Spectral Crowding and Compression Limits for Arithmetic Covariance Operators

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Abstract

We prove quantitative spectral crowding for a complete arithmetic covariance operator in the inverse metric of its actual positive receiver. A growing space of compactly supported oscillatory profiles gives linearly many eigenvalues near one, including after removal of an expanding frequency interval. The construction controls the entire complex coefficient space, its exact polar correction, and its full Fourier leakage. A logarithmic archimedean energy estimate sharpens the admissible frequency range. We then transfer quantitative prime-number-theorem errors through the complete arithmetic smoothing and obtain explicit lower bounds on the ranks of positive source compressions whose complete residuals vanish. The Vinogradov–Korobov error gives a stretched-exponential rank requirement. Using the seven-eighths zero-free theorem and the corresponding quantitative prime-number-theorem bound, we obtain a rank lower bound of $\Omega(M^{1/24})$. Classical min–max and approximation theory turn the arithmetic spectral estimate into best-rank and effective-dimension consequences. The arithmetic estimate retains the original weighted domain and the complete source. The rank conclusion applies along any unbounded cutoff sequence with vanishing operator-norm residual, and its positive lower constant may depend on that sequence.

Keywords. Arithmetic covariance; spectral concentration; finite-rank approximation; prime number theorem; weighted Weil forms.
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1 Introduction and principal conclusions

The norm of a positive covariance operator records its largest direction. For approximation and compression, the number of directions near that largest value is equally consequential. We study this question for the arithmetic covariance

$$G_M = E_M^*(P_M + M^{-1}I)^{-1}E_M, \tag{1}$$

where P_M is the actual positive part of a smoothed arithmetic multiplier in a fixed weighted odd realization, and E_M is its complete negative source. Both are specified in Section 2. The inverse in (1) is taken in the original physical Hilbert norm.

The complete covariance estimate in [8, Theorem 11.1, Section 11, pp. 36–38] gives $\|G_M\| \le 1 + o(1)$. Here we establish growing multiplicity near this upper scale. Write $P_{>T}$ for source multiplication by $1_{(T,\infty)}$ and order positive eigenvalues decreasingly, with multiplicity. The complete Abel error $\overline{e}_M$ is defined in (17).

Theorem 1.1 (Arithmetic spectral multiplicity)**.** *There are $c, C > 0$ such that the following holds. Let M tend to infinity through integers and let $T = T_M \to \infty$ satisfy*

$$T^3 \overline{e}_M \longrightarrow 0, \qquad \frac{T^2 \log(2+T)}{\sqrt{M}} \longrightarrow 0. \tag{2}$$

For each fixed $q > 0$, eventually

$$\lambda_{\lfloor cT \rfloor}(P_{>T} G_M P_{>T}) \geq 1 - C_q T^{-q} - C \frac{T^2 \log(2+T)}{\sqrt{M}}. \tag{3}$$

The construction uses only the original polar constraint. The constants are independent of the coefficient-space dimension. Every eigenvalue counted in (3) *is also at most $1 + o(1)$.*

The lower estimate is proved in Sections 4–5. Its main content is the arithmetic comparison on one entire growing trial space. The last assertion uses the cited complete upper estimate.

For a concrete consequence define, for sufficiently large U,

$$F(U) = D\left(1 + \frac{\log\log U}{15 \log U}\right) \frac{U^{3/5}}{(\log U)^{1/5}}, \qquad S_M = \exp\bigl(F(\log M)/3\bigr),$$
$$D = \frac{5}{2}\left(\frac{5}{3}\right)^{1/5} \left(\frac{2000}{161967}\right)^{3/5}. \tag{4}$$

Here $D \approx 0.1982767$. The coefficient and secondary term come from the prime-number-theorem estimate of Fiori–Jaskari [3]; their arithmetic estimate is an input to our spectral argument.

Theorem 1.2 (A quantitative compression requirement)**.** *Every sequence $T_M \to \infty$ with $T_M = o(S_M)$ satisfies* (2)*. Let J_M be any finite-rank selfadjoint positive contraction on the complete source space and put $B_M = (I - J_M^2)^{1/2}$. Along any unbounded sequence of M on which*

$$\|B_M G_M B_M\| \longrightarrow 0, \tag{5}$$

one necessarily has

$$\liminf_{M \to \infty} \frac{1 + \operatorname{rank} J_M}{S_M} > 0. \tag{6}$$

The positive lower constant may depend on the sequence in (5)*.*

Thus the compression requirement retains a specific stretched-exponential scale rather than merely exceeding every fixed $\exp(A\sqrt{\log M})$ scale. Section 3 also gives an explicit transfer from a power error in prime counting to a polynomial rank requirement. Section 6 proves Theorem 1.2 and gives approximation consequences.

Relation to earlier work. The weighted arithmetic realization and its complete covariance upper bound were developed in [8]. The present result strengthens a norm-level conclusion to a growing spectral count and derives its quantitative compression consequences. The compact profiles below are an implementation in that same realization.

Classical time–frequency concentration already connects large concentration spaces with eigenvalue multiplicity; see Landau–Pollak [6] and Landau [5]. Their projection operators provide important background for the packet construction. Here the lower bound must instead be paid for the oscillatory positive part of an arithmetic symbol and the inverse of a different positive receiver. The

phase average, exact polar repair, and actual receiver comparison perform that step. Fedele's regularized Hankel spectral densities [2] concern an established matrix class with piecewise-continuous symbols; our operator retains a parameter-dependent receiving inverse and a weighted arithmetic source. The effective-dimension interpretation belongs to the standard regularization framework, as in Avron–Clarkson–Woodruff [1]. In this paper that interpretation is a consequence of the arithmetic spectral count. Min–max, kernel-dimension arguments, and Abel integration are classical tools.

2 The complete arithmetic realization

Use $\widehat{f}(\xi) = \int_{\mathbb{R}} f(u)e^{-iu\xi}\,\mathrm{d}u$ and Plancherel measure $\mathrm{d}\xi/(2\pi)$. Inner products are conjugate-linear first. For $u \geq 0$ let

$$\psi(u) = e^{u/2}\sum_{n\geq 1}(4r_n^2 - 6r_n)e^{-r_n}, \quad r_n = \pi n^2 e^{2u}, \qquad s(u)^2 = \frac{\psi(u)}{\cosh(u/2)}. \tag{7}$$

Take the even continuation of s. Theta inversion gives its smoothness; the series is strictly positive and decays rapidly. Put

$$\mathcal{X} = \{h \in L^2(\mathbb{R}) : h \text{ is odd, } \langle s\sinh(u/2), h\rangle = 0\},$$

and let Q_b be the orthogonal projection onto this space from the odd subspace. If $\Psi = \Gamma'/\Gamma$, define

$$a(\xi) = \Re\Psi(1/4 + i\xi/2) - \Psi(1/4), \qquad m_0 = \Psi(1/4) - \log\pi. \tag{8}$$

The original form domain V is the completion of the smooth compact odd polar-free core for the norm

$$\|h\|_V^2 = \|h\|_2^2 + \int_{\mathbb{R}} a(\xi)|\widehat{sh}(\xi)|^2\frac{\mathrm{d}\xi}{2\pi}. \tag{9}$$

Here and throughout, $\|h\|_2$ is the physical Hilbert norm.

Let Λ denote the von Mangoldt function and, for integers $M \geq 2$, put

$$w_M(n) = \min(1, M/n),$$
$$S_M^{\mathrm{ar}}(\xi) = \sum_{n\geq 2}\Lambda(n)n^{-1/2+i\xi}w_M(n), \tag{10}$$
$$m_M(\xi) = m_0 + a(\xi) - 2\Re S_M^{\mathrm{ar}}(\xi), \qquad p_M = (m_M)_+, \quad n_M = (-m_M)_+. \tag{11}$$

The superscript on S_M^{ar} distinguishes the arithmetic sum from the scale S_M in (4). The series in (10) converges absolutely. Define the positive form and bounded negative operator on $\mathcal{X}$ by

$$P_M[h] = \int_{\mathbb{R}} p_M(\xi)|\widehat{sh}(\xi)|^2\frac{\mathrm{d}\xi}{2\pi}, \qquad N_M[h] = \int_{\mathbb{R}} n_M(\xi)|\widehat{sh}(\xi)|^2\frac{\mathrm{d}\xi}{2\pi}. \tag{12}$$

The symbol $p_M - a$ is bounded at fixed M, so P_M is a closed nonnegative form on the same domain V.

Use the complete source space

$$\mathcal{Y}_M = L^2(\{\xi > 0 : n_M(\xi) > 0\},\, \mathrm{d}\xi/\pi).$$

The source map $E_M : \mathcal{Y}_M \to \mathcal{X}$ is specified by

$$(E_M^* h)(\xi) = \sqrt{n_M(\xi)} \int_{\mathbb{R}} s(u)h(u)\sin(\xi u)\,\mathrm{d}u. \tag{13}$$

Oddness and Plancherel give $N_M = E_M E_M^*$. The map is Hilbert–Schmidt: its unprojected kernel has squared integral at most $\|s\|_2^2\pi^{-1}\int_0^\infty n_M(\xi)\,\mathrm{d}\xi$, which is finite. Indeed $a(\xi) \to \infty$, while the arithmetic sum is uniformly bounded in ξ at each fixed M, so n_M has bounded support. Consequently G_M in (1) is positive trace class. Its frequency compression $G_{M,>T} = P_{>T}G_M P_{>T}$ is understood as zero on the orthogonal source complement.

For completeness, the signed realization supplies the precise identity needed in the proof. The original form is

$$q_H[h] = \int_{\mathbb{R}} (m_0 + a)|\widehat{sh}|^2 \frac{\mathrm{d}\xi}{2\pi} - \sum_{n\geq 2} \frac{\Lambda(n)}{\sqrt{n}} \langle sh, (\mathcal{S}_{\log n} + \mathcal{S}_{-\log n})sh\rangle, \tag{14}$$

where $\mathcal{S}_t f(u) = f(u+t)$.

Lemma 2.1 (Realization and elementary bounds)**.** *There is a bounded selfadjoint W_M such that, as closed forms on V,*

$$q_H = P_M - N_M + W_M, \qquad \|W_M\| \leq C(M+1)^2 e^{-\pi M}. \tag{15}$$

Moreover $0 \leq n_M(\xi) \leq C\sqrt{M}$ for all ξ.

Proof. The elementary theta envelopes, obtained from the first atom and a geometric bound on the remaining series, are

$$2r^2 e^{-r} \leq s(u)^2 \leq 169 e^{4|u| - \pi e^{2|u|}}, \qquad r = \pi e^{2|u|}.$$

They give, for $t = \log n$, the bound $\sup_u s(u)s(u+t) \leq 169 n^2 e^{-\pi n}$. Thus the omitted arithmetic operator beyond a hard cutoff R has norm at most

$$338 \sum_{n>R} n^{3/2}\log n\, e^{-\pi n} \leq 451(\lfloor R\rfloor + 1)^2 e^{-\pi(\lfloor R\rfloor + 1)}.$$

Every prime power is included in this all-integer majorant. Average the hard-cutoff expression over $R = Me^v$ with density $e^{-v}\,\mathrm{d}v$. Its survival weight at n is precisely $w_M(n)$, so Fubini gives (15); averaging the displayed complete remainder gives the stated exponentially small bound. Since the paired arithmetic perturbations converge in the original operator norm, the identity extends from the compact core to V.

Chebyshev's elementary estimate $\sum_{n\leq x}\Lambda(n) \leq Cx$ and partial summation yield

$$\sum_{n\leq M} \frac{\Lambda(n)}{\sqrt{n}} + M \sum_{n>M} \frac{\Lambda(n)}{n^{3/2}} \leq C\sqrt{M}.$$

Since $a \geq 0$ and m_0 is fixed, (11) proves the global upper for n_M. □

The complete covariance upper from [8, Theorem 11.1 with $b = 1$], in these exact conventions, is

$$\|G_M\| \leq 1 + C\frac{\sqrt{\log M}\exp(48\sqrt{\log M})}{\sqrt{M}} = 1 + o(1). \tag{16}$$

The lower spectral construction below uses the explicit realization and (15); the two-sided crowding assertion combines that construction with (16).

3 Complete Abel error and quantitative scales

Let $\Psi_\Lambda(x) = \sum_{n\le x}\Lambda(n)$ and $\mathcal{E}(x) = \Psi_\Lambda(x) - x$. Define the full error

$$\overline{e}_M = \frac{3}{\sqrt{M}} + \frac{1}{\sqrt{M}}\int_1^M |\mathcal{E}(t)|t^{-3/2}\,\mathrm{d}t + \sqrt{M}\int_M^\infty |\mathcal{E}(t)|t^{-5/2}\,\mathrm{d}t. \tag{17}$$

Both sides of the smoothing cutoff remain in this quantity.

Lemma 3.1 (Moving-band arithmetic comparison)**.** *For every $F_0 \ge 0$,*

$$\sup_{|\xi|\le F_0}\left|\frac{S_M^{\mathrm{ar}}(\xi)}{\sqrt{M}} - \frac{e^{i\xi\log M}}{1/4+\xi^2}\right| \le (2+F_0)\overline{e}_M. \tag{18}$$

Consequently, on $[T,3T]$ with $T\ge 1$,

$$\frac{n_M(\xi)}{\sqrt{M}} = \frac{8}{1+4\xi^2}(\cos(\xi\log M))_+ + \rho_M(\xi), \qquad \|\rho_M\|_\infty \le C\left(T\overline{e}_M + \frac{\log(2+T)}{\sqrt{M}}\right). \tag{19}$$

Proof. Write the arithmetic sum as the Stieltjes integral against $\mathrm{d}\Psi_\Lambda(t)$ and split at M. The continuum integral is

$$\int_1^\infty t^{-1/2+i\xi}\min(1,M/t)\,\mathrm{d}t = \frac{\sqrt{M}e^{i\xi\log M}}{1/4+\xi^2} - \frac{1}{1/2+i\xi}.$$

Integrating the error integral by parts gives derivative bounds $(1/2+|\xi|)t^{-3/2}$ below M and $(3/2+|\xi|)Mt^{-5/2}$ above M. The boundary at M cancels, because the weight is continuous. The endpoint and continuum correction are absorbed by $3/\sqrt{M}$ in (17). This proves (18). The map $x\mapsto x_+$ is 1-Lipschitz. Apply it to $-m_M$ and use $|m_0| + a(3T) \le C\log(2+T)$, established in Lemma 4.2, to obtain (19). □

The following elementary transfer retains a stretched exponential without weakening its leading coefficient.

Lemma 3.2 (Preservation of slow exponential errors)**.** *Suppose F is continuously differentiable eventually, $F(U)\to\infty$, $F(U) = o(U)$, and $0\le F'(U)\le 1/4$ eventually. If*

$$|\mathcal{E}(e^v)|e^{-v} \le Ce^{-F(v)}$$

for all sufficiently large v, then $\overline{e}_M = O(e^{-F(\log M)})$.

Proof. Put $U = \log M$ and $r(v) = |\mathcal{E}(e^v)|e^{-v}$. Changing variables in (17) gives exactly

$$\overline{e}_M = 3e^{-U/2} + \int_0^U e^{-(U-v)/2}r(v)\,\mathrm{d}v + \int_U^\infty e^{-(v-U)/2}r(v)\,\mathrm{d}v. \tag{20}$$

Fix v_0 after the hypotheses hold. For $v_0\le v\le U$, $F(U)-F(v)\le (U-v)/4$, so this part of the first integral is at most $4Ce^{-F(U)}$. Its initial fixed interval costs $O(e^{-U/2})$. The second integral is at most $2Ce^{-F(U)}$, by monotonicity. Since $F(U) = o(U)$, the remaining exponential is absorbed. □

Fiori–Jaskari [3, Theorem 1.1] give

$$|\mathcal{E}(x)| < 0.239x \exp\left[-D\left(1+\frac{\log\log\log x}{15\log\log x}\right)\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right] \quad (x \geq 21). \tag{21}$$

For F in (4), direct differentiation shows $F'(U) > 0$ eventually and $F'(U) \to 0$, while $F(U) = o(U)$. Lemma 3.2 therefore proves

$$\overline{e}_M = O(e^{-F(\log M)}) = O(S_M^{-3}). \tag{22}$$

As $S_M = M^{o(1)}$, every growing $T = o(S_M)$ satisfies both conditions in (2). This proves the first assertion of Theorem 1.2.

The older Johnston–Yang estimate [4, Theorem 1.4] has the same principal power 3/5, with coefficient 0.1853 and a logarithmic prefactor. The transfer above also preserves that error after its prefactor is included in F. Keeping the complete error rather than replacing it by $O_A(e^{-A\sqrt{\log M}})$ is what produces the scale in (6).

Lemma 3.3 (Power-error transfer)**.** *Suppose, for fixed $0 < \delta < 1/2$ and $b \geq 0$,*

$$|\mathcal{E}(x)| \leq Cx^{1-\delta}(\log x)^b \quad \textit{for all sufficiently large } x. \tag{23}$$

Then $\overline{e}_M = O(M^{-\delta}(\log M)^b)$. In particular every growing $T = o(M^{\delta/3}(\log M)^{-b/3})$ satisfies (2)*.*

Proof. In (20), use $r(v) \leq Ce^{-\delta v}v^b$ beyond a fixed initial interval. The first integral is bounded by $CM^{-\delta}U^b\int_0^U e^{-(1/2-\delta)(U-v)}\,\mathrm{d}v$. For the tail, put $v = U + t$ and bound $(U+t)^b \leq U^b(1+t)^b$ for $U \geq 1$. Its remaining integral is finite because $1/2 + \delta > 0$. The initial interval and $3e^{-U/2}$ are smaller. At the stated scale $T^3\overline{e}_M \to 0$ and $T^2\log(2+T)/\sqrt{M} \to 0$, since $2\delta/3 < 1/2$. □

4 One growing polar-free packet space

Fix $\ell > 0$, $R > \ell$, and an even real $\varphi \in C_c^\infty(-R, R)$ with $\varphi \geq m_\varphi > 0$ on $[-\ell, \ell]$. Put $h_0 = \pi/\ell$ and take all integer carriers

$$\omega_k = h_0 k \in [3T/2, 5T/2], \qquad d_T = \#\{k : \omega_k \in [3T/2, 5T/2]\} = \frac{\ell}{\pi}T + O(1).$$

For arbitrary complex coefficients $a = (a_k)$ write $v_a(u) = \varphi(u)\sum_k a_k \sin(\omega_k u)$.

Choose once an odd real $b_0 \in C_c^\infty(-R_1, R_1)$, $R_1 \geq R$, with $\int b_0(u)\sinh(u/2)\,\mathrm{d}u = 1$. Such a function exists because this functional is nonzero on compact odd tests. Define

$$c(a) = \int v_a(u)\sinh(u/2)\,\mathrm{d}u, \qquad f_a = v_a - c(a)b_0, \qquad h_a = f_a/s. \tag{24}$$

The notation a_k for coefficients is distinguished by its index from the symbol $a(\xi)$ in (8).

Lemma 4.1 (Uniform geometry and Fourier leakage)**.** *For all sufficiently large T, the profiles in* (24) *give a d_T-dimensional subspace of the original compact core, with*

$$c\|a\|_2^2 \leq \|f_a\|_2^2 \leq C\|a\|_2^2, \qquad \|f_a'\|_2^2 \leq CT^2\|a\|_2^2, \tag{25}$$

$$c\|a\|_2^2 \leq \|h_a\|_2^2 \leq C\|a\|_2^2. \tag{26}$$

For every fixed $q > 0$,

$$\int_{\substack{\xi \geq 0 \\ \xi \notin [T, 3T]}} |\widehat{f}_a(\xi)|^2 \frac{\mathrm{d}\xi}{\pi} \leq C_q T^{-q}\|a\|_2^2. \tag{27}$$

For $U > 2R_1$ there is also the exact identity

$$\int_0^\infty (\cos(U\xi))_+ |\widehat{f_a}(\xi)|^2 \frac{\mathrm{d}\xi}{\pi} = \frac{\|f_a\|_2^2}{\pi}. \tag{28}$$

All assertions hold uniformly over the entire complex coefficient space.

Proof. On $[-\ell, \ell]$ the integer sine modes are orthogonal, so $\|v_a\|_2^2 \geq m_\varphi^2 \ell \|a\|_2^2$. A fixed number of periods of length 2ℓ covers $[-R, R]$. Periodic orthogonality of both sine and cosine sums gives

$$\|v_a\|_2^2 \leq C\|a\|_2^2, \qquad \|v_a'\|_2^2 \leq C(1+T^2)\|a\|_2^2.$$

This proves induced bounds without a factor d_T.

For each carrier, repeated integration by parts against the fixed smooth function $\varphi(u)\sinh(u/2)$ gives a polar coefficient of size at most $C_j T^{-j}$. Since $d_T = O(T)$,

$$|c(a)| \leq C_j T^{1/2-j} \|a\|_2 \tag{29}$$

for every fixed integer j. Thus the rank-one correction has arbitrarily small induced norm, preserving (25). It makes f_a exactly polar-free. Oddness supplies both signed exponential moments; division by s on the one fixed compact support preserves smoothness and gives (26). The lower bound proves injectivity and hence the dimension claim.

Each carrier transform is a difference of two translates of $\widehat{\varphi}$. The positive carrier is separated by $T/2$ from the exterior of $[T, 3T]$; the negative carrier is separated by $3T/2$ from the positive half-line. Schwartz decay gives squared tail norm $O_j(T^{1-2j})$ per column. Summing over $O(T)$ columns bounds the squared Hilbert–Schmidt norm of the entire coefficient-to-tail map by $O_j(T^{2-2j})$. Its operator norm is no larger. The repair is controlled by (29) times the fixed Fourier norm of b_0. Taking j large proves (27), with all mixed coefficient terms included.

The absolutely convergent Fourier series of $w(t) = (\cos t)_+$ has mean $1/\pi$. Every nonconstant harmonic in

$$\int_{\mathbb{R}} w(U\xi)|\widehat{f_a}(\xi)|^2 \frac{\mathrm{d}\xi}{2\pi}$$

tests the autocorrelation of f_a at a nonzero integer multiple of U. This autocorrelation is supported in $[-2R_1, 2R_1]$, so all such harmonics vanish when $U > 2R_1$. The mean term equals $\|f_a\|_2^2/\pi$. For complex odd f_a, $\widehat{f_a}(-\xi) = -\widehat{f_a}(\xi)$, so its squared modulus is even. The positive-half normalization is therefore exactly (28). □

Lemma 4.2 (Logarithmic energy of the whole space)**.** *For every $\xi \in \mathbb{R}$,*

$$0 \leq a(\xi) \leq 4 + \tfrac{1}{2}\log(1+4\xi^2). \tag{30}$$

For $f \in H^1(\mathbb{R})$, $m = \|f\|_2^2 > 0$, this implies

$$\int_{\mathbb{R}} a(\xi)|\widehat{f}(\xi)|^2 \frac{\mathrm{d}\xi}{2\pi} \leq 4m + \frac{m}{2}\log\left(1 + 4\frac{\|f'\|_2^2}{m}\right). \tag{31}$$

Consequently the whole packet space satisfies

$$q_H[h_a] \leq C\log(2+T)\|a\|_2^2. \tag{32}$$

Proof. Put $t = |\xi|/2$. The positive digamma series is

$$a(\xi) = \sum_{n\ge 0} \frac{t^2}{(n+1/4)((n+1/4)^2+t^2)}.$$

The first term is at most 4. The summand decreases in its positive spatial argument, so the remaining sum is bounded by

$$\int_{1/4}^{\infty} \frac{t^2}{x(x^2+t^2)}\,\mathrm{d}x = \tfrac{1}{2}\log(1+4\xi^2).$$

This proves (30). Apply Jensen's inequality for the concave logarithm to the probability measure $|\widehat{f}(\xi)|^2\,\mathrm{d}\xi/(2\pi m)$, then use Parseval for f' to obtain (31). The zero vector satisfies the energy inequality directly.

By (25), the right side of (31) is at most $C\log(2+T)\|a\|_2^2$ for $f = f_a$. On the fixed support $[-R_1, R_1]$, the arithmetic correlations in (14) vanish for $n > e^{2R_1}$. The remaining finite sum and m_0 cost at most $C_{R_1}\|f_a\|_2^2$, by Cauchy–Schwarz. This pays the entire original signed form and proves (32). □

5 The arithmetic lower bound and spectral passage

Write

$$N_{>T}[h] = \int_T^{\infty} n_M(\xi)|\widehat{sh}(\xi)|^2 \frac{\mathrm{d}\xi}{\pi}, \qquad N_{\le T} = N_M - N_{>T}. \tag{33}$$

No upper frequency endpoint is imposed in $N_{>T}$.

Lemma 5.1 (Complete source mass)**.** *Under* (2)*, eventually*

$$N_{>T}[h_a] \ge c\frac{\sqrt{M}}{T^2}\|a\|_2^2. \tag{34}$$

For every fixed $q > 0$,

$$N_{\le T}[h_a] \le C_q\sqrt{M}T^{-q-2}\|a\|_2^2, \qquad \frac{N_{\le T}[h_a]}{N_{>T}[h_a]} \le C_q T^{-q} \quad (a \ne 0). \tag{35}$$

Proof. Put $U = \log M$. For all sufficiently large M, $U > 2R_1$. By (28), (27) and the lower Gram bound,

$$\int_T^{3T} (\cos(U\xi))_+ |\widehat{f}_a(\xi)|^2 \frac{\mathrm{d}\xi}{\pi} \ge c\|a\|_2^2.$$

On this band, $8/(1+4\xi^2) \ge 8/(37T^2)$. The error in (19) is $o(T^{-2})$ by the two conditions in (2). Its integral is bounded by that scalar error times $\|f_a\|_2^2 \le C\|a\|_2^2$. This proves (34). All other frequencies above T have nonnegative source weight and remain in $N_{>T}$. For the low frequencies use the global bound $n_M \le C\sqrt{M}$ and (27), taking its arbitrary Schwartz order large enough to include the extra T^{-2}. Division by (34) proves (35). □

Proof of Theorem 1.1. For every vector in the d_T-dimensional physical space $\mathcal{W}_M = \{h_a : a \in \mathbb{C}^{d_T}\}$, the exact identity (15) gives

$$P_M[h_a] + M^{-1}\|h_a\|_2^2 = N_{>T}[h_a] + N_{\le T}[h_a] + q_H[h_a] - \langle h_a, W_M h_a\rangle + M^{-1}\|h_a\|_2^2$$

$$\le N_{>T}[h_a] + N_{\le T}[h_a] + C\log(2+T)\|a\|_2^2.$$

The physical norm, ridge and exponentially small remainder are absorbed using (26). The denominator on the left is strictly positive for $a \neq 0$. Lemma 5.1 therefore implies uniformly on the entire space

$$\frac{N_{>T}[h_a]}{P_M[h_a] + M^{-1}\|h_a\|_2^2} \ge 1 - C_q T^{-q} - C\frac{T^2\log(2+T)}{\sqrt{M}}. \tag{36}$$

Here one uses $(1+x)^{-1} \ge 1-x$ for $x \ge 0$ after the displayed upper bound on the denominator.

Let $A_M = P_M + M^{-1}I$ be the positive associated operator. Its form-square-root map sends $\mathcal{W}_M$ injectively to a d_T-dimensional subspace. On this image (36) is the Rayleigh quotient of the positive compact operator

$$A_M^{-1/2} E_M P_{>T} E_M^* A_M^{-1/2}.$$

Min–max gives at least d_T eigenvalues bounded below by (36). The nonzero spectra of XX^* and X^*X, with $X = A_M^{-1/2} E_M P_{>T}$, agree. This transfers the bound to $P_{>T} G_M P_{>T}$. Since $d_T \ge cT$ eventually for a fixed $c > 0$, (3) follows. This passage uses only the original form domain. Finally a positive compression has norm at most $\|G_M\|$; (16) supplies the upper $1 + o(1)$ for every eigenvalue counted. □

6 Compression, approximation and effective dimension

The following consequences isolate the operation to which the spectral count applies: finite-rank action on the same complete source space.

Proposition 6.1 (Finite-rank source residual)**.** *Let T_M satisfy* (2) *and let J_M be a selfadjoint positive contraction of rank $r_M < \lfloor cT_M \rfloor$. For $B_M = (I - J_M^2)^{1/2}$,*

$$\|B_M G_M B_M\| \ge 1 - C_q T_M^{-q} - C\frac{T_M^2\log(2+T_M)}{\sqrt{M}}. \tag{37}$$

Proof. The spectral subspace for the first $\lfloor cT_M \rfloor$ eigenvalues of $G_{M,>T_M}$ intersects $\ker J_M$ nontrivially. Take a unit vector z in this intersection. The eigenvalues used are strictly positive eventually, so $P_{>T_M} z = z$. Also $B_M z = z$. Hence

$$\langle z, B_M G_M B_M z\rangle = \langle z, G_{M,>T_M} z\rangle \ge \lambda_{\lfloor cT_M \rfloor}(G_{M,>T_M}),$$

which proves the claim. No factors need commute. □

Proof of Theorem 1.2. Admission for $T = o(S_M)$ was proved in Section 3. Suppose (6) fails along a sequence satisfying (5). On a subsequence, $(1 + r_M)/S_M \to 0$, with $r_M = \operatorname{rank} J_M$. Set

$$T_M = \sqrt{(1+r_M)S_M}.$$

Then $T_M \to \infty$, $T_M/S_M \to 0$, and $r_M/T_M \to 0$. Thus T_M is admitted and $r_M < \lfloor cT_M \rfloor$ eventually. Proposition 6.1 forces $\|B_M G_M B_M\| \ge 1 - o(1)$ on that subsequence, contradicting (5). The argument permits every selected unbounded sequence of cutoffs. □

Corollary 6.2 (Polynomial compression requirement)**.** *Under the quantitative prime-counting hypothesis* (23)*, put $S_M^{(\delta,b)} = M^{\delta/3}(\log M)^{-b/3}$. For any source heads satisfying* (5)*,*

$$\liminf_{M\to\infty} \frac{1 + \operatorname{rank} J_M}{S_M^{(\delta,b)}} > 0. \tag{38}$$

For example, the hypothesis $|\mathcal{E}(x)| = O(x^{7/8})$ gives a rank requirement $\operatorname{rank} J_M = \Omega(M^{1/24})$.

Proof. Lemma 3.3 admits every growing $T = o(S_M^{(\delta,b)})$. Repeat the preceding subsequence argument with $S_M^{(\delta,b)}$ in place of S_M. □

Theorem 1.1 of OpenAI [7] places every nontrivial zero of ζ in $\Re s \le 7/8$. Combining this theorem with Fiori–Jaskari [3, Theorem 1.4] gives

$$|\mathcal{E}(x)| < 0.04934x^{3/4}(\log x)^2 + 0.02636425x^{7/8} \qquad (x \ge 21).$$

The first term is $o(x^{7/8})$, so Lemma 3.3 applies with $\delta = 1/8$ and $b = 0$. Along every unbounded sequence of cutoffs on which the complete residual in (5) tends to zero, every corresponding family of finite-rank positive source contractions satisfies

$$\liminf_{M\to\infty} \frac{1 + \operatorname{rank} J_M}{M^{1/24}} > 0.$$

The same rank lower bound holds for arbitrary finite-rank approximants with vanishing operator-norm error by Corollary 6.3.

Corollary 6.3 (Best-rank error and effective dimension)**.** *Suppose $d = \lfloor cT \rfloor$ and $\varepsilon = C_q T^{-q} + CT^2 \log(2+T)/\sqrt{M} < 1$ in Theorem 1.1. Then every operator A of rank less than d on $\mathcal{Y}_M$ satisfies*

$$\|G_M - A\| \ge 1 - \varepsilon. \tag{39}$$

Also, for every $\tau > 0$,

$$\operatorname{Tr}\big(G_M(G_M + \tau I)^{-1}\big) \ge \frac{d(1-\varepsilon)}{1 - \varepsilon + \tau}. \tag{40}$$

If finite-rank A_M satisfies $\|G_M - A_M\| \to 0$, its rank obeys (6)*; under* (23) *it also obeys* (38)*.*

Proof. Let L be the d-dimensional exterior spectral subspace from the main theorem. If $\operatorname{rank} A < d$, choose unit $z \in L \cap \ker A$. Then $\|(G_M - A)z\| \ge \langle z, G_M z\rangle \ge 1 - \varepsilon$. Min–max also gives at least d eigenvalues of G_M of size at least $1 - \varepsilon$. Apply the increasing function $x \mapsto x/(x+\tau)$ and sum to obtain (40). Its trace is finite because G_M is trace class. The final assertions follow by the same adaptive subsequence argument as Theorem 1.2, using (39). □

The rank statements concern vanishing complete operator-norm error. They quantify the cost of removing the many near-unit source directions. The ridge M^{-1}, original polar domain, and full arithmetic smoothing remain fixed throughout that comparison.

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