Spectral transitions for concentrated zero-mean energy forms

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Luminary Oddities

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Abstract

We study the Riesz operators of concentrating signed potential forms in the derivative-energy space $H_0^1(a,b)$. For finitely many fixed separated centers and nonzero even zero-mean profiles with inverse-square tails, exactly one positive and one negative eigenvalue per center escape to infinity. Their leading coefficients are the square roots of the eigenvalues of a full Dirichlet Green matrix weighted by the squared L^2 norms of the profile primitives. The remaining spectrum is uniformly bounded. Equal critical tails give a common logarithmic correction, and mean perturbations at the square-root scale yield a complete finite Hermitian crossover model with an operator-norm remainder under explicit moving isometric embeddings. We apply the results to localized log-curvature forms, including a positive-Fourier-inverse example with different even and odd escape coefficients. For real zeros, a trace-class Hardy localization identifies the absolutely continuous multiplicities and explains strong convergence with an unavoidable norm gap under genuine root fusion. The results concern bounded-interval energy operators; they assert no full-line theta sign theorem or polynomial-degree rate.

1 Introduction and principal results

For a signed potential, concentration interacts with the Hilbert metric used to normalize the quadratic form. Here the metric is derivative energy, and the object is the compact selfadjoint operator C_ε defined by

$$e(C_\varepsilon u, z) = q_\varepsilon(u,z), \qquad e(u,z) = \int_a^b u'\overline{z'}\,dx, \qquad u, z \in H := H_0^1(a,b). \tag{1}$$

It is not the usual L^2 Schrödinger operator with the same potential. This distinction changes the spectral question, but does not by itself constitute a new mechanism. Weak-coupling primitive energies, trace removal, Hardy inequalities and critical-tail logarithms have classical predecessors [1, 2]. Our purpose is to give the complete finite-interval energy-space law, including its interaction matrix, bounded residual spectrum and critical mean crossover.

Fix $a < r_1 < \cdots < r_k < b$, with $1 \le k < \infty$. For each j let v_j be real, measurable and even, nonzero on a set of positive measure, and suppose that for constants $C_j < \infty$, $\eta_j > 0$, $\alpha_j \in \mathbb{R}$,

$$\int_{\mathbb{R}} v_j = 0, \qquad |v_j(t)| \le \frac{C_j}{1+t^2}, \qquad v_j(t) = \frac{\alpha_j}{t^2} + O(|t|^{-2-\eta_j}) \quad (|t| \to \infty). \tag{2}$$

Put

$$\begin{gathered} L_j(t) = \int_{-\infty}^t v_j(s)\,ds, \qquad d_j = \int_{\mathbb{R}} |L_j|^2 > 0, \qquad D = \operatorname{diag}(d_1, \dots, d_k), \\ G_{ij} = \frac{(\min(r_i, r_j) - a)(b - \max(r_i, r_j))}{b-a}. \end{gathered} \tag{3}$$

Let W_ε be real measurable with a common $L^\infty(a,b)$ bound, and set

$$q_\varepsilon(u,z) = \frac{1}{2}\int_a^b \left\{ \sum_{j=1}^{k} \varepsilon^{-2} v_j\left(\frac{x-r_j}{\varepsilon}\right) + W_\varepsilon(x) \right\} u\overline{z}\, dx, \qquad 0<\varepsilon<\tfrac{1}{2}. \tag{4}$$

Every C_ε is compact: for fixed ε the potential is bounded, and the embedding $H_0^1(a,b) \hookrightarrow L^2(a,b)$ is compact. Compactness is asserted separately for each ε.

Theorem 1.1 (Full Green-matrix escape law)**.** *Let $\rho_1 \geq \cdots \geq \rho_k > 0$ be the eigenvalues of $D^{1/2}GD^{1/2}$. There is $M < \infty$ such that, for all sufficiently small ε, C_ε has exactly k eigenvalues above M and exactly k below $-M$, counting multiplicity. All others lie in $[-M,M]$. The escaping branches, the negative ones ordered by decreasing absolute value, satisfy*

$$\lambda_j^{\pm}(\varepsilon) = \pm\frac{\sqrt{\rho_j}}{2\sqrt{\varepsilon}} + o(\varepsilon^{-1/2}), \qquad 1 \leq j \leq k. \tag{5}$$

If all α_j have one common value α, then

$$\lambda_j^{\pm}(\varepsilon) = \pm\frac{\sqrt{\rho_j}}{2\sqrt{\varepsilon}} - \frac{\alpha}{2}\log(1/\varepsilon) + O(1). \tag{6}$$

The bounds need not be uniform as centers approach one another or as k increases. No convergence of W_ε is needed.

The off-diagonal entries of G are part of the leading answer. Discarding them generally gives incorrect coefficients, as the exact parity example in Section 4.1 shows.

Theorem 1.2 (Critical mean crossover)**.** *In (4), replace v_j by $v_j + \sqrt{\varepsilon}\, w_j$, where each w_j is fixed, real, bounded, even and compactly supported. Put $\kappa_j = \int_{\mathbb{R}} w_j$, $\mathcal{K} = \operatorname{diag}(\kappa_j)$ and*

$$S = \tfrac{1}{2}(G^{1/2}DG^{1/2})^{1/2}, \qquad A_{\mathrm{det}} = \tfrac{1}{2}G^{1/2}\mathcal{K}G^{1/2}, \qquad \mathcal{L} = \begin{pmatrix} 0 & S \\ S & A_{\mathrm{det}} \end{pmatrix}. \tag{7}$$

There are isometries $J_\varepsilon : \mathbb{C}^{2k} \to H$ such that

$$\big\| \sqrt{\varepsilon}\, C_\varepsilon - J_\varepsilon \mathcal{L} J_\varepsilon^* \big\|_{\mathrm{op}} \longrightarrow 0. \tag{8}$$

The matrix $\mathcal{L}$ has inertia (k,k). Its eigenvalues are the nonzero limits of the $2k$ escaping eigenvalues multiplied by $\sqrt{\varepsilon}$. Every remaining unscaled eigenvalue stays uniformly bounded. The embeddings J_ε vary with ε.

1.1 Relation to earlier work

Simon [1, Theorems 2.4–2.5] identifies primitive-energy effects for one-dimensional weak binding. Blankenbecler, Goldberger and Simon [2, Sections 3–4] use a Dirichlet cut and Hardy bounds, and examine inverse-square-tail logarithmic corrections. These results concern Schrödinger eigenvalues; the cited critical-tail logarithm has nonzero negative mean. They are substantive antecedents to our proof. Golovaty and Hryniv [3] study spectral escape for singular ε^{-2} potentials despite norm-resolvent limits. The general phenomenon of escaping spectrum under concentration is established in the cited literature. Our stated results use the original energy-Riesz normalization and establish its complete Green geometry, paired counts, bounded bulk, critical crossover and two-term equal-tail law.

The real-zero comparison below uses classical Hardy–Mellin analysis [4, 5] and Kato–Rosenblum theory [6]. The paper [8] studies finite-degree localization and correction inertia for a full-line theta form; its later critical-tail successor [9] improves that finite-degree rate. Conversely our bounded-interval results do not give the original full-line theta sign, actual nonreal theta zeros or a degree-versus-collision law. The receiving identity, rather than a change of notation between two scalar moment problems, is stated explicitly in Section 4.

2 The trace decomposition and the escape law

All estimates below have constants depending on the fixed data in (2), the interval, center locations and common bound for W_ε. We write $\ell_\varepsilon = \log(1/\varepsilon)$.

Lemma 2.1 (Primitives)**.** *Each L_j is odd and absolutely continuous, with*

$$L_j(t) = O(\min(|t|, |t|^{-1})), \qquad L_j(t) = -\frac{\alpha_j}{t} + O(|t|^{-1-\eta_j}) \quad (|t| \to \infty).$$

Moreover $d_j > 0$, $\sup |tL_j(t)| < \infty$, and

$$J_j(t) := \int_0^t L_j(s)\, ds = -\alpha_j \log(1 + |t|) + O(1).$$

Proof. Evenness and zero total mass give $L_j(0) = 0$ and oddness. Boundedness of v_j gives the estimate at zero. On the positive half-line use $L_j(t) = -\int_t^\infty v_j$, and integrate the stated tail; oddness gives the negative half-line. Integrating this tail after subtracting $-\alpha_j/t$ proves the J_j formula, because $\eta_j > 0$. Finally, $d_j = 0$ would give $L_j = 0$ almost everywhere and hence $v_j = L_j' = 0$ almost everywhere, contrary to the hypothesis. □

Let $G_r \in H$ be the Riesz representer of evaluation at r: $e(u, G_r) = u(r)$. It is the Dirichlet Green tent, and $e(G_{r_i}, G_{r_j}) = G_{ij}$ with G from (3). Define

$$H_* := \{u \in H : u(r_j) = 0 \ (1 \le j \le k)\}, \qquad T := H_*^\perp.$$

Point evaluations at distinct points are independent, as disjoint compactly supported smooth test functions show. Thus $\dim T = k$, G is positive definite, and T is the span of the Green tents. Its elements are continuous functions linear between successive marked points $a, r_1, \dots, r_k, b$.

Take a real energy-orthonormal basis $e_1, \dots, e_k$ of T, and let $A_{jl} = e_l(r_j)$. The evaluation identity gives

$$AA^{\mathsf{T}} = G. \tag{9}$$

The canonical choice obtained from the Green tents has $A = G^{1/2}$. In $H = H_* \oplus T$, write

$$C_\varepsilon = \begin{pmatrix} B_\varepsilon & T_\varepsilon \\ T_\varepsilon^* & A_\varepsilon \end{pmatrix}. \tag{10}$$

Lemma 2.2 (Uniform reserve)**.** *There is a fixed M_0 with $\|B_\varepsilon\| \le M_0$.*

Proof. The profile envelope gives $|\varepsilon^{-2} v_j((x - r_j)/\varepsilon)| \le C_j/(x - r_j)^2$ away from r_j. For $u \in H_*$ the two one-sided Hardy inequalities give

$$\int_a^b \frac{|u|^2}{(x - r_j)^2}\, dx \le 4e(u, u).$$

The bounded W_ε term is controlled by Dirichlet Poincaré. This bounds $|q_\varepsilon(u, u)|$ on H_*; polarization gives the operator bound for its selfadjoint compression. All centers are tested on this same subspace and the complete form is retained. □

Lemma 2.3 (The complete coupling Gram)**.** *In any basis satisfying* (9)*,*

$$\|A_\varepsilon\| = O(\ell_\varepsilon), \qquad \varepsilon T_\varepsilon^* T_\varepsilon \longrightarrow \tfrac{1}{4} A^{\mathsf{T}} D A \tag{11}$$

in finite-dimensional operator norm. More precisely,

$$T_\varepsilon^* T_\varepsilon = \frac{A^{\mathsf{T}} D A}{4\varepsilon} + O(\ell_\varepsilon^2). \tag{12}$$

In the canonical basis the trace block has expansion

$$A_\varepsilon = -\tfrac{1}{2}\left(G^{-1/2}\,\mathrm{diag}(\alpha_j) G^{1/2} + G^{1/2}\,\mathrm{diag}(\alpha_j) G^{-1/2}\right)\ell_\varepsilon + O(1). \tag{13}$$

Proof. The lth column $b_{l,\varepsilon} \in H_*$ of T_ε is the Riesz representer of $z \mapsto q_\varepsilon(e_l, z)$ on H_*. On each marked subinterval it is the solution, with Dirichlet values at both ends, of

$$-b_{l,\varepsilon}'' = \tfrac{1}{2}\left\{\sum_j \varepsilon^{-2} v_j((x - r_j)/\varepsilon) + W_\varepsilon\right\} e_l.$$

For a center at an endpoint of that subinterval, write $e_l(x) = A_{jl} + c_l(x - r_j)$. A derivative primitive of its contribution is

$$-\frac{A_{jl}}{2\varepsilon} L_j(t) - \frac{c_l}{2} t L_j(t) + \frac{c_l}{2} J_j(t), \qquad t = (x - r_j)/\varepsilon. \tag{14}$$

Direct differentiation gives the required negative forcing. Subtract the subinterval mean of the sum of these primitives to enforce its two Dirichlet values. Terms from centers outside the closed subinterval and from W_ε have uniformly bounded forcing and bounded primitives: their fixed positive separation is used here. Consequently, throughout I,

$$b_{l,\varepsilon}' = -\sum_j \frac{A_{jl}}{2\varepsilon} L_j((x - r_j)/\varepsilon) + R_{l,\varepsilon}, \qquad \|R_{l,\varepsilon}\|_2 = O(\ell_\varepsilon). \tag{15}$$

Using the common expression for the leading sum in nonadjacent cells changes the remainder only by a bounded function.

The diagonal primitive-square integrals obey

$$\int_a^b \varepsilon^{-2} L_j((x - r_j)/\varepsilon)^2\, dx = \frac{d_j}{\varepsilon} + O(1). \tag{16}$$

Indeed the omitted tails after changing variables are $O(\varepsilon)$ in $L^2(\mathbb{R})$. For different centers the unscaled product of the two singular primitives has integral $O(\ell_\varepsilon)$: in a fixed small neighborhood of one center the other factor is bounded, the singular factor has L^1 norm $O(\ell_\varepsilon)$, and outside both neighborhoods both are bounded. These facts and (15) first give the second limit in (11); the squared remainder costs $O(\varepsilon \ell_\varepsilon^2)$ after scaling and its leading cross term costs $O(\sqrt{\varepsilon}\ell_\varepsilon)$.

For the sharper estimate retain (14). Subtract the constant logarithmic part of each J_j before subinterval centering; centering removes that subtraction exactly. By Lemma 2.1, the resulting uncentered remainder is bounded by

$$C\left(1 + \sum_j |\log(|x - r_j| + \varepsilon)|\right). \tag{17}$$

Its L^2 norm and means are uniformly bounded. Its cross integral with a singular primitive is $O(\ell_\varepsilon^2)$: for $|x-r_j| \le \varepsilon$ use $|L_j(t)| \le C|t|$, and outside that range use $|\varepsilon^{-1}L_j(t)| \le C/|x-r_j|$ and integrate $(1+|\log|x-r_j||)/|x-r_j|$. Centering the leading primitives changes their full squared and mixed Grams by $O(\ell_\varepsilon^2)$ because each mean is $O(\ell_\varepsilon)$. Together with (16) and the different-center estimate this proves (12), including all mixed entries.

For the trace block, integrate the primitive $\varepsilon^{-1}L_j((x-r_j)/\varepsilon)$ against $(e_l e_m)'$. Outer boundary terms vanish, and inner terms cancel because $e_l e_m$ is continuous. The bounded derivative on each side gives $O(\ell_\varepsilon)$. To compute its coefficient, let H_{jl} be the jump of e_l' at r_j. The canonical Green basis has $H = -G^{-1/2}$, since each Green tent has jump -1 at its node. The derivative jump of $e_l e_m$ is $A_{jl}H_{jm} + H_{jl}A_{jm}$. The two tail integrals therefore give the logarithmic coefficient

$$\tfrac{1}{2}\left(A^{\mathsf{T}}\operatorname{diag}(\alpha_j)H + H^{\mathsf{T}}\operatorname{diag}(\alpha_j)A\right).$$

Every remainder is integrable after the $-\alpha_j/t$ primitive tail is subtracted; $\eta_j > 0$ gives an $O(1)$ error. Substituting the canonical A, H proves (13). □

Proof of Theorem 1.1. The reference block $\left(\begin{smallmatrix} 0 & T_\varepsilon \\ T_\varepsilon^* & 0 \end{smallmatrix}\right)$ has eigenvalues equal to plus and minus the k singular values of T_ε, and zero otherwise. By (11) these singular values, multiplied by $\sqrt{\varepsilon}$, tend to $\sqrt{\rho_j}/2$. Here we used (9) and the equality of the nonzero spectra of $A^{\mathsf{T}}DA$ and $D^{1/2}GD^{1/2}$. The difference from (10) has norm $O(\ell_\varepsilon)$. Minmax/Weyl comparison proves (5).

The compression bound of Lemma 2.2 separately implies that at most k eigenvalues lie above M_0 and at most k below $-M_0$: an allegedly larger spectral subspace would meet H_* nontrivially and violate the Rayleigh bound there. The k divergences of each sign prove exact counts, and every other eigenvalue stays bounded. The $O(\ell_\varepsilon)$ comparison alone would not prove this bounded-bulk assertion.

For equal $\alpha_j = \alpha$, (13) reduces to $A_\varepsilon = -\alpha\ell_\varepsilon I + O(1)$. In the canonical basis write $S = \frac{1}{2}(G^{1/2}DG^{1/2})^{1/2} > 0$. The square-root perturbation estimate for positive definite finite matrices, applied to (12), gives

$$(T_\varepsilon^* T_\varepsilon)^{1/2} = \frac{S}{\sqrt{\varepsilon}} + O(\sqrt{\varepsilon}\,\ell_\varepsilon^2) = \frac{S}{\sqrt{\varepsilon}} + o(1).$$

This follows also from the Sylvester equation for the difference of two square roots, whose smallest eigenvalue sums are of order $\varepsilon^{-1/2}$; no eigenvector differentiability is required. The polar factor of T_ε identifies its range isometrically with $\mathbb{C}^k$. The complete operator differs by bounded norm from the finite block with off-diagonal $S/\sqrt{\varepsilon}$ and lower diagonal $-\alpha\ell_\varepsilon I$. Diagonalizing S reduces that block to k two-by-two matrices. If $s_j = \sqrt{\rho_j}/2$, their eigenvalues are

$$-\frac{\alpha}{2}\ell_\varepsilon \pm \sqrt{\frac{s_j^2}{\varepsilon} + \frac{\alpha^2}{4}\ell_\varepsilon^2} = \pm\frac{s_j}{\sqrt{\varepsilon}} - \frac{\alpha}{2}\ell_\varepsilon + o(1).$$

The bounded comparison proves (6) for every branch, including repeated eigenvalues. □

2.1 Proof of the mean crossover

Proof of Theorem 1.2. The added potential is $\varepsilon^{-3/2}w_j((x-r_j)/\varepsilon)$. Since w_j is bounded and compactly supported, it is bounded by $C/(1+t^2)$. Hardy therefore bounds its compression to H_* by $O(\sqrt{\varepsilon})$. In a trace-to-H_* pairing,

$$|z(r_j+\varepsilon t)| \le \sqrt{\varepsilon|t|}\,\|z'\|_2, \qquad z \in H_*.$$

After the change of variables the added coupling has norm $O(1)$. The complete coupling T_ε consequently has the same scaled Gram limit $\varepsilon T_\varepsilon^* T_\varepsilon \to S^2$.

The scaled trace block tends to A_{det}. Indeed the extra (l,m) entry multiplied by $\sqrt{\varepsilon}$ is

$$\frac{1}{2}\sum_j \int_{\mathbb{R}} w_j(t) e_l(r_j+\varepsilon t) e_m(r_j+\varepsilon t)\, dt = \frac{1}{2}(A^{\mathsf{T}}\mathcal{K}A)_{lm} + O(\varepsilon).$$

The old trace block contributes $O(\sqrt{\varepsilon}\ell_\varepsilon) = o(1)$. The piecewise-Lipschitz trace functions and compact support of w_j justify the error, including across the center.

For small ε, T_ε has rank k. Put

$$U_\varepsilon = T_\varepsilon (T_\varepsilon^* T_\varepsilon)^{-1/2} : \mathbb{C}^k \to H_*.$$

It is an isometry. Let J_ε be U_ε on the first k coordinates and the canonical trace basis on the last k. Their ranges are orthogonal. In these coordinates the scaled coupling is $(\varepsilon T_\varepsilon^* T_\varepsilon)^{1/2} \to S$, the trace block tends to A_{det}, and the discarded complete H_* block has norm $O(\sqrt{\varepsilon})$. This proves (8) on the full space.

The change $u \mapsto u + \frac{1}{2}S^{-1}A_{\mathrm{det}}z$ converts $2\Re(u^*Sz) + z^*A_{\mathrm{det}}z$ to the pure cross form. Because S is invertible, this gives inertia (k,k) and no zero eigenvalues. Norm comparison gives the scaled escaping eigenvalues; the codimension-k bounded compression again proves a bounded unscaled bulk. No commutation of $G, D, \mathcal{K}$ was used. □

Corollary 2.4 (One center)**.** *For $k=1$, put $\tau = G_{11} = (r-a)(b-r)/(b-a)$. The two escaping eigenvalues are simple for small ε, all others are uniformly bounded, and*

$$\lambda^{\pm}(\varepsilon) = \pm\frac{\sqrt{\tau}\,\|L\|_2}{2\sqrt{\varepsilon}} - \frac{\alpha}{2}\log(1/\varepsilon) + O(1).$$

3 Real roots: Hardy channels and fusion

This section describes the limiting trace-domain geometry. The main spectral input is classical; its role is to make the distinction between real fusion and complex-pair approach precise.

Suppose on the closed interval that

$$f(x) = h(x)\prod_{j=1}^{k}(x-r_j)^{\nu_j}, \qquad \nu_j \in \{1,2,\dots\}, \tag{18}$$

where h is real C^2 and nowhere zero. Then

$$V_f := \frac{f'^2 - ff''}{f^2} = \sum_j \frac{\nu_j}{(x-r_j)^2} + W, \qquad W = -(\log|h|)'' \in L^\infty. \tag{19}$$

The form $q_f(u,z) = \frac{1}{2}\int V_f u\bar{z}$ is bounded on $H_r = \{u \in H : u(r_j) = 0 \text{ for all } j\}$. Indeed

$$|q_f(u,u)| \le \left(2\sum_j \nu_j + \frac{\|W\|_\infty (b-a)^2}{2\pi^2}\right) e(u,u). \tag{20}$$

Let C_f be its energy-Riesz representative on H_r.

Proposition 3.1 (Polynomial completion)**.** *The energy closure of*

$$\{f(x)(x-a)(b-x)P(x) : P \text{ a polynomial}\}$$

is exactly H_r. The norm formed by adding $\int(1+|V_f|)|u|^2$ to derivative energy is equivalent to energy on H_r. With real data real polynomials suffice for the real space.

Proof. Every displayed function vanishes at the endpoints and roots, so its closure is contained in H_r. Conversely functions smooth and compactly supported away from all roots and endpoints are dense in H_r. To justify cutting off near a root, a cutoff with derivative $O(1/\delta)$ on $\delta < |x-r_j| < 2\delta$ has derivative-error term bounded by a constant times

$$\int_{|x-r_j|<2\delta} \frac{|u(x)|^2}{|x-r_j|^2}\,dx \longrightarrow 0;$$

Hardy ensures integrability. The other terms tend to zero by absolute continuity of the derivative integral. Apply ordinary Sobolev approximation on the remaining separated intervals. For such a smooth target u, the quotient $u/[f(x)(x-a)(b-x)]$ is C^1 after extending it by zero near every excluded point. Approximate that quotient and its first derivative uniformly by polynomials: approximate the derivative uniformly and integrate, fixing the value at one endpoint. Multiplication by the fixed C^1 factor gives energy convergence. Hardy and Poincaré give the asserted norm equivalence. □

In particular, a multiple zero imposes a value trace in the H^1 completion; additional pointwise derivative jets are not closed H^1 constraints. This does not conflict with stronger smooth-topology domains carrying higher jets.

Theorem 3.2 (Real-root spectral classification)**.** *For $k \geq 1$, C_f is unitarily equivalent, up to a trace-class perturbation, to two copies for every j of*

$$\frac{\nu_j}{2}\left(-\frac{d^2}{dz^2}+\frac{1}{4}\right)^{-1}_{D} \quad \text{on } L^2(0,\infty), \tag{21}$$

together with finitely many zero directions. Consequently

$$\sigma_{\mathrm{ess}}(C_f) = [0, 2\max_j \nu_j], \qquad m_{\mathrm{ac}}(q) = 2\#\{j : 0 < q < 2\nu_j\} \quad \text{for a.e. } q > 0. \tag{22}$$

There is no absolutely continuous spectrum off that interval. Point or singular continuous spectrum is not excluded. For $I - C_f$ the spectral variable is reflected by $q \mapsto 1-q$.

Proof. Place an additional Dirichlet cut at each midpoint between consecutive roots. Together with the existing root traces and interval endpoints, these cuts give exactly $2k$ one-sided root half-cells. The simultaneous extra-trace-zero subspace has codimension $k-1$ in H_r. In that subspace the complete derivative and potential forms split by disjoint supports. Restoring its finite-dimensional orthogonal complement adds a finite-rank operator, including its mixed block.

On a root half-cell $0 < t < l$ the potential is $\nu/t^2 + R(t)$ with R bounded. Set $t = le^{-z}$ and $u(t) = \sqrt{t}\,v(z)$. For compact targets, then by completion,

$$\int_0^l |u'|^2\,dt = \int_0^\infty \left(|v'|^2 + \tfrac{1}{4}|v|^2\right)dz,$$
$$\tfrac{1}{2}\int_0^l V_f|u|^2\,dt = \tfrac{1}{2}\int_0^\infty \left(\nu + l^2e^{-2z}R(le^{-z})\right)|v|^2 dz.$$

The condition at $t = l$ is Dirichlet at $z = 0$; the completed space is the form domain of $A = -d^2/dz^2 + 1/4$ on the half-line. The further energy-to-L^2 unitary $v \mapsto A^{1/2}v$ gives

$$\tfrac{\nu}{2}A^{-1} + \tfrac{1}{2}A^{-1/2}M_wA^{-1/2}, \qquad w(z) = l^2e^{-2z}R(le^{-z}).$$

The second term is trace class. The Dirichlet Green diagonal is $A^{-1}(z,z) = 1 - e^{-z}$, so factoring positive and negative parts through their Hilbert–Schmidt square roots gives

$$\left\|\tfrac{1}{2}A^{-1/2}M_wA^{-1/2}\right\|_1 \le \tfrac{1}{2}\int_0^\infty |w(z)|(1-e^{-z})\,dz \le \frac{l^2\,\|R\|_\infty}{12}.$$

This proves the trace-class comparison, including the finite-rank restoration. The sine transform diagonalizes A with multiplier $\xi^2 + 1/4$. Each reference operator has simple purely absolutely continuous spectrum $[0, 2\nu]$. Weyl's essential-spectrum invariance and Kato–Rosenblum [6, Theorems 13.2–13.3] give (22) and its multiplicities. The argument is the one-dimensional Hardy–Mellin mechanism, as in [4, 5]. □

The jumps in m_{ac} record the multiplicity histogram: the jump at $2n$ is twice the number of roots of multiplicity n. Also $\int_0^\infty m_{\mathrm{ac}}(q)\,dq = 4\sum_j \nu_j$.

Theorem 3.3 (Genuine real fusion)**.** *Let distinct $r_j(\varepsilon)$ tend to one interior point r_0, with fixed multiplicities ν_j, and let $W_\varepsilon \to W_0$ uniformly with a common bound. Put $m = \sum_j \nu_j$ and let C_ε act on $H_\varepsilon = \{u : u(r_j(\varepsilon)) = 0\}$ for the corresponding real-root potential. At zero use $V_0 = m/(x-r_0)^2 + W_0$ and $H_0 = \{u : u(r_0) = 0\}$. Extend each operator by zero on the energy orthogonal complement of its domain, writing $\widetilde{C}_\varepsilon$. Then $\widetilde{C}_\varepsilon \to \widetilde{C}_0$ strongly on H. Nevertheless every unitary $U_\varepsilon : H_\varepsilon \to H_0$ and compact selfadjoint T_ε on H_0 satisfy*

$$\|U_\varepsilon C_\varepsilon U_\varepsilon^* + T_\varepsilon - C_0\| \ge 2(m - \max_j \nu_j). \tag{23}$$

For $k \ge 2$ this is a positive norm obstruction. For $k = 1$ it is zero.

Proof. Let P_ε be the energy projection onto H_ε. Evaluation representers satisfy $\|G_r - G_s\|_e \le \sqrt{|r-s|}$, by the elementary evaluation inequality. Smooth functions compactly supported away from r_0 are dense in H_0 and eventually belong to every H_ε; thus $P_\varepsilon u \to u$ on H_0. Also G_{r_0} has distance tending to zero from the span of the $G_{r_j(\varepsilon)}$, so $P_\varepsilon G_{r_0} \to 0$. Hence $P_\varepsilon \to P_0$ strongly on H.

For a compact target u away from r_0, the ordinary functions $V_\varepsilon u$ converge in L^2 to $V_0 u$. Their unrestricted energy-Riesz representers converge, and projecting them with P_ε proves convergence of the extended operators on such targets. The uniform Hardy bound (20) extends convergence by density on H_0. On its one-dimensional orthogonal complement use $\left\|\widetilde{C}_\varepsilon u\right\| \le C\,\|P_\varepsilon u\|_e \to 0$.

For (23), Theorem 3.2 gives essential upper edges $2\max_j \nu_j$ before fusion and $2m$ afterward. Unitary conjugation and compact perturbation preserve essential spectrum. For bounded selfadjoint operators the Hausdorff distance of the essential spectra is at most the operator norm distance, by the corresponding spectral bound in the Calkin algebra. The missing upper interval gives (23). □

For example, $(x^2 - \varepsilon^2)e^{-x^2/2}$ on $(-1, 1)$ converges smoothly to $x^2e^{-x^2/2}$. The essential bands are respectively $[0, 2]$ and $[0, 4]$, and the norm gap is at least 2 under every identification and compact correction allowed above. Pole-collision sensitivity has substantial prior literature, including higher-dimensional inverse-square settings [7]; the assertion here is the precise one-dimensional energy-completion statement and its norm bound.

4 Original log-curvature forms and exact geometry

For a real C^2 function f on a bounded interval, define the localized original forms on vectors $u = f(x)(x-a)(b-x)P(x)$ by

$$E_f(P) = \frac{1}{4\pi}\int_a^b |u'|^2\,dx, \qquad Q_f(P) = \frac{1}{8\pi}\int_a^b V_f|u|^2\,dx, \qquad V_f = \frac{f'^2 - ff''}{f^2}.$$

At real zeros the product $V_f|u|^2$ has its removable integrable interpretation. Their quotient is exactly $q_f/e = Q_f/E_f$. The endpoint factors impose Dirichlet conditions for this localized receiver. These are not a statement that the full-line form has the same boundary conditions.

Suppose for $\varepsilon > 0$ that

$$f_\varepsilon(x) = h_\varepsilon(x)\prod_{j=1}^{k}((x-r_j)^2+\varepsilon^2)^{\nu_j}, \qquad \|-(\log|h_\varepsilon|)''\|_\infty \le C,$$

where h_ε is real nonzero and C^2. Direct logarithmic differentiation gives the exact potential in (4) with

$$v_j(t) = \frac{2\nu_j(t^2-1)}{(1+t^2)^2}, \quad L_j(t) = -\frac{2\nu_j t}{1+t^2}, \quad d_j = 2\pi\nu_j^2, \quad \alpha_j = 2\nu_j. \tag{24}$$

For every fixed $\varepsilon > 0$ the multiplier f_ε is nonzero, and the same polynomial-approximation argument as Proposition 3.1 gives the full completion H. Consequently Theorem 1.1 governs this original quotient. The leading squared-coefficient matrix is

$$\frac{\pi}{2}\,\mathrm{diag}(\nu_j)G\,\mathrm{diag}(\nu_j).$$

For one pair it gives

$$\lambda^{\pm}(\varepsilon) = \pm\nu\sqrt{\frac{\pi G(r,r)}{2\varepsilon}} - \nu\log(1/\varepsilon) + O(1). \tag{25}$$

For simple pairs, all critical tails equal 2, giving the common $-\log(1/\varepsilon)$ drift for every branch. With a nonvanishing C^2 limit of h_ε, the real-root limit has multiplicities $2\nu_j$ and the bounded Hardy essential band from Theorem 3.2. The asymptotic statements themselves require only the uniform log-curvature bound, not convergence of h_ε.

4.1 A positive-inverse two-center family

On $I = (-1,1)$ take centers $r_1 = -1/2$, $r_2 = 1/2$. Their exact Green matrix and symmetric/antisymmetric eigenvalues are

$$G = \begin{pmatrix} 3/8 & 1/8 \\ 1/8 & 3/8 \end{pmatrix}, \qquad \lambda_{\mathrm{even}}(G) = 1/2, \quad \lambda_{\mathrm{odd}}(G) = 1/4. \tag{26}$$

Consider the even entire function

$$f_\varepsilon(z) = e^{-z^2}\{\cos(2\pi z) + \cosh(2\pi\varepsilon)\}. \tag{27}$$

With the Fourier convention $f(z) = \int_{\mathbb{R}}\phi(s)e^{izs}\,ds$, put $B(s) = e^{-s^2/4}/(2\sqrt{\pi})$. Its inverse density is

$$\phi_\varepsilon(s) = \frac{B(s-2\pi)+B(s+2\pi)}{2} + \cosh(2\pi\varepsilon)B(s) > 0.$$

It is even and has Gaussian tails. The pairs inside the interval are $\pm 1/2 \pm i\varepsilon$. After their quadratic factors are removed, the remaining real factor is uniformly nonzero on the closed interval and has uniformly bounded logarithmic derivatives through order two for small ε. Locally this follows from

$$\frac{\cosh s - \cos t}{s^2 + t^2} = \tfrac{1}{2}\,\mathrm{sinhc}((s+it)/2)\,\mathrm{sinhc}((s-it)/2), \qquad \mathrm{sinhc}(z) = \frac{\sinh z}{z},$$

with its value 1 at zero. The remaining limiting zeros lie at a fixed positive distance from the interval, so the compact remainder is uniformly controlled.

Reflection preserves the complete form. The symmetric and antisymmetric trace vectors in (26) and their coupling vectors belong to the even and odd spaces, respectively. Applying the proof of Theorem 1.1 on each invariant parity gives exactly one escaping branch of each sign in each parity:

$$\lambda^{\pm}_{\mathrm{even}}(\varepsilon) = \pm\frac{\sqrt{\pi}}{2\sqrt{\varepsilon}} - \log(1/\varepsilon) + O(1), \tag{28}$$

$$\lambda^{\pm}_{\mathrm{odd}}(\varepsilon) = \pm\frac{\sqrt{\pi}}{2\sqrt{2\varepsilon}} - \log(1/\varepsilon) + O(1). \tag{29}$$

Replacing G by its diagonal would give two equal eigenvalues $3/8$, contradicting the actual coefficients. The clean exact control in `verify_green.py` checks this finite algebra, the Green inverse/jump identity and the polar model's characteristic polynomial.

4.2 Fixed degree and completion have different limits

At $\varepsilon = 0$,

$$f_0(x) = 2e^{-x^2}\cos^2(\pi x), \qquad V_0(x) = 2 + 2\pi^2\sec^2(\pi x) > 0$$

away from the two real zeros. Thus every nonzero localized finite-degree original Q form is strictly positive. The energy Gram is also positive definite. For each fixed degree and parity, the E and Q entries are analytic in ε^2: write Q without division using $(f_\varepsilon'^2 - f_\varepsilon f_\varepsilon'')$ and the endpoint factors, and integrate on the compact interval. The same holds for E by differentiating the complete product. Finite-dimensional continuity therefore preserves their strict positivity for sufficiently small ε.

For every fixed small $\varepsilon > 0$, however, (28)–(29) give a negative completed-space direction in each parity. Polynomial density and continuity of q_ε in energy yield a finite-degree negative polynomial direction of that parity. It follows that the least detecting degree tends to infinity in each parity as $\varepsilon \downarrow 0$. No rate is obtained.

This family is an order-two entire model, not the actual theta transform or a new generic positive-inverse counterexample class. Its role is the exact geometry-dependent spectral transition and the demonstrated difference between finite-degree and completed limits.

5 Scope and further questions

The concentration results keep the number and positive separations of the centers fixed. Joint coalescence, a growing number of centers and uniform estimates for such regimes require additional arguments. The equal-tail condition makes the logarithmic trace coefficient scalar; the paper does not diagonalize unequal logarithmic coefficients by assuming commutation with G or D. The mean-crossover statement is a leading-scale norm approximation with moving embeddings, not a two-term formula in the detuned case.

The positive-inverse example separates fixed-degree convergence from completed-space stability, but supplies no rate for the least detecting degree. No theorem here identifies the full-line actual-theta spectrum or decides its all-order polynomial positivity. Those are distinct receiving problems with global tails and arithmetic structure.

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