{\varepsilon,\mathbf{c}}=-\frac{1}{2}\int_A^B\varepsilon^{-1}F_{\mathbf{c}}((x-r)/\varepsilon)(e_r^2)'(x)\,\mathrm{d}x+O(1).$$

The outer endpoint terms vanish; the two inner terms cancel by continuity. On the left $\int\varepsilon^{-1}F\,\mathrm{d}x=\alpha^-\log(1/\varepsilon)+O(1)$, and on the right it equals $-\alpha^+\log(1/\varepsilon)+O(1)$. Replacing $(e_r^2)'$ by its one-sided value at r costs $O(1)$, since its difference is $O(|x-r|)$ and the primitive envelope bounds the product integrally. As $e_r(r)=\sqrt{\tau}$, $e_r'(r-)=\sqrt{\tau}/(r-A)$ and $e_r'(r+)=-\sqrt{\tau}/(B-r)$, it follows that

$$a_{\varepsilon,\mathbf{c}}=-\frac{\alpha^+(r-A)+\alpha^-(B-r)}{B-A}\log(1/\varepsilon)+O(1).$$

Every error is uniform in the compact offset set.

Deleting the uniformly bounded reserve compression leaves the scalar rank-two block with eigenvalues $\pm\|T\|+a/2+O(a^2/\|T\|)$. The last error is $O(\sqrt{\varepsilon}\log^2(1/\varepsilon))=o(1)$ uniformly. Restoring the reserve costs $O(1)$, proving the displayed law. The codimension-one compression bound allows at most one eigenvalue of each sign outside a fixed common interval. The two uniform divergences produce exactly those eigenvalues, counting multiplicity; hence they are simple and all other spectrum is bounded. □

The noncancellation condition is substantive. For $m=2$, take any admissible nonzero v and set $v_1=v$, $v_2=-v$, $c_1=c_2$. Then $F_{\mathbf{c}}=0$ and $V_{\mathbf{c}}=0$ identically, so only the uniformly bounded W remains and there is no escaping pair. More generally $D_{\mathbf{c}}=0$ holds precisely when the shifted profiles sum to zero almost everywhere. Compactness and continuity turn pointwise noncancellation on a compact offset set into the required positive uniform lower bound. Offsets affect the leading primitive energy, while the logarithmic coefficient depends on the summed two-sided tails and the interval geometry. The constants depend on the fixed number m. This result covers bounded collision ratios without reflection symmetry; its scalar precision is uniform on the specified compact set of collision ratios.

3 The bounded spectral bulk

After the escaping pair is removed, the bounded spectrum retains information about the merged profile. Bounded operator norm does not bound its total absolute spectral mass. Nonzero aggregate inverse-square tails produce logarithmically increasing counts inside their spectral bands

and absolute trace mass; the signed trace may cancel. Large-coupling compression limits also hold for indefinite finite-rank perturbations; see C. Koke, *Large Coupling Convergence Beyond Definiteness*, arXiv:2601.18055v2 (2026; https://arxiv.org/abs/2601.18055v2), Theorem 2.1 and Example 2.3. The bounded spectrum is described by a compression of the rescaled profile operator. Its critical tails determine the spectral interval; when their aggregate coefficients vanish, the convergence improves to trace norm.

Fix $a < r < b$ and a finite set of real measurable zero-mean profiles v_j with $|v_j(t)| \le C_j/(1+t^2)$ and two-sided tails $v_j(t) = \alpha_j^{\pm}/t^2 + O(|t|^{-2-\eta_j^{\pm}})$, $\eta_j^{\pm} > 0$. Put $P_j(t) = \int_{-\infty}^{t} v_j$. For offsets $\boldsymbol{c}$ in a fixed compact set $\mathcal{C}$, set

$$V_{\boldsymbol{c}}(t) = \sum_j v_j(t - c_j), \quad F_{\boldsymbol{c}}(t) = \sum_j P_j(t - c_j), \quad D_{\boldsymbol{c}} = \int_{\mathbb{R}} |F_{\boldsymbol{c}}|^2, \quad A^{\pm} = \sum_j \alpha_j^{\pm}.$$

Assume $\inf_{\mathcal{C}} D_{\boldsymbol{c}} > 0$ and take zero background. Let C_ε be the full energy operator on $H_0^1(a,b)$ for potential $\varepsilon^{-2} V_{\boldsymbol{c}_\varepsilon}((x-r)/\varepsilon)$, where $\boldsymbol{c}_\varepsilon \to \boldsymbol{c} \in \mathcal{C}$. Define the homogeneous anchor-zero space

$$X = \dot{H}_0^1(\mathbb{R}_-) \oplus \dot{H}_0^1(\mathbb{R}_+), \qquad e(K_{\boldsymbol{c}} u, z) = \tfrac{1}{2} \int_{\mathbb{R}} V_{\boldsymbol{c}} u \overline{z}\, \mathrm{d}t.$$

Here each half-line space is the derivative-energy completion of its compactly supported smooth functions. Let $w_{\boldsymbol{c}} \in X$ be specified by

$$w_{\boldsymbol{c}}(0) = 0, \qquad w_{\boldsymbol{c}}' = F_{\boldsymbol{c}}/\sqrt{D_{\boldsymbol{c}}}, \qquad P_{\boldsymbol{c}} = \text{energy projection onto } \mathbb{C} w_{\boldsymbol{c}}.$$

The derivative belongs to L^2, so this defines a unit vector in X.

The logarithmic change of variables is the energy form of the classical Hardy–Mellin reduction; compare [4].

Lemma 3.1 (A two-sided Hardy channel)**.** *Let U be real measurable with $|U(t)| \le C/(1+t^2)$ and $t^2 U(t) \to A^{\pm}$ as $t \to \pm\infty$. Its energy operator on X has essential spectrum*

$$[\min(0, 2A^-, 2A^+), \max(0, 2A^-, 2A^+)]$$

and essential norm $\max(2|A^-|, 2|A^+|)$*. It is compact exactly when $A^- = A^+ = 0$. On either half-line alone, its essential interval joins 0 to $2A^{\pm}$, and its essential norm is $2|A^{\pm}|$.*

Proof. Hardy at 0 bounds the form. On each half-line put $t = \sigma \mathrm{e}^y$, $u(\sigma \mathrm{e}^y) = \mathrm{e}^{y/2} f_\sigma(y)$, $\sigma \in \{-1, 1\}$. Integration by parts on compact targets gives

$$e(u,u) = \sum_\sigma \int_{\mathbb{R}} (|f_\sigma'|^2 + |f_\sigma|^2/4)\, \mathrm{d}y, \quad q(u,u) = \sum_\sigma \int_{\mathbb{R}} b_\sigma(y) |f_\sigma(y)|^2\, \mathrm{d}y,$$

where $b_\sigma(y) = \frac{1}{2}\mathrm{e}^{2y} U(\sigma \mathrm{e}^y)$. The mixed derivative integrates to zero. Completion gives an energy unitary, followed by the energy-to-L^2 unitary for $\mathcal{A} = -\mathrm{d}^2/\mathrm{d}y^2 + 1/4$. Thus the two half-line operators become $\mathcal{A}^{-1/2} M_{b_\sigma} \mathcal{A}^{-1/2}$. The multiplier tends to zero at $-\infty$ and to $\gamma_\sigma = A^\sigma/2$ at $+\infty$. Its difference from $T_\sigma = \gamma_\sigma \mathcal{A}^{-1/2} \mathbf{1}_{(0,\infty)} \mathcal{A}^{-1/2}$ is compact: truncate the vanishing multiplier difference to bounded intervals, use local Rellich compactness of $\mathcal{A}^{-1/2} : L^2 \to H^1$, and let the tails tend to zero in uniform norm.

The signed step operator has spectrum contained between 0 and $4\gamma_\sigma$, by positivity of the sandwich and $\left\|\mathcal{A}^{-1/2}\right\| = 2$. The following Weyl sequence also controls the nonlocal square root. Put $B = \mathcal{A}^{-1/2}$, $m = (\xi^2 + 1/4)^{-1/2}$, choose $\chi \in C_c^\infty(-1,1)$ with $\|\chi\|_2 = 1$, and set

$$g_n(y) = n^{-1/2} \chi((y - 3n)/n) \mathrm{e}^{i\xi y}, \qquad \eta_n = \|(B - m) g_n\|_2 \longrightarrow 0.$$

The limit follows by concentration of the Fourier transform and bounded continuity of the multiplier. These unit packets tend weakly to zero and have support in $(2n, 4n)$. Hence $\left\|\mathbf{1}_{(-\infty,0)}Bg_n\right\|_2 \le \eta_n$, and

$$\left\|(T_\sigma - \gamma_\sigma m^2)g_n\right\|_2 \le |\gamma_\sigma|(2+m)\eta_n \le 4|\gamma_\sigma|\eta_n.$$

Every $\gamma_\sigma/(\xi^2+1/4)$ is therefore essential; zero follows by closure (or is immediate when $\gamma_\sigma = 0$). The half-line essential intervals are $[\min(0, 2A^\sigma), \max(0, 2A^\sigma)]$. Their union is the stated two-sided interval. The compactness and essential-norm claims follow, including the case of two zero limits. □

To identify the full physical operator, let $e_r = G_r/\sqrt{\tau}$, $\tau = (r-a)(b-r)/(b-a)$. Write $u = z + \gamma e_r$ with $z(r) = 0$ and define the isometry

$$\mathcal{U}_\varepsilon u = (\varepsilon^{-1/2} z(r+\varepsilon t), \gamma) \in Y := X \oplus \mathbb{C},$$

extending the first coordinate by zero beyond $(-(r-a)/\varepsilon, (b-r)/\varepsilon)$. Here $\gamma = e(u, e_r) = u(r)/\sqrt{\tau}$. Let Π_ε be the actual spectral projection onto the two escaping branches of C_ε, and put

$$\widehat{D}_\varepsilon = \mathcal{U}_\varepsilon(I - \Pi_\varepsilon)C_\varepsilon(I - \Pi_\varepsilon)\mathcal{U}_\varepsilon^*.$$

Theorem 3.2 (Deflated asymmetric collision channel)**.** *The full operator has exactly one escaping eigenvalue of each sign, with uniformly bounded remaining spectrum. In Y one has*

$$\widehat{D}_\varepsilon \longrightarrow [(I - P_{\boldsymbol{c}})K_{\boldsymbol{c}}(I - P_{\boldsymbol{c}})] \oplus 0 \quad \textit{strongly.} \tag{6}$$

Moreover

$$\mathcal{U}_\varepsilon \Pi_\varepsilon \mathcal{U}_\varepsilon^* \longrightarrow P_{\boldsymbol{c}} \oplus I_{\mathbb{C}} \quad \textit{in operator norm.}$$

The essential spectrum of the limit in (6) *is*

$$[\min(0, 2A^-, 2A^+), \max(0, 2A^-, 2A^+)]. \tag{7}$$

If $A^- = A^+ = 0$, the limit is trace class and (6) *holds in trace norm. In particular the positive and negative traces, absolute trace, and signed Schatten p-masses converge for every $p \ge 1$. Otherwise every compact approximation to it has norm distance at least* $\max(2|A^-|, 2|A^+|)$.

Proof. The compact offset set gives uniform envelopes $|V_{\boldsymbol{c}}(t)| \le C/(1+t^2)$, $|F_{\boldsymbol{c}}(t)| \le C/(1+|t|)$, and uniform two-sided tails with coefficients $A^\pm$ and some common positive remainder exponent. Hardy at the anchor bounds $K_{\boldsymbol{c}}$ on X. Norm continuity in the offsets holds despite mere measurability: on $|t| < T$, $|u(t)|^2 \le |t|\,\|u\|_X^2$ controls the difference form by a weighted local L^1 translation difference; on $|t| > T$, the tail estimate bounds the weighted supremum difference by $C\,\|\boldsymbol{c}' - \boldsymbol{c}\|\,/T + CT^{-\eta}$. Use Hardy on that tail, then let $\boldsymbol{c}' \to \boldsymbol{c}$ and $T \to \infty$. Thus $\|K_{\boldsymbol{c}'} - K_{\boldsymbol{c}}\| \to 0$. Also $F_{\boldsymbol{c}'} \to F_{\boldsymbol{c}}$ in L^2 by translation continuity, so the normalized layer vectors converge in energy.

Let Q_ε project in X onto the zero-extended finite anchor-zero space with endpoints $-(r-a)/\varepsilon, (b-r)/\varepsilon$. These projections converge strongly to I. In the split $Y = X \oplus \mathbb{C}$ the embedded full operator has the form

$$\widehat{C}_\varepsilon = \begin{pmatrix} B_\varepsilon & -l_\varepsilon w_\varepsilon \\ -l_\varepsilon w_\varepsilon^* & a_\varepsilon \end{pmatrix}, \qquad B_\varepsilon = Q_\varepsilon K_{\boldsymbol{c}_\varepsilon} Q_\varepsilon, \tag{8}$$

where w_ε is unit in the finite anchor-zero subspace and $l_\varepsilon > 0$. The sign of the coupling is incorporated in w_ε. Direct integration of the primitive gives

$$l_\varepsilon = \frac{\sqrt{\tau D_{\boldsymbol{c}_\varepsilon}}}{2\sqrt{\varepsilon}} + O(1), \qquad |a_\varepsilon| = O(1 + \log(1/\varepsilon)), \qquad \|w_\varepsilon - w_{\boldsymbol{c}_\varepsilon}\|_X = O(\sqrt{\varepsilon}\log(1/\varepsilon)). \tag{9}$$

Here are direct estimates of these quantities. On either cell put $z = x - r$, $t = z/\varepsilon$, and write $e_r = \sqrt{\tau} + kz$. Let $J(t) = \int_0^t F_{\boldsymbol{c}_\varepsilon}(s)\,\mathrm{d}s$. Integration by parts shows that the reserve coupling has derivative

$$\mathsf{C}_I\left[-\frac{e_r F_{\boldsymbol{c}_\varepsilon}(t)}{2\varepsilon} + \frac{kJ(t)}{2}\right],$$

where C_I subtracts the physical cell mean. Indeed differentiation of the bracket equals $-e_r\varepsilon^{-2}V_{\boldsymbol{c}_\varepsilon}(t)/2$. The derivative of the difference from the leading vector is therefore $\frac{k}{2}\mathsf{C}_I[J(t) - tF_{\boldsymbol{c}_\varepsilon}(t)]$. The function $tF(t)$ is uniformly bounded. On either sign of t, $J(t) = -A^\pm\log(1 + |t|) + O(1)$. The variance of $\log(1 + |z|/\varepsilon)$ on a fixed physical cell is uniformly bounded: after subtracting $\log(1/\varepsilon)$ this follows from the uniform integrability of $|\log(|z| + \varepsilon)|^2$. Thus the difference has uniformly bounded energy norm. After dilation its leading derivative is $-\sqrt{\tau}\,\mathsf{C}_\varepsilon F_{\boldsymbol{c}_\varepsilon}/(2\sqrt{\varepsilon})$, where C_ε restricts and subtracts the mean on each scaled cell. Omitted squared tails cost $O(\varepsilon)$, while the two squared cell means cost $O(\varepsilon\log^2(1/\varepsilon))$. Hence $\|\mathsf{C}_\varepsilon F - F\|_2 = O(\sqrt{\varepsilon}\log(1/\varepsilon))$ and $\|\mathsf{C}_\varepsilon F\|_2^2 = D + O(\varepsilon\log^2(1/\varepsilon))$. The lower bound on D controls normalization. The bounded energy coupling error changes l_ε by $O(1)$ and its direction by $O(\sqrt{\varepsilon})$. The trace estimate follows by integrating the primitive against the bounded derivative of e_r^2. These establish (9). The bounded reserve and codimension one give at most one eigenvalue of either sign outside a fixed interval; $l_\varepsilon \to \infty$ and $a_\varepsilon/l_\varepsilon \to 0$ give exactly those two, with all remaining spectrum in a uniform bounded interval.

We next compare spectral deletion with the primitive-layer compression. Put P_ε for the projection onto w_ε in X, and decompose Y into the slow space $w_\varepsilon^\perp \oplus 0$ and the two-dimensional fast space $\operatorname{span}\{(w_\varepsilon, 0), (0, 1)\}$. The slow diagonal is $A_\varepsilon = (I - P_\varepsilon)B_\varepsilon(I - P_\varepsilon)$, its coupling to the fast space has uniformly bounded norm, and the fast diagonal is

$$\mathbf{F}_\varepsilon = \begin{pmatrix} \beta_\varepsilon & -l_\varepsilon \\ -l_\varepsilon & a_\varepsilon \end{pmatrix}, \qquad |\beta_\varepsilon| \le \|B_\varepsilon\| \le M.$$

Choose a fixed radius $R > M + 2$ enclosing all bounded spectrum. For $|z| = R$ the exact fast inverse has denominator $(z - \beta_\varepsilon)(z - a_\varepsilon) - l_\varepsilon^2$ and numerator entries of sizes $O(1 + |a_\varepsilon|)$, l_ε and $O(1)$. Since $a_\varepsilon = o(l_\varepsilon)$,

$$\sup_{|z|=R}\left\|(z - \mathbf{F}_\varepsilon)^{-1}\right\| = O(l_\varepsilon^{-1}).$$

The slow resolvent $(z - A_\varepsilon)^{-1}$ is uniformly bounded. The Schur complement formula therefore gives, uniformly on this circle,

$$(z - \widehat{C}_\varepsilon)^{-1} = [(z - A_\varepsilon)^{-1} \oplus 0_{\mathrm{fast}}] + O(l_\varepsilon^{-1}).$$

Integrating this identity and then its product with z around the circle yields respectively

$$\begin{aligned} \|\mathcal{U}_\varepsilon\Pi_\varepsilon\mathcal{U}_\varepsilon^* - (P_\varepsilon \oplus I_{\mathbb{C}})\| &= O(l_\varepsilon^{-1}), \\ \left\|\widehat{D}_\varepsilon - (A_\varepsilon \oplus 0_{\mathbb{C}})\right\| &= O(l_\varepsilon^{-1}) = O(\sqrt{\varepsilon}). \end{aligned} \tag{10}$$

In the last line A_ε is understood as zero on w_ε. The integration over the circle projects onto bounded spectrum; its complement is precisely the two escaping eigenspaces. Thus the estimate concerns spectral deflation, not a chosen trace compression.

Norm convergence $P_\varepsilon \to P_{\boldsymbol{c}}$ follows from (9) and L^2 continuity of F in its offsets. The reserve operators converge strongly to $K_{\boldsymbol{c}}$. Multiplying by these norm-convergent layer projections in (10) proves both convergence assertions. In particular no finite residual term arises from the fast block: its inverse tends uniformly to zero on the bounded spectral contour.

Apply Lemma 3.1 to $V_{\boldsymbol{c}}$. Compression by $I - P_{\boldsymbol{c}}$ changes $K_{\boldsymbol{c}}$ by finite rank, so its essential spectrum and essential norm are unchanged. Suppose both aggregate tail coefficients vanish. The translated power tails give $V_{\boldsymbol{c}}(t) = O(|t|^{-2-\gamma})$ uniformly on $\mathcal{C}$, for some $\gamma > 0$. The evaluation diagonal of X is $|t|$. For every real U with $\int |t||U(t)|\,\mathrm{d}t < \infty$, Hilbert–Schmidt factorization gives

$$\|K_U\|_1 \le \tfrac{1}{2} \int_{\mathbb{R}} |t||U(t)|\,\mathrm{d}t. \tag{11}$$

For completeness, the embedding $J_U : X \to L^2(|U|\,\mathrm{d}t)$ has squared Hilbert–Schmidt norm $\int |t||U(t)|\,\mathrm{d}t$, by Tonelli and the evaluation diagonal, while $K_U = \frac{1}{2} J_U^* M_{\operatorname{sgn} U} J_U$. Weighted local L^1 translation continuity and the uniformly integrable faster tails imply $K_{\boldsymbol{c}'} \to K_{\boldsymbol{c}}$ in trace norm. Strong two-sided compressions of a fixed trace-class operator converge in trace norm, as follows first for finite rank and then by trace-class approximation. Consequently $B_\varepsilon \to K_{\boldsymbol{c}}$ in trace norm. The norm convergence of P_ε yields the corresponding trace-norm limit of A_ε.

In the slow/fast split, $\widehat{C}_\varepsilon - (A_\varepsilon \oplus 0)$ has rank at most four: its range is contained in the two-dimensional fast space and the range of its coupling to the slow space, of dimension at most two. The removed spectral operator has rank two. Hence $\widehat{D}_\varepsilon - (A_\varepsilon \oplus 0)$ has rank at most six and (10) gives

$$\left\| \widehat{D}_\varepsilon - (A_\varepsilon \oplus 0) \right\|_1 = O(\sqrt{\varepsilon}).$$

This proves trace-norm convergence. Trace continuity and $\operatorname{tr} T_\pm = (\operatorname{tr}|T| \pm \operatorname{tr} T)/2$ give the trace assertions. For $p \ge 1$, trace-ideal eigenvalue comparison gives ℓ^1 convergence of the selfadjoint eigenvalue lists, and the functions $(x_\pm)^p$ are Lipschitz on a common bounded spectral interval. This proves the signed-mass convergence. Otherwise the essential norm of the limit is $\max(2|A^-|, 2|A^+|)$ and every $\widehat{D}_\varepsilon$ is compact, which proves the norm-distance obstruction. □

For an explicit compact-bulk example, fix $c > 0$ and take

$$v_0(t) = \frac{2(t^2-1)}{(1+t^2)^2}, \qquad V_c(t) = v_0(t-c) - v_0(t+c).$$

Its aggregate tails satisfy $A^- = A^+ = 0$, while the primitive correlation gives $D_c = 4\pi c^2/(1+c^2) > 0$. Thus its bounded spectral operator, after removal of the escaping pair and the stated transport, converges in trace norm to a trace-class compressed channel. The fast pair still escapes with leading magnitude $\sqrt{\pi\tau/\varepsilon}\, c/\sqrt{1+c^2}$.

Proposition 3.3 (Absolutely continuous inner channels)**.** *Under the power-tail hypotheses of the asymmetric deflated-channel theorem, the inner operator $K_{\boldsymbol{c}}$, and its codimension-one compression, are unitarily equivalent up to trace class to the two reference channels*

$$\frac{A^\pm}{2} \left(-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1}{4} \right)^{-1}_D \quad \text{on } L^2(0,\infty),$$

together with zero directions. Thus the absolutely continuous spectral multiplicity at almost every real $q \neq 0$ is

$$\sum_{\sigma \in \{-,+\}} \mathbf{1}_{(\min(0,2A^\sigma), \max(0,2A^\sigma))}(q).$$

There is no absolutely continuous spectrum off these intervals. Point and singular continuous spectrum within the essential set is not classified by this comparison; outside that closed set only discrete point spectrum can occur. If both aggregate tail coefficients vanish, the full inner operator and its compressed operator are trace class.

Proof. Cut each physical half-line at $|t| = 1$. The additional value traces have total codimension two in the homogeneous anchor-zero space. Restoring their finite-dimensional energy complement and its mixed blocks is finite rank. On either inner interval $(0, 1)$ the potential is bounded. Its energy sandwich is trace class, since the Dirichlet Green diagonal is $t(1-t)$ and

$$\left\| \tfrac{1}{2}(-\mathrm{d}_D^2)^{-1/2} M_U (-\mathrm{d}_D^2)^{-1/2} \right\|_1 \le \tfrac{1}{2} \int_0^1 |U(t)| t(1-t)\, \mathrm{d}t < \infty.$$

On an outer half-line use $|t| = \mathrm{e}^y$ and $u(\pm \mathrm{e}^y) = \mathrm{e}^{y/2} f(y)$, $y > 0$. The cut is Dirichlet at $y = 0$ and energy is the form of $\mathcal{A} = -\mathrm{d}^2/\mathrm{d}y^2 + 1/4$. The potential multiplier is $A^{\pm}/2 + m_{\pm}(y)$, where

$$m_{\pm}(y) = \tfrac{1}{2} \left[\mathrm{e}^{2y} V_{\boldsymbol{c}}(\pm \mathrm{e}^y) - A^{\pm} \right] = O(\mathrm{e}^{-\eta y}).$$

The compact offset shifts preserve a positive tail exponent, possibly reduced to $\min(1, \eta_j^{\pm})$. The remainder is bounded and integrable. Since $\mathcal{A}_D^{-1}(y, y) = 1 - \mathrm{e}^{-y}$, factoring its signed parts through Hilbert–Schmidt square roots yields

$$\| \mathcal{A}_D^{-1/2} M_{m_{\pm}} \mathcal{A}_D^{-1/2} \|_1 \le \int_0^{\infty} |m_{\pm}(y)| (1 - \mathrm{e}^{-y})\, \mathrm{d}y < \infty.$$

The sine transform gives simple purely absolutely continuous spectrum between 0 and $2A^{\pm}$ for each nonzero reference coefficient. The Kato–Rosenblum theorem [8] gives the stated multiplicities. Layer compression changes the bounded inner operator by finite rank and preserves this comparison. If both $A^{\pm} = 0$, both reference channels are zero, so all remaining terms are trace class. □

3.1 Signed eigenvalue counts at fixed thresholds

The preceding limit concerns a noncancelling cluster with zero background. The number and total mass of bounded eigenvalues also admit a description that allows cancellation and bounded backgrounds. The logarithmic oscillation mechanism is the inverse-square counting law of Kirsch–Simon [10]; the formulation below keeps the two tail signs and the finite-interval concentration parameter explicit.

Fix $a < r < b$. Let v_j be a finite collection of real measurable profiles with $|v_j(t)| \le C_j/(1+t^2)$ and

$$v_j(t) = \alpha_j^{\pm}/t^2 + O(|t|^{-2-\eta_j^{\pm}}), \qquad t \to \pm\infty, \quad \eta_j^{\pm} > 0.$$

For this subsection the means of the profiles may be arbitrary. Let $\boldsymbol{c}_\varepsilon$ range over a fixed compact set and let W_ε be real with $\sup_\varepsilon \|W_\varepsilon\|_\infty < \infty$. Set

$$V_\varepsilon(x) = \sum_j \varepsilon^{-2} v_j((x-r)/\varepsilon - c_{\varepsilon,j}) + W_\varepsilon(x), \qquad A^{\pm} = \sum_j \alpha_j^{\pm}. \tag{12}$$

Write C_ε for the energy operator of $\frac{1}{2}\int_a^b V_\varepsilon |u|^2$ on $H_0^1(a,b)$, and $x_+ = \max(x, 0)$.

Theorem 3.4 (Two-sided signed logarithmic count)**.** *For every fixed $t > 0$ and $\zeta \in \{-1, 1\}$,*

$$N_\varepsilon^\zeta(t) := \operatorname{rank} \mathbf{1}_{(t,\infty)}(\zeta C_\varepsilon) = \frac{\log(1/\varepsilon)}{\pi} \sum_{\sigma\in\{-,+\}} \sqrt{\left(\frac{\zeta A^\sigma}{2t} - \frac{1}{4}\right)_+} + O(1). \tag{13}$$

The remainder is uniform over the stated compact offset set and bounded background class, including thresholds $t = 2\zeta A^\sigma > 0$. For every fixed $M > 0$ there is also the uniform bound

$$N_\varepsilon^\zeta(t) \le C_M(1 + \log(1/\varepsilon))t^{-1/2}, \qquad 0 < t \le M, \tag{14}$$

for all sufficiently small ε.

Proof. By the spectral theorem, $N_\varepsilon^\zeta(t)$ is the negative index of

$$q_{\varepsilon,t}[u] = \int_a^b |u'|^2 - \frac{\zeta}{2t}\int_a^b V_\varepsilon |u|^2$$

on $H_0^1(a, b)$. Choose $d > 0$ with $[r - d, r + d] \subset (a, b)$, and choose a fixed R larger than twice the maximal allowed offset magnitude plus two. Impose zero values at $r-d, r-R\varepsilon, r+R\varepsilon, r+d$. These four conditions change the negative index by at most four and split the form into five Dirichlet interval forms.

The two outer intervals have uniformly bounded potentials. Their negative indices are bounded for fixed t. On the middle interval, $x = r + \varepsilon z$ gives a Dirichlet form on $(-R, R)$ with uniformly bounded rescaled potential $\sum_j v_j(z - c_{\varepsilon,j}) + \varepsilon^2 W_\varepsilon(r + \varepsilon z)$, so its negative index is bounded too.

On the right annulus put $z = x - r = \mathrm{e}^y$ and $u(x) = z^{1/2} f(y)$. Its endpoint values are zero and direct integration of the mixed derivative gives

$$\int |u'|^2 \,\mathrm{d}x = \int \left(|f'|^2 + \tfrac{1}{4}|f|^2\right) \mathrm{d}y, \qquad \int V_\varepsilon |u|^2 \,\mathrm{d}x = \int z^2 V_\varepsilon(r + z)|f|^2 \,\mathrm{d}y.$$

For $\gamma = \min(1, \eta_j^\pm) > 0$, the translated tails imply

$$z^2 V_\varepsilon(r + z) = A^+ + O((\varepsilon/z)^\gamma) + O(z^2).$$

The left annulus gives the same formula with A^-. Translating the logarithmic interval to $(0, L)$, where $L = \log(d/(R\varepsilon))$, the corresponding form is

$$\int_0^L \left(|f'|^2 - \kappa^2|f|^2 - b_\varepsilon(y)|f|^2\right) \mathrm{d}y, \qquad \kappa^2 = \frac{\zeta A^\sigma}{2t} - \frac{1}{4},$$

with

$$|b_\varepsilon(y)| \le C_t \left(\mathrm{e}^{-\gamma y} + \mathrm{e}^{-2(L-y)}\right).$$

Both $\int |b_\varepsilon|$ and $\int \min(y, L - y)|b_\varepsilon(y)| \,\mathrm{d}y$ are uniformly bounded.

If $\kappa^2 > 0$, use the real Pruefer substitution $f = \rho \sin\theta$, $f' = \kappa\rho\cos\theta$ for the solution with $f(0) = 0, f'(0) = 1$. It gives $|\theta' - \kappa| \le |b_\varepsilon|/\kappa$ almost everywhere. Thus $\theta(L) = \kappa L + O(1)$, and Sturm oscillation gives $\kappa L/\pi + O(1)$ strictly negative eigenvalues, with a possible zero endpoint eigenvalue changing the count by at most one.

If $\kappa^2 = -k^2 < 0$, replace b_ε by its positive part. The Dirichlet Green diagonal of $-\mathrm{d}^2 + k^2$ is bounded by $1/(2k)$, so the Birman–Schwinger trace bounds the negative index by $(2k)^{-1}\int (b_\varepsilon)_+ = O(1)$. At the exact threshold $\kappa = 0$, the Dirichlet inverse of $-\mathrm{d}^2$ on $(0, L)$ has diagonal $y(L - y)/L$.

Its trace against $(b_\varepsilon)_+$ is bounded by $\int \min(y, L-y)(b_\varepsilon)_+ = O(1)$. This proves (13), including the band edges.

For the uniform estimate as $t \downarrow 0$, use upper comparisons rather than a uniform Pruefer remainder. Constant-potential Dirichlet counts on the core and outer intervals are at most $C(1+t^{-1/2})$. On either annulus the envelope gives $|z^2 V_\varepsilon(r \pm z)| \le C$. The logarithmic form is therefore bounded below by $-\mathrm{d}^2 + 1/4 - C/(2t)$ on length L, whose negative index is at most $1 + CL/\sqrt{t}$. Add the five counts and the four trace conditions, and absorb the additive constants using $0 < t \le M$. This proves (14). □

3.2 Signed spectral mass and trace compensation

A bounded spectral bulk can be defined even when an escaping pair degenerates. The envelope in (12) gives

$$|V_\varepsilon(x)| \le C|x-r|^{-2} + \|W_\varepsilon\|_\infty \qquad (x \ne r)$$

uniformly in ε. Hardy on each side of r and interval Poincare therefore give a constant M_0 such that the energy compression to $\{u : u(r) = 0\}$ has norm at most M_0. Fix $M > M_0$ and define the spectral bulk

$$D_\varepsilon = C_\varepsilon \mathbf{1}_{[-M,M]}(C_\varepsilon). \tag{15}$$

Min–max on this codimension-one space shows that the deleted operator has rank at most two.

Theorem 3.5 (Signed Schatten masses)**.** *Under* (12)*, for every* $p > 1/2$ *and* $\zeta \in \{-1, 1\}$*,*

$$\frac{\mathrm{tr}[(\zeta D_\varepsilon)_+^p]}{\log(1/\varepsilon)} \longrightarrow \frac{2^{p-2}}{\sqrt{\pi}} \frac{\Gamma(p-1/2)}{\Gamma(p)} \sum_{\sigma \in \{-,+\}} (\zeta A^\sigma)_+^p. \tag{16}$$

In particular

$$\begin{aligned} \frac{\mathrm{tr}\, D_\varepsilon}{\log(1/\varepsilon)} &\longrightarrow \tfrac{1}{2}(A^+ + A^-), \\ \frac{\mathrm{tr}\,|D_\varepsilon|}{\log(1/\varepsilon)} &\longrightarrow \tfrac{1}{2}(|A^+| + |A^-|). \end{aligned} \tag{17}$$

Proof. The count bound (14) implies that D_ε belongs to every Schatten class $p > 1/2$. The spectral layer-cake identity gives

$$\mathrm{tr}[(\zeta D_\varepsilon)_+^p] = p \int_0^M t^{p-1} \,\mathrm{rank}\, \mathbf{1}_{(t,\infty)}(\zeta D_\varepsilon)\, \mathrm{d}t.$$

For each fixed $t > 0$, deletion changes the count by at most two. After division by $\log(1/\varepsilon)$ the integrand is dominated by $Ct^{p-3/2}$, which is integrable. Dominated convergence and (13) reduce the coefficient to

$$\frac{p}{\pi} \sum_\sigma \int_0^{2a_\sigma} t^{p-1} \sqrt{\frac{a_\sigma}{2t} - \frac{1}{4}}\, \mathrm{d}t, \qquad a_\sigma = (\zeta A^\sigma)_+.$$

The endpoint $2a_\sigma$ lies below M: tail wavepackets in the anchor-zero space, or the Hardy channel lemma, give $2|A^\sigma| \le M_0$. With $t = 2a_\sigma z$, the integral equals

$$\frac{p(2a_\sigma)^p}{2\pi} B(p - \tfrac{1}{2}, \tfrac{3}{2}) = \frac{2^{p-2} a_\sigma^p}{\sqrt{\pi}} \frac{\Gamma(p-1/2)}{\Gamma(p)}.$$

A zero coefficient contributes zero. This proves (16). At $p = 1$ the coefficient is $a_\sigma/2$; subtracting or adding the two signed limits gives (17). □

The geometry-dependent logarithmic sum of the escaping pair is consistent with this geometry-independent bulk trace. The next identity makes the compensation explicit.

Proposition 3.6 (Trace compensation)**.** *Suppose in addition that every profile has zero mean and that* $\inf_\varepsilon D_{\mathbf{c}_\varepsilon} > 0$*. Put* $L = r - a$*,* $R = b - r$*, and* $\ell = L + R$*. For all sufficiently small* ε*, the two eigenvalues deleted in* (15) *are the escaping pair, and*

$$\begin{aligned} \operatorname{tr} C_\varepsilon &= \frac{R-L}{2\ell}(A^+ - A^-)\log(1/\varepsilon) + O(1), \\ \lambda_\varepsilon^+ + \lambda_\varepsilon^- &= -\frac{A^+L + A^-R}{\ell}\log(1/\varepsilon) + O(1), \\ \operatorname{tr} D_\varepsilon &= \tfrac{1}{2}(A^+ + A^-)\log(1/\varepsilon) + O(1). \end{aligned} \tag{18}$$

These formulas allow the bounded backgrounds in (12)*.*

Proof. Let $g(x) = G_{(a,b)}(x,x) = (x-a)(b-x)/\ell$. The derivative-energy embedding into $L^2(|V_\varepsilon|\,\mathrm{d}x)$ is Hilbert–Schmidt, with squared norm $\int |V_\varepsilon| g$. Its signed factorization gives the exact trace identity

$$\operatorname{tr} C_\varepsilon = \tfrac{1}{2}\int_a^b V_\varepsilon(x) g(x)\,\mathrm{d}x.$$

This follows directly from any energy-orthonormal basis and $\sum_n |u_n(x)|^2 = g(x)$, with absolute domination.

Expand the exact quadratic g at r. For one profile the constant term is $O(1)$: its full mean is zero, and the omitted tails outside the physical endpoints have mass $O(\varepsilon)$ after the scaled profile integral. The first moment satisfies, uniformly for bounded offsets,

$$\int_{-L/\varepsilon}^{R/\varepsilon} t\, v_j(t - c_{\varepsilon,j})\,\mathrm{d}t = (\alpha_j^+ - \alpha_j^-)\log(1/\varepsilon) + O(1).$$

The second-order contribution is $O(1)$ because $\varepsilon \int_{-L/\varepsilon}^{R/\varepsilon} t^2 |v_j(t - c_{\varepsilon,j})|\,\mathrm{d}t = O(1)$. Since $g'(r) = (R-L)/\ell$ and bounded backgrounds have bounded trace, the first line of (18) follows.

For clarity the pair sum can also be obtained without identifying its individual logarithmic terms. Split off the unit trace vector $e_r = G_r/\sqrt{\tau}$. The reserve block has uniformly bounded norm. Zeroing that bounded reserve leaves the rank-two model $\left(\begin{smallmatrix} 0 & -l_\varepsilon \\ -l_\varepsilon & a_\varepsilon \end{smallmatrix}\right)$, where $a_\varepsilon = \frac{1}{2}\int V_\varepsilon e_r^2$. Its eigenvalue sum is a_ε, and restoring the reserve changes each escaping eigenvalue by $O(1)$. On the two sides of r,

$$(e_r^2)'(r-) = 2\tau/L, \qquad (e_r^2)'(r+) = -2\tau/R, \qquad \tau = LR/\ell.$$

In the corresponding first moments the negative tail contributes $-A^-\log(1/\varepsilon)$ and the positive tail contributes $A^+\log(1/\varepsilon)$. The constant and quadratic terms are $O(1)$ as above. Hence $a_\varepsilon = -(A^-R + A^+L)\log(1/\varepsilon)/\ell + O(1)$. The noncancellation assumption makes the coupling diverge, so these are exactly the two deleted eigenvalues for small ε. Subtracting their sum from the full trace leaves

$$\frac{(R-L)(A^+ - A^-) + 2(A^+L + A^-R)}{2\ell} = \frac{A^+ + A^-}{2},$$

which proves the last line. □

4 Cancellation and profile regularity

Exact cancellation removes the leading concentrating profile. Its response to small center displacements is then governed by the norm of the translated primitive. We first give a complete operator estimate for that primitive, and then show how a cusp produces a continuum of collision scales. Throughout this section the bounded background potential is zero. Write $I = (a, b)$, fix $r \in I$, and set

$$L = r - a, \qquad R = b - r, \qquad \tau = G_I(r, r) = \frac{LR}{L + R}.$$

All operator norms and isometries below refer to the derivative-energy inner product on $H_0^1(I)$.

4.1 A small primitive and the full operator

Lemma 4.1 (A primitive estimate)**.** *Let F be real and locally absolutely continuous, with $F \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$. Put*

$$M(F) = \|F\|_1 + \operatorname*{ess\,sup}_{t \in \mathbb{R}} |tF(t)|, \qquad s_{\varepsilon,F} = \frac{\sqrt{\tau}\, \|F\|_2}{2\sqrt{\varepsilon}}.$$

Assume $M(F) < \infty$. Let $C_{\varepsilon,F}$ be the energy-Riesz operator with form

$$q_{\varepsilon,F}(u, z) = \frac{1}{2} \int_I \varepsilon^{-2} F'((x - r)/\varepsilon) u(x) \overline{z(x)}\, \mathrm{d}x.$$

There is a constant C_I, depending only on I and r, such that, for $0 < \varepsilon \le 1$, there is an isometry $\mathcal{J}_{\varepsilon,F} : \mathbb{C}^2 \to H_0^1(I)$ satisfying

$$\left\| C_{\varepsilon,F} - \mathcal{J}_{\varepsilon,F} \begin{pmatrix} 0 & s_{\varepsilon,F} \\ s_{\varepsilon,F} & 0 \end{pmatrix} \mathcal{J}_{\varepsilon,F}^* \right\| \le C_I M(F). \tag{19}$$

In particular $|\, \|C_{\varepsilon,F}\| - s_{\varepsilon,F}| \le C_I M(F)$, and at most one eigenvalue of either sign has magnitude exceeding $C_I M(F)$.

Proof. Set $K(x) = \varepsilon^{-1} F((x - r)/\varepsilon)$ and $e_r = G_I(\,\cdot\,, r)/\sqrt{\tau}$. Decompose

$$H_0^1(I) = \mathbb{C} e_r \oplus \mathcal{H}_*, \qquad \mathcal{H}_* = \{z \in H_0^1(I) : z(r) = 0\}.$$

Integration by parts gives

$$q_{\varepsilon,F}(u, z) = -\frac{1}{2} \int_I K(u' \overline{z} + u \overline{z'})\, \mathrm{d}x. \tag{20}$$

There are no endpoint terms because u, z vanish at a, b. The bound $|(x - r)K(x)| \le M(F)$ and the Hardy inequality on the two anchor intervals give

$$\|C_{\varepsilon,F}|_{\mathcal{H}_* \times \mathcal{H}_*}\| \le C_I M(F).$$

Here and below a restriction to two subspaces denotes the corresponding compressed form. The scalar trace entry is

$$q_{\varepsilon,F}(e_r, e_r) = -\int_I K e_r e_r'\, \mathrm{d}x, \qquad |q_{\varepsilon,F}(e_r, e_r)| \le C_I \|F\|_1 .$$

On each anchor interval, $e_r(x) - \sqrt{\tau}$ is a fixed multiple of $x - r$. Applying this fact and Hardy to (20) shows that the coupling on $\mathcal{H}_*$ differs by at most $C_I M(F)$ from

$$z \longmapsto -\frac{\sqrt{\tau}}{2} \int_I K \overline{z'} \, \mathrm{d}x.$$

Indeed the two error terms contain respectively $K(e_r - \sqrt{\tau})\overline{z'}$ and $K e_r' \overline{z}$; the first is bounded in L^2 by $C_I M(F)$, and the second uses $\|z/(x-r)\|_2 \le 2\,\|z'\|_2$.

Let Π subtract the separate means on (a, r) and (r, b). The derivative map identifies $\mathcal{H}_*$ isometrically with the $L^2(I)$ functions of mean zero on both intervals. The preceding functional is represented there by the vector b_F with $b_F' = -\sqrt{\tau}\,\Pi K/2$. Its norm t_F satisfies

$$|t_F - s_{\varepsilon,F}| \le C_I M(F).$$

To see this without a lower bound on either norm, the omitted tail obeys

$$\varepsilon^{-1} \int_{\mathbb{R} \setminus (-L/\varepsilon, R/\varepsilon)} |F(t)|^2 \, \mathrm{d}t \le M(F)^2 (L^{-1} + R^{-1}),$$

and subtracting the two cell means costs at most $\|F\|_1^2\,(L^{-1} + R^{-1})$ in squared norm. Apply $|\sqrt{x} - \sqrt{y}| \le \sqrt{|x - y|}$. The operator with coupling b_F, zero trace entry and zero reserve is supported on $\operatorname{span}(e_r, b_F)$ and has eigenvalues $\pm t_F$. Replacing t_F by $s_{\varepsilon,F}$ gives (19); when $b_F = 0$, choose any unit vector in $\mathcal{H}_*$. The full error bounds all three discarded blocks. The spectral conclusions follow from selfadjoint norm perturbation and the min–max principle. □

Proposition 4.2 (Measurable profiles near cancellation)**.** *Let $v_1, \dots, v_m$ be fixed real measurable profiles with*

$$\int_{\mathbb{R}} v_j = 0, \qquad |v_j(t)| \le C_j/(1 + t^2), \qquad P_j(t) = \int_{-\infty}^{t} v_j(u) \, \mathrm{d}u.$$

Suppose $\boldsymbol{c}^0 \in \mathbb{R}^m$ satisfies $\sum_j P_j(t - c_j^0) = 0$ for every t. For $\boldsymbol{c}_\varepsilon \to \boldsymbol{c}^0$, put

$$\delta_\varepsilon = \left\| \boldsymbol{c}_\varepsilon - \boldsymbol{c}^0 \right\|_\infty, \quad F_\varepsilon(t) = \sum_j P_j(t - c_{\varepsilon,j}), \quad s_\varepsilon = \frac{\sqrt{\tau}\,\|F_\varepsilon\|_2}{2\sqrt{\varepsilon}}.$$

The full operator for $\varepsilon^{-2} \sum_j v_j((x-r)/\varepsilon - c_{\varepsilon,j})$ has the paired model (19) with error $O(\delta_\varepsilon)$. If $\delta_\varepsilon > 0$ and

$$\boldsymbol{c}_\varepsilon = \boldsymbol{c}^0 + \delta_\varepsilon \boldsymbol{\theta} + o(\delta_\varepsilon), \qquad G = -\sum_j \theta_j v_j(\,\cdot - c_j^0) \neq 0 \quad \text{in } L^2(\mathbb{R}),$$

then its first positive and negative eigenvalues satisfy

$$\lambda_\varepsilon^{\pm} = \pm \frac{\sqrt{\tau}\,\|G\|_2}{2} \frac{\delta_\varepsilon}{\sqrt{\varepsilon}} [1 + o(1)]. \tag{21}$$

Every other eigenvalue is $O(\delta_\varepsilon)$. Thus $\delta_\varepsilon/\sqrt{\varepsilon}$ determines whether the full operator tends to zero, has a finite nonzero limiting pair, or has an escaping pair. Critical displacements of the physical centers are of order $\varepsilon^{3/2}$.

Proof. For offsets in a fixed bounded neighborhood of $\boldsymbol{c}^0$, absolute continuity and the profile envelope imply

$$|P_j(t - c_j) - P_j(t - c_j^0)| \le \frac{C|c_j - c_j^0|}{1 + t^2}.$$

Indeed the integral is over a segment of length $|c_j - c_j^0|$; on this bounded family of translates, $(1+u^2)^{-1} \le C(1+t^2)^{-1}$. Exact cancellation therefore gives

$$|F_\varepsilon(t)| \le C\delta_\varepsilon/(1+t^2), \qquad M(F_\varepsilon) \le C\delta_\varepsilon.$$

Lemma 4.1 proves the full operator statement. Each P_j belongs to $H^1(\mathbb{R})$, since $v_j \in L^2$ and its zero-mean primitive is $O((1+|t|)^{-1})$. Translation is differentiable in L^2 for H^1 functions, so $F_\varepsilon/\delta_\varepsilon \to G$ in L^2. When $G \neq 0$, the ratio of the operator error to s_ε is $O(\sqrt{\varepsilon})$, which tends to zero independently of the size of $\delta_\varepsilon/\sqrt{\varepsilon}$. The two signed eigenvalues consequently exist and obey (21), including when they tend to zero. The norm approximation proves the three stated regimes. □

4.2 An anchored cusp and a continuum of scales

Fix an even nonnegative $\chi \in C_c^\infty(\mathbb{R})$ equal to one near zero. For a fixed $\beta \ge 1$ set

$$P_\beta(t) = \chi(t)|t|^\beta, \qquad v_\beta = P_\beta', \qquad F_{\beta,\delta}(t) = P_\beta(t+\delta) - 2P_\beta(t) + P_\beta(t-\delta).$$

The derivative v_β is interpreted almost everywhere. It is a bounded compactly supported real zero-mean profile. The potential

$$\varepsilon^{-2}\left[v_\beta((x-r)/\varepsilon+\delta) - 2v_\beta((x-r)/\varepsilon) + v_\beta((x-r)/\varepsilon-\delta)\right] \tag{22}$$

uses three fixed profiles at the offsets $-\delta, 0, \delta$. The aggregate primitive and its first L^2 displacement derivative vanish at $\delta = 0$. The next theorem determines the spectral effect of its second difference, including the borderline between a cusp and an H^2 primitive.

Theorem 4.3 (Anchored cusp cancellation)**.** *Let $C_{\varepsilon,\delta}^{(\beta)}$ denote the full energy operator for (22), and put*

$$s_{\varepsilon,\delta}^{(\beta)} = \frac{\sqrt{\tau}}{2\sqrt{\varepsilon}}\,\|F_{\beta,\delta}\|_2\,.$$

For all sufficiently small $\varepsilon, \delta > 0$, independently, there are isometries $\mathcal{J}_{\varepsilon,\delta} : \mathbb{C}^2 \to H_0^1(I)$ such that

$$\left\| C_{\varepsilon,\delta}^{(\beta)} - \mathcal{J}_{\varepsilon,\delta} \begin{pmatrix} 0 & s_{\varepsilon,\delta}^{(\beta)} \\ s_{\varepsilon,\delta}^{(\beta)} & 0 \end{pmatrix} \mathcal{J}_{\varepsilon,\delta}^* \right\| \le C_\beta \delta^2. \tag{23}$$

In particular all but at most one eigenvalue of either sign are $O(\delta^2)$. Along every path $\varepsilon, \delta \downarrow 0$, the two signed eigenvalues exist and have the following asymptotics.

1. *If $1 \le \beta < 3/2$, let*

$$H_\beta(z) = |z+1|^\beta - 2|z|^\beta + |z-1|^\beta, \qquad c_\beta = \int_{\mathbb{R}} H_\beta(z)^2\,\mathrm{d}z.$$

Then $0 < c_\beta < \infty$, $c_1 = 8/3$, and

$$\lambda_{\varepsilon,\delta}^{\pm} = \pm\frac{\sqrt{\tau c_\beta}}{2}\,\frac{\delta^{\beta+1/2}}{\sqrt{\varepsilon}}[1+o(1)]. \tag{24}$$

2. *At $\beta = 3/2$,*

$$\left\|F_{3/2,\delta}\right\|_2^2 = \frac{9}{8}\delta^4\log(1/\delta) + O(\delta^4),$$

and

$$\lambda_{\varepsilon,\delta}^{\pm} = \pm\frac{3\sqrt{\tau}}{4\sqrt{2}}\,\frac{\delta^2\sqrt{\log(1/\delta)}}{\sqrt{\varepsilon}}[1+o(1)]. \tag{25}$$

3. For fixed $\beta > 3/2$, $P_\beta \in H^2(\mathbb{R})$ and

$$\lambda^{\pm}_{\varepsilon,\delta} = \pm \frac{\sqrt{\tau}\,\left\|P''_\beta\right\|_2}{2}\,\frac{\delta^2}{\sqrt{\varepsilon}}[1+o(1)]. \tag{26}$$

In each case the displayed scale determines the full norm's vanishing, finite-pair and escaping-pair regimes; the remaining spectrum tends to zero. All constants and limits are for fixed β and χ.

Proof. The distribution P''_β is a finite signed measure. Near zero it equals $2\delta_0$ when $\beta = 1$, and $\beta(\beta-1)|t|^{\beta-2}\,\mathrm{d}t$ when $\beta > 1$; all cutoff terms are smooth away from zero. If $h_\delta(t) = (\delta - |t|)_+$, the distributional second-difference identity gives

$$F_{\beta,\delta} = h_\delta * P''_\beta, \qquad \|F_{\beta,\delta}\|_1 \le \delta^2 \left\|P''_\beta\right\|_{\mathrm{TV}}.$$

Moreover

$$\sup_t |tF_{\beta,\delta}(t)| \le C_\beta \delta^2. \tag{27}$$

For $|t| \le 2\delta$, homogeneity in the neighborhood where $\chi = 1$ gives $|F_{\beta,\delta}(t)| \le C_\beta \delta^\beta$, so the weighted product is $O(\delta^{\beta+1}) = O(\delta^2)$. For $2\delta < |t|$ in a smaller fixed cusp neighborhood, the integral second-difference formula gives $|F_{\beta,\delta}(t)| \le C_\beta \delta^2 |t|^{\beta-2}$; multiplication by $|t|$ remains bounded for fixed $\beta \ge 1$. On the cutoff annulus the function is smooth and its second difference is $O(\delta^2)$; outside a fixed compact set it vanishes. This proves (27), hence $M(F_{\beta,\delta}) \le C_\beta \delta^2$. Lemma 4.1 now proves the full operator estimate (23).

For $1 \le \beta < 3/2$, H_β is locally bounded and nonzero. For $\beta > 1$ its large-$|z|$ expansion is

$$H_\beta(z) = \beta(\beta-1)|z|^{\beta-2} + O(|z|^{\beta-4}).$$

Its square is integrable in this range. At $\beta = 1$, $H_1(z) = 2(1-|z|)_+$, whose squared integral is $8/3$. On a fixed neighborhood of zero, $F_{\beta,\delta}(\delta z) = \delta^\beta H_\beta(z)$. More globally,

$$F_{\beta,\delta}(t) = \chi(t)\delta^\beta H_\beta(t/\delta) + R_\delta(t), \qquad \|R_\delta\|_2 = O(\delta^2).$$

The remainder vanishes near zero and is $O(\delta^2)$ on a fixed compact annulus, by Taylor expansion of the smooth factors there. Scaling and dominated convergence therefore give

$$\|F_{\beta,\delta}\|_2^2 = c_\beta \delta^{2\beta+1}[1+o(1)].$$

The $O(\delta^2)$ remainder is smaller than $\delta^{\beta+1/2}$ because $\beta < 3/2$.

At $\beta = 3/2$, choose a fixed $t_0 > 0$ inside the region where $\chi = 1$. On $2\delta \le |t| \le t_0$,

$$F_{3/2,\delta}(t) = \frac{3}{4}\delta^2 |t|^{-1/2} + O(\delta^4 |t|^{-5/2}).$$

The squared leading term integrates over both sides to $(9/8)\delta^4 \log(1/\delta) + O(\delta^4)$. The cross term and squared error have integrals $O(\delta^4)$: use respectively $\delta^6 \int_{2\delta}^{t_0} t^{-3}\,\mathrm{d}t$ and $\delta^8 \int_{2\delta}^{t_0} t^{-5}\,\mathrm{d}t$. The region $|t| < 2\delta$ contributes $O(\delta^4)$ by homogeneity, and the fixed outer annulus contributes $O(\delta^4)$ by smoothness. This proves the asserted logarithmic norm formula.

For $\beta > 3/2$, the local second derivative is square integrable, so $P_\beta \in H^2(\mathbb{R})$. The exact integral identity

$$\delta^{-2} F_{\beta,\delta}(t) = \int_{-1}^{1} (1-|u|) P''_\beta(t+\delta u)\,\mathrm{d}u$$

and continuity of translations in L^2 give convergence to P_β''. Its norm is positive: a compactly supported nonzero function cannot have identically zero second derivative.

It remains to compare these primitive asymptotics with the full operator. In the three cases, the ratio of its $O(\delta^2)$ error to the principal pair size is respectively

$$O(\sqrt{\varepsilon}\,\delta^{3/2-\beta}), \qquad O(\sqrt{\varepsilon}/\sqrt{\log(1/\delta)}), \qquad O(\sqrt{\varepsilon}).$$

Each tends to zero on every path $\varepsilon, \delta \downarrow 0$. Selfadjoint perturbation proves the signed laws and existence of the pair, even when its magnitude vanishes. The same norm approximation proves the three full-operator regimes. □

The critical offset and physical displacement orders are collected in Table 1. For example,

Table 1: Critical offset and physical displacement orders for the anchored cusp family.

Primitive exponent	Critical offset δ	Physical displacement $\varepsilon\delta$
$1 \le \beta < 3/2$	$\varepsilon^{1/(2\beta+1)}$	$\varepsilon^{1+1/(2\beta+1)}$
$\beta = 3/2$	$\varepsilon^{1/4}(\log(1/\varepsilon))^{-1/4}$	$\varepsilon^{5/4}(\log(1/\varepsilon))^{-1/4}$
$\beta > 3/2$	$\varepsilon^{1/4}$	$\varepsilon^{5/4}$

$\beta = 1$ gives the physical order $\varepsilon^{4/3}$ and $\beta = 5/4$ gives $\varepsilon^{9/7}$. At the logarithmic endpoint, $\delta \sim (4\varepsilon/\log(1/\varepsilon))^{1/4}$ makes $\delta^2\sqrt{\log(1/\delta)}/\sqrt{\varepsilon} \to 1$. The first-order displacement vanishes for every member of this family, but the subsequent scale changes with the regularity of the primitive.

5 Scalar matching for identical even profiles

When two centers merge, the trace geometry degenerates and distinct spectral scales emerge. For identical even profiles the parity reduction gives estimates uniform through this changing geometry. We take $W_\varepsilon = 0$ throughout this section. Fix $\ell > 0$ and a real measurable even profile v, nonzero on a set of positive measure, with

$$\int_{\mathbb{R}} v = 0, \qquad |v(t)| \le C/(1+t^2), \qquad v(t) = \alpha/t^2 + O(|t|^{-2-\eta}), \quad |t| \to \infty, \tag{28}$$

where $C, \eta > 0$ and $\alpha \in \mathbb{R}$. Set

$$P(t) = \int_{-\infty}^{t} v(z)\,\mathrm{d}z, \qquad J(t) = \int_0^t P(z)\,\mathrm{d}z, \qquad d = \int_{\mathbb{R}} P^2 > 0. \tag{29}$$

Evenness and zero mean give $P(0) = 0$ and oddness. Boundedness of v gives $P(t) = O(|t|)$ near zero; integrating the tails gives boundedness of $tP(t)$ and $P(t) = -\alpha/t + O(|t|^{-1-\eta})$ at infinity. The condition on v implies $d > 0$. More precisely, with $P_0(t) = -2t/(1+t^2)$ and $J_0(t) = -\log(1+t^2)$, write

$$P = (\alpha/2)P_0 + r, \qquad r \in L^1(\mathbb{R}), \qquad J = (\alpha/2)J_0 + R, \quad \|R\|_\infty \le \|r\|_1. \tag{30}$$

Indeed the remainder at infinity is $O(|t|^{-1-\min(\eta,2)})$ and is bounded on compact sets. All constants below may depend on this fixed profile and ℓ. On $H_0^1(-\ell,\ell)$ consider

$$q_{\varepsilon,s}(u,u) = \frac{1}{2}\int_{-\ell}^{\ell} \varepsilon^{-2}\left[v\left(\frac{x-s}{\varepsilon}\right) + v\left(\frac{x+s}{\varepsilon}\right)\right]|u|^2\,\mathrm{d}x, \qquad 0 \le s \le \ell/2. \tag{31}$$

Write $c = s/\varepsilon$ and

$$Q_c = \frac{1}{2}\int_{\mathbb{R}} [P(t-c) + P(t+c)]^2 \, \mathrm{d}t.$$

Reflection splits its energy-Riesz operator into even and odd parts.

Theorem 5.1 (Uniform coalescence laws)**.** *There is a constant M, independent of s and ε, with the following properties for sufficiently small ε. The even part has exactly one eigenvalue above M and one below $-M$; all its remaining spectrum lies in $[-M, M]$. Uniformly for $0 \le s \le \ell/2$,*

$$\lambda^{\pm}_{e,\varepsilon,s} = \pm\frac{\sqrt{(\ell-s)Q_c}}{2\sqrt{\varepsilon}} - \frac{\alpha}{2}\left[\log(1/\varepsilon) + \log\frac{1}{s+\varepsilon}\right] + O(1). \tag{32}$$

Whenever $c \to \infty$, the odd part has exactly one eigenvalue of each sign escaping beyond a fixed bounded interval, and

$$\lambda^{\pm}_{o,\varepsilon,s} = \pm\sqrt{\frac{ds(\ell-s)}{4\ell\varepsilon}} - \frac{\alpha}{2}\log c + O(1). \tag{33}$$

The remainder in (33) is uniform for $0 < s \le \ell/2$ and c sufficiently large; all other odd spectrum is uniformly bounded. If instead c stays bounded, the entire odd spectrum is bounded, uniformly on each bounded set of ratios. Consequently, a mesoscopic merger $s \to 0$, $\varepsilon/s \to 0$ has exactly two escaping eigenvalues of each sign and bounded remaining spectrum, whereas bounded collision ratios have only one escaping pair.

For bounded c, (32) becomes

$$\lambda^{\pm}_{e,\varepsilon,s} = \pm\sqrt{\frac{\ell D_c}{8\varepsilon}} - \alpha\log(1/\varepsilon) + O(1), \qquad D_c = 2Q_c. \tag{34}$$

At $s = 0$ the two profiles coincide. In the mesoscopic regime the even branch is of order $\varepsilon^{-1/2}$ and the odd branch of order $\sqrt{s/\varepsilon}$. The exact primitive overlap in Q_c and the extra coalescence logarithm are retained uniformly. The positive lower bound for Q_c depends on the fixed profile.

The complete uniform parity proof is given in Appendix A. It keeps the exact overlap in the even channel and rescales the odd channel before estimating the slow pair.

5.1 The inner channel and the critical-tail dichotomy

Let $X = \dot{H}^1_0(\mathbb{R}_+)$ be the derivative-energy completion of $C^\infty_c(\mathbb{R}_+)$, and define the bounded selfadjoint operator K_c by

$$e(K_c u, z) = \frac{1}{2}\int_0^\infty [v(t-c) + v(t+c)]u(t)\overline{z(t)}\,\mathrm{d}t. \tag{35}$$

Hardy at 0 gives boundedness, locally uniformly in c.

Proposition 5.2 (The surviving collision channel)**.** *For each fixed $c \ge 0$,*

$$\sigma_{\mathrm{ess}}(K_c) = [\min(0, 4\alpha), \max(0, 4\alpha)]. \tag{36}$$

If $\varepsilon \to 0$ and $s/\varepsilon \to c < \infty$, rescaling the odd finite-interval operator by $x = \varepsilon t$ and extending by zero to X gives strong operator convergence to K_c. If $\alpha \neq 0$, every compact approximation has distance at least $4|\alpha|$ from K_c, so this convergence is not in norm. If $\alpha = 0$, K_c is compact and

the convergence is in norm. As $c \to \infty$, exactly one eigenvalue of each sign of K_c escapes beyond a fixed bounded interval, with

$$\nu_c^{\pm} = \pm\sqrt{dc}/2 - \frac{\alpha}{2}\log c + O(1), \tag{37}$$

and all its remaining spectrum stays uniformly bounded.

For identical even profiles the aggregate primitive is odd, so its anchor-zero layer vector is even. The rank-one removal in Theorem 3.2 therefore leaves the odd sector unchanged. The following half-line realization retains its large-collision-ratio law.

Proof. For $R > 0$ let P_R denote the energy-orthogonal projection in X onto the zero-extended $H_0^1(0,R)$. These spaces have dense increasing union, so $P_R \to I$ strongly. The normalized dilation $u(\varepsilon t)/\sqrt{\varepsilon}$ identifies the odd operator with the compression $P_{\ell/\varepsilon}K_{s/\varepsilon}P_{\ell/\varepsilon}$, acting as zero on the orthogonal complement. For the present measurable profile we verify norm continuity of K_c without assuming pointwise or uniform continuity of v. If c' and c belong to a fixed bounded set, put $U_c(t) = v(t-c) + v(t+c)$. On $0 < t < T$, point evaluation in X gives $|u(t)| \le \sqrt{t}\,\|u\|_X$, so the difference form is bounded by

$$\frac{1}{2}\|u\|_X^2 \int_0^T t|U_{c'}(t) - U_c(t)|\,\mathrm{d}t.$$

This tends to zero by local L^1 translation continuity of v. On $t > T$, the profile tail gives, uniformly for bounded c, c',

$$\sup_{t>T} t^2|U_{c'}(t) - U_c(t)| \le C\frac{|c'-c|}{T} + CT^{-\eta}.$$

Hardy bounds the tail difference form by twice this supremum times $\|u\|_X^2$. First let $c' \to c$ and then $T \to \infty$. The selfadjoint form bound gives $\|K_{c'} - K_c\| \to 0$. Together with the strong convergence of P_R this proves the strong collision limit.

For the essential spectrum apply the one-half-line assertion proved in Lemma 3.1 to $U_c(t) = v(t-c) + v(t+c)$. Its tail coefficient is 2α, so the essential interval joins 0 to 4α and the operator is compact exactly when $\alpha = 0$. This proves (36). The essential norm of K_c is $4|\alpha|$. Each finite-R compression is compact, so its distance from K_c is at least $4|\alpha|$. When $\alpha = 0$, compactness instead gives $\|P_R K_c P_R - K_c\| \to 0$; together with the proved norm continuity in c this proves the norm collision limit in that case.

For the large-c assertion rescale the half-line by $t = cy$. The center is 1 and the width is $1/c$. The one-trace tent is $\min(y, 1)$, with unit energy; on the infinite right cell it is constant, so the affine correction vanishes and there is no mean subtraction there. The left-cell estimates in (42) remain unchanged. On the right the primitive is in $L^2(1,\infty)$, and its full square gives the remaining half of d/δ. Hence the same computation now gives

$$\|T\|^2 = \frac{dc}{4} + O(1 + \log^2 c), \qquad A_o = -\alpha\log c + O(1).$$

The outside-center term is bounded on all of X by Hardy at 0, and the trace-zero reserve is uniformly bounded by Hardy at 1. The scalar rank-two calculation proves (37). The codimension-one variational bound excludes further spectrum outside a common bounded interval; it applies to this bounded noncompact selfadjoint operator as well, and the two escaping spectral points are isolated eigenvalues. This proves the full assertion, including the uniformly bounded essential band. □

The inner operator records the part of the odd spectrum that does not escape when the collision ratio is bounded. When $\alpha \neq 0$, its nonzero essential interval explains why compact finite-interval approximants cannot converge in operator norm. For $\alpha = 0$ this obstruction disappears and the inner channel is compact. Its large-c law, in turn, supplies the slow pair of the mesoscopic regime. The finite-interval coefficient in (33) retains the far-boundary factor $\sqrt{(\ell - s)/\ell}$: replacing it by 1 need not preserve a bounded remainder along an arbitrary mesoscopic path.

5.2 The rational profile as an explicit example

For $v_0(t) = 2(t^2 - 1)/(1 + t^2)^2$ one has $P_0(t) = -2t/(1 + t^2)$, $d_0 = 2\pi$ and $\alpha_0 = 2$. With Fourier convention $\widehat{f}(\xi) = \int \mathrm{e}^{-it\xi} f(t)\,\mathrm{d}t$, the elementary transform $\widehat{(1 + t^2)^{-1}} = \pi \mathrm{e}^{-|\xi|}$ and its L^2 derivative give $\widehat{P}_0 = 2\pi i \operatorname{sgn}(\xi)\mathrm{e}^{-|\xi|}$. Parseval therefore gives

$$\int_{\mathbb{R}} P_0(t - c)P_0(t + c)\,\mathrm{d}t = 2\pi \int_{\mathbb{R}} \mathrm{e}^{-2|\xi|} \cos(2c\xi)\,\mathrm{d}\xi = \frac{2\pi}{1 + c^2}.$$

Thus $Q_c = 2\pi(1 + (1 + c^2)^{-1})$. The general formulas recover the rational even and odd laws, and the inner essential interval is $[0, 8]$.

5.3 A compactly supported profile

A nonrational example is

$$P(t) = t(1 - t^2)^2 \mathbf{1}_{\{|t|<1\}}, \qquad v(t) = (1 - t^2)(1 - 5t^2)\mathbf{1}_{\{|t|<1\}}.$$

The primitive vanishes continuously at both support endpoints, so $P' = v$ almost everywhere without boundary point masses. The profile is even, bounded and has zero mean; here $\alpha = 0$ and

$$d = 2\int_0^1 t^2(1 - t^2)^4\,\mathrm{d}t = \frac{256}{3465}.$$

For $c \geq 1$ the two translated primitive supports are disjoint up to null endpoints, hence $Q_c = d$ exactly. Its inner collision operator is compact and the collision convergence is in norm. This example also shows why the non-norm assertion needs the condition $\alpha \neq 0$. Replacing any admissible v by $-v$ keeps d and Q_c unchanged, reverses both logarithmic drifts and reflects the inner essential interval. The paired leading magnitudes are unchanged.

6 Further directions

The uniform finite matrix model accommodates larger clusters, while scalar logarithmic extraction for multiple nested Green scales requires corresponding mixed-block estimates. At a degenerate cancellation, primitive regularity determines the next scale; the anchored cusp family suggests extensions to more general moduli of smoothness. A further quantitative problem is phase information beyond the fixed-threshold logarithmic count, with simultaneous control of collision and tail parameters.

A Uniform parity estimates

This appendix proves Theorem 5.1. The marked trace at the positive center has different endpoint conditions in the two parity sectors. Keeping those conditions gives the all-ratio even law and the uniform resolved odd law.

A.1 Uniform reserve and the primitive overlap

The normalized restriction $u \mapsto \sqrt{2}\,u|_{(0,\ell)}$ identifies the odd space with $H^1_0(0,\ell)$ and the even space with $\{u \in H^1(0,\ell) : u(\ell) = 0\}$. The half-interval form still has the factor $1/2$ in (31). In either space impose $u(s) = 0$. For $s > 0$ the two one-sided Hardy inequalities at s apply. The envelope

$$|\varepsilon^{-2} v((x-s)/\varepsilon)| \le \frac{C}{(x-s)^2}, \qquad |\varepsilon^{-2} v((x+s)/\varepsilon)| \le \frac{C}{(x+s)^2} \le \frac{C}{(x-s)^2}$$

therefore bounds the reserve compression uniformly. At $s = 0$ use Hardy at the left endpoint in the even reserve $u(0) = 0$. For $s > 0$ each reserve has codimension one; at $s = 0$ the even reserve has codimension one and Hardy bounds the entire odd space. The variational principle implies that at most one eigenvalue of each sign can lie outside a common fixed interval in each parity sector. Thus constructing one escaping pair also proves an exact count for the full sector, not only for a finite matrix.

The overlap identity is

$$Q_c = d + \int_{\mathbb{R}} P(t-c)P(t+c)\,\mathrm{d}t, \qquad 0 < \inf_{c \ge 0} Q_c \le Q_c \le 2d. \tag{38}$$

Translation continuity in L^2 makes Q_c continuous. The inner product of a fixed L^2 function with a translate tends to zero as the translation tends to infinity: first approximate by compactly supported functions, whose sufficiently separated translates have zero inner product, and then use Cauchy–Schwarz. Thus $Q_c \to d$ as $c \to \infty$. If $Q_c = 0$, then $P(t+2c) = -P(t)$ almost everywhere. For $c > 0$ the modulus would be periodic and integrable in square on the line, forcing $P = 0$. For $c = 0$ the same conclusion is immediate. Both contradict $d > 0$. Positivity, continuity and the positive limit give the lower bound in (38); its constant depends on the fixed profile. In particular,

$$\int_0^\infty \left\{\varepsilon^{-1}[P((x-s)/\varepsilon) + P((x+s)/\varepsilon)]\right\}^2 \mathrm{d}x = \frac{Q_c}{\varepsilon}. \tag{39}$$

Indeed the primitive sum is odd in x, so its half-line squared integral is half its whole-line squared integral.

A.2 The even channel: a sharp estimate for every ratio

Put $B = \ell - s$, $C = \ell + s$, and

$$e_s(x) = \frac{\ell - \max(x,s)}{\sqrt{B}}, \qquad F(x) = \varepsilon^{-1}[P((x-s)/\varepsilon) + P((x+s)/\varepsilon)].$$

The tent e_s has unit energy and value $\sqrt{B}$ at s. In the reserve/trace decomposition write the even operator as $\left(\begin{smallmatrix} R & T \\ T^* & A_e \end{smallmatrix}\right)$, where $\|R\|$ is uniformly bounded and T is a single column. The trace block is

$$A_e = \int_s^\ell F(x)\frac{\ell - x}{B}\,\mathrm{d}x.$$

To check the boundary terms, integrate $F' = V$ against $e_s^2/2$. At ℓ the tent vanishes, at 0 one has $F(0) = 0$, and the two terms at s cancel by continuity. The affine weight in the last display is bounded by 1. Consequently (30) gives $A_e = (\alpha/2)A_e^{(0)} + O(\|r\|_1)$, uniformly in s, ε, where $A_e^{(0)}$ is the trace block for P_0. Its explicit formula (43) below, with $B \ge \ell/2$, yields

$$A_e/2 = -\frac{\alpha}{2}[\log(1/\varepsilon) + \log(1/(s+\varepsilon))] + O(1), \qquad |A_e| = O(1 + \log(1/\varepsilon)). \tag{40}$$

The reduction error is uniform because each scaled translate of r has L^1 norm at most $\|r\|_1$ on the half interval.

The sharp coupling estimate is

$$\|T\|^2 = \frac{BQ_c}{4\varepsilon} + O(\log^2(1/\varepsilon)), \tag{41}$$

uniformly down to $s = 0$. Here are the uniform error costs. On $(0, s)$ the coupling derivative is exactly $-\sqrt{B}F/2$: the natural endpoint condition is derivative zero at 0, paid by $F(0) = 0$. There is no mean subtraction on this free-endpoint cell. On (s, ℓ) integrate the affine forcing. Before subtracting its mean, the derivative equals $-\sqrt{B}F/2 + E$; after removal of constant terms the correction E is a bounded linear combination of

$$(x-s)\varepsilon^{-1}P((x-s)/\varepsilon), \quad (x-s)\varepsilon^{-1}P((x+s)/\varepsilon), \quad J((x-s)/\varepsilon), \quad J((x+s)/\varepsilon).$$

The first two expressions are uniformly bounded; for the second use $(x-s)/(x+s) \le 1$. Adding the constant $\alpha \log(1/\varepsilon)$ to each J shows that

$$|E(x)| \le C_\ell[1 + |\log(x - s + \varepsilon)| + |\log(x + s + \varepsilon)|].$$

Its L^2 norm and its right-cell mean are uniformly bounded. Its cross integral with either singular primitive is $O(\log^2(1/\varepsilon))$: the primitive has L^1 norm $O(\log(1/\varepsilon))$ there, and each logarithm is pointwise $O(1 + \log(1/\varepsilon))$. The squared correction costs $O(1)$. The right-cell mean of the leading primitive costs $O(\log^2(1/\varepsilon))$ in the Gram, since the cell length is at least $\ell/2$. These are the only centering costs. Finally, (39) gives the raw leading Gram, while truncating its integral at ℓ costs $O(1)$: for $x \ge \ell$ and $s \le \ell/2$ each primitive is bounded by C/x. This proves (41) without a constant depending on the short-cell length.

Deleting R changes the operator by a uniformly bounded amount. The only nonzero eigenvalues of the resulting operator are those of $\left(\begin{smallmatrix} 0 & \|T\| \\ \|T\| & A_e \end{smallmatrix}\right)$. Since BQ_c is bounded below by a positive fixed constant,

$$\|T\| = \frac{\sqrt{BQ_c}}{2\sqrt{\varepsilon}} + O(\sqrt{\varepsilon}\log^2(1/\varepsilon)),$$

and the two scalar eigenvalues equal $\pm\|T\| + A_e/2 + O(A_e^2/\|T\|)$. The last error is $O(\sqrt{\varepsilon}\log^2(1/\varepsilon)) = o(1)$, uniformly. Restoring R and using the codimension-one count proves (32) and its assertion about the entire remaining spectrum.

A.3 The odd channel: a uniform expanding-interval estimate

The outside-center term is a uniformly bounded operator on the entire odd space: its absolute potential is bounded by $C/(x+s)^2 \le C/x^2$, so Hardy at 0 applies. It therefore suffices, up to bounded operator error, to keep only the center s. For $s > 0$ rescale $x = sy$, $u(sy) = \sqrt{s}f(y)$. The new interval is $(0, N+1)$ with $N = (\ell - s)/s \ge 1$, center 1, and width $\delta = \varepsilon/s = 1/c$. The normalized trace tent has value $A = \sqrt{N/(N+1)}$, left slope A, and right slope $-A/N$. We claim uniformly for every $N \ge 1$ and $c \to \infty$ that

$$\|T\|_o^2 = \frac{A^2 d}{4\delta} + O(1 + \log^2 c), \qquad A_o = -\alpha \log c + O(1). \tag{42}$$

The reserve norm is uniform by Hardy at 1.

On a cell where the tent slope is k, an exact uncentered coupling derivative is

$$-\frac{A}{2\delta}P(t)-\frac{k}{2}[tP(t)-J(t)],\qquad t=(y-1)/\delta.$$

Subtract the cell mean to impose zero values at its two endpoints. The left cell has length 1 and $|k|\le 1$, so its squared correction and leading/correction cross integrals cost $O(1+\log^2 c)$, exactly by integrating $(1+|\log(z+\delta)|)/(z+\delta)$ for $0<z<1$. On the right, of length N, one has $|k|\le 1/N$. After removing constants its correction is bounded by

$$\frac{C}{N}[1+|\log((y-1+\delta)/N)|].$$

Its squared L^2 norm is $O(1/N)$, and its cross integral with the leading singular primitive is at most $C[1+\log^2(Nc)]/N$. The squared right-cell mean of the leading primitive obeys the same bound, while the left-cell mean costs $O(\log^2 c)$. These estimates are uniform because

$$\frac{\log^2(Nc)}{N}\le 2\log^2 c+2\frac{\log^2 N}{N}\le C(1+\log^2 c).$$

Omitted primitive-square tails cost $O(1)$, since both cell lengths are at least 1. This proves the first formula in (42).

For the trace block use (30) in $-\int\delta^{-1}Pee'$. On the left $|ee'|\le 1$, and on the right $|ee'|\le 1/N\le 1$. The r term therefore costs at most $2\,\|r\|_1$, uniformly. For P_0 direct integration gives

$$\begin{aligned}A_o^{(0)}=&-A^2\log(1+c^2)-\frac{A^2}{N}\log(1+N^2c^2)\\&+2A^2[1-\delta\arctan c]+\frac{2A^2}{N}[1-(\delta/N)\arctan(Nc)].\end{aligned}$$

The bounded terms are uniform and its logarithmic part is $-2\log c-2\log N/(N+1)+O(1)$. Multiplication by $\alpha/2$ and restoration of the integrable remainder prove $A_o=-\alpha\log c+O(1)$ uniformly in $N\ge 1$.

Since $A^2\ge 1/2$, the scalar root and quadratic trace errors are $O((1+\log^2 c)/\sqrt{c})=o(1)$. Deleting and restoring the reserve, and restoring the bounded outside-center operator, gives

$$\lambda_o^{\pm}=\pm\frac{A\sqrt{d}}{2\sqrt{\delta}}-\frac{\alpha}{2}\log c+O(1).$$

Substituting $A^2=N/(N+1)=(\ell-s)/\ell$ proves (33). Codimension one proves the exact odd count. For bounded c, on the other hand, $|v(t-c)+v(t+c)|\le C_S/(1+t^2)\le C_S/t^2$ for $0\le c\le S$; after $x=\varepsilon t$, Hardy at 0 bounds the entire odd operator. This finishes the proof of Theorem 5.1.

A.4 The rational reference trace

For completeness its trace primitive, used in the general reduction above, is exactly

$$\begin{aligned}A_e^{(0)}=&-\log(1+B^2/\varepsilon^2)-\frac{C}{B}\log\frac{C^2+\varepsilon^2}{4s^2+\varepsilon^2}+4\\&-\frac{2\varepsilon}{B}[\arctan(B/\varepsilon)+\arctan(C/\varepsilon)-\arctan(2s/\varepsilon)],\qquad B=\ell-s,\quad C=\ell+s.\end{aligned}\tag{43}$$

To derive it, the integral of $\varepsilon^{-1}P_0((x-s)/\varepsilon)(\ell-x)/B$ on (s,ℓ), with $y=x-s$, equals $-\log(1+B^2/\varepsilon^2)+2-(2\varepsilon/B)\arctan(B/\varepsilon)$. The translated term, with $z=x+s$, contributes the remaining logarithm, $+2$ and the remaining arctangents. All bounded terms are uniform since $B\ge\ell/2$. Also $\sqrt{4s^2+\varepsilon^2}$ is comparable to $s+\varepsilon$, and $(C/B-1)|\log(s+\varepsilon)|=O(s|\log(s+\varepsilon)|)=O(1)$. Consequently $A_e^{(0)}/2=-\log(1/\varepsilon)-\log(1/(s+\varepsilon))+O(1)$, as used above.

References

[1] R. Blankenbecler, M. L. Goldberger and B. Simon, *The bound states of weakly coupled long-range one-dimensional quantum Hamiltonians*, Ann. Physics **108**(1) (1977), 69–78, https://doi.org/10.1016/0003-4916(77)90351-7; author-hosted text.

[2] P. Exner, H. Neidhardt and V. A. Zagrebnov, *Potential approximations to δ': an inverse Klauder phenomenon with norm-resolvent convergence*, Commun. Math. Phys. **224**(3) (2001), 593–612, https://doi.org/10.1007/s002200100567.

[3] S. Fassari, M. Gadella, L. M. Nieto and F. Rinaldi, *Analysis of a one-dimensional Hamiltonian with a singular double well consisting of two nonlocal δ' interactions*, Eur. Phys. J. Plus **139**(2) (2024), article 132, https://doi.org/10.1140/epjp/s13360-024-04923-8.

[4] F. Gesztesy, L. L. Littlejohn, I. Michael and R. Wellman, *On Birman's sequence of Hardy–Rellich-type inequalities*, J. Differential Equations **264**(4) (2018), 2761–2801, https://doi.org/10.1016/j.jde.2017.11.002; author version arXiv:1710.06955v3, Section 6, Theorem 6.6.

[5] Y. Golovaty, *1D Schrödinger operators with short range interactions: two-scale regularization of distributional potentials*, Integral Equations Operator Theory **75** (2013), 341–362, https://doi.org/10.1007/s00020-012-2027-z; author version arXiv:1202.4711v3.

[6] Y. Golovaty, *On coupling constant thresholds in one dimension*, Carpathian Math. Publ. **13**(1) (2021), 22–38, https://doi.org/10.15330/cmp.13.1.22-38; author version arXiv:1905.10766v2.

[7] Y. Golovaty and R. Hryniv, *On negative eigenvalues of 1D Schrödinger operators with δ'-like potentials*, arXiv:2504.14934v2 (2025), Front. Appl. Math. Stat. **11** (2025), article 1615447, https://doi.org/10.3389/fams.2025.1615447.

[8] T. Kato, *Perturbation of continuous spectra by trace class operators*, Proc. Japan Acad. **33**(5) (1957), 260–264, https://doi.org/10.3792/pja/1195525063.

[9] D. Kielty, *Singular limits of sign-changing weighted eigenproblems*, Asymptotic Analysis **122** (2021), nos. 1–2, 165–200, https://doi.org/10.3233/ASY-201615.

[10] W. Kirsch and B. Simon, *Corrections to the classical behavior of the number of bound states of Schrödinger operators*, Ann. Physics **183**(1) (1988), 122–130, https://doi.org/10.1016/0003-4916(88)90248-5; author-hosted text.

[11] B. Simon, *The bound state of weakly coupled Schrödinger operators in one and two dimensions*, Ann. Physics **97**(2) (1976), 279–288, https://doi.org/10.1016/0003-4916(76)90038-5; author-hosted text.

[12] A. V. Zolotaryuk, *Families of one-point interactions resulting from the squeezing limit of the sum of two- and three-delta-like potentials*, J. Phys. A: Math. Theor. **50**(22) (2017), article 225303, https://doi.org/10.1088/1751-8121/aa6dc2.

Statements and Declarations

Funding. This work received no external funding.

Competing interests. The author's organization, Luminary Oddities, operates the Luminary Research Archive hosting the preprint. The author declares no other relevant competing interests.

AI assistance. ChatGPT (OpenAI) was used to generate and revise manuscript drafts, propose proof developments and strengthenings, compare literature arguments, develop verification code, and assist with internal proof review. The author directed the research and revision decisions and is responsible for the mathematical content and presentation.

Data availability. No empirical datasets were analysed. The finite algebraic verification results can be reproduced using the code in the accompanying source package.

Code availability. The manuscript contains the mathematical proofs. Editable LaTeX sources and standard-library finite-verification code accompany the manuscript.