Small Dirichlet apertures and the screen limit: edge laws, sharp rates and many-window interactions

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Abstract

Two Dirichlet cylinders joined through an aperture of diameter of order ϵ have a positive relative heat trace. We determine its behaviour on the natural parabolic time scale $t = \epsilon^2\tau$. The trace converges to that of the same window in an infinite flat Dirichlet screen between two half-spaces; bounded repulsive contacts disappear to leading order, and their first correction is the contact-weighted heat kernel on the window. The limiting screen trace is the Laplace transform of a positive measure with a strictly positive continuous density at every energy, expressed through compressed Fourier multipliers of the two half-spaces. At short times the window area gives the leading term and a C^2 edge contributes the classical local coefficient $-1/16$; at long times the leading coefficient is the exterior-Dirichlet half-Laplacian torsion of the window, with a logarithmic second term in one tangential dimension. The convergence of the energy measures is quantified sharply: cumulative distributions converge at order ϵ and densities in weak L^2 at order $\sqrt{\epsilon}$ in every dimension, a rate attained by identical round sections, while no Lorentz space $L^{2,q}$ with finite q is possible near an active threshold. For round transverse sections we determine the leading strong-norm error exactly: the forward and reflected sides of the dense transverse thresholds produce a Hurwitz zeta profile, with even and odd spherical harmonics offset by half a period, which gives the exact L^p constants for $1 \le p < 2$ and the exact exponentially weighted total-variation limit for arbitrary bounded apertures; for two different radii the constant is an average over two independent phases, or over a resonance circle when the radii are rationally related, and a contact at the critical square-root scale adds a deterministic phase derivative. For rectangular sections the threshold lattice is arithmetic: a trace-norm lattice discrepancy improves the density error to $O(r^{-d/(d+1)})$, quadratic Gauss sums give exact oscillatory tests at order r^{-1}, including critical aspect-ratio detuning, cubes have nonzero rational subharmonics that rule out fixed-period strong profiles, and for square sections the high density spikes have squared weak-L^2 mass of order $r^{-2}\log r$. Finally, for finitely many windows whose centres separate at scale L, the connected part of the heat trace, obtained by inclusion–exclusion, is a sum of time convolutions over directed Hamiltonian cycles through the windows, weighted by the product of their fractional torsions. It is strictly positive, although its spectral density changes sign.

Keywords. Dirichlet screens, shrinking apertures, relative heat traces, fractional torsion, spectral measures, finite propagation, Hamiltonian cycles.
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Overview

Sections 2–5 prove the parabolic screen limit (Theorem 2.1), the short-time edge law (Theorem 3.1), the structure of the limiting spectral measure (Theorem 4.1) and its long-time torsion

expansion (Theorem 5.1). The next sections give the sharp convergence rates of the energy densities (Theorem 7.1 and Proposition 13.1), and Sections 14–18 identify the exact threshold-oscillation constants for round sections (Theorems 14.1 and 14.2), with the arithmetic rectangular case in Sections 19–23. The final sections treat several distant windows (Theorem 25.1). Positivity of the relative measures for a fixed aperture, and the complementary shrinking limit at fixed time, are proved in the companion paper [1].

1 Introduction

A window between two cylinders can be examined at a fixed time or at the diffusion time across the window. At fixed time the transverse modes of each cylinder still enter the leading correction [1]. We take time proportional to the square of the aperture size. On this scale the lateral walls recede to infinity, and both arms become half-spaces. The object that survives is a relative heat trace for the window in a flat Dirichlet screen.

The screen geometry occurs in the theory of Benedicks domains. Collet, Martínez, and San Martín [2, Theorems 1 and 2] prove pointwise heat-kernel and survival asymptotics under their boundary regularity convention. The relative trace studied here integrates a difference over both noncompact half-spaces, including starting points whose distances grow with the diffusion scale. Chandler-Wilde and Hewett [3, Section 8] give Fourier boundary operators for bounded apertures in infinite screens in their acoustic scattering formulation. Their hypersingular symbol [3, Theorem 4.1 and eq. (61)] has the same propagating and evanescent square roots, with conjugation and scalar normalization set by the scattering convention. We use the square roots to obtain an integrated heat spectral measure, its all-energy density, and its threshold expansion.

Grubb's mixed-boundary Krein formula [4, Section 4] and the boundary-logarithm trace formulas of Behrndt, Gesztesy, and Nakamura [5, Theorem 4.1] are precedents for the boundary-operator approach. The half-space variational problem of Felli, Noris, and Ognibene [6, Lemma 3.1 and eqs. (3.2)–(3.7)] gives the one-arm counterpart for a disappearing Neumann window. There the coefficient controls discrete eigenvalue shifts. In our setting it controls the continuum measure at zero energy.

Brownian bridges give the parabolic limit and the short-time expansion. Exhaustion by product domains then gives the spectral measure, whose expansion at zero yields the long-time laws.

2 Parabolic scaling and contact

In tangential dimension $d \geq 1$, let $\Omega_\pm$ be bounded Lipschitz open subsets of $\mathbb{R}^d$, each containing a ball about y_0, and let $K \subset \mathbb{R}^d$ be a nonempty bounded open set. Connect the half-cylinders with sections $\Omega_\pm$ through $y_0 + \epsilon K$ in the separating plane, for ϵ small enough that the aperture closure lies in both sections. All boundaries are Dirichlet except at the aperture. Let $D_\epsilon(t)$ be the connected-minus-cut relative heat trace in the two infinite arms, defined by imposing Dirichlet caps and taking their distances to infinity. The screen model uses full half-spaces with the open set

$$\mathcal{O}_K = \mathbb{R}^{d+1} \setminus ((\mathbb{R}^d \setminus K) \times \{0\})$$

and cut domain $\mathcal{O}_\varnothing$. Write p_G for the Dirichlet heat kernel of an open set G for generator Δ, extended by zero off G, and put

$$D_K^{\mathrm{screen}}(\tau) = \int_{\mathbb{R}^{d+1}} \left[p_{\mathcal{O}_K}(\tau; q, q) - p_{\mathcal{O}_\varnothing}(\tau; q, q) \right] dq.$$

A contact of strength ρ_ϵ means adding $\int_{y_0+\epsilon K} \rho_\epsilon(y)|f(y,0)|^2\,dy$ to the connected Dirichlet energy.

Theorem 2.1. *For each nonempty bounded open aperture K, the screen trace is finite and $D_\epsilon(\epsilon^2\tau) \to D_K^{\mathrm{screen}}(\tau)$ locally uniformly for $\tau > 0$. The limit is the Laplace transform of a positive locally finite measure ν_K on $[0,\infty)$. The positive relative measures of the cylinders, pushed forward by $u \mapsto \epsilon^2 u$, converge vaguely to ν_K. These assertions hold with a uniformly bounded nonnegative contact. If the pulled-back contact profiles $\rho_\epsilon(y_0+\epsilon y)$ converge uniformly to $\rho(y)$ on K, then at each fixed $\tau > 0$*

$$D_\epsilon(\epsilon^2\tau) = D_K^{\mathrm{screen}}(\tau) - \epsilon\tau \int_K \rho(y) p_{\mathcal{O}_K}(\tau;(y,0),(y,0))\,dy + o(\epsilon). \tag{1}$$

Proof. After spatial dilation the aperture becomes K and the sections become $(\Omega_\pm - y_0)/\epsilon$. Use coordinates $q = (y,x)$, with y tangential and x normal to the plane. For $x \neq 0$, consider the bridge for generator Δ, multiplying probabilities by the free diagonal factor $(4\pi\tau)^{-(d+1)/2}$. In the absence of contact the connected-minus-cut diagonal is the free diagonal times the probability that the bridge survives in the connected domain and hits the plane. Every such bridge hits $K \times \{0\}$. Finite cylinders and their limiting kernels obey the same identity by increasing cap exhaustion. For starting points outside a domain, its heat kernel is extended by zero. For a sequence $\epsilon \to 0$, every fixed surviving bridge in the full screen domain has a bounded tangential range eventually contained in both rescaled sections, so the indicators converge. The difference is bounded by the probability that the bridge travels at least its initial distance to $K \times \{0\}$. A $(d+1)$-dimensional reflection and union bound majorizes this by

$$2(d+1)\exp[-\operatorname{dist}(q, K\times\{0\})^2/((d+1)\tau)].$$

It is integrable uniformly on compact positive time intervals. For local uniform convergence, each rescaled section contains a ball of radius c/ϵ. For a starting point inside half that ball's tangential radius, a missed surviving bridge must move tangentially at least $c/(2\epsilon)$. Combine the tail of this event with the displayed aperture-hitting bound using $\min(a,b) \le \sqrt{ab}$. For starts outside that radius, integrate the aperture-hitting bound directly. After spatial integration, both errors are uniformly $O(e^{-c_0/\epsilon^2})$ for some $c_0 > 0$.

For a bounded repulsive contact, dilation converts the surface potential coefficient to at most $C\epsilon$. The flat surface has Gaussian time integral $O(\sqrt{t})$ as $t \downarrow 0$, also after killing. The contact is a Kato measure perturbation in the framework of [7]. Its Feynman–Kac factor is the exponential of minus the integrated contact against plane local time. The cut kernel remains unchanged. On a bridge that never hits the plane the two contributions still cancel. The difference from the contact-free case is bounded by the free diagonal times

$$\mathbb{E}_q^\tau[\mathbf{1}_{\{\text{hit } K\times\{0\}\}}(1 - e^{-C'\epsilon L_\tau})],$$

where L_τ is the local time at zero of the normal bridge. The aperture contact error is $O(\epsilon)$ uniformly on compact positive time intervals. The tangential bridge visiting K is necessary for a nonzero contact-exposure term, and is independent of L_τ. The normal bridge density at zero at time s is $(4\pi s(\tau-s)/\tau)^{-1/2}\exp[-x^2\tau/(4s(\tau-s))]$; thus $\mathbb{E}L_\tau \le C\sqrt{\tau}e^{-x^2/\tau}$. The tangential visiting probability supplies integrable Gaussian decay in y. To prove (1), take local time normalized as the occupation density against ds for the normal process of generator Δ, so the surface Feynman–Kac weight uses the form coefficient without a factor of two. The difference from lateral truncation is exponentially small in ϵ^{-2} on compact time intervals by the preceding bounds. Divide the contact difference by ϵ and use dominated convergence, with the first-moment

Gaussian integrable bound just derived. The limiting insertion is the integral in s over a surface point $z = (y, 0)$ of $\rho(y)p_{\mathcal{O}_K}(s; q, z)p_{\mathcal{O}_K}(\tau - s; z, q)$. This is the Markov bridge insertion formula for plane occupation density. For that formula one can approximate occupation density on a narrow normal strip for continuous test functions compactly supported in K, then use the resulting equality of positive surface measures for bounded measurable profiles. The integral in q equals $p_{\mathcal{O}_K}(\tau; z, z)$ by the semigroup property, independent of s, giving the stated trace.

The cylindrical aperture theorem [1] supplies a positive measure ν_ϵ with Laplace transform $D_\epsilon(t)$. Write $\widetilde{\nu}_\epsilon$ for its pushforward by $u \mapsto \epsilon^2 u$. For each $t_0 > 0$ the pushed-forward measures are uniformly bounded against $e^{-t_0 u}$ by heat convergence. For $\tau > t_0$ the estimate

$$\int_{[R,\infty)} e^{-\tau u}\, d\widetilde{\nu}_\epsilon(u) \le e^{-(\tau - t_0)R} D_\epsilon(\epsilon^2 t_0)$$

passes their Laplace transforms to any vague subsequential limit. Uniqueness of the Laplace transform identifies the positive limit ν_K. □

3 Short times: aperture area and edge

Theorem 3.1. *If K is bounded Lipschitz, then*

$$D_K^{\mathrm{screen}}(\tau) = \frac{|K|}{2(4\pi\tau)^{d/2}} + O_K(\tau^{-(d-1)/2}) \qquad (\tau \downarrow 0).$$

If K has compact C^2 boundary, with K locally on one side, then

$$D_K^{\mathrm{screen}}(\tau) = \frac{|K|}{2(4\pi\tau)^{d/2}} - \frac{\mathcal{H}^{d-1}(\partial K)}{16(4\pi\tau)^{(d-1)/2}} + O(\tau^{-(d-2)/2}).$$

Proof. The tangential and normal bridges are independent. The normal crossing probability from x back to x is $e^{-x^2/\tau}$. If $y \in K$ and the tangential bridge stays within distance $\delta(y) = \mathrm{dist}(y, \partial K)$ of y, every normal crossing survives the screen. If $y \notin K$ and the bridge stays within the same distance, none survives. The absolute difference from $\mathbf{1}_K(y)e^{-x^2/\tau}$ is bounded by $e^{-x^2/\tau}$ times the tangential deviation probability, at most $2de^{-\delta(y)^2/(d\tau)}$. A bounded Lipschitz set has a two-sided tubular volume $O_K(r)$ for small r; together with polynomial volume growth at large radii this yields a tangential integral $O_K(\sqrt{\tau})$. The normal integral is $\sqrt{\pi\tau}$, giving both the leading coefficient and the error.

For the edge term, the local model is the Dirichlet slit plane. Reflection across the screen gives an odd Dirichlet half-plane and an even half-plane with a Dirichlet–Neumann junction. The cut pair gives two Dirichlet half-planes, so the odd sector cancels. The junction coefficient is classical. Avramidi gives $-1/16$ for the regular Zaremba condition in [8, eq. (169)]. Related polygonal corner and mixed-wedge formulas appear in [9] and [10, Appendix B, eq. (B.8)]. We compute the coefficient directly in the relative slit model and control the error in transferring it to the curved edge.

Inside the signed-distance tube write $y = p + \sqrt{\tau}\, s\, n_p$, where $p \in \partial K$ and n_p is the inward unit normal, and write the longitudinal coordinate as $\sqrt{\tau}\, x$. The Jacobian after the two scaled integrations is $\tau(1 + O(\sqrt{\tau}\,|s|))$. Away from this fixed tube the preceding Gaussian estimate makes the area-subtracted integral exponentially small. In the tube, divide the relative diagonal defect by the free diagonal factor and rescale each bridge to time one. The tangent model uses the half-space aperture bounded by the tangent plane at p. The corresponding area-subtracted

probability has uniform bound $2de^{-x^2-s^2/d}$, as does the rescaled curved probability. We compare their integrals using a uniform tangent graph chart at each p. On bridges that stay inside the chart while $|s|$ ranges over a smaller fixed-width physical tube, the boundary graph in unit coordinates is bounded by $C\sqrt{\tau}\sup|\beta_{\tan}|^2$, with $\beta_{\tan}$ the unit tangential bridge increments parallel to the tangent plane. Given the bridge path and a nonempty set Z of normal zero times, write $m = \min_{j\in Z}\beta_{\mathrm{in}}(j)$, where β_{in} is the remaining signed lateral increment. The curved and tangent survival indicators can disagree only for $|s+m| \le C\sqrt{\tau}\sup|\beta_{\tan}|^2$, an interval of length at most $2C\sqrt{\tau}\sup|\beta_{\tan}|^2$. The expected supremum squared is finite and the crossing probability e^{-x^2} is independent of lateral increments. Integrating over s, x therefore gives $O(\sqrt{\tau})$ after scaling. Chart exits require a lateral deviation at least a constant times $\tau^{-1/2}$ in unit coordinates and have exponentially small total effect after the $O(\tau^{-1/2})$ scaled tube length. Factoring $(4\pi\tau)^{-(d-1)/2}$ leaves the two-dimensional time-one defect integral, including its free diagonal factor $(4\pi)^{-1}$. The Jacobian error is also $O(\sqrt{\tau})$ by Gaussian domination with a factor $|s|$.

To compute the last number without an endpoint assumption on polygon formulas, let W be the plane slit along the nonpositive horizontal half-axis, the Dirichlet wedge of angle 2π, and let W_{cut} be the two Dirichlet half-planes. Subtract the local area baseline $(4\pi)^{-1}\mathbf{1}_{s>0}e^{-x^2}$ from the time-one relative diagonal. Its integral converges absolutely by the same tangential and normal bridge bound. Regulate all terms by $e^{-\lambda(s^2+x^2)/2}$ and let $\lambda \downarrow 0$. Write $\cosh a = 1+\lambda$. The ordinary wedge kernel separated in polar angle has Bessel indices $k\pi/\theta$, $k \ge 1$, for Dirichlet opening θ. After angular integration and the substitution $v = (s^2+x^2)/2$, the radial heat integral at time one is $\frac{1}{2}\sum_{k\ge1}\int_0^\infty e^{-(1+\lambda)v}I_{k\pi/\theta}(v)\,dv$. This uses orthonormal angular eigenfunctions and the radial heat factor $\frac{1}{2}e^{-v}I_{k\pi/\theta}(v)$. Using $\int_0^\infty e^{-v\cosh a}I_\nu(v)\,dv = e^{-a\nu}/\sinh a$ gives the wedge-minus-cut value

$$\frac{1}{\sinh a}\left[\frac{1}{2(e^{a/2}-1)} - \frac{1}{e^a-1}\right].$$

The separately integrated regulated baseline is $1/(4\sqrt{\lambda(2+\lambda)}) = 1/(4\sinh a)$. The remaining bracket after subtracting $1/4$ is $-a/16+O(a^3)$. The limit is $-1/16$, which proves the theorem. □

4 The continuum spectral measure

Let $E : L^2(K) \to L^2(\mathbb{R}^d)$ denote zero extension. With Parseval factor $(2\pi)^{-d}$, use the convention

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot y}f(y)\,dy.$$

Let F be the exterior-Dirichlet half-Laplacian on K, the form compression of the whole-space multiplier $|D|$, with symbol $|\xi|$, to the closure of $EC_c^\infty(K)$ in $H^{1/2}(\mathbb{R}^d)$. This form domain is compact in $L^2(K)$ by the fractional Rellich theorem and boundedness of K.

Theorem 4.1. *For $u \ge 0$ let $U(u)$ be the form compression of $2\sqrt{(|\xi|^2-u)_+}$ to the same domain and let $V(u)$ be the bounded compression of $2\sqrt{(u-|\xi|^2)_+}$. Then $U(u)$ has a positive lower bound and $V(u)$ is positive trace class. The distribution of ν_K is*

$$\Xi(u) := \nu_K([0,u]) = \frac{1}{\pi}\operatorname{tr}\arctan\left(V(u)^{1/2}U(u)^{-1}V(u)^{1/2}\right). \tag{2}$$

The measure has no atom at zero and has a strictly positive continuous density on $(0,\infty)$.

Proof. Approximate the full arms by cylinders with identical transverse Dirichlet cubes $Q_L = (-L,L)^d$ and the same aperture K, for L sufficiently large. Their positive relative measures

ν_L have heat transforms tending locally uniformly to the screen transform by the same bridge bound. The measures thus converge vaguely with exponential tail control. Let $N_L(z)$ denote twice the compression to K of $\sqrt{-\Delta_{Q_L,D}+z}$. For real $u \geq 0$, write upper-bank $N_L(-u+i0) = U_L(u) + iV_L(u)$. The multiplier difference $D_L(u) = U_L(u) - N_L(0)$ has norm at most $2\sqrt{u}$. The variational Poisson extension characterization gives $N_L(0) \geq 2F$.

For each fixed u, the positive form operator $U_L(u)$ has compact resolvent because its difference from $N_L(0)$ is bounded. Its kernel is trivial: if a zero extension of an aperture function has zero $U_L(u)$-energy, it is a finite combination of cube eigenfunctions of eigenvalue at most u. This analytic function vanishes on the open part of the cube disjoint from $\overline{K}$, hence is zero. The continuum operator $U(u)$ is the form compression of $2\sqrt{(|\xi|^2-u)_+}$. It is a bounded perturbation of $2F$ and thus also has compact resolvent. A zero-extension function in its kernel must be compactly supported and bandlimited, hence zero by analyticity. It follows that $U(u)$ is strictly positive at every finite u.

In each Galerkin compression the accumulated relative measure is the upper-bank determinant phase divided by π, by the logarithmic-derivative Stieltjes representation in [1]. Fix $R < \infty$. On $[0, R]$ the real part has zero phase and is invertible. The resulting cumulative measure is

$$\Xi_L(u) = \nu_L([0,u]) = \frac{1}{\pi} \operatorname{tr} \arctan\bigl(V_L(u)^{1/2} U_L(u)^{-1} V_L(u)^{1/2}\bigr).$$

Indeed, the phase of the determinant is the sum of arctan of the nonnegative eigenvalues after factoring the positive real part. Only finitely many transverse modes are open in each cube. Their compressed V_L has finite rank and is continuous in u in trace norm. Uniform lower bounds let Galerkin inverses converge strongly, and convergence of the finite-rank products in trace norm gives its pointwise limit. The higher relative resolvents of [1] give weighted weak convergence of the Galerkin measures, hence convergence against compactly supported continuous functions. The primitives converge at continuity points of the limiting primitive. Trace-norm convergence identifies it with the displayed continuous function at those points. Squeezing with continuity gives equality everywhere and excludes atoms.

As $L \to \infty$, $N_L(0)$ converges in strong resolvent sense on $L^2(K)$ to $2F$. Here the aperture form domain means the closure of $C_c^\infty(K)$ in the norm of the ambient $H^{1/2}$ zero extensions, not merely supported functions at an irregular K. This is the same as the cube trace-form domain by the interior localization from [1]. For one arm, the decreasing energy infimum of the Poisson extension in $Q_L \times (0,\infty)$ is at least the half-space energy. To obtain the reverse limit, multiply the free half-space Poisson extension by a full spatial cutoff equal to one near K. The half-space Poisson formula for compactly supported trace $f \in L^1(K)$ gives $O(R^{-d})$ on large annuli, so the cutoff term in the energy goes to zero. The limiting forms agree for all trace data and the monotone form theorem gives strong convergence of resolvents. The bounded real perturbation from $N_L(0)$ converges strongly to the compressed Fourier multiplier $2(\sqrt{(|\xi|^2-u)_+} - |\xi|)$. For example, extend the Dirichlet cube functional calculus by zero to all of $\mathbb{R}^d$, and use strong resolvent convergence of the Dirichlet Laplacians on the exhausting cubes. The bounded continuous multiplier in the spectral variable decays at infinity.

These statements give coercivity of $U_L(u)$, uniformly for large L and u in any fixed $[0, R]$. By monotonicity in u it suffices to prove the lower bound at R. Otherwise take unit vectors g_L of vanishing $U_L(R)$-energy along a diverging sequence of cubes. Since $N_L(0) \geq 2F$ and $D_L(R)$ is uniformly bounded, the g_L are F-form bounded. A subsequence converges in $L^2(K)$ by compact embedding, to a unit vector g. The strong convergence of $D_L(R)$ and lower semicontinuity of the form of F imply $0 \geq \langle g, U(R) g\rangle$, a contradiction to strict positivity. The bounded-perturbation resolvent identity with the uniform positivity now gives $U_L(u)^{-1} \to U(u)^{-1}$ strongly.

The finite-rank positive operators $V_L(u)$ converge in trace norm on $L^2(K)$ to

$$V(u) = \frac{2}{(2\pi)^d} \int_{|\xi|<\sqrt{u}} (u - |\xi|^2)^{1/2} |e^{i\xi\cdot y}\rangle\langle e^{i\xi\cdot y}| \, d\xi.$$

In a cube, the products of normalized centered Dirichlet sine and cosine modes give Riemann sums for this kernel. In each coordinate the even and odd modes have equal limiting frequency density, and their kernel sums combine by $\cos(py)\cos(py') + \sin(py)\sin(py') = \cos(p(y-y'))$. For bounded y, y' the convergence is uniform at fixed u, including the diagonal. This gives weak convergence and trace convergence for positive operators, hence trace-norm convergence by finite-rank compression and the positive cross-block estimate. Consequently $V_L^{1/2} \to V^{1/2}$ in Hilbert–Schmidt norm, and strong convergence of the inverses gives convergence of the sandwiched operators in trace norm. The eigenvalue trace tr arctan is continuous here, for instance by the trace-norm eigenvalue variation bound. The sandwiched operators therefore give the right-hand side of (2). The multiplier and integral formulas make this limit continuous on bounded energy intervals. Vague convergence identifies (2) at its continuity points and then everywhere by squeezing. In particular $V(0) = 0$ and right continuity give zero mass at zero.

For $u > 0$, $V(u)$ is continuously differentiable in trace norm and the bounded perturbation $U(u) - 2F$ is continuously differentiable in operator norm. The derivative multipliers are respectively $(u - |\xi|^2)^{-1/2}\mathbf{1}_{|\xi|^2<u}$ and $-(|\xi|^2 - u)^{-1/2}\mathbf{1}_{|\xi|^2>u}$. On compact frequency balls both are locally integrable, continuously in $u > 0$ in the L^1 norm. This follows from the integrable square-root singularity in the radial variable. At high frequencies the real derivative is continuous in multiplier norm. Compression converts the L^1 bound at bounded frequencies to operator norm, or to trace norm for V, by the rank-one Fourier integral and $\|e^{i\xi\cdot y}\|^2_{L^2(K)} = |K|$. This justifies differentiating under the compressed integrals.

Equivalently $\Xi(u) = \pi^{-1} \operatorname{Im} \log\det(I + iU^{-1}V)$, with the continuous phase from $u = 0$. The nonzero eigenvalues of $U^{-1}V$ are the eigenvalues of the positive sandwiched operator, so this is the same phase as the arctangent trace. Logarithmic differentiation of the Fredholm determinant shows that the measure is absolutely continuous on $(0, \infty)$ with nonnegative continuous density

$$\Xi'(u) = \frac{1}{\pi} \operatorname{Im} \operatorname{tr}\left[(I + iU^{-1}V)^{-1} i(U^{-1}V' - U^{-1}U'U^{-1}V)\right].$$

The density is strictly positive there. Put $S = U^{-1/2}VU^{-1/2}$, $Q_1 = U^{-1/2}V'U^{-1/2}$, and $P_1 = -U^{-1/2}U'U^{-1/2}$. The first two are positive trace-class operators and P_1 is bounded and positive. Apply $(I + iU^{-1}V)^{-1}U^{-1/2} = U^{-1/2}(I + iS)^{-1}$ and cyclicity with bounded factors. Taking the real part after removing i gives

$$\pi\Xi'(u) = \operatorname{tr}((I + S^2)^{-1}Q_1) + \operatorname{tr}(S(I + S^2)^{-1}P_1) > 0.$$

Here $Q_1 \neq 0$ because V' is the positive integral over a nonempty frequency ball and K has positive measure.

□

5 Long times: fractional torsion and the next term

Put $A_0 = (2F)^{-1}$ and $T = \langle \mathbf{1}, A_0\mathbf{1}\rangle$. The number T is the torsional rigidity for the exterior-Dirichlet form with operator $2F$.

Theorem 5.1. *For any nonempty bounded open K, when $d = 1$,*

$$D_K^{\mathrm{screen}}(\tau) = \frac{T}{2\pi\tau} + \frac{T^2 \log \tau}{2\pi^2\tau^2} + O_K(\tau^{-2}).$$

For $d \geq 2$, define $W = E^*|D|^{-1}E$ as the compressed Fourier multiplier interpreted as a bounded form, and put

$$M = \langle A_0\mathbf{1}, WA_0\mathbf{1}\rangle, \qquad Q = \sum_{i=1}^{d} (\langle y_i, A_0 y_i\rangle - \langle \mathbf{1}, A_0 y_i^2\rangle).$$

Then

$$D_K^{\mathrm{screen}}(\tau) = \frac{\tau^{-(d+1)/2}}{\sqrt{\pi}(4\pi)^{d/2}} \left[T + \frac{(d+3)M + Q}{2\tau} + o(\tau^{-1})\right].$$

Proof. As $u \downarrow 0$, $U(u)^{-1} \to (2F)^{-1}$ in operator norm, with error $O(\sqrt{u})$, and $\|V(u)\|_1 = O(u^{(d+1)/2})$. Thus the leading term of the arctangent trace is linear and

$$\Xi(u) \sim \frac{2T}{\pi(2\pi)^d} \int_{|\xi|<\sqrt{u}} (u - |\xi|^2)^{1/2}\, d\xi.$$

To see the rank-one coefficient, $e^{i\xi\cdot y} \to \mathbf{1}$ in $L^2(K)$ uniformly over the shrinking frequency ball. Integrate by parts against $e^{-\tau u}$ on the low-energy interval and use the fixed-time exponential bound on the remaining tail. The coefficient is evaluated by

$$\tau \int_0^\infty e^{-\tau u} \int_{|\xi|<\sqrt{u}} (u - |\xi|^2)^{1/2}\, d\xi\, du = \Gamma(3/2)\pi^{d/2}\tau^{-(d+1)/2}.$$

This gives the leading term $T\tau^{-(d+1)/2}/[\sqrt{\pi}(4\pi)^{d/2}]$.

For the next term when $d = 1$, on $|\xi| \leq \sqrt{u}$ the multiplier $2(\sqrt{(|\xi|^2 - u)_+} - |\xi|)$ has L^1 norm $O(u)$. For $\sqrt{u} < |\xi| < 1$ it equals $-u/|\xi| + O(u^2/|\xi|^3)$, with $O(u)$ integral of the error. Replacing the Fourier rank-one kernel there by $|\mathbf{1}\rangle\langle\mathbf{1}|$ costs $O(u)$ in operator norm, since K is bounded. On $|\xi| \geq 1$ the compressed multiplier has norm $O(u)$. With the Fourier normalization this gives

$$U(u) = 2F - \frac{u \log(1/u)}{2\pi} |\mathbf{1}\rangle\langle\mathbf{1}| + O(u).$$

The inverse gains the same rank-one perturbation sandwiched with A_0. In the trace against $V(u)$, the even Fourier rank-one integral replaces the leading oscillatory kernel by its constant-rank term with relative error $O(u)$. Linearizing the arctangent costs $O(\|V(u)\|_1^3)$. Consequently $\Xi(u) = Tu/(2\pi) + T^2u^2 \log(1/u)/(4\pi^2) + O(u^2)$. The Laplace integration of $u^2 \log(1/u)$ produces $2\tau^{-2}\log\tau + O(\tau^{-2})$ for the transform $\tau \int_0^\infty e^{-\tau u}\Xi(u)\, du$.

For $d \geq 2$, the low part of W is bounded by the L^1 norm of its symbol times the squared uniform bound on Fourier transforms of functions supported in K. The high part is a bounded multiplier. The same estimates show $U(u) = 2F - uW + o(u)$ in the norm of their bounded difference. On $|\xi| < 1$ apply dominated convergence after dividing the multiplier difference by u, dominated by $2/|\xi|$. On $|\xi| \geq 1$ convergence is uniform. Thus $U(u)^{-1} = A_0 + uA_0WA_0 + o(u)$. In $\langle e^{i\xi y}, A_0 e^{i\xi y}\rangle$, the linear Taylor term vanishes since A_0 is real and self-adjoint, and the trace of the quadratic coefficient matrix is Q. Under the radial weight $(u - |\xi|^2)^{1/2}$ on the frequency ball each ξ_i^2 has mean $u/(d+3)$, by the beta integral. The cross terms integrate to zero. Consequently the bracket in the low-energy phase is $T + u(M + Q/(d+3)) + o(u)$. Laplace integration multiplies the next coefficient relative to the leading one by $(d+3)/2$. □

6 Sharp rates: introduction

At time of order ϵ^2, a shrinking window between two half-cylinders has the limiting heat trace of the same window in an infinite flat screen Sections 2–5. At this scale the discrete transverse thresholds approach a continuum, and the associated relative energy measures converge vaguely. The density of a finite-section measure retains inverse-square-root spikes at individual thresholds. We estimate the difference from the continuous screen density in the distribution norm of weak L^2, including the region near zero energy.

Safarov [11, Section 3.1 and Corollary 3.1] derives the scalar $O(R^{-1})$ interior spectral-function bound on bounded frequency ranges using wave locality up to the clearance time R. Li and Strohmaier [12, Sections 2–3] apply the same locality and Tauberian method to heat kernels. Our projection comparison places the free and finite-section compressions in a common Loewner interval with trace-class width $O(R^{-1})$. The bandlimited bounds use the classical interval approximation strategy discussed by Carneiro, Littmann, and Vaaler [13, Introduction]. In the aperture problem the comparison gives control in trace norm, including the off-diagonal information needed for the differentiated boundary phase.

Boundary-operator logarithms give spectral shift formulas in Behrndt, Gesztesy, and Nakamura [5, Theorem 4.1]. Large-volume limits for conventional spectral shifts include the vague and weighted weak results of Hislop and Müller [14, Theorem 1.4] and Gesztesy and Nichols [15, Theorem 3.13]. The convergence here concerns the derivative of an accumulated aperture phase. The fixed-aperture results of [1] and the threshold expansions for compactly perturbed guides of Richard and Tiedra de Aldecoa [16, Proposition 3.2] describe singular behavior at fixed transverse thresholds. We estimate the cumulative effect as those thresholds become dense. The fixed-time shrinking-aperture result [1] also gives weighted total variation under its mass normalization, with transverse thresholds held fixed.

7 Setting and main theorem

Let $d \geq 1$. The bounded Lipschitz transverse sections Ω_+ and Ω_- of the two half-cylinders in $\mathbb{R}^{d+1}$ contain a ball about the same point y_0. For a bounded nonempty open $K \subset \mathbb{R}^d$, the two arms join at $y_0 + \epsilon K$ in the intervening plane for small ϵ. The boundaries are Dirichlet away from the window. A uniformly bounded nonnegative contact profile ρ_ϵ may add its surface energy on the aperture. The connected-minus-cut heat trace is the integral of the heat kernel difference on the diagonal, equivalently the difference obtained with receding Dirichlet caps [1]. Its positive relative energy measure is denoted by ν_ϵ.

Let $\widetilde{\nu}_\epsilon$ be the pushforward of ν_ϵ by $u \mapsto \epsilon^2 u$. Write ν for the corresponding limiting relative energy measure on the open screen domain

$$\mathcal{O}_K = \mathbb{R}^{d+1} \setminus ((\mathbb{R}^d \setminus K) \times \{0\}),$$

relative to the two cut Dirichlet half-spaces. The measures have cumulative distributions $\Xi_\epsilon(u) = \widetilde{\nu}_\epsilon([0,u])$ and $\Xi(u) = \nu([0,u])$. The screen density g is continuous at positive energies Sections 2–5. For functions with values in a normed space X, use the weak-L^2 distribution quasi-norm

$$\|f\|_{L^{2,\infty}(I;X)} = \sup_{b>0} b\,|\{u \in I : \|f(u)\|_X > b\}|^{1/2}.$$

For scalar functions omit X.

Theorem 7.1. *For every $E > 0$, the scaled measures $\widetilde{\nu}_\epsilon$ have no atom at zero and are absolutely continuous on $[0, E]$ for all sufficiently small ϵ. With density g_ϵ there,*

$$\sup_{0 \le u \le E} |\Xi_\epsilon(u) - \Xi(u)| \le C_E \epsilon, \tag{3}$$

$$\|g_\epsilon - g\|_{L^{2,\infty}(0,E)} \le C_E \sqrt{\epsilon}. \tag{4}$$

In particular $g_\epsilon \to g$ in $L^p(0, E)$ for $1 \le p < 2$. For every $\tau_0 > 0$,

$$\int_0^\infty e^{-\tau_0 u} \, |d\widetilde{\nu}_\epsilon - d\nu|(u) \le C_{\tau_0} \sqrt{\epsilon}. \tag{5}$$

For any fixed bounded interior energy interval $I \subset (0, \infty)$ of positive length with $\inf I > 0$, the difference $g_\epsilon - g$ fails to lie in $L^{2,q}(I)$ for every $0 < q < \infty$, for all sufficiently small ϵ. When both transverse sections are identical balls centered at y_0, in every $d \ge 1$,

$$\|g_\epsilon - g\|_{L^{2,\infty}(I)} \ge c_I \sqrt{\epsilon} \tag{6}$$

for all sufficiently small ϵ.

The rate in (4) is a norm rate on bounded energy bands. Equation (5) allows atoms of the scaled measures outside any fixed band.

After dilation the transverse sections become $U_{\epsilon,j} = (\Omega_j - y_0)/\epsilon$, $j \in \{+, -\}$. Each contains an R-neighborhood of K with $R \asymp \epsilon^{-1}$. The contact becomes $\widehat{\rho}_\epsilon(y) = \epsilon \rho_\epsilon(y_0 + \epsilon y)$. We work on $\mathcal{H} = L^2(K)$. Write $\mathcal{S}_1$ and $\mathcal{B}$ for its trace class and bounded operators. For a section U, denote zero extension to U by E_U. The positive operator-valued measure $P_U(da)$ on transverse energy is the compression by E_U of the Dirichlet spectral resolution of $-\Delta_{U,D}$. Write $P_\infty(da)$ for the free resolution compressed to K. A subscript ϵ on an aperture operator will denote both scaled arms, and a missing subscript denotes the two free arms.

8 A trace-norm local projection estimate

Lemma 8.1. *Suppose a bounded section U contains the R-neighborhood of K, with $R \ge 1$. Write P_U^λ and P_∞^λ for the compressed spectral projections in frequency $\lambda = \sqrt{a}$. For every interval $J \subset [0, M]$,*

$$\|P_U^\lambda(J) - P_\infty^\lambda(J)\|_{\mathcal{S}_1} \le C_{K,d}(1 + M)^{d-1} R^{-1}. \tag{7}$$

The estimate holds with arbitrary endpoint inclusion, including a singleton.

Proof. For $|t| < R$ the compression to K of the Dirichlet cosine propagator equals that of the free cosine propagator. Indeed, initial data supported compactly in K propagate in free space by at most $|t|$, so the restriction solves the Dirichlet wave equation until time R. Density gives the compressed identity on $L^2(K)$. Cosine inversion therefore identifies the two compressions of any even multiplier with integrable Fourier transform supported in $(-R, R)$.

Let $\chi = \mathbf{1}_{J \cup (-J)}$ on $\mathbb{R}$, with the specified endpoint values. Smooth χ by convolution with $r\kappa(r\,\cdot)$, where κ is an even nonnegative Schwartz function of integral one with compactly supported Fourier transform. Such a kernel is obtained by squaring and normalizing the real even inverse transform of a smooth even bump. Take r a sufficiently small fixed multiple of R. The pointwise smoothing error is bounded by $C_m \sum_b (1 + r|\lambda - b|)^{-2m}$, where b runs over at most four endpoints of the symmetric intervals. For each jump the error is bounded by the integral of the mollifier outside the distance to that jump. The same estimate covers assigned indicator values

at the jumps. For a singleton χ has zero convolution and its assigned values are covered by the endpoint bounds.

The even positive bandlimited function

$$b_m(x) = \left(\frac{\sin x}{x}\right)^{2m} + \left(\frac{\sin(x-\pi/2)}{x-\pi/2}\right)^{2m} + \left(\frac{\sin(x+\pi/2)}{x+\pi/2}\right)^{2m}$$

with continuous values at removable points is comparable to $(1+|x|)^{-2m}$. The sine and cosine factors have complementary zeros, with comparable denominators at infinity. Add and subtract a sufficiently large multiple of the sum of $b_m(r(\lambda-b))$ over the endpoints to obtain even multipliers $\phi_+ \geq \chi \geq \phi_-$. Both have Fourier support in $(-R, R)$ by the choice of r. Their Fourier transforms are integrable by compact support and Plancherel.

By spectral order and the compressed short-wave identity, both projectors lie between the same free compressions of ϕ_- and ϕ_+. Their difference lies between $-W$ and W, where W is the positive free compression of $\phi_+ - \phi_-$. With $2m > d+1$,

$$\operatorname{tr} W = (2\pi)^{-d}|K| \int_{\mathbb{R}^d} (\phi_+ - \phi_-)(|\xi|)\, d\xi \leq C_{K,d}(1+M)^{d-1}R^{-1}.$$

Indeed the integral has radial weight λ^{d-1}, and each endpoint envelope gives this bound by the substitution $x = r(\lambda - b)$. The plane waves restricted to K have squared norm $|K|$, giving the trace formula. The difference of the projectors is self-adjoint trace class. On each of its positive and negative spectral subspaces, the corresponding compressed part of W bounds its absolute trace. Adding proves (7). □

In particular the same estimate holds for energy intervals a in a fixed bounded range. A frequency interval of length $O(R^{-1})$ in a bounded range carries free trace mass $O(R^{-1})$, because the radial frequency density is a multiple of λ^{d-1}. Lemma 8.1 gives the same for each scaled section. We will also use the following short-wave smoothing consequence. If an even scalar multiplier is Lipschitz and in $L^2(\mathbb{R})$, its convolution with $r\kappa(r\,\cdot)$ changes it in uniform norm by at most CR^{-1} times its Lipschitz constant. The convolved function has an integrable bandlimited Fourier transform, so the difference of the finite and free compressions in operator norm is at most twice this error.

9 Two derivative estimates on a bounded band

At energy $u \geq 0$, let $U_\epsilon(u) + iV_\epsilon(u)$ be the upper-bank boundary response on $\mathcal{H}$. In terms of the transverse projections, U_ϵ compresses the real multipliers $\sqrt{(a-u)_+}$ and adds $\widehat{\rho}_\epsilon$, and V_ϵ compresses the imaginary multipliers $\sqrt{(u-a)_+}$, each summed over both arms. For U, V use the free projectors and zero contact. All the real forms have a common domain, the closure of the aperture tests extended by zero in $H^{1/2}(\mathbb{R}^d)$. The trace form of each section agrees with this closure by interior localization, as in [1] and Sections 2–5. In particular $U(0) = 2F_K$, where F_K is the exterior-Dirichlet half-Laplacian.

We estimate the positive forward and reflected derivative kernels

$$\tfrac{1}{2}\mathbf{1}_{\{u>a\}}(u-a)^{-1/2}, \qquad \tfrac{1}{2}\mathbf{1}_{\{a>u\}}(a-u)^{-1/2}.$$

They give V_ϵ' and $-U_\epsilon'$, respectively. Their values at equality can be assigned arbitrarily for the almost-everywhere derivative. On bounded energies they are integrable in u uniformly in a,

and the reflected high-energy part is a bounded multiplier when the input is separated from the output band. Integrating the scalar derivative formulas and then the projection measures proves norm absolute continuity of V_ϵ, V in trace class and of the bounded differences $U_\epsilon(u) - U_\epsilon(0)$, $U(u) - U(0)$ in operator norm. For the low part this uses Tonelli and finite trace mass, and for the high part it follows by uniform differentiation away from the threshold.

Put $\eta = R^{-1}$ and write $v = \sqrt{u}, \lambda = \sqrt{a}$. Cut either kernel into a near part, supported where $|v - \lambda| \le 2\eta$, and its complement. The cutoff is one for $|v - \lambda| \le \eta$ and monotone between η and 2η. Partition $(0, E)$ into disjoint intervals I_j obtained by taking the squares of consecutive frequency intervals of length η, truncating the last interval. For $u \in I_j$, the near part receives mass only from an enlarged frequency interval J_j of length $O(\eta)$, with bounded overlap in the input frequency.

On any energy interval I_j, a translated one-sided inverse square root has uniformly bounded weak L^2 quasi-norm. The equivalent norm $\sup_A |A|^{-1/2} \int_A \|f(u)\|\, du$, over measurable subsets $A \subset I_j$ of positive length, gives the weak bound for an integral of these kernels against a positive trace-class measure. Indeed the norm of each positive integral is bounded by its trace, and Tonelli then gives the bound $C\mu(J_j)$, where $\mu = \operatorname{tr} P^\lambda$. This is the measure version of the weak Lorentz convolution bound [17]. Summing superlevel-set lengths over the disjoint I_j gives

$$\|\text{near integral}\|_{L^{2,\infty}(0,E;\mathcal{S}_1)} \le C \left(\sum_j \mu(J_j)^2 \right)^{1/2} = O_E(\sqrt{\eta}). \tag{8}$$

Here $\sup_j \mu(J_j) = O_E(\eta)$ by Lemma 8.1 and the free frequency density. Bounded overlap and finite mass on the bounded input range give $\sum_j \mu(J_j) = O_E(1)$. This holds for both finite and free sections. It also bounds the near reflected kernel in operator norm.

For the far part with bounded input energy, pair by integration by parts with the trace-norm discrepancy of the cumulative compressed projections. For a fixed u, its multiplier in a has variation at most $C[\eta(v + \eta)]^{-1/2}$. On the far support $|a - u| = |\lambda - v|(\lambda + v)$. If $\lambda \ge v + \eta$ this is at least $\eta(2v + \eta)$. The opposite support is empty unless $v \ge \eta$, in which case $|a - u| \ge \eta v$. On each side the far cutoff and the inverse root vary monotonically, so their product has variation bounded by a constant times its maximal size. A fixed smooth high-energy cutoff contributes at most the same order. The integration by parts bound, including any finite endpoint term, is $C_E\sqrt{\eta}(v + \eta)^{-1/2}$. Its square is at most $C_E\eta u^{-1/2}$, so the far difference is $O_E(\sqrt{\eta})$ in ordinary $L^2(0, E; \mathcal{S}_1)$ on bounded input energies.

For the reflected high-energy remainder, use an input cutoff supported at $a > E + 1$. As an even function of λ, the remaining multiplier is uniformly Lipschitz and in $L^2(\mathbb{R})$. Its short-wave smoothing comparison costs $O_E(\eta)$ in operator norm, uniformly in the output. The forward kernel has no high-energy part. Combining the three estimates gives

$$\|V_\epsilon' - V'\|_{L^{2,\infty}(0,E;\mathcal{S}_1)} + \|U_\epsilon' - U'\|_{L^{2,\infty}(0,E;\mathcal{B})} \le C_E\sqrt{\epsilon}. \tag{9}$$

The free derivative norms in these spaces are bounded. For bounded input ranges this follows from the same set-average inequality using the full trace mass, and for the reflected high input range from its uniform multiplier bound. Thus (9) also bounds the individual scaled derivative norms uniformly.

10 Phase estimates and cumulative convergence

First consider the undifferentiated boundary operators. The imaginary kernel $\sqrt{(u - a)_+}$ has uniformly bounded variation in a on $0 \le u \le E$ and vanishes above E. Integration by parts

using (7) gives $\|V_\epsilon(u) - V(u)\|_{\mathcal{S}_1} = O_E(\eta)$, uniformly. The zero-energy form for one arm is bounded below by its free counterpart by the minimum-energy characterization of harmonic extension. For the reverse estimate, choose a spatial cutoff of the free Poisson extension at radius comparable to R, equal to one near K, supported in the section. For aperture trace f the extension and its gradient are bounded on the cutoff annulus by $CR^{-d}\|f\|_1$ and $CR^{-d-1}\|f\|_1$, respectively, by the Poisson kernel. Its energy cost on that annulus is $O(R^{-d-1})\|f\|_2^2$, using the annulus volume and $|K| < \infty$. Outside the cutoff region the omitted energy has the favorable sign. This trial comparison is valid in the homogeneous energy norm, and the cutoff trial belongs to the inhomogeneous form domain. Thus the difference of zero-energy forms without contact is bounded on $\mathcal{H}$ with norm $O(R^{-d-1})$. The contact contributes $O(\epsilon)$.

For the real energy change, consider the multiplier $\sqrt{(a-u)_+} - \sqrt{a}$. On bounded input energies it decreases to $-\sqrt{u}$ and increases thereafter toward zero, so its variation is uniformly bounded on a fixed output band. After a smooth low-energy cutoff its compressed discrepancy is $O_E(\eta)$ in trace norm by (7). The complementary high-energy symbol is even in frequency, uniformly Lipschitz and in $L^2(\mathbb{R})$. Apply short-wave smoothing to obtain $O_E(\eta)$ in operator norm. It follows that

$$\sup_{0\le u\le E} (\|V_\epsilon - V\|_{\mathcal{S}_1} + \|U_\epsilon - U\|_{\mathcal{B}}) \le C_E\epsilon, \tag{10}$$

where the real difference is the bounded difference of the form operators.

The free $U(u)$ is bounded below at each u by the screen result. In fact it is a bounded perturbation of the compact-resolvent operator $2F_K$, and a null vector would give a compactly supported bandlimited function, which vanishes by analyticity. Monotonicity in u makes its bound uniform on $[0, E]$. Equation (10) therefore gives a positive lower bound for U_ϵ on that band for small ϵ, also uniform in ϵ. With $A = U^{-1}$ and $A_\epsilon = U_\epsilon^{-1}$, the inverse identity gives $\|A_\epsilon - A\|_{\mathcal{B}} = O_E(\epsilon)$.

The cumulative distribution on this band is the normalized Fredholm phase

$$\Xi_\epsilon(u) = \pi^{-1} \operatorname{Im} \log\det(I + iA_\epsilon V_\epsilon), \qquad \Xi(u) = \pi^{-1} \operatorname{Im} \log\det(I + iAV), \tag{11}$$

with the continuous lifts from zero. For the screen this is the arctangent formula of Sections 2–5. In the finite sections use the Galerkin phase of [1]. A positive real part implies that the finite-dimensional phase is the sum of arctangents of the eigenvalues of its normalized imaginary part. On $[0, E]$, the real Galerkin inverses converge strongly by coercivity on the common form domain, and the open-channel part is finite rank. More explicitly, if P_n projects onto the finite aperture form subspace, the inverse of the real form restricted to $P_n\mathcal{H}$, extended by zero, converges strongly to A_ϵ. After sandwiching with $V_\epsilon^{1/2}$, it converges in trace norm. The arctangent trace is continuous in this norm by eigenvalue variation. The higher relative resolvents of [1] give weighted weak convergence of the Galerkin measures, hence convergence of the primitives at continuity points. The limiting arctangent is continuous across channel openings, which identifies the primitive everywhere by squeezing. Both primitives vanish at zero since both imaginary parts vanish there.

Set $Z = (I + iAV)^{-1}$, $Z_\epsilon = (I + iA_\epsilon V_\epsilon)^{-1}$. To control their norms write $R_0 = U^{-1/2}$ and $S = R_0 V R_0 \ge 0$. The identity

$$Z = I - iR_0(I + iS)^{-1}R_0V$$

gives $\|Z\| \le 1 + \|A\|\|V\|$, and likewise for Z_ϵ. Equation (10) and the inverse identities give $\|Z_\epsilon - Z\|_{\mathcal{B}} = O_E(\epsilon)$. The perturbations $C_\epsilon = iA_\epsilon V_\epsilon$ and $C = iAV$ differ by $O_E(\epsilon)$ in trace norm. For small ϵ, all $I + C + t(C_\epsilon - C)$, $0 \le t \le 1$, have uniformly bounded inverses by a Neumann

estimate around $I+C$. Integrating the logarithmic Fredholm determinant derivative in t bounds the log difference by $C_E\|C_\epsilon - C\|_{\mathcal{S}_1}$. The interpolation integral is continuous in u, and zero at zero. Its exponential is the determinant ratio, so its imaginary part is exactly the difference of the phase lifts in (11). This proves (3).

The trace-class perturbations in (11) are norm absolutely continuous by the derivative integrability in Section 9 and the inverse identity. The determinants are nonzero because the nonzero eigenvalues of AV and $A_\epsilon V_\epsilon$ agree with those of their positive sandwiches. The phase lifts are consequently absolutely continuous. Neither measure has an atom at zero. Differentiating gives

$$\pi g = \operatorname{Re}\operatorname{tr}[Z(AV' - AU'AV)], \tag{12}$$

and the same formula with subscripts for g_ϵ. The coefficient of V' under the trace is bounded in operator norm, and that of U' is $AVZA$ in trace class, by cycling bounded factors. The analogous coefficients converge at order $O_E(\epsilon)$ in their respective norms by (10) and the inverse estimates. Multiplying by the weak-L^2 input norms and using (9) proves (4). The embedding into L^p, $p<2$, on a bounded interval follows by integrating the distribution bound.

For use in the lower estimates we also give the positive form of the derivative. With R_0 and S as above, let $Q = R_0V'R_0$, $J = -R_0U'R_0$. They are positive and Q is trace class. In (12) use $ZR_0 = R_0(I+iS)^{-1}$ and cycle the bounded factors. This gives

$$\pi g = \operatorname{tr}\left((I+S^2)^{-1}Q\right) + \operatorname{tr}\left(S(I+S^2)^{-1}J\right), \tag{13}$$

with the same identity for g_ϵ. Each trace is nonnegative.

11 Sharpness from discrete channels

Fix a compact interior subinterval of the energy interval I in Theorem 7.1, with positive length. Its free compressed spectral projection has positive trace. By (7), for all sufficiently small ϵ one arm has a nonzero compressed projection $P_\epsilon(\{a\})$ at a threshold a there. In the upper-side density, its summand in the first trace of (13) is $(u-a)^{-1/2}$ times a strictly positive continuous coefficient at a. Positivity follows from injectivity of $U_\epsilon(a)^{-1/2}$, positivity and invertibility of $I + S_\epsilon(a)^2$, and the nonzero positive spectral projection. Continuity follows from norm continuity of the inverse and imaginary part. Since every other summand and the second trace are nonnegative, this term gives a lower bound on g_ϵ on a small right interval. The screen density g is bounded there. Thus the decreasing rearrangement of $|g_\epsilon - g|$ is bounded below by $c_\epsilon s^{-1/2}$ for all sufficiently small $s>0$. For every finite $q>0$, the integral of $[s^{1/2}|g_\epsilon - g|^*(s)]^q\,ds/s$ diverges. This proves the endpoint exclusion.

For the matching lower rate, let both scaled sections be the ball of radius $r_\epsilon \asymp \epsilon^{-1}$ centered at the origin. Use a fixed closed frequency interval of positive length whose squared energies lie strictly inside I. There are $\asymp r_\epsilon$ distinct radial mode frequencies $j_{\alpha,k}/r_\epsilon$ in this interval, where $\alpha = d/2-1$ and $j_{\alpha,k}$ is the k-th positive zero of J_α. The fixed-order zero asymptotics and normalization [18, Sections 10.21 and 10.22] give

$$\int_0^{r_\epsilon} rJ_\alpha(j_{\alpha,k}r/r_\epsilon)^2\,dr = \tfrac{1}{2}(r_\epsilon)^2 J_{\alpha+1}(j_{\alpha,k})^2 \asymp r_\epsilon.$$

The radial eigenfunction on K is therefore a scalar of magnitude comparable to $(r_\epsilon)^{-1/2}$ times the profile $h_\lambda(y) = |y|^{-\alpha}J_\alpha(\lambda|y|)$, with a fixed angular normalization. In $d=1$ these profiles are the even Dirichlet modes of an interval. The profiles form a compact continuous family in

$L^2(K)$ on the chosen frequency interval and are nonzero: each has a nonzero analytic extension as a radial function.

In the first trace of (13), the profile coefficient is a positive multiple of the quadratic form of $U_\epsilon^{-1/2}(I+S_\epsilon^2)^{-1}U_\epsilon^{-1/2}$ at $u=\lambda^2$. Its free counterpart is strictly positive on each h_λ by injectivity. It has a positive minimum on this compact frequency interval. The uniform norm convergence of the inverses gives uniform norm convergence of their square roots by continuous functional calculus, and (10) gives convergence of the other factors. Thus each selected radial threshold supplies an upper-side singular coefficient at least c/r_ϵ. Coincident nonradial levels only increase the coefficient.

Fix r_ϵ. For sufficiently large height b, the superlevel set of $|g_\epsilon - g|$ has an interval of length at least $c'(r_\epsilon b)^{-2}$ above each selected threshold, by continuity of its coefficient and boundedness of g. The finitely many intervals lie in I and are disjoint for large enough b. Hence b^2 times the superlevel length is bounded below by c''/r_ϵ, and taking the supremum over b gives (6).

12 Exponentially weighted total variation

We now track the dependence on the output energy bound E. The Nazarov–Jaming uncertainty inequality [19, 20] applied to a function supported in K and to the frequency ball of radius $\sqrt{E+1}$ gives

$$(2\pi)^{-d}\int_{|\xi|^2>E+1}|\widehat{f}(\xi)|^2\,d\xi \geq c_K e^{-C_K\sqrt{1+E}}\|f\|_2^2. \tag{14}$$

For $d\geq 2$, use the mean-width term in the exponent of Jaming's estimate, since the width of a frequency ball is proportional to its radius. In $d=1$, use the one-dimensional Nazarov estimate. The harmless Fourier scaling is absorbed into the constants. The symbol of $U(E)$ is at least two on the exterior region in (14). By monotonicity this gives the same lower bound, up to constants, for every $U(u)$ with $0\leq u\leq E$.

The estimates before inverting the boundary operators involve polynomial factors in $1+E$. Indeed the projection bound (7) gives a polynomial prefactor at $M=\sqrt{E+2}$. The cell masses and total mass used in (8) have polynomial prefactors by the free radial density. The far derivative integrates the square bound $C\eta u^{-1/2}$ up to E, also with a polynomial prefactor. Choose the smooth high-energy cutoff in a with a fixed-width transition between $E+1$ and $E+2$. The high-energy reflected derivative and real dispersion symbols in frequency λ then have Lipschitz constants bounded by a polynomial in $1+E$. On their supports $a-u\geq 1$, and frequency differentiation of the cutoff brings only a factor of order $\sqrt{1+E}$ on its transition. The bounded-input real dispersion has variation $O(\sqrt{E})$ before cutting off, and the imaginary root has variation at most $\sqrt{E}$. The zero-energy form estimate is independent of E. These bounds show that (10), (9), and the input derivative norm bounds can be taken with polynomial prefactors. For the undifferenced free low-input derivative norm one may use the same set-average estimate with the full trace mass on $[0,E+2]$, which is polynomial in $1+E$.

In particular the bounded real difference in (10) has norm at most $C\epsilon(1+E)^M$. Whenever this is at most $\frac{1}{2}c_K e^{-C_K\sqrt{1+E}}$, the finite real part has at least half the uniform positive lower bound of the free part. The inverse norms then grow at most as $C'e^{C'\sqrt{1+E}}$. In the coefficient estimates for (12), each term involves only a fixed number of these inverses and their accompanying Z factors. The bound $\|Z\|\leq 1+\|A\|\|V\|$ and the inverse identities show that the coefficient norms, and the constants in their $O(\epsilon)$ differences, grow at most polynomially times $e^{C'\sqrt{1+E}}$. Since $\|V\|_{\mathcal{S}_1}$

is polynomial on the band, (12) gives

$$\|g_\epsilon - g\|_{L^{2,\infty}(0,E)} \le C'\sqrt{\epsilon}\,(1+E)^{M'} e^{C'\sqrt{1+E}}$$

under the indicated smallness condition.

Choose $E_\epsilon = \gamma^2 \log^2(1/\epsilon)$ with a fixed sufficiently small $\gamma > 0$. The smallness condition holds for all $E \le E_\epsilon$ and small ϵ. On $[0, E_\epsilon]$, bound the local L^1 error on each unit interval by the weak norm at its upper endpoint, using the weak-L^2 embedding on intervals of length at most one. The truncated last interval obeys the same estimate. With the weight $e^{-\tau_0 u}$, summing gives $O_{\tau_0}(\sqrt{\epsilon})$, since

$$\sum_{k\ge 0} e^{-\tau_0 k}(2+k)^{M'} e^{C'\sqrt{2+k}} < \infty.$$

Beyond the cutoff, positivity bounds the weighted variation by $e^{-\tau_0 E_\epsilon/2}$ times the sum of the two heat transforms at $\tau_0/2$. Those transforms are uniformly bounded by the parabolic heat convergence of Sections 2–5. This tail is $o(\sqrt{\epsilon})$ and completes the proof of (5).

13 Strong norm sharpness for interval sections

Proposition 13.1. *In tangential dimension one, take identical interval sections centered at y_0 and zero contact. On every fixed bounded positive-energy interval I with $\inf I > 0$ and positive length, for all sufficiently small ϵ,*

$$\|g_\epsilon - g\|_{L^1(I)} \ge c_I\sqrt{\epsilon}.$$

In this example the same lower order holds in $L^p(I)$ for $1 \le p < 2$, and in the exponentially weighted total variation of (5) for every $\tau_0 > 0$.

Proof. After dilation the sections are $(-r_\epsilon, r_\epsilon)$, with $r_\epsilon \asymp \epsilon^{-1}$. Write $r = r_\epsilon$. The transverse frequencies are $n\pi/(2r)$, with squared energies a_n. The compressed one-arm projection P_n of a normalized sine or cosine mode satisfies $\|P_n\|_{\mathcal{S}_1} \le |K|/r$. Choose a compact positive-energy subinterval strictly inside I. Its $\asymp r$ thresholds have spacings comparable to r^{-1}. For $u = a_n + x$ with $0 < x \le 4\delta/r$, where $\delta > 0$ is a small fixed fraction of the least scaled spacing there, the two-arm V_ϵ' has n-th term $x^{-1/2}P_n$. Denote the remainder by V'_{reg}. Uniformly on these intervals,

$$\|V'_{\mathrm{reg}}\|_{\mathcal{S}_1} + \|U'_\epsilon\|_{\mathcal{B}} = O(1),$$
$$\|(V'_{\mathrm{reg}})'\|_{\mathcal{S}_1} + \|U''_\epsilon\|_{\mathcal{B}} = O(\sqrt{r}).$$

Indeed the other nearby thresholds have distance bounded below by $c|k-n|/r$. The two bounds follow by summing $Cr^{-1}(r/|k-n|)^{1/2}$ or $Cr^{-1}(r/|k-n|)^{3/2}$ over $O(r)$ bounded modes. The separated bounded-input part has bounded total projected mass, and the reflected high-input part is bounded in operator norm by its scalar multipliers and their derivatives.

By (12), $g_\epsilon(a_n + x) = c_n(a_n + x)x^{-1/2} + b_n(a_n + x)$, where $c_n = \pi^{-1}\operatorname{Re}\operatorname{tr}(Z_\epsilon A_\epsilon P_n)$. As in the radial proof, the compact nonzero sine and cosine profile families and the positive sandwich give $c_n(a_n) \ge c_0/r$. Inverse differentiation gives $A'_\epsilon = O(1)$, and $Z'_\epsilon = O(1 + r^{-1}x^{-1/2})$, because V'_ϵ has just the one extra singular term on this interval. Thus

$$|c_n(a_n + x) - c_n(a_n)| \le Cr^{-1}(x + r^{-1}\sqrt{x}).$$

In particular $c_n \geq c_0/(2r)$ on these intervals for large r. On $[\delta/r, 4\delta/r]$, the bounds on Z'_ϵ and A'_ϵ are $O(1)$ for $r \geq 1/\delta$. Differentiating the background trace and using the regular bounds above gives $|b'_n| \leq C\sqrt{r}$ there, and $|c'_n| \leq C/r$, with constants independent of the small fixed δ.

The screen density is locally Lipschitz on positive energies for $d = 1$. The free compressed spectral measure in a has trace-class density given by the two plane-wave rank-one kernels with scalar factor proportional to $a^{-1/2}$. It is smooth in trace norm on compact positive intervals. In each singular derivative integral use $a = u \pm v$ with a smooth cutoff near $v = 0$. This moves the next derivative to the smooth density, with integrable factor $v^{-1/2}$. Away from $v = 0$, the differentiated multipliers are bounded on the low input range and in operator norm at high energies. Applying (12) gives the local Lipschitz bound.

Put $h = g_\epsilon - g$. For $x \in [\delta/r, 2\delta/r]$, compare $u = a_n + x$ to $v = u + 2\delta/r$. The drop $x^{-1/2} - (x + 2\delta/r)^{-1/2}$ is bounded below by $c\sqrt{r/\delta}$. With its positive coefficient this contributes at least $c/\sqrt{r\delta}$. Moving the coefficient between the two points costs $O(\sqrt{\delta}\, r^{-3/2})$, and the background and free variations cost $O(\delta/\sqrt{r})$. For small fixed δ and large r we obtain $h(u) - h(v) \geq c_\delta r^{-1/2}$. Integrating $|h(u)| + |h(v)|$ over the first interval therefore contributes at least $c'_\delta r^{-3/2}$ per threshold. The interval pairs are disjoint by the spacing choice. Summing over $\asymp r$ levels proves the lower bound. Hölder gives the $L^p(I)$ claim and the positive minimum of $e^{-\tau_0 u}$ on I gives the weighted variation claim. □

14 Threshold oscillations: setting and main result

The parabolic screen limit of Sections 2–5 comes from two Dirichlet half-cylinders joined through a small window. In dilated coordinates their transverse sections grow, while the aperture becomes a fixed bounded nonempty open set $K \subset \mathbb{R}^d$. We take the two dilated sections to be balls about the common origin, of radii r_-, r_+, and allow K to be off-center. We assume $d \geq 2$. Write ν_r for the relative energy measure in the dilated problem and ν for the relative measure in the infinite screen at zero contact. Let g_r, g be their densities on a fixed compact positive-energy band I of positive length. Here $r \to \infty$ is a common reference scale and $r_i/r \to a_i > 0$ on each side. The sections contain a neighborhood of K of radius comparable to r, so the positive-band density and uniform coefficient estimates of Sections 6ff. apply. The nonnegative scaled contact M_r can have size $O(r^{-1})$, as for bounded physical contact, or vary at the larger critical scale $r^{-1/2}$.

Richard and Tiedra de Aldecoa [16] study the resolvent and scattering matrix at fixed waveguide thresholds. The thresholds here move and become dense, and we estimate the derivative of the relative aperture phase. Theorem 7.1 gives a sharp weak-L^2 rate and proves the strong lower rate for one-dimensional intervals. Our leading term resolves the dense threshold oscillation for round sections and supplies the strong and globally weighted constants in higher dimensions. The scalar endpoint quadrature coefficient is classical [27, Theorem 2.1]. The main steps are its application on both sides of the threshold, uniform control of the high angular degrees in the phase derivative, and the averaging of the two radius-dependent lattices.

At energy $u = v^2 > 0$ write $U + iV$ for the free boundary response of the two arms on $L^2(K)$, with U, V the compressions of $2\sqrt{(|D|^2 - u)_+}$ and $2\sqrt{(u - |D|^2)_+}$, respectively. Put $A = U^{-1}$, $S = A^{1/2} V A^{1/2}$, and $Z = (I + iAV)^{-1}$. Let $F_\sigma(v)dv$, $\sigma = 0, 1$, be the one-arm free compressed spectral measure in frequency on the even and odd angular degrees about the common origin.

The compression is performed after taking the angular sector on the full transverse space. Define

$$\begin{aligned} a_\sigma(v) &= \operatorname{tr}[(I+S^2)^{-1}A^{1/2}F_\sigma(v)A^{1/2}],\\ b_\sigma(v) &= \operatorname{tr}[S(I+S^2)^{-1}A^{1/2}F_\sigma(v)A^{1/2}],\\ \Psi(v,\theta) &= \frac{1}{\sqrt{2\pi v}}\sum_{\sigma=0}^{1}\Big[a_\sigma(v)\zeta\Big(\tfrac12,\{\theta-\sigma/2\}\Big)+b_\sigma(v)\zeta\Big(\tfrac12,1-\{\theta-\sigma/2\}\Big)\Big]. \end{aligned}$$

The weights are nonnegative and continuous on positive bands. Fractional parts at integer arguments may be assigned any value for the almost-everywhere formulas.

Theorem 14.1. *Put $s_d=(d-3)/4$ and $\theta_{i,r}(u)=\{r_i\sqrt{u}/\pi-s_d\}$. For $\|M_r\|_\infty=O(r^{-1})$, every compact positive band I, and $1\le p<2$,*

$$\left\|\sqrt{r}(g_r-g)(u)-\sum_{i=\pm}\frac{\sqrt{r/r_i}}{2}\Psi(\sqrt{u},\theta_{i,r}(u))\right\|_{L^p(I)}\longrightarrow 0.$$

The formula also holds for nonnegative contact with $\sqrt{r}\,M_r\to M_*$ in $L^\infty(K)$ after adding the following term to the approximation:

$$G_{M_*}(u)=-\frac{1}{\pi}\frac{d}{du}\operatorname{tr}[S(I+S^2)^{-1}A^{1/2}M_*A^{1/2}].$$

Theorem 14.2. *Write $\Phi(v,\theta)=\sum_{i=\pm}(2\sqrt{a_i})^{-1}\Psi(v,\theta_i)$. If a_-/a_+ is irrational, let μ_v be uniform measure on the two-dimensional torus. If $a_-/a_+=m/n$ with coprime positive integers m,n, set $\tau_r=nr_--mr_+$. For $|\tau_r|\to\infty$, again use uniform torus measure. For $\tau_r\to\tau\in\mathbb{R}$, use normalized Haar measure on the circle*

$$n\theta_- - m\theta_+ = \frac{\tau v}{\pi}-(n-m)s_d \pmod 1.$$

Then, writing $\Phi_*(\sqrt{u},\theta)=\Phi(\sqrt{u},\theta)+G_{M_*}(u)$, with $G_{M_*}=0$ for contact $O(r^{-1})$, one has

$$\begin{aligned} \lim_{r\to\infty} r^{p/2}\|g_r-g\|_{L^p(I)}^p &= \int_I\int|\Phi_*(\sqrt{u},\theta)|^p\,d\mu_{\sqrt{u}}(\theta)\,du,\\ \lim_{r\to\infty}\sqrt{r}\int_0^\infty e^{-tu}\,|d\nu_r-d\nu|(u) &= \int_0^\infty e^{-tu}\int|\Phi_*(\sqrt{u},\theta)|\,d\mu_{\sqrt{u}}(\theta)\,du \qquad (t>0). \end{aligned}$$

Both limits are finite and strictly positive. The contact hypothesis is that of Theorem 14.1. In the rational case the formulas apply subsequentially if τ_r has no limit of the specified types.

15 Phase linearization and angular tails

Set $V_r'=dV_r/du$ and $U_r'=dU_r/du$, with the same conventions in the free problem. The phase differentiation identity of Sections 6ff. is

$$\pi g=\operatorname{Re}\operatorname{tr}[Z(AV'-AU'AV)].$$

For each arm V' uses $\frac12(u-a)^{-1/2}$ on $a<u$, and $-U'$ uses $\frac12(a-u)^{-1/2}$ on $a>u$, integrated against the compressed transverse energy projections. With contact $O(r^{-1})$, the finite-to-free differences of ZA and $AVZA$ are $O(r^{-1})$ on the band in operator and trace norm, respectively. The individual derivative factors have bounded weak-L^2 norms in the corresponding norms. Thus

$g_r - g$ equals π^{-1} times the free linear functionals on $V_r' - V'$ and $(-U_r') - (-U')$, up to $O(r^{-1})$ in $L^p(I)$. Cycling the factors via $ZA^{1/2} = A^{1/2}(I + iS)^{-1}$ gives the weights a_σ, b_σ when the derivative inputs are F_σ. The unsymmetrized identity gives the coefficient comparison without estimating a difference of inverse square roots at order r^{-1}.

Choose a fixed centered ball $K_0 \supset K$ and extend aperture test functions by zero to K_0. On K_0, each transverse spectral measure respects angular degree. Write Q_L for the projection onto degrees above L. Decompose each spectral input on K_0 by Q_L and its complementary projection before compressing to K. Compression contracts the trace norm and the operator norm. The response operators on K need no angular symmetry.

On any fixed frequency band $[0, M+1]$, the Q_L part of the free compressed spectral density on K_0 has trace at most $\delta_L \to 0$ uniformly in frequency. The plane-wave formula for that trace integrates the squared norms of $Q_L e^{iv\omega\cdot x}$ times a bounded radial weight. The Taylor polynomial of degree L in x has angular degrees at most L on each sphere, and its uniform remainder on K_0 tends to zero on this band. For larger frequencies the total free trace density has polynomial bound $C(1+v)^{d-1}$.

The projection comparison of Sections 6ff. uses an identical Loewner interval for the finite and free compressions of an input frequency interval. The width is the free compression of a sum of polynomial envelopes of width $O(r^{-1})$ about its endpoints. Compress both sides by Q_L before taking the trace. For endpoints in $[0, M]$, the total envelope contribution from $[0, M+1]$ is $O(\delta_L/r)$; beyond $M+1$ it is $O(r^{-N})$ for any prescribed N by choosing the polynomial envelope order. Thus the Q_L-compressed projector discrepancy has trace norm at most $C(\delta_L + o(1))/r$ uniformly in endpoint positions, with constant independent of L. The same bound holds for the trace of the compressed spectral mass of cells of width $O(r^{-1})$, and the total bounded-band mass is $O(\delta_L + o(1))$.

Apply the near and far derivative estimates of Sections 6ff. to these compressed inputs. The near weak-L^2 norm is bounded by the square root of the sum of squared input cell masses, hence $O((\delta_L + o(1))/\sqrt{r})$. The far bounded-input term uses integration by parts against the projector discrepancy and gives the same order in $L^2(I)$. The high reflected part is $O(r^{-1})$ in operator norm by the short-wave comparison for Lipschitz multipliers, also after compression. With the bounded phase coefficients this makes the contribution of Q_L to the normalized density error in $L^p(I)$ at most $C_{I,p}\delta_L$ in the limsup. We can therefore compute first at fixed angular degree and then sum.

16 Shifted lattices and the threshold profile

At angular degree l, put $\nu_l = l + d/2 - 1$ and let Y_{lm} be normalized spherical harmonics. The eigenfrequencies of a ball of radius r_i are $\lambda_{lk} = j_{\nu_l,k}/r_i$, where $j_{\nu,k}$ is the k-th positive Bessel zero. The unnormalized profiles restricted to K are

$$h_{lm}(\lambda, x) = |x|^{1-d/2} J_{\nu_l}(\lambda|x|) Y_{lm}(x/|x|).$$

The Hankel transform gives $F_l(\lambda) = \lambda \sum_m |h_{lm}(\lambda)\rangle\langle h_{lm}(\lambda)|$ for the free one-arm sector. The squared normalization of a ball eigenfunction is $2/[r_i^2 J_{\nu_l+1}(j_{\nu_l,k})^2]$. At fixed degree and on a compact positive frequency band, the Bessel asymptotics [18, Sections 10.21–10.22] yield

$$\lambda_{lk} = \mu_{lk} + O(r_i^{-2}), \qquad \mu_{lk} = \Delta_i(k + s_d + l/2), \qquad \Delta_i = \pi/r_i.$$

The associated compressed projection is $\Delta_i F_l(\lambda_{lk}) + O(\Delta_i^2)$ in trace norm, since $J_{\nu_l+1}(j)^2 = 2/(\pi j)(1 + O(j^{-1}))$ at these zeros. The frequency derivatives of F_l are bounded in trace norm on the band.

Use a smooth compact frequency cutoff equal to one near the output frequencies and supported in the positive axis. Outside it the response derivatives are separated from their threshold; on bounded inputs use the projector discrepancy and bounded variation, and at high reflected frequencies use the operator-norm Lipschitz multiplier comparison. The resulting difference is $O(r^{-1})$. Inside the cutoff and at fixed degree the weight errors $O(\Delta_i^2)$ sum to $O(\Delta_i)$ in $L^p(I)$ with values in the trace class. This uses the uniformly bounded L^p norm of a single threshold factor and $O(\Delta_i^{-1})$ frequencies there.

Replacing the actual threshold by its lattice position costs $o(\sqrt{\Delta_i})$ in the same norm. It suffices to work in output frequency $v = \sqrt{u}$. On disjoint neighborhoods of radius $\Delta_i/4$ of the lattice slots, the nearest singular term has shift $O(\Delta_i^2)$. The p-th power of the inverse-root translation difference on each neighborhood is bounded by $C(\Delta_i^2)^{1-p/2}$, by integrating separately within twice the shift and beyond it, where the derivative is bounded by a constant times distance to the power $-3/2$. Include its weight $O(\Delta_i)$ and sum the disjoint neighborhoods to get $O(\Delta_i)$ in p-th power, hence $o(\sqrt{\Delta_i})$ in norm. All far shifts, at distances comparable to Δ_i or larger, sum pointwise to $O(\Delta_i^{3/2})$ by the same derivative bound. Changes of the smooth factors, such as $(v+\lambda)^{-1/2}$ and $F_l(\lambda)$, cost $O(\Delta_i^2)$ after summation of their weighted kernel norms. The reflected inverse root obeys the same estimates.

Fix a lattice and let $a = \{v/\Delta_i - s_d - l/2\} \in (0,1)$. Its left distances from v are $\Delta_i(j+a)$, $j \geq 0$, and its right distances are $\Delta_i(j+1-a)$. After extracting the factor $1/2$ from one arm's derivative, the left Riemann sum minus integral has leading term

$$\frac{\sqrt{\Delta_i}}{\sqrt{2v}} F_l(v)\zeta(1/2, a)$$

with trace-norm error $O(\Delta_i)$ uniformly off the null set of threshold positions. On the right replace a by $1-a$. To see this, split off the value of the smooth weight at zero distance to the endpoint, with a fixed small distance cutoff. The subtracted numerator is $O(t)$ at distance t, and after multiplication by $t^{-1/2}$ its variation is bounded uniformly. The corresponding shifted mesh error is $O(\Delta_i)$ by cellwise variation. For the constant part use $\sum_{j=0}^{N}(j+a)^{-1/2} = 2\sqrt{N+1+a} + \zeta(1/2,a) + O((1+N)^{-1/2})$, uniformly for $0 < a < 1$ by the integral comparison of the large-index tail. This is the first singular Euler–Maclaurin coefficient [27]. The normalization at the endpoint is $F_l(v)/\sqrt{2v}$.

For $l \leq L$ these profiles depend on degree only through parity. The normalized phase factor is $\sqrt{r}/\pi$, so the coefficient of one arm is $\sqrt{r/r_i}/2$ times the profile in Theorem 14.1. The phase functions $\zeta(1/2, \{\theta\})$ and their reflected and shifted versions have uniformly bounded $L^p(I)$ norms when evaluated at $\theta_{i,r}(u)$. This follows by changing variables to v, partitioning into periods, and integrating the one-sided singular bound $C(1 + a^{-1/2})$. The truncation error in the coefficients of the full parity profiles is bounded by $C\delta_L$ uniformly in v. Let $L \to \infty$ and use the angular tail derivative bounds. This proves the oscillating formula.

17 Critical contact

For nonnegative contact M_r of size $O(r^{-1/2})$, first compare the finite sections with the free screen carrying the same contact. The form discrepancy comparing these two is $O(r^{-1})$, since the contacts cancel, and the derivative inputs are unchanged. The corresponding free phase coefficients differ from their transparent counterparts by $O(\|M_r\|_\infty)$. The derivative differences are $O(r^{-1/2})$ in $L^p(I)$ by Sections 6ff.; hence coefficient replacement only costs $O(r^{-1})$, and the same oscillation applies.

For the free comparison, perturb U to $U+M$. Uniformly on the band,

$$(U+M)^{-1} = A - AMA + O(\|M\|_\infty^2), \qquad (I + i(U+M)^{-1}V)^{-1} = Z + ZiAMAVZ + O(\|M\|_\infty^2).$$

The expansions can be differentiated in u with the same remainder bounds in the phase trace norms. Indeed U' is bounded in operator norm and V' in trace norm on compact positive bands, by their singular kernels integrated against the smooth free compressed spectral density near the output and by separated multiplier estimates away from it. The inverse derivative product rules therefore preserve the error bounds. Applying first variation to $\pi^{-1}\operatorname{Im}\log\det(I+iAV)$, the linear term is $-\pi^{-1}\operatorname{Re}\operatorname{tr}(ZAMAV)$. Cycling factors with S gives the negative trace in Theorem 14.1 before differentiation. Its derivative is the first variation of the density, by the same product rules applied to the differentiated trace. Thus for $\sqrt{r}\,M_r \to M_*$ the free difference contributes G_{M_*} after scaling. This argument also gives a continuous representative of G_{M_*} on positive compact bands, using continuity of the free derivatives there.

18 Radius resonance and weighted variation

Change variables to $v = \sqrt{u}$ and first test the phase trajectory against a continuous function of (v, θ_-, θ_+) on a compact positive band times the torus. Approximate uniformly by a finite Fourier sum in the angles with continuous coefficients in v. The character $k = (k_-, k_+)$ gives integrals of the form

$$e^{-2\pi i(k_-+k_+)s_d} \int f_k(v) e^{2iv(k_- r_- + k_+ r_+)}\, dv,$$

where the coefficient includes any weight and the Jacobian $2v$. If $k \cdot (a_-, a_+) \neq 0$, this vanishes by the Riemann–Lebesgue lemma. The irrational ratio leaves only the zero character. In the rational case the remaining characters are $k = j(n, -m)$, with frequencies $j\tau_r$. If $|\tau_r| \to \infty$ these also vanish for $j \neq 0$. If $\tau_r \to \tau$ is finite, their limits equal the corresponding characters of the circle in Theorem 14.2; all other characters of that Haar measure vanish. This establishes the conditional angle averages for continuous tests.

For singular Φ_*, approximate each Hurwitz factor by continuous periodic functions away from small neighborhoods of its singularities. The trajectory $L^p(I)$ errors are bounded by constants times the one-period L^p errors, by partition in v into periods for each arm. Under either limiting angle law, the coordinate marginals are uniform. For the rational circle this follows because its direction (m, n) has both entries nonzero. Thus the truncation error is also small in the conditional averages. Let the discarded neighborhoods shrink, using $p < 2$. With Theorem 14.1 and the convergence of r_i/r, this proves the strong-norm limit. The singular weights are bounded below on a compact band, and the even forward coefficient $a_0(v)$ is strictly positive by the nonzero analytic radial mode on K and the positive inverse sandwich. All Hurwitz terms are bounded below on their phase intervals. A neighborhood of the positive divergence for one arm therefore forces a strictly positive norm under the uniform marginal even if some other singularities coincide. This also shows that a bounded deterministic contact correction cannot cancel the oscillation.

At zero contact and $a_- = a_+ = 1$ the exactly equal radii case averages $|\Psi(v, \cdot)|^p$. If $|r_- - r_+| \to \infty$, the independent average of $|(\Psi(v, \theta_-) + \Psi(v, \theta_+))/2|^p$ is strictly smaller. For $p > 1$ this is strict convexity and nonconstancy; for $p = 1$ it follows from the triangle inequality on opposite signs. In fact the periodic mean of Ψ is zero, since $\int_0^1 \zeta(1/2, a)\, da = 0$ by integrating its partial-sum definition. Each $\Psi(v, \cdot)$ is nonzero and thus has both signs. A finite nonzero detuning $r_- - r_+ \to \tau$ also gives a strictly smaller coefficient. Its circle uses an angle difference $\tau v/\pi$,

irrational modulo one for almost every v. For $p > 1$ equality in the convexity bound would make Ψ invariant under this irrational rotation. For $p = 1$ equality would make its positive set invariant up to null sets. To see the latter, the zero set has measure zero because the profile is real analytic and nonzero on each half period, with a forward singularity there from a nonzero mode of the corresponding parity. Irrational rotation preserves no nontrivial measurable set, as follows by taking Fourier coefficients of its indicator. Thus neither equality can hold. This proves strict change from the coincident case on every positive band. The same strict comparison applies after positive exponential weighting.

To pass from interior bands to the global weighted variation, the growing-band bounds in Sections 6ff. apply uniformly for comparable radii. For critical contact, the bounded real discrepancy from the transparent comparison is at most $C[(1 + E)^A/r + r^{-1/2}]$ on $[0, E]$ for fixed constants C, A, by the unchanged geometric and contact bounds. The free real form has lower bound $ce^{-C\sqrt{1+E}}$ there by the same uncertainty estimate. Choose the small coefficient in the growing cutoff $E_r = \gamma^2 \log^2 r$ so that the finite form is coercive with at least half that bound for every $E \le E_r$. The derivative discrepancies have size $O(r^{-1/2})$ in weak L^2 with polynomial energy factors, independently of the contact, which has no u-derivative. By the inverse identities the phase coefficient differences have the same size with at most polynomial times $e^{C'\sqrt{1+E}}$ factors. Thus the proof of Sections 6ff. gives

$$\sqrt{r}\,\|g_r - g\|_{L^{2,\infty}(0,E)} \le C'(1 + E)^{A'} e^{C'\sqrt{1+E}}, \qquad E \le E_r.$$

In particular the normalized mass of the variation on $[0, \delta]$ is $O(\sqrt{\delta})$, and the positive-band phase argument gives no atom at zero. For large $u \le E_r$, sum the bounds over unit intervals with the exponential weight. After normalization the resulting bound has summable tail $C'e^{-tk}(2 + k)^{A'} e^{C'\sqrt{2+k}}$. Beyond E_r use positivity to bound the weighted variation by $e^{-tE_r/2}$ times the sum of the two heat transforms at $t/2$. They are uniformly bounded, since restricting the transverse domains or adding nonnegative contact only reduces the all-hit heat gain from that of the infinite transparent screen Sections 2–5. This remains valid for atoms beyond the cutoff. The high tail thus vanishes uniformly at the required scale. The local $p = 1$ result with continuous weight and exhaustion prove the asserted weighted limit, including at critical contact.

19 Rectangular sections: setting and results

We use the parabolic relative energy measures of Sections 2–5 and Sections 6ff.. In dilated coordinates the transverse sections are boxes about the origin, with half-lengths R_{ij}, where $i = \pm$ specifies the arm and $j = 1, \dots, d$ the coordinate. Assume $d \ge 2$, $cr \le R_{ij} \le Cr$, and $r \to \infty$. The aperture $K \subset \mathbb{R}^d$ is bounded, nonempty and open. It is fixed in these coordinates. Write ν_r, ν for the finite-section and infinite transparent screen relative energy measures, and g_r, g for their densities on bounded energy bands. The relative reference is the cut screen, with both half-cylinders closed at the aperture. A scaled nonnegative contact M_r on the aperture may have norm $O(r^{-1})$ in the upper estimates below. The arithmetic results use zero contact.

The round sections considered in Sections 14–18 have Bessel threshold lattices and sharp density error of order $r^{-1/2}$ in strong norms. Rectangular sections instead have quadratic-form thresholds. Hlawka's lattice-point estimate [28] and later improvements such as [29] concern scalar counts of lattice points in smooth curved bodies. We use the smoothing estimate with reflections, local plane-wave weights, and kernel derivatives, to control the aperture-compressed projections in trace norm. Waveguide scattering expansions such as [16] analyze fixed thresholds.

Marklof's work on rectangular billiard form factors [30] and on pair correlations of quadratic-form values [31] gives related arithmetic spectral observables. Here the quantity is the connected-minus-cut phase derivative and its one-point oscillatory tests, with operator weights from the aperture.

Theorem 19.1. *Put $s = 2d/(d+1)$. For every $E > 0$ and $t > 0$,*

$$\|g_r - g\|_{L^{2,\infty}(0,E)} = O_E(r^{-s/2}), \qquad \int_0^\infty e^{-tu} |d\nu_r - d\nu|(u) = O_t(r^{-s/2}).$$

Thus the local L^p upper bound is $O_E(r^{-s/2})$ for $1 \le p < 2$. At zero contact the cumulative density discrepancy on $[0, E]$ is uniformly $O_E(r^{-s})$.

We give the arithmetic tests after introducing their aperture weights in Section 22. On a compact positive band I, for identical cubes of half-length r, they give $c_I r^{-1}$ lower bounds in all $L^p(I)$, $1 \le p < 2$, and exclude finite sums of fixed-period phase functions with continuous energy weights as strong asymptotics at order r^{-1}. The stronger endpoint lower bound for $d = 2$ is $c_I r^{-1}\sqrt{\log r}$ in weak $L^2(I)$. These lower bounds leave a gap to Theorem 19.1.

20 Compressed projections in a rectangular box

In one box put $\lambda_{n,j} = \pi n_j/(2R_j)$, $n \in \mathbb{Z}^d$. The compressed Dirichlet projection onto frequencies at most v has kernel on $K \times K$

$$\frac{1}{\prod_j 4R_j} \sum_{|\lambda_n| \le v} \prod_j [\cos(\lambda_{n,j}(x_j - y_j)) - (-1)^{n_j} \cos(\lambda_{n,j}(x_j + y_j))].$$

This is the normalized product of sines in the centered interval coordinates, using the product-to-sum identity and signed-index summation; the modes with a zero coordinate cancel. Expand the reflections, indexed by $e \in \{0,1\}^d$. The invariance under every sign change of n_j replaces the product of cosines by the single exponential $e^{i\lambda_n \cdot (x - D_e y)}$, where $D_e y = ((-1)^{e_j} y_j)_j$. Each summand carries the reflection sign $(-1)^{|e|}$ and twist $(-1)^{e \cdot n}$.

Smooth the indicator of the frequency ball by convolution with a nonnegative smooth compactly supported probability mollifier of width h, where $hr \le 1$. Poisson summation for a given reflection gives the dual grid $4R_j(m_j + e_j/2)$ in coordinate j, evaluated after a bounded shift by the aperture variables. Only the direct term $e = 0$ has a zero dual point. It gives the free kernel with the smoothed multiplier. At nonzero dual points, the Fourier transform of a ball and each fixed derivative obey $C_M |\eta|^{-(d+1)/2}$, uniformly for $0 \le v \le M$ and $|\eta| \ge 1$. This follows by the radial Bessel expression and dilation, including small v, where the factor $v^d (1 + v|\eta|)^{-(d+1)/2}$ suffices. When differentiating the Fourier transform the factor $v^{|\gamma|}$ for a fixed derivative of order $|\gamma|$ preserves this bound. The mollifier yields an additional arbitrarily decaying factor in $h|\eta|$, also after any fixed derivative. Summing the absolute nonzero dual terms and their spatial derivatives by dyadic annuli therefore costs

$$C_M r^{-(d+1)/2} (hr)^{-(d-1)/2} = C_M r^{-d} h^{-(d-1)/2}.$$

The constants at any fixed derivative order grow polynomially in $1 + M$.

For the sharp indicator, first estimate the unweighted number of lattice points in its transition annulus by smooth upper and lower ball indicators with radii shifted by h, taking the lower one to be zero near radius zero. Their main free terms differ by $O_M(h)$. The preceding

untwisted estimate bounds the normalized annular count by $O_M(h + r^{-d}h^{-(d-1)/2})$. The same bound controls the sharp-to-smooth difference for any twist and its spatial derivatives, since the phase has modulus one and the frequency weights of fixed derivative order are bounded at these radii. The free smooth-to-sharp change is $O_M(h)$. Choose $h = r^{-s}$ to balance the two errors. Thus the full kernel discrepancy and all fixed-order spatial derivatives on a bounded enclosing neighborhood of $K \times K$ are $O_M(r^{-s})$. Boundary frequency points are in the same layer, so any inclusion convention at endpoints is covered.

Choose smooth compact cutoffs equal to one on that aperture neighborhood and periodically extend the cutoff kernel on a larger box in each variable. Its double Fourier coefficients have sum of absolute values $O_M(r^{-s})$ by integration by parts with fixed sufficiently many derivatives. Each summand restricts to a rank-one plane-wave kernel on K of uniformly bounded trace norm. This proves the compressed projection bound

$$\|P_r(J) - P_\infty(J)\|_1 = O_M(r^{-s}) \tag{15}$$

for all frequency intervals $J \subset [0, M]$, including singletons, where P_∞ is the one-arm free compressed resolution. The bound has polynomial prefactors in $1 + M$.

21 From projection discrepancy to the density rate

On the Hilbert space $L^2(K)$, write $U + iV$ for the free boundary response at energy u; the finite response is $U_r + iV_r$. Their real and imaginary parts come from the compressed square roots of $(|\xi|^2 - u)_+$ and $(u - |\xi|^2)_+$ summed over the two arms, with the contact added to U_r. The input spectral derivatives of each arm have kernels $\frac{1}{2}\mathbf{1}_{a<u}(u-a)^{-1/2}$ for V' and $\frac{1}{2}\mathbf{1}_{a>u}(a-u)^{-1/2}$ for $-U'$. The phase derivative formula of Sections 6ff. is

$$\pi g = \operatorname{Re}\operatorname{tr}[Z(AV' - AU'AV)], \qquad A = U^{-1}, \quad Z = (I + iAV)^{-1}, \tag{16}$$

and likewise for finite radius on its coercive bands.

Use the near/far derivative argument of Sections 6ff. with frequency mesh $\delta = r^{-s}$. On the near part, each enlarged input cell has compressed mass $O_E(\delta)$ by (15), and the bounded range of cells has total mass $O_E(1)$ under bounded overlap. The weak-L^2 cell estimates give $O_E(\sqrt{\delta})$ after summing squared masses. For the far part on bounded input energy, at $v = \sqrt{u}$ the cutoff kernel has variation $O_E([\delta(v+\delta)]^{-1/2})$. Integration by parts using (15) gives a bound $O_E(\sqrt{\delta}(v+\delta)^{-1/2})$, with $L^2(0, E)$ norm $O_E(\sqrt{\delta})$.

The high reflected multiplier, separated from the energy band, is even in frequency and has bounded second derivative. Convolve with the even bandlimited Schwartz approximate identity from Sections 6ff., of width comparable to r^{-1} and Fourier support within the common propagation time. Its zero first moment and Taylor's inequality give a uniform $O_E(r^{-2})$ error in the multiplier norm. The smoothed finite and free compressions agree by finite propagation. The same refinement applies to the separated high part of the real energy displacement $U(u) - U(0)$. On bounded inputs the undifferentiated square roots have bounded variation and yield $O_E(r^{-s})$ using (15). The real zero-energy comparison costs $O(r^{-d-1})$ by the Poisson extension argument in Sections 6ff.. Thus the real and imaginary coefficient errors before inversion are $O_E(r^{-s})$ at zero contact and $O_E(r^{-s}+r^{-1})$ with permitted contact. The free coercivity on bounded bands and the inverse identities preserve these rates in the phase coefficients. In (16) the derivative discrepancies give $O_E(r^{-s/2})$ in weak L^2, and the coefficient errors times bounded derivative norms are smaller. Comparing the undifferentiated Fredholm phases also gives the stated cumulative estimate at zero contact.

For the global estimate use the growing-band argument of Sections 6ff.. The projection, derivative, and separated-symbol bounds above have polynomial prefactors in the band edge E, taking a fixed-width high-input cutoff in energy. The uncertainty estimate bounds the free inverse by $Ce^{C\sqrt{1+E}}$, and the finite real perturbation is small compared with the corresponding coercivity when $E \le E_r = \gamma^2 \log^2 r$ for $\gamma > 0$ sufficiently small. The local estimate multiplied by $r^{s/2}$ then has subexponential growth $C(1+E)^A e^{C\sqrt{1+E}}$. Sum the weak-L^2 estimates on unit intervals with the exponential weight. Beyond E_r positivity bounds the entire weighted variation, including atoms, by $e^{-tE_r/2}$ times the sum of the heat traces at $t/2$. Those are uniformly dominated by the infinite transparent screen heat gain by the killed all-hit representation. This proves Theorem 19.1.

22 Arithmetic tests and Gauss coefficients

Put $S = A^{1/2} V A^{1/2}$ and let $C_\pm(u)$ be the positive phase coefficients

$$C_+ = A^{1/2}(I + S^2)^{-1} A^{1/2}, \qquad C_- = A^{1/2} S (I + S^2)^{-1} A^{1/2}.$$

They are the self-adjoint real parts of ZA and $AVZA$, respectively. Thus the linearized density discrepancy is the trace of C_+ against the forward derivative input and C_- against the reflected one, divided by π.

Assume zero contact and $R_{ij} = r A_{ij,r}$, with $A_{ij,r} \to A_{ij} > 0$. Set $\eta_{ij,r} = \pi^2/(4A_{ij,r}^2)$, $\eta_{ij} = \pi^2/(4A_{ij}^2)$, and allow the critical detuning

$$r^2(\eta_{ij,r} - \eta_{ij}) \longrightarrow D_{ij} \in \mathbb{R}.$$

For $e = 0, 1$, let $G_e(b)$ be the mean over integers of $e^{2\pi i b n^2 + i\pi e n}$. It is zero for irrational b by quadratic equidistribution [32]; for $b = h/q$ it is the finite mean over a period of length $\operatorname{lcm}(q, 2)$.

For $\omega \neq 0$, define the complex trace-class-valued measure $F_{i,\omega}(du)$ on bounded bands by summing, over $e \in \{0,1\}^d$, the kernel

$$\frac{(-1)^{|e|} \prod_j G_{e_j}(\omega \eta_{ij})}{(2\pi)^d} \int_{\{|\xi|^2 \in du\}} e^{i\xi \cdot (x - D_e y)} e^{2\pi i \omega \sum_j D_{ij} \xi_j^2 / \eta_{ij}} \, d\xi.$$

The integral means the pushforward of the frequency measure by $|\xi|^2$. It is an integral of bounded rank-one kernels on K, including when the reflected second argument lies outside K.

Theorem 22.1. *For any $\omega \in \mathbb{R} \setminus \{0\}$ and any smooth compactly supported test χ on the positive energy axis,*

$$\lim_{r \to \infty} r \int e^{2\pi i \omega r^2 u} \chi(u)(g_r - g)(u)\, du = \frac{1}{2\pi\sqrt{2|\omega|}} \sum_{i=\pm} \int \chi(u) \operatorname{tr}\Big[\big(e^{i\pi \operatorname{sgn}(\omega)/4} C_+(u) + e^{-i\pi \operatorname{sgn}(\omega)/4} C_-(u)\big) F_{i,\omega}(du) \Big].$$

In particular, an arm contributes zero if any $\omega \eta_{ij}$ is irrational.

Proof. At zero contact the coefficient error in (16) contributes $O(r^{-s})$ in L^1 on the support of χ by the response comparison, the inverse identities, and the uniformly bounded derivative norms. This is $o(r^{-1})$. We may use the free C_+, C_- as linear functionals of the input derivative discrepancies.

Set $\omega_r = 2\pi\omega r^2$. At fixed input energy a, first integrate the forward kernel in u, with amplitude χC_+ and exponential $e^{i\omega_r u}$. Without the derivative factor $1/2$ it gives

$$e^{i\omega_r a}\sqrt{\pi}\,|\omega_r|^{-1/2}e^{i\pi\,\mathrm{sgn}(\omega)/4}\chi(a)C_+(a) + O(|\omega_r|^{-1}).$$

The reflected input reverses the angle and uses C_-. The estimate is uniform for bounded input energies. Subtract the endpoint amplitude with a fixed smooth distance cutoff. The remainder times the inverse square root has uniformly integrable derivative, because the amplitude is uniformly Lipschitz and compactly supported in a positive band. Integration by parts gives the remainder. The constant term is the Fourier integral of the positive half-line inverse square root, and its remote cutoff error is $O(|\omega_r|^{-1})$. The response amplitudes have the requisite bounded derivatives in operator or trace norm, since the free projected spectral density is smooth on positive bounded bands and the separated high reflected part is a bounded multiplier. Reflected input far above the band can be integrated by parts directly in u with those bounded multiplier norms, without summing its trace measure. All errors vanish after multiplication by r.

For free input, the factor $e^{i\omega_r a}$ in the main integral averages to zero by the Riemann–Lebesgue lemma. In a finite box it equals $e^{2\pi i\omega\sum_j \eta_{ij,r}n_j^2}$. The residual phase after replacing $\eta_{ij,r}$ by η_{ij} converges uniformly on bounded frequencies to $e^{2\pi i\omega\sum_j D_{ij}\xi_j^2/\eta_{ij}}$, since $\xi_j = \sqrt{\eta_{ij,r}}\,n_j/r$. The remaining integer phase factors by coordinate, including the reflection twist. For each fixed compactly supported continuous frequency weight, the normalized twisted lattice sum tends to its integral multiplied by $\prod_j G_{e_j}(\omega\eta_{ij})$. Prove it for indicators of rectangles by one-dimensional interval means and then approximate the continuous weight uniformly by steps. The means on intervals with endpoints proportional to r follow by subtracting prefix averages, with negative indices covered by the even phase. The exact normalizing frequency volume divided by $(2\pi)^d$ is $(\prod_j 4R_{ij})^{-1}$.

Apply the preceding limit to the continuous frequency weight formed by pairing the plane-wave kernel with $\chi(|\xi|^2)C_\pm(|\xi|^2)$. This gives the specified measure and detuning phase. The factor $1/2$ for each input arm and the $1/\pi$ in (16) give $r\sqrt{\pi}/(2\pi\sqrt{|\omega_r|}) = 1/(2\pi\sqrt{2|\omega|})$. This proves the theorem. □

23 Cube tests and the square spike mass

Take both boxes equal to $(-r, r)^d$. Put $\eta = \pi^2/4$ and $\Delta_e = \eta/r^2$. The thresholds lie in $\Delta_e\mathbb{N}$. For $\omega = j/\eta$ with an integer $j \neq 0$, the Gauss factors are $G_0 = 1, G_1 = 0$, so both arms give the free one-arm measure F. Write

$$a(u) = \mathrm{tr}[C_+(u)F'(u)], \qquad b(u) = \mathrm{tr}[C_-(u)F'(u)].$$

Here $a(u) > 0$ since $F'(u)$ is a nonzero positive operator and the positive $C_+(u)$ is injective. Theorem 22.1 gives the limit of the j-th oscillating test at phase $\{u/\Delta_e\}$ as the integral against χ of

$$\frac{e^{i\pi\,\mathrm{sgn}(j)/4}a(u) + e^{-i\pi\,\mathrm{sgn}(j)/4}b(u)}{2\sqrt{2|j|}}.$$

For $j = 0$ the limit is zero by integration by parts against the cumulative $O(r^{-s})$ bound. Equivalently these tests see the distributional profile

$$\Theta(u,\theta) = \tfrac{1}{2}\big[a(u)\zeta(1/2,\theta) + b(u)\zeta(1/2,1-\theta)\big].$$

Indeed, for $j \neq 0$, the first Hurwitz function has positive-exponential coefficient

$$\int_0^1 \zeta(1/2,\theta) e^{2\pi i j\theta}\, d\theta = \frac{e^{i\pi \operatorname{sgn}(j)/4}}{\sqrt{2|j|}}.$$

Sum its partial-sum definition over periods and use the same half-line Fourier integral at $2\pi j$. Its zero coefficient is zero by direct integration of the partial sums. Hence the test limit holds for finite sums of smooth compactly supported energy functions times trigonometric polynomials in phase. By Hölder applied before taking limits, periodic averaging of the test norm, and duality over these tests, we obtain

$$\liminf_{r\to\infty} r\,\|g_r - g\|_{L^p(I)} \geq \|\Theta\|_{L^p(I\times(0,1))} > 0 \qquad (1 \leq p < 2).$$

For $p = 1$ approximate the sign of the integrable profile by bounded continuous tests and then uniformly by trigonometric polynomials in phase on the band. Positivity of the norm follows from $a(u) > 0$.

At the rational probe $\omega = (h/q)/\eta$ with $q > 1$ odd and coprime h, q, $G_1(h/q) = 0$, while $|G_0(h/q)| = q^{-1/2}$. For the latter identity expand its squared modulus and use the invertible change of variables to difference and sum modulo q. Thus the preceding cube test with $j = h/q$ is multiplied by the nonzero factor $G_0(h/q)^d$. It is nonzero for every nonnegative nonzero χ, because the two nonnegative response coefficients have positive real weighted combination before the Gauss factor. At $h/q = 1/2$, instead $G_0 = 0, G_1 = 1$ and the all-coordinate reflection is selected.

These subharmonics exclude a strong approximation to $r(g_r - g)$ by $\Theta(u, \{u/\Delta_e\})$, or by a finite sum of continuous slow coefficients times any prescribed-period L^p functions of u/Δ_e. For period $T > 0$ choose an odd prime q with $T/q \notin \mathbb{Z}$. Multiplication by the $1/q$ harmonic and grouping into T-cells gives zero in the limit for such candidate functions. Freeze the slow coefficients first on macroscopic subintervals and use the bounded geometric sum of the cell phases, then refine the partition; the integrable periodic amplitudes give uniform errors. Strong L^p approximation would imply the same zero for the actual test, contradicting the Gauss coefficient. The single-period profile above is therefore distributional even for the strong lower bound.

Theorem 23.1. *In $d = 2$, for two identical square sections with zero contact and any compact positive interior band I of positive length, define*

$$\mathcal{S}_r(I) = \lim_{b\to\infty} b^2 |\{u \in I : |g_r(u) - g(u)| > b\}|.$$

The limit exists for all sufficiently large r and satisfies

$$c_I r^{-2} \log r \leq \mathcal{S}_r(I) \leq C_I r^{-2} \log r.$$

Consequently the weak-$L^2(I)$ norm is at least $c_I' r^{-1}\sqrt{\log r}$.

Proof. At a given radius there are finitely many thresholds on the band. Just to the right of one, the only singular derivative comes from its forward inverse root, and to the left from the reflected inverse root. Their continuous coefficients have nonnegative limits A_N^+, A_N^- at threshold $\Delta_e N$, by the positive phase formula of Sections 6ff.. Away from all such points the density difference is bounded on I. With x the distance from a threshold on an allowed side, the density difference is $x^{-1/2}(A_N^\pm + o(1))$, since the continuum density and the other terms are bounded there at fixed r. The asymptotic superlevel lengths therefore give $\mathcal{S}_r(I) = \sum (A_N^\pm)^2$, taking only the sides lying

in the band for endpoint thresholds. The trace norm of the single normalized mode projection is $O(r^{-2})$, by the explicit product eigenfunctions on the fixed aperture. The finite phase weights are uniformly bounded on I; hence each amplitude is at most Cr^{-2} times the number of positive two-square representations of its index.

For a lower estimate, restrict to positive mode vectors in a small angular cone about the diagonal and a compact annular subband of I. Their coordinate frequencies lie in a compact subset of $(0, \infty)$. The mode profiles on the aperture, after multiplying by r, are products of sines or cosines from this compact frequency family, up to unit signs. Each product is nonzero on any open aperture by analyticity. The positive forward inverse sandwich of the free response has a strictly positive minimum on this family, uniformly over the compact output band. Convergence of the finite phase coefficients, including their continuous threshold values, gives a lower contribution cr^{-2} per retained mode to A_N^+.

We give an elementary pair count retaining this angular constraint. Let $p = (a, b)$ range over primitive positive integer vectors in a narrower cone about the diagonal. Put $m = s_0 p$ and $n = t_0 p^\perp$ for positive integers s_0, t_0, and set $v = m + n$, $w = m - n$. The resulting pair has equal norm and determines p, s_0, t_0 uniquely. Choose s_0 in a fixed radial subinterval times $r/|p|$ and t_0 in a smaller positive interval times $r/|p|$. The intervals can be chosen so that both output vectors lie in the retained cone and annulus. If $|p| \le c_0 r$ with c_0 small, there are at least $c(r/|p|)^2$ such pairs per primitive p. In each sufficiently large dyadic annular sector, primitive vectors have positive area density, by applying Möbius inversion to the lattice count there. The area contribution is multiplied by $1/\zeta(2)$, and summing the perimeter errors gives $O(D \log D)$ at sector size D. Thus the sum of $|p|^{-2}$ over the eligible primitive vectors is at least $c \log r$. This gives at least $cr^2 \log r$ equal-length ordered pairs within the retained modes.

For the matching upper count, take any positive equal-length vectors v, w of norm at most Cr. Write $v + w = s_0 p$ with p positive primitive. Then $v - w = t_0 p^\perp$ for some integer t_0, by orthogonality and primitivity. The choices per p are $O((r/|p|)^2)$, including $t_0 = 0$, since $|p| \le 2Cr$. Sum over dyadic shells, each with $O(D^2)$ vectors, to get $O(r^2 \log r)$. This agrees in order with the unrestricted classical second moment of the representation function [33]. Multiplying the lower and upper counts by r^{-4} proves the two bounds for $\mathcal{S}_r(I)$. The supremum defining the weak norm is at least the square root of the large-height limit. □

24 Distant windows: introduction

Opening one window in a Dirichlet screen allows a heat path to reach the other side and return. For several windows, inclusion–exclusion isolates the part of this gain that requires every window. At fixed window shapes, separation proportional to L and time proportional to L^2, the leading connected interaction depends on each shape through one scalar torsion and on the centers through propagation kernels for the legs of directed cycles.

The screen geometry occurs in the pointwise heat-kernel analysis of Benedicks domains [2, Theorem 1]. The acoustic aperture problem of Chandler-Wilde and Hewett [3, Theorem 4.1 and Section 8] uses the same closure-of-tests trace space and square roots of the tangential symbol, with the scattering normalization. We use the continuum phase formula and the integrated relative heat kernel of Sections 2–5. In coupled Dirichlet strips, Borisov and Exner [21, Theorems 2.2–2.3] derive exponentially small shifts and splittings of discrete eigenvalues. Their symmetric Neumann-window analysis also treats a threshold resonance [22]. Here the half-spaces give a continuum from zero, with an algebraic interaction on the diffusion time scale.

Schaden [23, eqs. (15)–(18), (22)–(23)] gives the all-hit Brownian-bridge identity and a short-

est closed route bound for irreducible many-body heat functions. With all windows open as the background, closing selected windows gives the same inclusion–exclusion mechanism, with the parity sign converted by complementing the subsets. Normalizing a relative boundary determinant by the individual blocks has a rigorous obstacle-scattering antecedent in Hanisch, Strohmaier, and Waters [24, Theorems 1.6–1.7]. Emig, Graham, Jaffe, and Kardar [25, Sections II.C, III.C, and IV.D and eq. (III.27)] resolve multiple-scattering paths and low-frequency shape factors for scalar vacuum energies. Fang and Strohmaier [26, Theorems 3.7 and 4.1] identify shortest-orbit amplitudes in relative wave traces. We evaluate the entire diffusion-time profile for windows in a single screen, including the coefficients at both ends of that scale.

25 Screen trace and main theorem

For bounded open $K \subset \mathbb{R}^d$, let

$$\mathcal{O}_K = \mathbb{R}^{d+1} \setminus ((\mathbb{R}^d \setminus K) \times \{0\}).$$

A bounded measurable contact $\rho \geq 0$ on K adds $\int_K \rho(y)|f(y,0)|^2\,dy$ to the Dirichlet energy on $H^1_0(\mathcal{O}_K)$. Write $D_{K,\rho}(t)$ for the integrated diagonal of its heat kernel minus that of the two cut Dirichlet half-spaces. Both kernels are extended by zero outside their domains. The free heat generator is Δ, with diagonal factor $(4\pi t)^{-(d+1)/2}$. Set $D_{\varnothing,0}(t) = 0$ and abbreviate $D_{K,0} = D_K$.

Let F_K be the exterior-Dirichlet half-Laplacian: compress the Fourier multiplier $|\xi|$ to the form closure of zero extensions of $C_c^\infty(K)$ in $H^{1/2}(\mathbb{R}^d)$. We use the convention $e^{-i\xi\cdot y}$ with Parseval factor $(2\pi)^{-d}$. For nonempty K, the inverse below is bounded and the normalized torsion is strictly positive:

$$T_{K,\rho} = \langle \mathbf{1}, (2F_K + \rho)^{-1}\mathbf{1}\rangle_{L^2(K)}.$$

Fix $m \geq 2$, distinct centers $p_i \in \mathbb{R}^d$, and nonempty bounded open windows K_i with bounded fixed contacts $\rho_i \geq 0$. For $S \subseteq [m] = \{1,\dots,m\}$, set $K_{L,S} = \bigcup_{i\in S}(Lp_i + K_i)$ and denote the contact obtained by translating each included profile by $\rho_{L,S}$. For large L the translated closures are disjoint. Define

$$\mathcal{I}_L(t) = \sum_{S\subseteq[m]} (-1)^{m-|S|} D_{K_{L,S},\rho_{L,S}}(t). \tag{17}$$

For $\ell > 0$, put $C_d = 1/[\sqrt{\pi}(4\pi)^{d/2}]$ and

$$k_{d,\ell}(t) = C_d t^{-(d+3)/2} e^{-\ell^2/(4t)}, \qquad t > 0.$$

Write $\ell_{ij} = |p_i - p_j|$ and $T_i = T_{K_i,\rho_i}$. The set $\mathcal{H}_m$ consists of the directed Hamiltonian cycles on the labels, modulo cyclic rotation. Both orientations occur for $m \geq 3$, and there is one cycle for $m = 2$.

Theorem 25.1. *In every tangential dimension $d \geq 1$, as $L \to \infty$ locally uniformly for $\theta > 0$,*

$$L^{m(d+1)}\mathcal{I}_L(L^2\theta) \longrightarrow \Phi(\theta) := \left(\prod_{i=1}^m T_i\right)\theta \sum_{\gamma\in\mathcal{H}_m} \left(*_{(i,j)\in\gamma} k_{d,\ell_{ij}}\right)(\theta). \tag{18}$$

The convolutions are in positive time. At every fixed positive separation of the windows, $\mathcal{I}_L(t) > 0$ for $t > 0$. Its Laplace representation is by a signed measure with no atom at zero and a continuous density taking both signs on $(0,\infty)$. The limiting profile Φ is strictly positive.

The legs in (18) are propagators between windows in the boundary operator. The short- and long-time formulas below describe the profile after the diffusive limit. For zero contact, $D_{\lambda K}(t) = D_K(t/\lambda^2)$ by spatial dilation. Thus (18) also describes windows $p_i + L^{-1}K_i$ at fixed positive time.

26 Bridge representation and spectral sign

A bridge based off the plane contributes to the heat gain when it survives in the open domain and hits the plane. Its plane-hit positions must lie in the windows. The cut domain cancels paths that avoid the plane. A surviving bridge with contact acquires the positive Feynman–Kac weight of the contact integrated against normal local time in the occupation-density convention. The Gaussian time integral of flat-surface measure is $O(\sqrt{t})$, so this is a Kato measure perturbation [7] and Sections 2–5.

Let J be the nonempty set of labels visited at the plane-hit times of a surviving all-window bridge. It contributes to S precisely when $S \supseteq J$. Its contact weight is then the same for every such S, since plane local time is carried by plane-hit times. The weight $\sum_{S \supseteq J} (-1)^{m-|S|}$ is one for $J = [m]$ and zero otherwise. A bridge killed in the all-window domain contributes nowhere. Consequently the connected gain counts just the bridges visiting every window. A polygonal loop can cross through an interior ball in each aperture, repeating one crossing if m is odd. It stays off the plane between crossings. Its compact image has a small tube inside the domain that forces all the visits. Bridge support gives this tube positive probability for a neighborhood of starting points. The contact weight is strictly positive almost surely, which proves the fixed-time strict sign.

The maximum excursion of a free generator-Δ bridge from q is bounded by

$$\mathbb{P}_q^t\big(\sup_r |B_r - q| \ge b\big) \le 2(d+1)e^{-b^2/((d+1)t)}.$$

For a bridge that reaches a given window in the plane one may take b to be the initial distance to that window. The resulting Gaussian envelope is integrable in q and invariant in its integral under translations of the window. At any fixed $t_0 > 0$ the sum of these bounds controls all subset gains uniformly in S, L.

For a fixed separated configuration, hitting two windows a distance $b > 0$ apart requires a tangential excursion of at least $b/2$. The geometric mean of this excursion bound and the integrable window-hit bound controls the all-hit probability. Including the free diagonal and integrating gives $O(t^{-(d+1)/2}e^{-c/t})$ as $t \downarrow 0$ for some $c > 0$. Hence $\mathcal{I}_L(0+) = 0$. The trace is positive at positive times, whereas a completely monotone function is nonincreasing.

Section 27 gives a positive locally finite measure for every subset, with no zero atom, continuous density on $(0, \infty)$, and finite Laplace transforms at positive times. Their alternating sum represents $\mathcal{I}_L$ and has exponentially weighted finite total variation at every positive time. An everywhere nonpositive density would make the trace nonpositive, and an everywhere nonnegative density would make it completely monotone. Its continuous density therefore takes both signs.

27 Screen phase and separated kernels

For $z \in \mathbb{C} \setminus (-\infty, 0]$, let $N_{K,\rho}(z)$ be the form compression of $2\sqrt{|D|^2 + z}$ plus multiplication by ρ. On the upper bank write $N_{K,\rho}(-u + i0) = U(u) + iV(u)$. These parts are the compressions of $2\sqrt{(|\xi|^2 - u)_+}$ and $2\sqrt{(u - |\xi|^2)_+}$, with ρ added to U. The real part has a positive lower bound

at each finite energy, and V is positive trace class. The continuum screen theorem Sections 2–5 gives, for zero contact,

$$\Xi_{K,\rho}(u) := \nu_{K,\rho}([0,u]) = \frac{1}{\pi}\,\mathrm{tr}\arctan\big(U(u)^{-1/2}V(u)U(u)^{-1/2}\big),$$
$$D_{K,\rho}(t) = t\int_0^\infty e^{-tu}\Xi_{K,\rho}(u)\,du. \tag{19}$$

To extend the formula to contact, exhaust the two half-spaces by half-cylinders with identical transverse Dirichlet cubes, keeping ρ fixed. Normal and transverse exhaustion converge in the heat diagonal integral by the bridge bound and the Feynman–Kac weight bounded by one. The cylindrical aperture theorem [1] supplies positive measures for the approximants, also with contact. Compactness on finite energy intervals and control of the tails by a heat transform at an earlier positive time give vague convergence to a positive screen measure. In the cube phase proof of Sections 2–5, adding $\rho \geq 0$ preserves the uniform coercivity of the real forms on each bounded energy interval. Their strong resolvent convergence persists under the bounded perturbation, and coercivity yields strong inverse convergence. The imaginary parts still converge in trace norm. Their square roots converge in Hilbert–Schmidt norm, so the sandwiched operators and their arctangent traces converge in trace norm and in trace, respectively. Using continuity of the limiting phase identifies the vague limit at every energy. At zero $V(0) = 0$, so there is no atom. Integration of the cumulative measure by Tonelli gives the second formula of (19).

For $u > 0$, V is C^1 in trace norm and $U-(2F_K+\rho)$ is C^1 in operator norm, by the compressed locally integrable derivative multipliers of Sections 2–5. The phase is the imaginary logarithm of $\det(I+iU^{-1}V)$ continued from zero, so differentiation gives a continuous density. The derivative signs $V' \geq 0$ and $U' \leq 0$ are unchanged by ρ. This proves the properties of the individual measures used in Section 26.

For separated windows identify the components with $L^2(K_i)$ by translations. The aperture form space decomposes as the direct sum of the individual form spaces by smooth cutoffs separating the windows. Contact acts diagonally. Away from its diagonal the kernel of $2\sqrt{|D|^2+z}$ is

$$n_z(x) = -C_d\int_0^\infty t^{-(d+3)/2}e^{-zt-|x|^2/(4t)}\,dt \qquad (x \neq 0,\ \mathrm{Re}\,z > 0). \tag{20}$$

Indeed the representation $\sqrt{a} = (2\sqrt{\pi})^{-1}\int_0^\infty(1-e^{-ta})t^{-3/2}\,dt$ has no off-diagonal contribution from its identity term. The factor two for the arms and the tangential heat kernel give the stated normalization. Put $\alpha = (d+1)/2$. The integral equals $2(4z/|x|^2)^{\alpha/2}K_\alpha(|x|\sqrt{z})$, with K_α the modified Bessel function of the second kind. Its principal branches continue the kernel to the slit plane. Uniformly on compact spatial annuli away from zero,

$$|n_{-s+i0}(x)| \leq C(1+s)^{d/4}, \qquad s \geq 0.$$

The kernel is bounded for small s and has large-s bound $O(s^{\alpha/2-1/4})$ by the fixed-order Bessel asymptotic. It is smooth off $x = 0$ at each fixed s.

Let N_{ij} be the block with target i and source j, $i \neq j$. Changing variables in (20) and taking upper-bank limits gives

$$n_{-s/L^2+i0}\big(L(p_i-p_j)+y-y'\big) = L^{-(d+1)}n_{-s+i0}\big(p_i-p_j+(y-y')/L\big).$$

In Hilbert–Schmidt norm, denoted by $\|\cdot\|_2$, this gives

$$\|N_{ij}(-s/L^2+i0)\|_2 \leq CL^{-(d+1)}(1+s)^{d/4}, \tag{21}$$
$$L^{d+1}N_{ij}(-s/L^2+i0) \longrightarrow n_{-s+i0}(p_i-p_j)|\mathbf{1}_{K_i}\rangle\langle\mathbf{1}_{K_j}| \quad \text{at fixed } s. \tag{22}$$

Also, for fixed L, u, the off-diagonal block is trace class between these spaces. Multiply its smooth kernel by smooth cutoffs equal to one on the two windows in separated neighborhoods, and extend the resulting kernel periodically on enclosing boxes. Its double Fourier coefficients are absolutely summable. The blocks are restrictions of trace-norm sums of rank-one operators.

28 Boundary determinant and the time profile

At $u \geq 0$ write $N_i = U_i + iV_i$ for the isolated boundary block of window i. On $0 \leq u \leq E$ with $E > 0$ sufficiently small, the inverses are uniformly bounded and tend in norm to $A_i = (2F_{K_i} + \rho_i)^{-1}$ at zero. This follows from the uniform $O(\sqrt{u})$ bound on the square-root perturbation and the positive zero-energy gap. Normalize the subset form by $H_S^{-1/2}$, where $H_S = \mathrm{diag}_{i \in S}(U_i)$. The diagonal blocks become $B_i = I + iU_i^{-1/2} V_i U_i^{-1/2}$, and the other blocks become $Z_{ij} = U_i^{-1/2} N_{ij} U_j^{-1/2}$. Put $X_{ij} = B_i^{-1} Z_{ij}$ for $i \neq j$ and $X_{ii} = 0$, with $X[S]$ the principal submatrix on S. Equation (21) gives

$$\|X[S]\|_2 \leq C L^{-(d+1)} (1+s)^{d/4} = O(L^{-d/2-1}) \quad (0 \leq s \leq EL^2). \tag{23}$$

For the unnormalized subset form use U_S, V_S for its real and imaginary parts. In normalized coordinates let $R = H_S^{-1/2} U_S H_S^{-1/2}$ and $J = H_S^{-1/2} V_S H_S^{-1/2}$. Then $R - I$ and J are trace class, and R is uniformly positive on the cutoff interval by the off-diagonal norm bound. Factoring R out of the Fredholm determinant $\det(R + iJ)$ gives a positive factor $\det R$ times $\det(I + iR^{-1/2} J R^{-1/2})$. The variational inverse identity $U_S^{-1} = H_S^{-1/2} R^{-1} H_S^{-1/2}$ gives

$$\det(I + iR^{-1/2} J R^{-1/2}) = \det(I + iV_S^{1/2} U_S^{-1} V_S^{1/2}).$$

The nonzero spectra of the positive sandwiches in this identity and in (19) agree by cyclic products. Thus the normalized determinant has continuous phase $\pi \Xi_{K_{L,S}, \rho_{L,S}}(u)$ from zero.

Factoring instead the diagonal blocks B_i and using determinant multiplicativity yields

$$\Xi_{K_{L,S}, \rho_{L,S}}(u) - \sum_{i \in S} \Xi_{K_i, \rho_i}(u) = \pi^{-1} \operatorname{Im} \log \det(I + X[S]). \tag{24}$$

Equation (23) keeps the logarithm on the near-identity branch. At $u = 0$ the ratio is positive real, so continuity fixes the phase in (24). For a pair, block elimination reduces $\det(I + X[\{1,2\}])$ to $\det(I - X_{21} X_{12})$.

Let $\Delta_L(u) = \sum_{S \subseteq [m]} (-1)^{m-|S|} \Xi_{K_{L,S}, \rho_{L,S}}(u)$. The single-window phases cancel. In the series

$$\log \det(I + X[S]) = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \operatorname{tr}(X[S]^k),$$

a trace is a sum of rooted closed walks. Each term survives the subset sum exactly when its walk visits every label. Degrees $k < m$ therefore vanish. At $k = m$ each walk visits every label once before closure and has m possible roots. This cancels $1/m$ and leaves the cycles $\mathcal{H}_m$. For $k \geq 2$, $|\operatorname{tr} X[S]^k| \leq \|X[S]\|_2^2 \|X[S]\|^{k-2} \leq \|X[S]\|_2^k$. Summing the geometric bound gives

$$|L^{m(d+1)} \Delta_L(s/L^2)| \leq C (1+s)^{md/4} \quad (0 \leq s \leq EL^2).$$

For fixed s all degrees above m vanish after scaling.

Cyclicity removes the normalizations in a rooted product since $U_i^{-1/2}B_i^{-1}U_i^{-1/2} = N_i^{-1}$. By (22) and the isolated inverse limit, the rank-one contractions give one T_i per label. Each directed edge gives $n_{-s+i0}(p_i - p_j)$. Both directions use the upper bank and the scalar is radial. Define $f_\gamma(z) = \prod_{(i,j)\in\gamma} \mathcal{L}(k_{d,\ell_{ij}})(z)$. Equation (20) gives a minus sign per edge. Together with $(-1)^{m+1}$ from the logarithm, this yields

$$L^{m(d+1)}\Delta_L(s/L^2) \longrightarrow -\frac{\prod_i T_i}{\pi} \sum_{\gamma\in\mathcal{H}_m} \operatorname{Im} f_\gamma(-s+i0). \tag{25}$$

Set $u = s/L^2$ in the heat transform in (19). The prefactor becomes θ. The polynomial bound controls the integral over $s \le EL^2$ against $\theta e^{-\theta s}$, uniformly on compact positive θ-intervals. Above the cutoff, positivity gives $\Xi_{K_{L,S},\rho_{L,S}}(u) \le e^{t_0 u} D_{K_{L,S},\rho_{L,S}}(t_0)$. The uniform fixed-time bridge bound makes the scaled tail exponentially small there. Thus the limiting heat is the locally uniform transform of (25).

The function f_γ is the Laplace transform of the time convolution along γ. To invert it, close the Bromwich line to the left in a rectangle with left side at $-R$ and imaginary parts $\pm R$, excluding the slit and a small circle around zero. On the horizontal sides, $\operatorname{Re}\sqrt{z} \ge c\sqrt{R}$, so the product of Bessel factors has exponential decay, up to a polynomial. On the left side the Laplace inversion factor gives $e^{-\theta R}$. Boundedness at zero permits shrinking the circle. The upper bank of the excluded slit runs toward zero. Subtracting the two slit integrals from the contour and using conjugation gives

$$\left(*_{(i,j)\in\gamma} k_{d,\ell_{ij}}\right)(\theta) = -\frac{1}{\pi}\int_0^\infty e^{-\theta s} \operatorname{Im} f_\gamma(-s+i0)\, ds.$$

Combining this identity with the transformed phase proves (18). Each convolution is strictly positive for positive time.

29 Travel lengths and the late-time coefficient

For a cycle γ, list its edge lengths as $\ell_1, \dots, \ell_m > 0$, put $P_\gamma = \sum_j \ell_j$, and write

$$\Phi_\gamma(\theta) = \theta(*_{j=1}^m k_{d,\ell_j})(\theta), \qquad c_d = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}, \qquad \mu_j = \frac{2c_d}{\ell_j^{d+1}}.$$

Proposition 29.1. *As $\theta \downarrow 0$,*

$$\Phi_\gamma(\theta) \sim C_d^m (4\pi)^{(m-1)/2} \frac{P_\gamma^{(md+2)/2}}{\prod_j \ell_j^{(d+2)/2}}\, \theta^{-(md+1)/2} e^{-P_\gamma^2/(4\theta)}. \tag{26}$$

In the sum (18) *the leading terms come from all cycles of minimum P_γ. As $\theta \to \infty$,*

$$\Phi_\gamma(\theta) \sim C_d\, \theta^{-(d+1)/2} \sum_{j=1}^m \prod_{q\ne j} \mu_q. \tag{27}$$

Proof. For small θ, set the convolution times equal to θv_j, with $\sum_j v_j = 1$, $v_j > 0$. Put $a = (d+3)/2$. The prefactor is $C_d^m \theta^{m-ma}$, and the simplex integral has amplitude $\prod_j v_j^{-a}$ and phase

$\frac{1}{4}\sum_j \ell_j^2/v_j$. Cauchy–Schwarz gives the unique minimum $P_\gamma^2/4$ at $v_j = \ell_j/P_\gamma$. In the first $m-1$ coordinates, the Hessian there is

$$\operatorname{diag}_{j<m}\left(\frac{P_\gamma^3}{2\ell_j}\right) + \frac{P_\gamma^3}{2\ell_m}\mathbf{1}\mathbf{1}^{\mathsf{T}}.$$

The determinant is $(P_\gamma^3/2)^{m-1}P_\gamma/\prod_j \ell_j$ by the rank-one determinant lemma. The amplitude at the minimum is $P_\gamma^{ma}/\prod_j \ell_j^a$. The stronger exponential at every simplex face controls the singular amplitude there: away from the minimizer retain a fixed part of the phase in the exponential to get an integrable envelope with an exponentially smaller bound. Laplace's method then gives (26), including the factor $(2\pi\theta)^{(m-1)/2}$ in the Gaussian integral. There are finitely many cycles, and every shortest one contributes its prefactor.

For large θ, k_{d,ℓ_j} has mass μ_j by $v = \ell_j^2/(4t)$ and tail $C_d t^{-a}$. For two integrable kernels with power tails of exponent $-a < -1$, split their convolution at a fixed cutoff and its reflection about $\theta/2$. On the two endpoint regions one kernel contributes its mass as the cutoff increases and the other contributes its tail constant. In the middle, for $s \le \theta/2$ the factor at $\theta - s$ is bounded by a constant times θ^{-a}, and integration of the other factor gives a tail mass tending to zero as the cutoff increases. On $s \ge \theta/2$ reverse the two roles. Each convolution has mass equal to the mass product, a tail of the same power, and a bound by a constant times that power at large times. Induction therefore gives the sum in (27): one edge time supplies the long tail, and the remaining edges contribute their masses. □

30 Zero-energy paths and two-window constants

Define the connected torsion by

$$\mathcal{T}_L = \sum_{S\subseteq[m]} (-1)^{m-|S|} T_{K_{L,S},\rho_{L,S}}, \qquad T_{\varnothing,0} = 0.$$

Let $\mathcal{P}_m$ denote the directed Hamiltonian paths (ordered lists of all m distinct labels). Put $\mu_{ij} = 2c_d/|p_i - p_j|^{d+1}$.

Proposition 30.1. *The connected torsion satisfies*

$$L^{(m-1)(d+1)}\mathcal{T}_L \longrightarrow \left(\prod_i T_i\right) \sum_{\sigma\in\mathcal{P}_m} \prod_{(i,j)\in\sigma} \mu_{ij}. \tag{28}$$

The expression on the right, multiplied by C_d, is the coefficient of $\theta^{-(d+1)/2}$ in the large-θ law for Φ.

Proof. At zero energy (20) gives $n_0(x) = -2c_d|x|^{-(d+1)}$. Factor the zero-energy form of each subset by the diagonal of its isolated real forms. Its inverse is given by a Neumann series for large L, since the normalized off-diagonal blocks have norm $O(L^{-(d+1)})$. The torsion pairs that inverse with the constant vector in each component, producing open block walks. The alternating signs of the inverse series cancel those of the negative edge kernels. Inclusion–exclusion retains the walks visiting all labels, whose least possible degree is $m-1$. At that degree they are exactly the paths in $\mathcal{P}_m$. The rank-one limit of each scaled edge, with the minus sign canceled, has scalar μ_{ij}. The endpoint pairings and interior contractions give one T_i per label. Higher degrees vanish at the scale in (28) by the geometric norm bound. Closing a path gives a directed

Hamiltonian cycle with its closing edge distinguished, and deleting a distinguished edge reverses this operation. The product along the path omits just the distinguished mass from the cycle. This matches the sum in (27), including $m = 2$. □

For two windows with $p_2 - p_1 = e$ a unit vector, (18) reduces to

$$L^{2(d+1)}\Big[D_{K_{L,\{1,2\}},\rho_{L,\{1,2\}}}(L^2\theta) - D_{K_1,\rho_1}(L^2\theta) - D_{K_2,\rho_2}(L^2\theta)\Big] \longrightarrow T_1T_2\,\theta(k_{d,1} * k_{d,1})(\theta).$$

The two ends of the unweighted profile are

$$\begin{aligned}
&\theta(k_{d,1} * k_{d,1})(\theta) \sim 2^{\,2-d}\pi^{-d-1/2}\theta^{-d-1/2}e^{-1/\theta} && (\theta \downarrow 0),\\
&\theta(k_{d,1} * k_{d,1})(\theta) \sim 4c_dC_d\,\theta^{-(d+1)/2} && (\theta \to \infty).
\end{aligned}$$

The two oriented paths each give $2c_dT_1T_2$ at zero energy. In this pair case, the difference between the off-diagonal kernel and its constant rank-one approximation is $O(L^{-(d+2)})$ in operator norm, by smoothness of $|x|^{-(d+1)}$ near e and boundedness of the windows. Higher inverse products are $O(L^{-2(d+1)})$. This gives the more precise pair formula

$$T_{K_{L,\{1,2\}},\rho_{L,\{1,2\}}} = T_1 + T_2 + 4c_dT_1T_2L^{-(d+1)} + O(L^{-(d+2)}).$$

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