Small-ball estimates for Hölder centers of the Kloosterman process

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Luminary Oddities

Abstract

We prove uniform-norm small-ball estimates with negative logarithm of order ϵ^{-2} around every admissible half-Hölder center of the Kloosterman limit process, including boundary coefficients and endpoints. We also count entire prime Kloosterman paths within radius $D_L(\log p)^{-1/4}$ of every center in a fixed admissible class: with both nonzero parameters averaged, their proportion is at least $\frac{1}{2}\exp[-c_L\sqrt{\log p}]$ for all sufficiently large primes. A direct polynomial sandwich transfers these rare events without an additive Lipschitz error. Higher moments control the path truncation, while conditional binomial noise gives exponentially accurate indicator approximation with an explicit character budget. Separately, the full-path Wasserstein convergence rate is $O(\sqrt{\log\log p/\log p})$ in the uniform metric. For admissible γ-Hölder classes, the worst-case small-ball power is $\max(2, 1/\gamma)$. Extensions to bounded symmetric unimodal coefficient laws determine noncritical roughness–density power thresholds. For a critically flat boundary density, a fixed center has negative logarithm between $\epsilon^{-2}\log^2(1/\epsilon)$ and $\epsilon^{-2}\log^3(\mathrm{e}/\epsilon)$; the exact logarithmic order remains open. A positive-mixture obstruction explains why optimizing the present Fejér error payment alone cannot remove that gap.

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1 Statement and context

The two main results concern approximation by the Kloosterman limit process and quantitative convergence of arithmetic paths to that process. First, we determine the sharp small-ball scale around all admissible 1/2-Hölder centers, including the full coefficient boundary and endpoint range. Second, we give a Wasserstein convergence rate for entire prime Kloosterman paths with both nonzero parameters averaged, using the uniform path metric. A direct rare-event transfer further gives uniform shrinking-target counts at radius $D_L(\log p)^{-1/4}$, with proportions bounded below by $\frac{1}{2}\exp[-c_L\sqrt{\log p}]$. This improves the radius $O_L((\log\log p)^{-1/2})$ obtainable here from Wasserstein convergence and a Lipschitz bump. The count proportions differ because the new balls are smaller; no relative equidistribution assertion is made.

Small-ball probabilities around a prescribed path involve two competing costs: confining the centered random Fourier tail and forcing coefficients toward the boundary of their allowed support. Graded and fractional Fejér approximations balance these costs. The later Hölder-class and coefficient-law results explain how roughness and boundary density affect this balance; they extend the probabilistic analysis to other bounded coefficient laws.

Let $(X_h)_{h\in\mathbb{Z}}$ be independent real random variables of density

$$f(x) = \frac{\sqrt{4-x^2}}{2\pi}\,\mathbf{1}_{[-2,2]}(x). \tag{1.1}$$

Set $v_0(t) = t$, and, for $h \neq 0$, set

$$v_h(t) = \frac{\mathrm{e}^{2\pi i h t} - 1}{2\pi i h}, \qquad 0 \le t \le 1.$$

The Kloosterman process is

$$K(t) = \sum_{h \in \mathbb{Z}} X_h v_h(t), \tag{1.2}$$

where the nonzero frequencies are summed symmetrically. Almost sure uniform convergence is known from the work of Kowalski–Sawin [1]; a proof in the form needed here is included in Section 2. We use $\|w\|_\infty = \sup_{0 \le t \le 1} |w(t)|$ for complex-valued paths, and $\widehat{u}(h) = \int_0^1 u(t)\mathrm{e}^{-2\pi i h t}\, dt$ for Fourier coefficients.

For $L \ge 1$, let $\mathcal{H}_L$ be the class of continuous $g : [0,1] \to \mathbb{C}$ with the following properties:

1. $g(0) = 0$ and $a = g(1)$ belongs to the real interval $[-2, 2]$;
2. $|g(t) - g(s)| \le L\sqrt{|t - s|}$ for all $s, t \in [0,1]$;
3. with $u(t) = g(t) - at$, the numbers

$$\beta_h = 2\pi i h \widehat{u}(h), \qquad h \in \mathbb{Z} \setminus \{0\}, \tag{1.3}$$

   are real and belong to $[-2, 2]$.

The endpoint and Fourier conditions are the established support characterization of Kowalski–Sawin [2, Theorem 1.1]; we add a uniform Hölder bound. The proof below uses the displayed conditions directly and does not assume the support theorem.

Theorem 1.1. *For every $L \ge 1$, every $g \in \mathcal{H}_L$, and $0 < \epsilon \le 1/4$,*

$$\mathbb{P}(\|K - g\|_\infty < \epsilon) \ge \exp\{-26000(L+2)^2 \epsilon^{-2}\}. \tag{1.4}$$

Let

$$p = 1 - \pi^{-1}, \qquad \delta = \frac{p}{128\pi^2}, \qquad \epsilon_0 = \sqrt{\delta/2}, \qquad c = \frac{p\delta}{16}. \tag{1.5}$$

Then, for every deterministic $g \in C([0,1], \mathbb{C})$,

$$\mathbb{P}(\|K - g\|_\infty < \epsilon) \le \exp\{-c\epsilon^{-2}\}, \qquad 0 < \epsilon \le \epsilon_0. \tag{1.6}$$

Thus, for fixed L, the negative logarithm of the probability has order ϵ^{-2}, uniformly on $\mathcal{H}_L$. The constants are deliberately inefficient; the theorem determines the order, not an asymptotic leading constant. Both $a = \pm 2$ and individual $\beta_h = \pm 2$ are allowed. The upper bound needs none of the endpoint, Fourier or Hölder restrictions imposed on the lower bound.

A further main result determines the worst-case scale for every Hölder exponent. Let $\mathcal{H}_{\gamma,L}$ retain the same endpoint and Fourier restrictions, with Hölder exponent γ in place of $1/2$. Theorem 7.1 proves that

$$-\log \inf_{g \in \mathcal{H}_{\gamma,L}} \mathbb{P}(\|K - g\|_\infty < \epsilon) \asymp_{\gamma,L} \epsilon^{-\max(2,1/\gamma)}.$$

For $\gamma < 1/2$, a Rudin–Shapiro signed train of narrow triangular spikes supplies the matching upper estimate. This is a worst-case family whose center can vary with ϵ, rather than a sharp law asserted at every fixed rough center. The signs and their use in Kloosterman support are established tools, including in [3]; the claim here is their constrained approximation-probability consequence.

The arithmetic main result, Theorem 6.1, bounds Wasserstein distance between the entire two-parameter prime path law and its limit by $O(\sqrt{\log\log p/\log p})$, for the uniform path metric. Theorem 6.5 instead transfers the rare small-ball events directly, with exponentially accurate polynomial sandwiches. Corollary 6.7 records the weaker-radius consequence of the Wasserstein estimate for comparison. The arithmetic input is the classical normalized Kloosterman sheaf and trace/Euler/weight estimates, with all representation-degree dependence retained. Growing moments control dyadic Fourier blocks. An unbiased tensor Bernstein approximation allocates degrees according to the squared Fourier weights; conditional concentration pays the path-functional error while a bounded-polynomial character expansion controls the arithmetic comparison. Quantitative path work by S. Pitte is noted in the current author manuscript of Kowalski–Untrau [4]. Recent scalar endpoint Wasserstein estimates of Rahimi [5] concern the scalar law; the present estimates concern the joint full path in the uniform metric.

The closest quantitative predecessor is Kyselov [3, Theorem 1.2], whose version 1 gives

$$\mathbb{P}(\|K-g\|_\infty<\epsilon)\geq\exp\{-C_L\epsilon^{-2}\log^2(2/\epsilon)\}$$

for closed centers $g(0)=g(1)=0$ with $\beta_h\in[-1,1]$ and the same Hölder exponent. The lower estimate (1.4) removes both logarithmic factors, extends the coefficient interval to its full permitted range, and allows every permitted endpoint. Estimate (1.6) shows that the power ϵ^{-2} cannot be improved uniformly, or even at an individual admissible center, to a smaller logarithmic order.

Our contribution is specific to this bounded-coefficient Fourier process and this path norm. Shifted small-deviation estimates for non-Gaussian series form an established subject; for example, the p-exponential framework in [6] uses unbounded coefficient laws. The classical comparison ingredients below are instances of Prékopa marginal log-concavity [7], product peakedness associated with Kanter [8, 9], and Gaussian translation comparison [10]. The point at which coefficient boundaries matter is Section 4: the Fejér weights supply different interior margins at different frequencies, and their product retains a factorial.

The norm and coefficient law distinguish these estimates from other small-deviation results. Uniform-series priors in [11, Section 4.2] use a coordinate-supremum sequence topology rather than the uniform norm of this Fourier image. The non-Gaussian Fourier estimates in [12, Section 2.6.4] concern L^2 small deviations; an L^2 lower bound does not by itself give a uniform-norm lower bound. Section 8 isolates a sufficient boundary condition for other bounded symmetric unimodal coefficient laws. This direct extension clarifies the scope of the matching argument for the stated bounded symmetric unimodal coefficient laws. For the family $f(x)\propto\exp[-A(2-|x|)^{-a}]$, Sections 9 and 10 determine the lower-order property for $0<\gamma\leq1/2$: it holds precisely when $a<1/\gamma$. The failure at equality uses a single fixed admissible boundary center. Theorem 9.2 also supplies a uniform critical lower estimate at $\gamma=1/2, a=2$, leaving one logarithm between the two proved bounds at its fixed witness. Section 11 completes the noncritical power orders for $0<\gamma<1$: the worst-case negative logarithm has order $\epsilon^{-\max(2,1/\gamma,a)}$, excluding only $a=1/\gamma$ when $\gamma\leq1/2$. In particular the smoother classes $1/2<\gamma<1$ have order $\epsilon^{-\max(2,a)}$ for every $a>0$; $a=2$ causes no extra logarithm there. These are extensions within the same result family. Supremum-norm small-ball estimates for independent symmetric unimodal wavelet series were established by Reiss and Schmidt-Hieber [13, Lemma 4.1]; their localized-wavelet proof does not directly give the full-boundary harmonic Fourier result considered here, and a change of basis need not preserve coefficient independence. Regularity-dependent shifted concentration for Gaussian processes is also established through approximation in their reproducing-kernel Hilbert spaces [14, 15]. Those precedents motivate comparison, but do not give the bounded Sato–Tate harmonic-class worst-case theorem above.

2 Convergence and a centered comparison

Lemma 2.1. *If the independent coefficients W_h are either all distributed as (1.1) or all distributed as $N(0,4)$, the symmetric partial sums $\sum_{|h|\le H} W_h v_h$ converge almost surely uniformly on $[0,1]$, and in expected uniform norm. The limit is continuous. A finite set of coefficients is independent of the continuous series formed from its complement.*

Proof. For integers $n \ge 1$ and $n \le H \le 2n$, put

$$P_{n,H}(t) = \sum_{n<|h|\le H} \frac{W_h \mathrm{e}^{2\pi i h t}}{2\pi i h}.$$

A centered variable in $[-2,2]$ satisfies $\mathbb{E}\mathrm{e}^{\lambda X} \le \cosh(2\lambda) \le \mathrm{e}^{2\lambda^2}$: convexity bounds $\mathrm{e}^{\lambda x}$ by its endpoint chord, whose expectation is $\cosh(2\lambda)$, and $\log\cosh y \le y^2/2$ follows by integrating $|\tanh y| \le |y|$. This is the elementary bounded-variable exponential estimate. The last bound is an equality for $N(0,4)$. Independence and $\sum_{n<|h|\le 2n} h^{-2} \le C/n$ therefore show that both real and imaginary parts of $P_{n,H}(t)$ have subgaussian tails of variance parameter at most C/n. Exponential Markov inequality, followed by a union over the two parts, gives

$$\mathbb{P}(|P_{n,H}(t)| > v) \le 4\mathrm{e}^{-c_1 n v^2} \tag{2.1}$$

with absolute constants, uniformly in t, H.

For completeness, a deterministic grid may be chosen with only polynomially many points. If P is a trigonometric polynomial of degree d, each Fourier coefficient is at most $\|P\|_\infty$ in absolute value, whence

$$\|P'\|_\infty \le 2\pi \sum_{|k|\le d} |k| \, \|P\|_\infty = 2\pi d(d+1) \, \|P\|_\infty .$$

A grid of spacing at most $[4\pi d(d+1)]^{-1}$ gives $\|P\|_\infty \le 2 \max_{t \text{ on grid}} |P(t)|$, by applying the derivative bound between a maximizer and a grid point. For $d = 2n$ this uses $O(n^2)$ points. Combining (2.1) for these points and all H gives

$$\mathbb{P}\left(\max_{n\le H\le 2n} \|P_{n,H}\|_\infty > 2v \right) \le \min(1, C_1 n^3 \mathrm{e}^{-c_1 n v^2}).$$

Integrating this inequality in v yields

$$\mathbb{E} \max_{n\le H\le 2n} \|P_{n,H}\|_\infty \le C_2 \sqrt{\frac{\log(2n)}{n}}.$$

Subtracting $P_{n,H}(0)$ changes the bound by at most a factor two and produces the blocks of $\sum W_h v_h$. On $n = 1, 2, 4, \ldots$, the expected maximal block norms have a finite sum. Tonelli's theorem gives almost sure summability of these maxima, hence convergence of complete and incomplete symmetric blocks. The same estimates give convergence in expected norm. The finite low-frequency terms cause no difficulty. Measurability as a uniform limit and independence of disjoint coefficient families give the last assertion. □

Write $Z \succeq Y$ for symmetric scalar variables when $\mathbb{P}(|Z| \le r) \ge \mathbb{P}(|Y| \le r)$ for all $r \ge 0$. This is the usual central peakedness order in one dimension.

Lemma 2.2 (Classical peakedness comparison in the required form)**.** *Let $Z_1, \dots, Z_m$ be independent symmetric variables with log-concave densities, and let Y_j be independent $N(0,4)$ variables. If $Z_j \succeq Y_j$ for each j, then for every real linear map $T : \mathbb{R}^m \to \mathbb{R}^d$ and convex symmetric Borel set $A \subseteq \mathbb{R}^d$,*

$$\mathbb{P}(TZ \in A) \geq \mathbb{P}(TY \in A).$$

The conclusion applies to the inverse image of a closed uniform-norm ball under a finite continuous-function map.

Proof. Replace one coordinate at a time. For the coordinate being replaced, the conditional acceptance function is

$$F(a) = \int \mathbf{1}_A \left(aT(e_j) + \sum_{k \neq j} z_k T(e_k) \right) \prod_{k \neq j} f_k(z_k)\, dz.$$

The integrand is jointly log-concave: its indicator is that of a convex inverse image, with the extended convention for zero values. Prékopa's marginal theorem [7, Theorem 6] makes F log-concave. Reflection of all coordinates makes F even. An even log-concave function is nonincreasing in absolute value, since every $a \in [-b, b]$ is a convex combination of b and $-b$. Its strict superlevel sets are central intervals. Scalar peakedness and layer cake therefore give $\mathbb{E}F(Z_j) \geq \mathbb{E}F(Y_j)$. Each intermediate product still has symmetric log-concave factors, so iteration proves the result. No full-rank assumption on T is used. For a function map, its closed-ball inverse image in coefficient space is convex and symmetric, so the same proof applies directly. □

If X has density (1.1), its central mass on $[0, 2]$ is concave in the radius and joins 0 to 1. Thus

$$\mathbb{P}(|X| \leq r) \geq r/2 \quad (0 \leq r \leq 2).$$

For $Y \sim N(0, 4)$, the density bound gives $\mathbb{P}(|Y| \leq r) \leq r/\sqrt{2\pi} \leq r/2$. For $r \geq 2$ the first probability is one. Also

$$\frac{d^2}{dx^2} \log f(x) = -\frac{4 + x^2}{(4 - x^2)^2} < 0 \quad (|x| < 2),$$

so X has the symmetry, log-concavity and peakedness required in Lemma 2.2. A uniform variable on $[-r_j, r_j]$, $0 < r_j \leq 1/2$, has the same properties and dominates Y: for $0 \leq s \leq r_j$ its central mass is $s/r_j \geq 2s$, and beyond r_j it is one. All these laws have densities, so open or closed central intervals give the same probabilities.

Lemma 2.3 (The bounded symmetric unimodal case)**.** *Let $Z_1, \dots, Z_m$ be independent variables whose densities are even, bounded, supported on $[-2, 2]$, and nonincreasing on $[0, 2)$. Let Y_j be independent $N(0, 4)$ variables. For every real linear map T and convex symmetric Borel set A,*

$$\mathbb{P}(TZ \in A) \geq \mathbb{P}(TY \in A).$$

The conclusion also applies to the inverse image of a closed uniform-norm ball under a finite continuous-function map.

Proof. This is a special case of classical unimodal product peakedness [9, Theorem 3.6]; the following mixture argument records the hypotheses needed here. Choose the right-continuous monotone representative of one density f on $[0, 2)$ and set $f(2) = 0$. Let μ be its positive Stieltjes measure on $(0, 2]$, so that $\mu((x, 2]) = f(x)$ for $0 \leq x < 2$. Put $\nu(ds) = 2s\mu(ds)$. Tonelli's theorem gives

$$\nu((0, 2]) = 2 \int_0^2 \mu((x, 2])\, dx = 2 \int_0^2 f(x)\, dx = 1.$$

Furthermore,

$$f(x) = \int_{(0,2]} \frac{\mathbf{1}_{\{|x|<s\}}}{2s}\, \nu(ds) \quad \text{for almost every } x.$$

Thus each coordinate is a scale mixture of centered uniforms of half-width at most two. Choose its mixing scale independently of the other coordinates. Conditional on all scales, the variables are independent, symmetric and log-concave. A uniform variable on $[-s, s]$, $0 < s \le 2$, is more peaked than $N(0, 4)$: its central mass is at least $r/2$ for $0 \le r \le 2$, with value one once $r \ge s$, whereas the Gaussian central mass is at most $r/\sqrt{2\pi}$. Lemma 2.2 therefore gives the claimed inequality conditionally on every scale vector. Integrating over that vector proves the result. The finite path-ball assertion follows from its convex symmetric inverse image exactly as before. □

3 A Gaussian reference and uniform tail control

Let Y_h be independent $N(0, 4)$ variables and put $G = \sum_h Y_h v_h$, using Lemma 2.1.

Lemma 3.1. *For $0 < r \le 1$,*

$$\mathbb{P}(\|G\|_\infty \le r) \ge \exp(-64/r^2). \tag{3.1}$$

For every finite $I \subset \mathbb{Z}$, with $G_I = \sum_{h\in I} Y_h v_h$,

$$\mathbb{P}(\|G_I\|_\infty \le r) \ge \mathbb{P}(\|G\|_\infty \le r). \tag{3.2}$$

Consequently, for every $M \ge 0$ and $R_M = \sum_{|h|>M} X_h v_h$,

$$\mathbb{P}(\|R_M\|_\infty \le r) \ge \exp(-256/r^2), \qquad 0 < r \le 1. \tag{3.3}$$

Proof. For $h \ge 1$ put $U_h = (Y_h + Y_{-h})/\sqrt{2}$ and $V_h = (Y_h - Y_{-h})/\sqrt{2}$. These are mutually independent $N(0, 4)$ variables, independent of Y_0. Direct expansion gives

$$\operatorname{Re} G(t) = tY_0 + \sum_{h\ge1} \frac{U_h \sin(2\pi ht)}{\sqrt{2}\pi h},$$
$$\operatorname{Im} G(t) = \sum_{h\ge1} \frac{V_h(1 - \cos(2\pi ht))}{\sqrt{2}\pi h}.$$

Consider the continuous centered Gaussian process

$$B(t) = \sum_{h\ge1} \frac{U_h \sin(2\pi ht) + V_h(\cos(2\pi ht) - 1)}{\sqrt{2}\pi h}.$$

The preceding convergence proof also applies to its partial sums. Its covariance is

$$\frac{2}{\pi^2} \sum_{h\ge1} \frac{\cos(2\pi h(t-s)) - \cos(2\pi hs) - \cos(2\pi ht) + 1}{h^2}.$$

The identity

$$\sum_{h\ge1} \frac{\cos(2\pi hx)}{h^2} = \pi^2(x^2 - x + 1/6), \qquad 0 \le x \le 1, \tag{3.4}$$

follows by computing the Fourier coefficients of the displayed quadratic; the absolutely convergent series represents that continuous periodic function. For $0 \le s \le t \le 1$, substitution gives

$\mathrm{Cov}(B(s), B(t)) = 4s(1-t)$. Thus B is a variance-four Brownian bridge, since a centered Gaussian law is determined by its covariances. Its sine and cosine-minus-one parts are respectively $(B(t) - B(1-t))/2$ and $(B(t) + B(1-t))/2$, each of norm at most $\|B\|_\infty$. Hence

$$\|G\|_\infty \le |Y_0| + 2\,\|B\|_\infty . \tag{3.5}$$

Here is an explicit confinement estimate, with its normalization recorded. Variance-four Brownian motion has generator $2\,d^2/dx^2$ and free density $1/\sqrt{8\pi}$ at $(1; 0, 0)$. The Dirichlet eigenfunction expansion on $(-b, b)$ gives its killed density there as

$$p_D(1; 0, 0) = \frac{1}{b} \sum_{j \ge 0} \exp\left\{ -\frac{(2j+1)^2 \pi^2}{2b^2} \right\}.$$

Indeed the odd indexed eigenfunctions have squared value $1/b$ at zero, and the other ones vanish there. Dividing by the free endpoint density is the probability that the bridge stays in $(-b, b)$. Keeping the first positive term yields

$$\mathbb{P}(\|B\|_\infty < b) \ge \frac{\sqrt{8\pi}}{b} \exp\{-\pi^2/(2b^2)\} \ge \exp\{-\pi^2/(2b^2)\}, \qquad 0 < b \le 1.$$

This is a bound from a positive kernel expansion, not an asymptotic formula. On $[-r/3, r/3]$, the $N(0, 4)$ density is at least $\mathrm{e}^{-1/72}/(2\sqrt{2\pi})$ for $r \le 1$. Independence, with $b = r/3$, and (3.5) give

$$\mathbb{P}(\|G\|_\infty \le r) \ge c_0 r \exp\{-9\pi^2/(2r^2)\}, \qquad c_0 = \frac{\mathrm{e}^{-1/72}}{3\sqrt{2\pi}}.$$

Since $9\pi^2/2 < 45$, $\log(1/c_0) < 4$, and $\log(1/r) \le r^{-2}$ on this range, (3.1) follows.

To prove (3.2), first evaluate paths on a finite time grid and identify complex values with a real vector. Its radius-r acceptance set A is a product of disks, hence convex and symmetric. If z is the finite Gaussian coefficient vector of G_I with evaluation map T, then

$$F(y) = \int \mathbf{1}_A(Tz + y)\phi(z)\,dz$$

is log-concave by Prékopa and even by reflection. Therefore $F(y) \le F(0)$. Conditioning on the independent evaluation vector of $G - G_I$ proves that adding this complement cannot increase acceptance. Increasing the grid to a countable dense set proves the assertion for closed uniform balls, by path continuity and continuity of probability from above. This argument allows singular evaluation maps and an infinite independent complement; it is the required instance of the classical Gaussian translation comparison.

Finally let $S_{M,H} = \sum_{M<|h|\le H} X_h v_h$. Lemma 2.2 and (3.2) show

$$\mathbb{P}(\|S_{M,H}\|_\infty \le r') \ge \mathbb{P}(\|G\|_\infty \le r').$$

For $r' < r$, almost sure uniform convergence implies

$$\limsup_{H\to\infty} \{\|S_{M,H}\|_\infty \le r'\} \subseteq \{\|R_M\|_\infty \le r\}$$

outside a null set. Reverse Fatou for bounded indicators gives the probability of this limsup at least the limsup of the probabilities. Take $r' = r/2$ and apply (3.1) to obtain (3.3). The radius slack avoids any assumption about atoms on a norm-sphere. □

4 Fejér approximation with graded margins

Fix $g \in \mathcal{H}_L$, $a = g(1)$ and $u = g - at$. The function u vanishes at both endpoints and has Hölder constant at most $L + 2$ on $[0, 1]$. As a periodic function, it satisfies

$$|u(t) - u(s)| \le \sqrt{2}(L + 2)\sqrt{d(t - s, \mathbb{Z})}. \tag{4.1}$$

For a wrapped pair this follows by passing through the common zero endpoint and using $\sqrt{x}+\sqrt{y} \le \sqrt{2(x + y)}$; otherwise use the ordinary Hölder estimate.

The degree-M Fejér kernel has integral one, is nonnegative, and obeys

$$F_M(t) \le M + 1, \qquad F_M(t) \le \frac{1}{4(M + 1)d(t, \mathbb{Z})^2}.$$

These bounds follow from $F_M(t) = \sin^2((M + 1)\pi t)/((M + 1)\sin^2(\pi t))$ and $\sin(\pi d) \ge 2d$ for $0 \le d \le 1/2$. Splitting the integral at $1/(M + 1)$ gives

$$\int_0^1 F_M(t)\sqrt{d(t, \mathbb{Z})}\, dt \le \frac{7}{3\sqrt{M + 1}}, \qquad M \ge 1. \tag{4.2}$$

In fact, the inner part is at most $4/(3\sqrt{M + 1})$, and the outer part is at most $1/\sqrt{M + 1}$, by direct integration of the two kernel bounds. If $\sigma_M u = F_M * u$, equations (4.1) and (4.2) imply

$$\|\sigma_M u - \sigma_M u(0) - u\|_\infty \le \frac{14\sqrt{2}(L + 2)}{3\sqrt{M + 1}}.$$

For $0 < \epsilon \le 1/4$ choose

$$M = \left\lceil 4096(L + 2)^2\epsilon^{-2} \right\rceil. \tag{4.3}$$

The last approximation error is at most $\epsilon/8$. Set

$$b_0 = (1 - \epsilon/8)a, \qquad x_h = \frac{|h|}{M + 1}, \qquad b_h = (1 - x_h)\beta_h \quad (0 < |h| \le M). \tag{4.4}$$

The pinned polynomial

$$q_M(t) = b_0 t + \sigma_M u(t) - \sigma_M u(0) = \sum_{|h| \le M} b_h v_h(t)$$

satisfies

$$\|q_M - g\|_\infty \le \epsilon/4 + \epsilon/8 = 3\epsilon/8. \tag{4.5}$$

The equality follows directly from the Fourier formula for a Fejér mean and (1.3).

Lemma 4.1. *For $T_M z = \sum_{|h| \le M} z_h v_h$ and the coefficients in (4.4),*

$$\mathbb{P}(\|T_M(X - b)\|_\infty \le r) \ge c_e \epsilon^{3/2} \exp\{-6M - 64/r^2\}, \qquad 0 < r \le 1, \tag{4.6}$$

where $c_e = \sqrt{3/8}/(8\pi) > \mathrm{e}^{-5}$.

Proof. For $0 < |h| \le M$ put $r_h = \min(1/2, x_h)$ and let Z_h be independent uniform variables on $[-r_h, r_h]$. If $|z| \le r_h$, then

$$|b_h + z| \le 2 - 2x_h + r_h \le 2 - x_h, \qquad f(b_h + z) \ge \frac{\sqrt{3x_h}}{2\pi}.$$

As $2r_h \ge x_h$, comparison of this shifted density with the uniform density pays a factor at least $c_* x_h^{3/2}$, where $c_* = \sqrt{3}/(2\pi)$. Multiplying over both signs of each frequency gives

$$c_*^{2M} \left(\frac{M!}{(M+1)^M} \right)^3 \ge \mathrm{e}^{-6M}. \tag{4.7}$$

Indeed,

$$\begin{aligned} \log \frac{M!}{(M+1)^M} &\ge \int_1^M \log t \, dt - M \log(M+1) \\ &= -M + 1 - M \log(1 + 1/M) \ge -M, \end{aligned}$$

and $3 + 2\log(2\pi/\sqrt{3}) < 6$. Retaining the differing margins x_h in this product is essential: replacing all of them by their smallest value would introduce an unnecessary $M \log M$ loss.

For $h = 0$, take $r_0 = \epsilon/8$ and Z_0 uniform on $[-r_0, r_0]$. The inequalities

$$|b_0 + z| \le 2 - \epsilon/8, \qquad f(b_0 + z) \ge \frac{\sqrt{3\epsilon/8}}{2\pi}$$

hold on this interval, including when $a = \pm 2$. Its density factor is at least $c_e \epsilon^{3/2}$. Integrating the finite product density only on the box $\prod_{|h| \le M} [-r_h, r_h]$ gives

$$\mathbb{P}(\|T_M(X - b)\|_\infty \le r) \ge c_e \epsilon^{3/2} \mathrm{e}^{-6M} \mathbb{P}(\|T_M Z\|_\infty \le r).$$

All uniform widths are positive and at most $1/2$. Lemma 2.2, followed by (3.2) and (3.1), bounds the last probability below by e^{-64/r^2}. This proves (4.6). The shifts were handled by a finite density comparison before applying the centered comparison; no translated peakedness or infinite product of shifted densities was used. □

5 Proof of the two-sided bounds

Proof of the lower bound in Theorem 1.1. The low-frequency event in Lemma 4.1 with $r = \epsilon/4$ has probability at least

$$c_e \epsilon^{3/2} \exp\{-6M - 1024\epsilon^{-2}\}.$$

The independent high-frequency event $\|R_M\|_\infty \le \epsilon/4$ has probability at least $\exp(-4096\epsilon^{-2})$ by (3.3). Together with (4.5), these events imply

$$\|K - g\|_\infty \le \epsilon/4 + \epsilon/4 + 3\epsilon/8 = 7\epsilon/8 < \epsilon.$$

Their product gives the strict-ball lower bound $c_e \epsilon^{3/2} \exp\{-6M - 5120\epsilon^{-2}\}$. For the allowed parameters,

$$M \le 4097(L+2)^2 \epsilon^{-2}, \qquad \log \frac{1}{c_e \epsilon^{3/2}} \le 7\epsilon^{-2}.$$

Therefore its negative logarithm is at most

$$[24582(L+2)^2 + 5127]\epsilon^{-2} \le 26000(L+2)^2 \epsilon^{-2},$$

which proves (1.4). □

Proof of the upper bound in Theorem 1.1. For an arbitrary $g \in C([0,1],\mathbb{C})$, set $a = g(1)$, $u = g - at$ and $\beta_h = 2\pi i h \widehat{u}(h)$, which may now be complex. On the strict small-ball event, $K(1) = X_0$ gives $|X_0 - a| < \epsilon$. Consequently

$$D(t) = K(t) - tX_0 - u(t) = (K(t) - g(t)) + t(a - X_0)$$

has norm less than 2ϵ. Uniform convergence justifies its nonzero Fourier coefficients $\widehat{D}(h) = (X_h - \beta_h)/(2\pi i h)$. Parseval, retaining only positive frequencies through N, gives

$$\sum_{h=1}^{N} |X_h - \beta_h|^2 \le N^2 \sum_{h=1}^{N} \frac{|X_h - \beta_h|^2}{h^2} \le 16\pi^2 N^2 \epsilon^2. \tag{5.1}$$

Since $f \le 1/\pi$, an interval of length one has probability at most $1/\pi$. The independent indicators $I_h = \mathbf{1}_{\{|X_h - \operatorname{Re}\beta_h| \ge 1/2\}}$ have success probabilities at least $p = 1 - 1/\pi$. On the small-ball event,

$$\sum_{h=1}^{N} I_h \le 4 \sum_{h=1}^{N} |X_h - \operatorname{Re}\beta_h|^2 \le 4 \sum_{h=1}^{N} |X_h - \beta_h|^2 \le 64\pi^2 N^2 \epsilon^2.$$

With $N = \lfloor \delta \epsilon^{-2} \rfloor$ and $\delta = p/(128\pi^2)$, the right-hand side is at most $pN/2$. For clarity, exponential Markov inequality at $\lambda = \log 2$ yields

$$\begin{aligned} \mathbb{P}\left(\sum I_h \le pN/2\right) &\le \exp\{\lambda pN/2 + pN(\mathrm{e}^{-\lambda} - 1)\} \\ &= \exp\{-pN(1 - \log 2)/2\} \le \exp(-pN/8). \end{aligned}$$

Here each success probability is at least p, so $\mathbb{E}\mathrm{e}^{-\lambda I_h} \le \exp(p(\mathrm{e}^{-\lambda} - 1))$. If $\epsilon^2 \le \delta/2$, then $N \ge \delta/(2\epsilon^2)$, giving (1.6) with the constants (1.5). □

6 Effective arithmetic and shrinking targets

For an odd prime p and $(A, B) \in (\mathbb{F}_p^\times)^2$, let $K_p(\cdot; A, B)$ be the polygonal path with vertices

$$K_p\left(\frac{j}{p-1}; A, B\right) = p^{-1/2} \sum_{1 \le x \le j} \exp\left(\frac{2\pi i (Ax + B\overline{x})}{p}\right), \qquad 0 \le j \le p-1. \tag{6.1}$$

Here $\overline{x}$ is the inverse of x modulo p, and both parameters are chosen independently and uniformly nonzero. Write μ_p for this path law and $\mu = \mathrm{Law}(K)$. On probability measures with finite first uniform moment, W_1 is the Wasserstein distance for the cost $\|x - y\|_\infty$, equivalently the supremum of differences of integrals of real 1-Lipschitz functions.

The qualitative convergence $\mu_p \Rightarrow \mu$ is known [1, Theorem 1.5]. We quantify it for this two-parameter ensemble. The sheaf-theoretic machinery below is classical: in particular the rank, puncture and Swan trace estimate is the curve estimate explained in [16, Remark 4.2]. The proof records its dependence on growing representation degree. Growing-moment bounds on dyadic Fourier blocks then sharpen the path approximation, before an unbiased tensor Bernstein comparison of path functionals. The current author version of Kowalski–Untrau [4, Section 1.3(4)] notes related quantitative path work by S. Pitte.

For $M \ge 1$ put

$$a_0 = 1, \qquad a_h = \frac{1}{\pi |h|} \ (0 < |h| \le M), \qquad S_M = \sum_{|h| \le M} a_h = 1 + \frac{2}{\pi} \sum_{h=1}^{M} \frac{1}{h}.$$

For integers $2 \le M \le N < p/2$, put $d = 2M+1$, $r = \lceil \log(16M) \rceil$, and allocate degrees

$$n_0 = M, \qquad n_h = \left\lceil \frac{M^2}{h^2} \right\rceil \quad (0 < |h| \le M). \tag{6.2}$$

Define

$$D_B(p,M,N) = 1152\sqrt{r/M} + 2048N^{-1/4} + 384r(N^2/p)^{1/(4r)} + p^{-1/2} + \frac{8M}{p} + \frac{8S_M(d+1)}{\sqrt{p}} \prod_{|h|\le M} (n_h+2)^2. \tag{6.3}$$

The last product is at most $\exp(16M)$.

Theorem 6.1. *For every odd prime p and integers $2 \le M \le N < p/2$,*

$$W_1(\mu_p, \mu) \le D_B(p,M,N).$$

In particular,

$$W_1(\mu_p, \mu) = O\left(\sqrt{\frac{\log\log p}{\log p}}\right). \tag{6.4}$$

A coarse explicit version is $W_1(\mu_p,\mu) \le 20000\sqrt{\log\log p/\log p}$ for $\log p \ge 10^6$.

Corollary 6.2 (Support-size obstruction)**.** *For any path law ν supported on at most $q \ge 1$ points,*

$$W_1(\mu,\nu) \ge \tfrac{1}{2}\min\left\{\epsilon_0, \sqrt{c/\log(2q)}\right\},$$

where c, ϵ_0 are the constants in (1.5). In particular for every odd prime p,

$$W_1(\mu_p,\mu) \ge \tfrac{1}{2}\min\left\{\epsilon_0, \sqrt{c/\log(2(p-1)^2)}\right\} \ge \frac{\sqrt{c}}{2\sqrt{3\log p}}.$$

Proof. Let S contain the at most q support points and put $r = \min\{\epsilon_0, \sqrt{c/\log(2q)}\}$. The universal-center upper bound of Theorem 1.1 and a union bound show that the union of their open radius-r balls has μ-mass at most $qe^{-c/r^2} \le 1/2$. Outside it the distance to S is at least r. Every coupling with ν therefore has mean cost at least $r/2$. The arithmetic law has at most $(p-1)^2$ atoms, regardless of collisions. Finally $\log(2(p-1)^2) \le 3\log p$ and $c/\epsilon_0^2 = (1-\pi^{-1})/8 < 1$ imply the last inequality for $p \ge 3$. □

Thus the upper rate in Theorem 6.1 is within a factor $\sqrt{\log\log p}$ of the support-size lower order. This does not show that the extra factor is necessary, or identify an optimal constant.

6.1 Path approximation with a uniform expected modulus

For $I \subseteq \mathbb{F}_p^\times$, $H = |I|$, set

$$T_I = p^{-1/2} \sum_{x \in I} \exp(2\pi i(Ax + B/x)/p).$$

Orthogonality over all $A, B \in \mathbb{F}_p$ gives

$$\sum_{A,B \in \mathbb{F}_p} |T_I|^4 = N_I,$$

where N_I counts quadruples with equal sums and equal inverse sums. If the common sum is nonzero, the inverse-sum condition forces equal products; each first ordered pair has at most two second pairs. If it is zero, both pairs are opposite and there are at most H^2 quadruples. Thus $N_I \le 3H^2$. Restricting this nonnegative sum to nonzero parameters yields

$$\mathbb{E}_{A,B\neq 0}|T_I|^4 \le \frac{3H^2}{(p-1)^2}. \tag{6.5}$$

This use of both parameters is essential to the present argument.

Define Q_p on the mesh j/p using the same partial sums through $j = p-1$, and make its last mesh segment constant. Then $K_p(t) = Q_p((p-1)t/p)$ and

$$\|K_p - Q_p\|_\infty \le p^{-1/2}, \tag{6.6}$$

because Q_p is deterministically $\sqrt{p}$-Lipschitz. For $\delta = |t-s| < p^{-1}$ this bound gives $|Q_p(t) - Q_p(s)|^4 \le \delta^2$. For $\delta \ge p^{-1}$, decompose an increment into its complete interior mesh intervals and at most two fractional endpoint increments. The interior set has at most $p\delta$ elements, and each endpoint increment is at most $p^{-1/2}$. Using (6.5), $p/(p-1) \le 3/2$ and $|z_1+z_2+z_3|^4 \le 27\sum |z_j|^4$ gives

$$\mathbb{E}|Q_p(t) - Q_p(s)|^4 \le 1024\delta^2. \tag{6.7}$$

The same estimate holds for K. Indeed

$$\sum_{h\in\mathbb{Z}} |v_h(t) - v_h(s)|^2 = \delta$$

by (3.4). The bounded-variable exponential estimate in Lemma 2.1, applied to the real and imaginary parts and then integrated, gives the displayed fourth-moment bound. Finite truncations converge in L^4 at fixed times, so the estimate passes to the series.

Here is the expected-modulus consequence with constants. For a continuous X satisfying (6.7) and $X(0) = 0$, let A_j be the largest adjacent dyadic increment of level j. The fourth moment and a union bound give $\mathbb{E}A_j \le 6\,2^{-j/4}$. If $\delta \le 2^{-m} < 2\delta$, dyadic chaining and continuity give

$$\omega_X(\delta) \le A_m + 2\sum_{j>m} A_j, \qquad \mathbb{E}\omega_X(\delta) \le 100\delta^{1/4}.$$

Also $\mathbb{E}|X(1)| < 6$. Pin $u_X(t) = X(t) - tX(1)$ and extend it periodically. For a wrapped pair, split at the common zero endpoint. The expected circular modulus is at most $208\delta^{1/4}$.

For the ordinary Fejér kernel, direct integration of its two bounds in Section 4 gives

$$\int_{-1/2}^{1/2} |t|^{1/4} F_M(t)\,dt \le 3(M+1)^{-1/4}.$$

With $\mathcal{P}_M X = tX(1) + F_M * u_X - (F_M * u_X)(0)$, it follows that

$$\mathbb{E}\,\|X - \mathcal{P}_M X\|_\infty \le 2048M^{-1/4}, \qquad X = Q_p,\ K. \tag{6.8}$$

The dyadic sums also prove finite first uniform moments. These are expected-modulus estimates.

6.2 Arithmetic coefficients and character correlations

Put

$$W_h = p^{-1/2} \sum_{x \in \mathbb{F}_p^\times} \exp(2\pi i((A-h)x + B/x)/p) = \mathrm{Kl}_2(B(A-h); p).$$

These variables are real and $|W_h| \le 2$, including the singular value $W_h = -p^{-1/2}$ when $A = h$. Integrating each mesh derivative against $\mathrm{e}^{-2\pi i h t}$ shows that the normalized pinned coefficient of Q_p is $c_{p,h} W_h$, where

$$c_{p,h} = \mathrm{e}^{\pi i h/p} \frac{\sin(\pi h/p)}{\pi h/p}, \qquad |c_{p,h} - 1| \le 2\pi |h|/p \quad (0 < |h| < p/2).$$

The endpoint is W_0. On $[-2,2]^{2M+1}$ define

$$\mathcal{A}_M(x) = x_0 t + \sum_{0 < |h| \le M} \left(1 - \frac{|h|}{M+1}\right) x_h v_h(t).$$

Then

$$\|\mathcal{P}_M Q_p - \mathcal{A}_M(W)\|_\infty \le 8M/p, \qquad \mathcal{P}_M K = \mathcal{A}_M((X_h)_{|h| \le M}). \tag{6.9}$$

For the first inequality each nonzero frequency contributes at most $2|c_{p,h} - 1|/(\pi|h|) \le 4/p$.

Lemma 6.3. *Define $\chi_k(2\cos\theta) = \sin((k+1)\theta)/\sin\theta$ for $k \ge 0$, using continuous endpoint values. Let h_i be distinct modulo an odd prime p, and let $k_i \ge 0$ not all vanish. Put $s = \#\{i : k_i > 0\}$ and $D = \prod_i (k_i + 1)$. For each fixed $B \ne 0$,*

$$\left| \mathbb{E}_{A \ne 0} \prod_i \chi_{k_i}(W_{h_i}) \right| \le \frac{4(s+1)D}{\sqrt{p}}. \tag{6.10}$$

The corresponding expectation for independent Sato–Tate variables is zero. The trivial character has zero discrepancy.

Proof. The polynomial χ_k is the trace character of the kth symmetric power of the standard SL_2 representation. It has dimension $k+1$ and $|\chi_k(x)| \le k+1$ on $[-2,2]$. Also

$$\mathbb{E}\chi_k(X) = \frac{2}{\pi} \int_0^\pi \sin((k+1)\theta) \sin\theta \, d\theta = \mathbf{1}_{\{k=0\}}.$$

We now specify the classical sheaf inputs and their quantitative use. The normalized rank-two Kloosterman sheaf is pure of weight zero, has trivial determinant, trace $-\mathrm{Kl}_2$, geometric and arithmetic monodromy SL_2, tame nontrivial unipotent ramification at zero, and sole infinity break $1/2$, of Swan conductor one; see [17, proof of Proposition 3.2]. The change $y = BA$ reduces the active factors to distinct translations $y \mapsto y - Bh_i$. Their direct sum has full product geometric monodromy SL_2^s by the Goursat–Kolchin–Ribet argument in [1, Lemma 2.4]. Distinct finite singularities exclude rank-one-twisted geometric isomorphisms: nontrivial unipotent local monodromy cannot become unramified by a scalar twist.

Tensor the symmetric powers with positive k_i. The resulting rank-D sheaf $\mathcal{F}$ on $U = \mathbb{A}^1 \setminus \{Bh_i : k_i > 0\}$ has no geometric invariants: each nontrivial symmetric power is an irreducible nontrivial representation of its own factor of the product. Irreducibility here is in characteristic zero; the usual raising and lowering operators connect its monomial basis. At finite punctures it is tame. Every infinity upper ramification group of index greater than $1/2$ acts trivially on every rank-two

factor, hence on tensor powers, symmetric-power subquotients and their tensor product. Thus $\mathrm{Swan}_\infty(\mathcal{F}) \le D/2$, independently of the degrees except for the explicit rank.

The trace/Euler/weight formulas [18, Theorems 4.1, 4.2, 4.6 and (4.2)], also summarized in the curve estimate [16, Remark 4.2], give

$$H_c^0 = H_c^2 = 0, \qquad \dim H_c^1 = D(s-1) + \mathrm{Swan}_\infty(\mathcal{F}) \le Ds.$$

The sheaf is pure of weight zero, so the full U trace sum has absolute value at most $Ds\sqrt{p}$ by Deligne's theorem. Its trace is $(-1)^{\sum k_i} \prod_i \chi_{k_i}(W_{h_i})$; this sign does not affect the zero-main-term bound.

At most $s+1$ points are changed when passing from $U(\mathbb{F}_p)$ to $\mathbb{F}_p^\times$. Each product character there is at most D in absolute value, including at singular values $W_h = -p^{-1/2}$. Therefore

$$\left| \mathbb{E}_{A \ne 0} \prod_i \chi_{k_i}(W_{h_i}) \right| \le D\,\frac{s\sqrt{p}+s+1}{p-1} \le \frac{4(s+1)D}{\sqrt{p}}.$$

The proof is uniform in B and the distinct shifts. This is an explicit use of the classical character trace bound, rather than an extrapolation of a fixed-degree implicit constant. □

6.3 Growing moments and a sharper path approximation

For distinct residues h, complex scalars b_h, and an integer $r \ge 1$, put $\sigma^2 = \sum_h |b_h|^2$ and $\ell = \sum_h |b_h|$. Then

$$\left(\mathbb{E} \left| \sum_h b_h W_h \right|^{2r} \right)^{1/(2r)} \le 8\sqrt{r}\,\sigma + 8\ell p^{-1/(4r)}. \tag{6.11}$$

Here and below the arithmetic expectation averages both nonzero parameters. To see the degree dependence explicitly, the character $x^m = \chi_1(x)^m$ decomposes as $\sum_k c_{m,k}\chi_k(x)$ with nonnegative integer multiplicities and $\sum_k c_{m,k}(k+1) = 2^m$. This follows by induction from $\chi_1\chi_k = \chi_{k+1} + \chi_{k-1}$ for $k \ge 1$ and $\chi_1\chi_0 = \chi_1$. Thus Lemma 6.3 bounds the discrepancy of any monomial of total degree $2r$ by $4(2r+1)2^{2r}p^{-1/2}$: the trivial character cancels, and every other term involves at most $2r$ active shifts. Expanding the absolute $2r$th power gives

$$\mathbb{E}|\sum_h b_h W_h|^{2r} \le \mathbb{E}|\sum_h b_h X_h|^{2r} + 4(2r+1)(2\ell)^{2r}p^{-1/2}.$$

For the independent variables, the bounded-variable exponential inequality gives $\mathbb{P}(|\sum b_h X_h| > u) \le 4\exp[-u^2/(16\sigma^2)]$. Integration bounds its $2r$th moment by $4(16\sigma^2)^r r!$. Taking roots, using $r! \le r^r$ and $2[4(2r+1)]^{1/(2r)} \le 8$, proves (6.11). The second term is absent for independent coefficients. This argument does not require a sheaf with all available shifts active simultaneously.

For a trigonometric polynomial T of degree at most H, the Bernstein inequality $\|T'\|_\infty \le 2\pi H\|T\|_\infty$ and the grid of $J = \lceil 4\pi H \rceil \le 14H$ equally spaced points give

$$\|T\|_\infty \le 2 \max_{0 \le j < J} |T(j/J)|.$$

For complex T, the Bernstein inequality follows from its real form by rotating the value of the derivative at a maximizing point and taking the real part. Applying (6.11) at the grid points, with no independence assumption on those points, yields

$$\mathbb{E}\|T - T(0)\|_\infty \le 32(14H)^{1/(2r)}(\sqrt{r}\,\sigma + \ell p^{-1/(4r)}), \quad T(t) = \sum_{|h| \le H} b_h W_h \mathrm{e}^{2\pi i h t}. \tag{6.12}$$

Write $w_J(h) = (1 - |h|/(J+1))_+$. The difference $\mathcal{P}_N Q_p - \mathcal{P}_M Q_p$ has zero endpoint term and coefficients

$$b_h = \frac{[w_N(h) - w_M(h)]c_{p,h}}{2\pi i h}, \qquad 0 < |h| \le N,$$

in front of $W_h(\mathrm{e}^{2\pi i h t} - 1)$. In particular $|c_{p,h}| \le 1$. Split the frequencies into $1 \le |h| \le M$ and the subsequent blocks $H_{j-1} < |h| \le \min(H_j, N)$, where $H_j = 2^j M$ and $1 \le j \le J_* := \lceil \log_2(N/M) \rceil$. The first block has $|b_h| \le 1/(2\pi M)$, hence $\ell \le 1/\pi < 1$ and $\sigma^2 \le 1/(2\pi^2 M) < 1/M$. Each later block has $|b_h| \le 1/(\pi H_j)$ and at most $2H_j$ coefficients, so $\ell < 1$ and $\sigma^2 < 1/H_j$ again. The last block has degree at most $H_{J_*} < 2N$, but its nonzero frequencies still lie in $[-N, N]$ and are distinct modulo p.

Set $r = \lceil \log(16M) \rceil \ge 4$. The first terms in (6.12) sum to at most

$$32\sqrt{r}\,(14M)^{1/(2r)} M^{-1/2} \sum_{j \ge 0} 2^{j(-1/2+1/(2r))} \le 512\sqrt{r/M},$$

since $(14M)^{1/(2r)} \le \sqrt{\mathrm{e}} < 2$ and the sum is at most $(1 - 2^{-3/8})^{-1} < 5$. The second terms sum to at most $384r(N^2/p)^{1/(4r)}$, using $(1 - 2^{-1/(2r)})^{-1} \le 6r$ and $28^{1/(2r)} < 2$. The former inequality follows from $1 - \mathrm{e}^{-x} \ge x/2$ for $0 \le x \le 1$, with $x = \log 2/(2r)$. Apply (6.8) only at the larger cutoff N to obtain

$$\begin{aligned} \mathbb{E}\|Q_p - \mathcal{P}_M Q_p\| \le 512\sqrt{r/M} + 2048 N^{-1/4} \\ + 384 r (N^2/p)^{1/(4r)}. \end{aligned} \tag{6.13}$$

For K the block proof has no arithmetic error. Letting $N \to \infty$ and using the L^1 convergence in (6.8) gives

$$\mathbb{E}\|K - \mathcal{P}_M K\| \le 512\sqrt{r/M}. \tag{6.14}$$

6.4 Unbiased Bernstein approximation of path functionals

Use the degrees in (6.2), and retain $d = 2M+1$ and $r = \lceil \log(16M) \rceil$. For $x \in [-2,2]^d$, conditionally choose independent binomial variables

$$B_h \sim \mathrm{Bin}\left(n_h, \frac{x_h + 2}{4}\right), \quad Y_h = 4B_h/n_h - 2, \quad \Delta_h = Y_h - x_h.$$

For a real 1-Lipschitz path functional φ with $\varphi(0) = 0$, write $G(x) = \varphi(\mathcal{A}_M(x))$ and define its tensor Bernstein polynomial $P(x) = \mathbb{E}[G(Y) \mid x]$. This is a polynomial of degree at most n_h in coordinate h, and $|P(x)| \le 2S_M$ on the cube.

Lemma 6.4 (Uniform functional approximation)**.** *For every such φ,*

$$\sup_{x \in [-2,2]^d} |P(x) - G(x)| \le 64\sqrt{r/M}.$$

Proof. The usual binomial representation of Bernstein polynomials is used; we give the concentration argument to retain all dimension dependence. For a Bernoulli variable of mean q, the second derivative of its centered log moment-generating function is a Bernoulli variance and is at most $1/4$. Its value and derivative at zero vanish, so its log moment-generating function is at most $u^2/8$. Consequently, uniformly over every conditioning vector x,

$$\mathbb{E}[e^{u\Delta_h} \mid x] \le e^{2u^2/n_h}, \qquad \mathbb{E}[\Delta_h \mid x] = 0.$$

This includes $x_h = \pm 2$, where the coordinate is deterministic.

Put $b_h = (1 - |h|/(M+1))/(2\pi i h)$ for $h \neq 0$ and $T(t) = \sum_{0<|h|\le M} b_h \Delta_h e^{2\pi i h t}$. The variance proxy satisfies

$$\sigma^2 = \sum_{0<|h|\le M} \frac{|b_h|^2}{n_h} \le \frac{1}{2\pi^2 M} < \frac{1}{M}.$$

The real and imaginary exponential estimates therefore give

$$\Pr(|T(t)| > u \mid x) \le 4e^{-u^2/(16\sigma^2)}.$$

Use the trigonometric Bernstein grid with $J = \lceil 4\pi M \rceil \le 14M$, for which

$$\|T\|_\infty \le 2 \max_{j<J} |T(j/J)|.$$

If $V = \max_{j<J} |T(j/J)|$, integrate its union tail at $a = 4\sigma\sqrt{\log(4J)}$ to obtain

$$\mathbb{E}[V \mid x] \le 4\sigma\sqrt{\log(4J)} + \frac{2\sigma}{\sqrt{\log(4J)}} \le 6\sigma\sqrt{\log(4J)}.$$

Indeed $\int_a^\infty e^{-u^2/(16\sigma^2)} du \le 8\sigma^2 a^{-1} e^{-a^2/(16\sigma^2)}$. Pinning adds at most a factor two, so $\mathbb{E}\|T - T(0)\|_\infty \le 24\sigma\sqrt{\log(56M)}$. Also $\mathbb{E}|\Delta_0| \le 2/\sqrt{M}$. Since $\log(56M) \le 2r$ and $r \ge 4$, these estimates imply

$$\mathbb{E}\|\mathcal{A}_M(Y - x)\|_\infty \le 64\sqrt{r/M}.$$

The Lipschitz property and Jensen's inequality for absolute values prove the asserted approximation bound, uniformly in x. No independence of arithmetic coefficients or of process increments is needed: independence here is only that of the auxiliary binomials under a fixed x. □

Lift $x_h = 2\cos\theta_h$. Since P is bounded by $2S_M$ on the cube, every complex Fourier coefficient of its even angular lift has modulus at most $2S_M$. Its degree in coordinate h is at most n_h. Define $C_j(2\cos\theta) = \cos(j\theta)$. The exact identities

$$C_0 = \chi_0, \qquad C_1 = \chi_1/2, \qquad C_j = (\chi_j - \chi_{j-2})/2 \quad (j \ge 2)$$

show that the absolute character mass weighted by representation dimension is at most $j+1$ for C_j. Grouping Fourier sign orbits into cosine products costs no more than summing the original complex Fourier modes. It follows that

$$P(x) = \sum_{\mathbf{k}} b_{\mathbf{k}} \prod_h \chi_{k_h}(x_h), \qquad \sum_{\mathbf{k}} |b_{\mathbf{k}}| \prod_h (k_h + 1) \le 2S_M \prod_h (n_h^2 + 3n_h + 1) \le 2S_M \prod_h (n_h + 2)^2.$$

This coefficient estimate uses the bounded polynomial on the cube, not its generally large monomial coefficients. The latter are not inserted into the sheaf estimate.

The allocated degrees have only exponential total character cost:

$$\prod_h (n_h + 2)^2 \le 4M^2 \frac{4^{4M} M^{8M}}{(M!)^8} \le e^{16M}.$$

Indeed $n_0 + 2 \le 2M$ and $n_h + 2 \le 4M^2/h^2$ for $h \neq 0$. Using $\log M! \ge M\log M - M$ bounds the logarithm by $(8 + 4\log 4)M + 2\log M + \log 4 \le 16M$ for $M \ge 2$. For an integer-only coarser bound, $e < 3$ and $4M^2 \le 4^M$ give $\prod_h (n_h + 2)^2 \le 2^{23M}$, since $4 \cdot 6^8 < 2^{23}$.

Applying Lemma 6.3, with the trivial character discrepancy omitted as before, yields

$$|\mathbb{E}G(W) - \mathbb{E}G(X)| \le 128\sqrt{r/M} + \frac{8S_M(d+1)}{\sqrt{p}} \prod_h (n_h+2)^2.$$

Adding (6.13), (6.14), (6.6) and (6.9), and taking the supremum over φ, gives

$$W_1(\mu_p, \mu) \le D_B(p, M, N),$$

which proves the finite part of Theorem 6.1. The previously established finite first moments justify Kantorovich–Rubinstein duality on the separable path space. All high-degree dependence remains in the explicit rank-weighted character bound of Lemma 6.3.

Rate optimization. For $t = \log p \ge 10^6$, choose

$$M = \lfloor t/128 \rfloor, \qquad N = \lfloor p^{1/8} \rfloor, \qquad r = \lceil \log(16M) \rceil.$$

Then $M \ge t/256$, $r \le \log t$, $2 \le M \le N < p/2$ and $16M \le t/8$. The principal term is at most $18432\sqrt{\log t/t}$. The other terms are bounded by

$$2500e^{-t/32} + 384 \log t \, e^{-3t/(16 \log t)} + 9te^{-t/2} + 8t^2 e^{-3t/8} \le \sqrt{\log t/t}.$$

For the last inequality each term is at most one quarter of the right side at $t = 10^6$, and its ratio decreases thereafter. For the second ratio its logarithmic derivative is

$$\frac{1}{2t} + \frac{1}{2t \log t} - \frac{3(\log t - 1)}{16(\log t)^2} < 0;$$

the other three derivatives are immediately negative. Thus

$$W_1(\mu_p, \mu) \le 20000\sqrt{\frac{\log\log p}{\log p}} \quad (\log p \ge 10^6),$$

and in particular $W_1 = O(\sqrt{\log\log p/\log p})$.

This proves (6.4) with the stated coarse constant.

6.5 Direct transfer of rare path events

The preceding Wasserstein bound is useful for general Lipschitz functionals. For small balls whose probabilities are exponentially small in ϵ^{-2}, an additive Lipschitz approximation error becomes too large. We instead approximate an indicator by a bounded polynomial with exponentially small noise tails. The auxiliary noise is independent only conditionally on its coefficient vector; the arithmetic coefficients remain dependent.

Put

$$C_L = 26000(L+2)^2, \quad \Lambda_L = 64C_L, \quad C_0 = 2^{36}, \quad D_L = [1024C_0(\Lambda_L+1)]^{1/4}.$$

Theorem 6.5. *Fix $L \ge 1$. For every sufficiently large prime p, uniformly for $g \in \mathcal{H}_L$, with $\epsilon_p = D_L(\log p)^{-1/4}$,*

$$\mu_p(B(g, \epsilon_p)) \ge \tfrac{1}{2} \exp[-\Lambda_L \epsilon_p^{-2}]. \tag{6.15}$$

Consequently the number of pairs $(A, B) \in (\mathbb{F}_p^\times)^2$ whose polygonal paths lie in this open uniform ball is at least

$$\tfrac{1}{2}(p-1)^2 \exp[-\Lambda_L D_L^{-2}\sqrt{\log p}].$$

The threshold is effective. Centers may depend deterministically on p while remaining in one fixed class. No optimal radius or one-parameter assertion is claimed.

We prove an explicit finite inequality first, so that all probability errors and degree restrictions are visible.

For integers $2 \le M \le N < p/2$ and $r \ge \max(4, \lceil \log(16M) \rceil)$, define

$$\begin{aligned} B_r &= 512\sqrt{r/M} + 384r(N^2/p)^{1/(4r)}, \\ T_{\mathrm{lim}} &= \left[\frac{4096\sqrt{r}}{\epsilon\sqrt{M}}\right]^{2r}, & T_{\mathrm{ar}} &= \frac{32768N^{-1/4}}{\epsilon} + \left[\frac{16B_r}{\epsilon}\right]^{2r}, \\ T_{\mathrm{noise}} &= (56M+2)\exp(-M\epsilon^2/65536), & T_{\mathrm{char}} &= \frac{4(2M+2)}{\sqrt{p}}\exp(16M). \end{aligned} \tag{6.16}$$

Proposition 6.6. *For $0 < \epsilon \le 1/4$ and these integer parameters, if*

$$p^{-1/2} + 8M/p \le \epsilon/16, \tag{6.17}$$

then

$$\inf_{g \in \mathcal{H}_L} \mu_p(B(g,\epsilon)) \ge e^{-\Lambda_L \epsilon^{-2}} - T_{\mathrm{lim}} - T_{\mathrm{ar}} - 2T_{\mathrm{noise}} - T_{\mathrm{char}}. \tag{6.18}$$

A negative right side is permitted.

Proof. First upgrade the growing-moment block estimate to a path-norm moment. For a polynomial with the coefficients of (6.12), the L^{2r} norm of its maximum on a J-point grid is at most $J^{1/(2r)}$ times the largest pointwise L^{2r} norm: sum the $2r$th powers before taking roots. Grid reconstruction and pinning contribute a factor four. Thus (6.12) remains valid with its left side replaced by the L^{2r} norm of the uniform norm. Minkowski then sums the dyadic blocks used above. The same geometric sums, now for arbitrary $r \ge \max(4, \lceil \log(16M) \rceil)$, give

$$\left(\mathbb{E}\,\|\mathcal{P}_N Q_p - \mathcal{P}_M Q_p\|_\infty^{2r}\right)^{1/(2r)} \le B_r, \tag{6.19}$$

$$\left(\mathbb{E}\,\|K - \mathcal{P}_M K\|_\infty^{2r}\right)^{1/(2r)} \le 512\sqrt{r/M}. \tag{6.20}$$

Indeed $(14M)^{1/(2r)} < 2$, the first geometric sum is at most five, and the second at most $6r$, as in the previous proof. For the limit process the arithmetic term is absent. Apply the finite-block bound through any larger cutoff; the L^1 convergence of $\mathcal{P}_N K$ to K supplies an almost surely convergent subsequence, and Fatou proves (6.20). No L^{2r} convergence is assumed. Markov at $\epsilon/8$ gives

$$\mathbb{P}(\|K - \mathcal{P}_M K\|_\infty > \epsilon/8) \le T_{\mathrm{lim}}.$$

Split $Q_p - \mathcal{P}_M Q_p$ at $\mathcal{P}_N Q_p$, allocating $\epsilon/16$ to each part. Equations (6.8) and (6.19) give

$$\mathbb{P}(\|Q_p - \mathcal{P}_M Q_p\|_\infty > \epsilon/8) \le T_{\mathrm{ar}}.$$

Use the allocated degrees (6.2) and the conditional binomial variables Y_h of Lemma 6.4. Their centered differences $\Delta_h = Y_h - x_h$ obey the same exponential bound uniformly in x. At the endpoint,

$$\mathbb{P}(|\Delta_0| > \epsilon/16 \mid x) \le 2e^{-M\epsilon^2/2048}.$$

For $T(t) = \sum_{h\neq 0} b_h \Delta_h e^{2\pi i h t}$ as in that lemma, the variance proxy is $< 1/M$, so each grid value has tail $4e^{-Mu^2/16}$. There are at most $14M$ grid points and $\|T - T(0)\|_\infty \le 4 \max_{\mathrm{grid}} |T|$. At $u = \epsilon/64$, the nonzero-frequency part exceeds $\epsilon/16$ with probability at most $56Me^{-M\epsilon^2/65536}$. Adding the endpoint shows

$$\mathbb{P}(\|\mathcal{A}_M(Y - x)\|_\infty > \epsilon/8 \mid x) \le T_{\mathrm{noise}}. \tag{6.21}$$

This includes boundary coordinates, where the auxiliary binomial is deterministic.

Fix g and set

$$F(y) = \mathbf{1}_{\{\|\mathcal{A}_M(y)-g\|_\infty < \epsilon/2\}}, \qquad P(x) = \mathbb{E}[F(Y) \mid x].$$

Despite the discontinuity of F, its conditional average is an ordinary tensor Bernstein polynomial: it is a finite sum of fixed grid values times binomial probability polynomials. Its degree in coordinate h is at most n_h, and $0 \le P \le 1$ everywhere on the cube. By (6.21),

$$\begin{aligned} P(x) &\ge 1 - T_{\mathrm{noise}} && \text{if } \|\mathcal{A}_M(x) - g\|_\infty < \epsilon/4, \\ P(x) &\le T_{\mathrm{noise}} && \text{if } \|\mathcal{A}_M(x) - g\|_\infty \ge 3\epsilon/4. \end{aligned}$$

The radius slack avoids any norm-sphere atom assumption.

The angular-lift character calculation above applies to every polynomial bounded by one with these separate degrees. It gives rank-weighted character mass at most $\prod_h (n_h + 2)^2 \le e^{16M}$. Lemma 6.3, with its trivial-character discrepancy zero and at most $2M + 1$ active shifts, therefore gives

$$|\mathbb{E}P(W) - \mathbb{E}P(X)| \le T_{\mathrm{char}}.$$

The coefficients of the discontinuous F are never inserted into the trace estimate; only the bounded polynomial P is used.

Put $v = e^{-\Lambda_L \epsilon^{-2}}$. Theorem 1.1 at radius $\epsilon/8$, and the limit residual bound, yield

$$\mathbb{P}(\|\mathcal{A}_M(X) - g\|_\infty < \epsilon/4) \ge v - T_{\mathrm{lim}}.$$

If $T_{\mathrm{noise}} \le 1$, the two polynomial implications and the character comparison imply

$$\begin{aligned} \mathbb{P}(\|\mathcal{A}_M(W) - g\|_\infty < 3\epsilon/4) &\ge (1 - T_{\mathrm{noise}})(v - T_{\mathrm{lim}}) - T_{\mathrm{char}} - T_{\mathrm{noise}} \\ &\ge v - T_{\mathrm{lim}} - 2T_{\mathrm{noise}} - T_{\mathrm{char}}. \end{aligned}$$

If $T_{\mathrm{noise}} > 1$, the final lower bound is nonpositive and trivial. Outside an event of probability at most T_{ar}, (6.6), (6.9) and (6.17) give $\|K_p - \mathcal{A}_M(W)\|_\infty \le 3\epsilon/16$. The finite-coefficient ball of radius $3\epsilon/4$ therefore lands inside the strict final ball, since $3/4 + 3/16 = 15/16 < 1$. Subtract T_{ar} to prove (6.18). Every estimate is uniform in g. □

Proof of Theorem 6.5. Set $t = \log p$, $\epsilon = D_L t^{-1/4}$ and choose

$$r = \lceil 4(\Lambda_L + 1)\epsilon^{-2} \rceil, \qquad M = \lceil C_0 r \epsilon^{-2} \rceil, \qquad N = \lfloor p^{1/8} \rfloor.$$

Then r is asymptotic to a positive constant times $\sqrt{t}$, and $M \sim t/256$. Eventually $M \le t/128$, all finite hypotheses hold, and (6.17) holds. Write $v = e^{-\Lambda_L \epsilon^{-2}}$. We verify that every error in (6.18) is $o(v)$.

Since $M \ge C_0 r \epsilon^{-2}$, $T_{\mathrm{lim}} \le 64^{-2r}$, and $r \ge 4(\Lambda_L + 1)\epsilon^{-2}$ makes its ratio to v tend to zero. The first part of T_{ar} is a fixed polynomial in t times $e^{-t/32}$, up to the harmless floor constant in N, and is also $o(v)$. The first part of B_r is at most $\epsilon/512$; its second part is at most

$$384r \exp[-3t/(16r)] = o(\epsilon).$$

Thus eventually $B_r \le \epsilon/256$, so the second part of T_{ar} is at most $16^{-2r} = o(v)$. Furthermore

$$M\epsilon^2/65536 \ge 2^{20}r,$$

which makes $T_{\mathrm{noise}}/v \to 0$ despite its polynomial prefactor. Finally $16M \le t/8$ gives $T_{\mathrm{char}} \le 4(2M+2)e^{-3t/8} = o(v)$. Their sum, including the factor two on T_{noise}, is at most $v/2$ eventually. Proposition 6.6 proves (6.15). All comparisons are explicit elementary inequalities in t, giving an effective threshold for fixed L. □

The radius gain is paid by probability-scale accuracy: r of order ϵ^{-2} controls the genuine and auxiliary tails, M of order $r\epsilon^{-2} = \epsilon^{-4}$ pays uniform path concentration, and the character budget allows M of order $\log p$. This does not assert an optimal shrinking scale or a relative count.

6.6 Uniform arithmetic neighborhoods

Put $A_L = 104000(L+2)^2$. A bump equal to one on the closed $\epsilon/2$-ball, linear on the annulus and zero outside the open ϵ-ball has Lipschitz constant $2/\epsilon$. Theorem 1.1 and Theorem 6.1 yield

$$\inf_{g\in\mathcal{H}_L} \mu_p(B(g,\epsilon)) \ge \exp(-A_L\epsilon^{-2}) - \frac{2D_B(p,M,N)}{\epsilon}, \qquad 0<\epsilon\le 1/4. \tag{6.22}$$

This is a finite-prime inequality, even when its right side is negative. The balls are open; using a closed inner ball needs no norm-sphere atom assumption.

Corollary 6.7. *Fix $L \ge 1$ and $0 < \kappa < 1/2$, and put*

$$\epsilon_p = \sqrt{\frac{A_L}{\kappa \log\log p}}.$$

For all sufficiently large primes, with an effective threshold,

$$\inf_{g\in\mathcal{H}_L} \#\{(A,B)\in(\mathbb{F}_p^\times)^2 : \|K_p(\cdot;A,B)-g\|_\infty < \epsilon_p\} \ge \tfrac{1}{2}(p-1)^2(\log p)^{-\kappa}.$$

In particular $\kappa = 1/4$ gives $\epsilon_p = \sqrt{4A_L/\log\log p}$.

Proof. The main term in (6.22) is $(\log p)^{-\kappa}$. The error is $O_L((\log p)^{-1/2}\log\log p)$, eventually at most half the main term. The explicit estimate (6.3) and $\epsilon_p \le 1/4$ give an effective threshold. □

The bounds permit deterministic prime-dependent centers in one fixed class. They give very slowly shrinking radii and enormous thresholds; they are not practical small-prime estimates.

The following fixed-radius consequence is also useful independently of the quantified estimate. We include its compactness proof for later use with other fixed regularity classes.

Corollary 6.8. *Fix $L \ge 1$ and $0 < \epsilon \le 1/4$. Then*

$$\liminf_{\substack{q\to\infty \\ q \text{ prime}}} \inf_{g\in\mathcal{H}_L} \frac{1}{(q-1)^2}\#\{(a,b)\in(\mathbb{F}_q^\times)^2 : \|K_q(\cdot;a,b)-g\|_\infty < \epsilon\} \ge \exp\{-104000(L+2)^2\epsilon^{-2}\}. \tag{6.23}$$

In particular the same bound holds for any deterministic choice of centers $g_q \in \mathcal{H}_L$.

Proof. The class $\mathcal{H}_L$ is compact in uniform norm. Indeed $|g(t)| \le L\sqrt{t} \le L$ and its common Hölder bound gives equicontinuity, hence relative compactness by Arzelà–Ascoli. It is closed: endpoints and Hölder inequalities pass to uniform limits, and for each fixed $h \neq 0$ the map

$$g \longmapsto 2\pi i h \int_0^1 [g(t) - tg(1)]\mathrm{e}^{-2\pi i h t}\, dt$$

is continuous. Its constraint to the closed real interval $[-2, 2]$ is therefore preserved. The intersection of these constraints is closed.

Choose a finite $\epsilon/4$-net $g_1, \dots, g_J$ from the class. For every $g \in \mathcal{H}_L$ some j satisfies $B(g_j, \epsilon/2) \subset B(g, \epsilon)$, with open uniform balls. It follows that

$$\inf_{g \in \mathcal{H}_L} \mu_q(B(g, \epsilon)) \ge \min_{1 \le j \le J} \mu_q(B(g_j, \epsilon/2)).$$

Portmanteau and Theorem 1.1 give, for each j,

$$\liminf_q \mu_q(B(g_j, \epsilon/2)) \ge \mu(B(g_j, \epsilon/2)) \ge \exp\{-104000(L+2)^2\epsilon^{-2}\}.$$

For finitely many sequences, the liminf of their minimum is at least the minimum of their liminfs: take the largest of their finitely many eventual starting indices. This proves (6.23). The deterministic moving-center assertion follows from the infimum bound. □

7 The exact worst-case scale for Hölder centers

For $0 < \gamma \le 1$ and $L \ge 1$, let $\mathcal{H}_{\gamma,L}$ have the same endpoint and normalized Fourier restrictions as $\mathcal{H}_L$, with its regularity condition replaced by $|g(t) - g(s)| \le L|t - s|^\gamma$. Thus $\mathcal{H}_{1/2,L} = \mathcal{H}_L$. Define

$$F_{\gamma,L}(\epsilon) = \inf_{g \in \mathcal{H}_{\gamma,L}} \mathbb{P}(\|K - g\|_\infty < \epsilon).$$

Theorem 7.1. *For every fixed $0 < \gamma \le 1$ and $L \ge 1$, there are constants $0 < c_\gamma, C_{\gamma,L} < \infty$ and $\epsilon_\gamma > 0$ such that*

$$\exp[-C_{\gamma,L}\epsilon^{-\max(2,1/\gamma)}] \le F_{\gamma,L}(\epsilon) \le \exp[-c_\gamma \epsilon^{-\max(2,1/\gamma)}], \qquad 0 < \epsilon \le \epsilon_\gamma. \tag{7.1}$$

For $0 < \gamma < 1/2$, one may take $c_\gamma = 64^{-1/\gamma}/2048$.

The rough upper bound is a worst-case statement: the center used to bound the infimum can change with ϵ. Every center constructed below is piecewise linear, hence individually 1/2-Hölder with a nonuniform constant. The uniform bound on the weaker γ-Hölder norm is what makes them one admissible family. Theorem 1.1 retains its stronger each-center conclusion at exponent 1/2.

Proof. First suppose $0 < \gamma < 1/2$. Put

$$E_\gamma = \frac{2}{\gamma + 1} + \frac{1}{2(1 - \gamma)}, \qquad D_\gamma = 2^{2-\gamma} E_\gamma.$$

For $u = g - at$, the circular Hölder constant is at most $2^{1-\gamma}(L + 2)$, by passing through the zero endpoint as in (4.1). The Fejér bounds in Section 4, with $n = M + 1$, give

$$\int_0^1 F_M(t) d(t, \mathbb{Z})^\gamma\, dt \le E_\gamma n^{-\gamma}.$$

Indeed the two inner intervals contribute at most $2n\int_0^{1/n} t^\gamma\,dt = 2n^{-\gamma}/(\gamma+1)$, and the outer intervals contribute at most $(2n)^{-1}\int_{1/n}^{1/2} t^{\gamma-2}\,dt \le n^{-\gamma}/(2(1-\gamma))$. Pinning at zero therefore gives error at most $D_\gamma(L+2)(M+1)^{-\gamma}$. Choose

$$M = \left\lceil [8D_\gamma(L+2)]^{1/\gamma}\epsilon^{-1/\gamma} \right\rceil.$$

This error is at most $\epsilon/8$. The finite shifted-density proof of Lemma 4.1 uses only the endpoint and Fourier bounds, so it applies to this cutoff without change. Together with the independent tail event, it gives

$$\mathbb{P}(\|K-g\|_\infty < \epsilon) \ge c_e\epsilon^{3/2}\exp[-6M - 5120\epsilon^{-2}].$$

Since $1/\gamma > 2$, the required lower bound follows for $0 < \epsilon \le 1/4$, for example with

$$C_{\gamma,L} = 6\{[8D_\gamma(L+2)]^{1/\gamma} + 1\} + 5127. \tag{7.2}$$

We now construct centers giving the opposite estimate. For every dyadic $J = 2^m$, there are classical Rudin–Shapiro signs $\sigma_1, \dots, \sigma_J$ with

$$\sup_{|z|=1}\left|\sum_{j=1}^{J}\sigma_j z^j\right| \le \sqrt{2J}. \tag{7.3}$$

The elementary construction, also used in [3], starts with $P_0 = Q_0 = 1$ and takes

$$P_{m+1} = P_m + z^{2^m}Q_m, \qquad Q_{m+1} = P_m - z^{2^m}Q_m.$$

The coefficients are signs on disjoint concatenated supports, and on the unit circle the parallelogram identity gives $|P_m|^2 + |Q_m|^2 = 2^{m+1} = 2J$. Multiplying P_m by z proves (7.3). These signs and their cancellation are classical.

Set

$$\lambda = \frac{1}{8\sqrt{2}}, \quad N = \lceil J^{1/(2\gamma)}\rceil, \quad A = \lambda J^{-1/2}, \quad t_j = \frac{j}{4J}, \quad \phi(x) = (1-|x|)_+,$$

and choose J large enough that $N \ge 16J$. Define

$$g_J(t) = A\sum_{j=1}^{J}\sigma_j\{\phi(N(t-t_j)) - \phi(N(t-(1-t_j)))\}. \tag{7.4}$$

All supports are disjoint and interior to $[0,1]$: the left centers lie in $[1/(4J), 1/4]$, their spacing is $1/(4J)$ and their radii are at most $1/(16J)$; the remaining supports are their reflections. Thus g_J is continuous, real, zero at both endpoints, and odd under $t \mapsto 1-t$. It has norm at most A and global Lipschitz constant at most AN, because its piecewise linear slopes have that bound. It follows that

$$|g_J(t) - g_J(s)| \le \min(AN|t-s|, 2A) \le 2^{1-\gamma}AN^\gamma|t-s|^\gamma \le 2\lambda|t-s|^\gamma.$$

The last step uses $N \le 2J^{1/(2\gamma)}$.

The full-line Fourier transform of ϕ is $\widehat{\phi}(\xi) = (\sin(\pi\xi)/(\pi\xi))^2$. Since the bumps are interior, their periodic Fourier coefficients give

$$\widehat{g}_J(h) = -\frac{2iA}{N}\widehat{\phi}(h/N)\sum_{j=1}^{J}\sigma_j\sin(2\pi h t_j), \qquad h \ne 0.$$

Consequently their normalized coefficients are real and equal to

$$\beta_h = 4\pi A \frac{h}{N} \widehat{\phi}(h/N) \sum_{j=1}^{J} \sigma_j \sin(2\pi h t_j). \tag{7.5}$$

The inequality $\sin^2 x \le \min(x^2, 1) \le |x|$ gives $|\xi\widehat{\phi}(\xi)| \le 1/\pi$. The sine sum is an imaginary part of the polynomial in (7.3), so $|\beta_h| \le 4A\sqrt{2J} = 1/2$. Thus every g_J lies in the single class $\mathcal{H}_{\gamma,1}$, and hence in $\mathcal{H}_{\gamma,L}$ for $L \ge 1$.

The upper probability estimate uses orthogonal increments in coefficient space, not independent process increments. Write, for $0 \le t \le 1/2$,

$$\operatorname{Re} K(t) = \sum_h c_h(t) X_h, \qquad c_0(t) = t, \quad c_h(t) = \frac{\sin(2\pi h t)}{2\pi h} \ (h \ne 0).$$

For $0 \le s \le t \le 1/2$, identity (3.4) and product-to-sum show

$$\sum_h c_h(s) c_h(t) = st + \frac{1}{2\pi^2} \sum_{h \ge 1} \frac{\sin(2\pi h s)\sin(2\pi h t)}{h^2} = \frac{s}{2}. \tag{7.6}$$

For clarity, the sine sum is half the difference of the cosine series at $t - s$ and $t + s$, and equals $\pi^2(s - 2st)$. Let $s_j = t_j - 1/N$. The intervals $[s_j, t_j]$ are disjoint and inside $(0, 1/2)$. For

$$S = \sum_{j=1}^{J} \sigma_j \{\operatorname{Re} K(t_j) - \operatorname{Re} K(s_j)\} = \sum_h d_h X_h,$$

equation (7.6) gives $\sum_h d_h^2 = J/(2N)$: each diagonal contributes half the interval length, and each disjoint off-diagonal pair contributes zero. The independent bounded scalar coefficients and their exponential estimate, followed by Fatou through the convergent symmetric sums, yield $\mathbb{E}e^{uS} \le \exp(u^2 J/N)$.

If $\|K - g_J\|_\infty < \epsilon$ and $\epsilon \le A/4$, each signed target increment is A and its two endpoint errors total less than 2ϵ. Hence $S \ge JA/2$. Exponential Markov inequality at $u = AN/4$ proves

$$\mathbb{P}(\|K - g_J\|_\infty < \epsilon) \le \exp(-JA^2 N/16) = \exp(-\lambda^2 N/16). \tag{7.7}$$

To cover every sufficiently small radius, let J_γ be a dyadic integer at least $16^{1/(1/(2\gamma)-1)}$. For $0 < \epsilon \le \lambda/(4\sqrt{2J_\gamma})$, choose the largest dyadic $J \le (\lambda/(4\epsilon))^2$. Then $J \ge J_\gamma$, $4\epsilon \le A < 4\sqrt{2}\epsilon$, and

$$N \ge J^{1/(2\gamma)} > \left(\frac{\lambda}{4\sqrt{2}}\right)^{1/\gamma} \epsilon^{-1/\gamma}.$$

Taking the infimum in (7.7) proves the rough upper bound with $c_\gamma = (\lambda^2/16)(\lambda/(4\sqrt{2}))^{1/\gamma} = 64^{-1/\gamma}/2048$.

Finally, if $1/2 \le \gamma \le 1$, then $\mathcal{H}_{\gamma,L} \subseteq \mathcal{H}_L$, since $|t - s|^\gamma \le |t - s|^{1/2}$. Theorem 1.1 supplies the uniform lower bound. The zero center belongs to every class, and its upper bound in that theorem bounds the infimum above by $\exp(-c\epsilon^{-2})$. This completes (7.1). □

For fixed γ, L, the class $\mathcal{H}_{\gamma,L}$ is compact by the same equicontinuity and closed-coefficient argument used in Corollary 6.8. Hence its lower estimate also gives

$$\liminf_{q\to\infty} \inf_{g \in \mathcal{H}_{\gamma,L}} \mu_q(B(g,\epsilon)) \ge \exp[-2^s C_{\gamma,L} \epsilon^{-s}], \qquad s = \max(2, 1/\gamma),$$

for each fixed sufficiently small ϵ: repeat that corollary's finite $\epsilon/4$-net proof with radius $\epsilon/2$. This fixed-radius display follows from the qualitative finite-net argument. The stronger effective and shrinking-radius statements for the half-Hölder class are proved in Section 6; the law here remains the same two-parameter prime ensemble.

8 A weighted boundary condition for other coefficient laws

The same matching mechanism extends beyond Sato–Tate. Fractional Fejér tapers improve the sufficient condition on how rapidly a bounded symmetric unimodal density can vanish at its boundary. The positive-kernel representation used below is classical. This section concerns specified random Fourier series and supplies no arithmetic convergence theorem for their laws.

Proposition 8.1. *Let f be an even probability density supported on $[-2,2]$, nonincreasing on $[0,2)$, positive in its interior, and bounded above by $B<\infty$. Use its right-continuous monotone representative on $[0,2)$ and even extension, and put*

$$\Psi(s)=\max\{0,-\log f(2-s)\},\qquad 0<s\le 1.$$

Suppose that for some fixed $1/2<r<1$,

$$I_r=\int_0^1 \Psi(t^r)\,dt=\frac{1}{r}\int_0^1 \Psi(s)s^{1/r-1}\,ds<\infty. \tag{8.1}$$

Let the independent X_h have density f, and define $K_f=\sum_h X_h v_h$ with the same basis and symmetric summation as before. For every $L\ge 1$, $g\in\mathcal{H}_L$, and $0<\epsilon\le 1/4$,

$$\mathbb{P}(\|K_f-g\|_\infty<\epsilon)\ge\exp\{-C_{f,L,r}\epsilon^{-2}\}, \tag{8.2}$$

where one possible constant is

$$T_r=3+\frac{2r(1-r)}{r-1/2},\qquad D_r=\frac{14\sqrt{2}}{3}T_r,\qquad Q_r=64D_r^2+1,$$
$$C_{f,L,r}=(2r+2I_r)Q_r(L+2)^2+5120+\log 4+1+8^{1/r}I_r.$$

For an upper estimate, put

$$\eta=\frac{1}{4B},\qquad p_0=\frac{1}{2},\qquad \delta_0=\frac{p_0\eta^2}{32\pi^2},\qquad c_0=\frac{p_0\delta_0}{16},\qquad \epsilon_f=\sqrt{\delta_0/2}.$$

Then, for every deterministic $g\in C([0,1],\mathbb{C})$,

$$\mathbb{P}(\|K_f-g\|_\infty<\epsilon)\le\exp(-c_0\epsilon^{-2}),\qquad 0<\epsilon\le\epsilon_f. \tag{8.3}$$

The class $\mathcal{H}_L$ retains its exact definition from Section 1; no separate support characterization for f is needed.

Proof. We first record the approximation kernel and its full estimate. For an integer $N\ge 2$ let $w_h=(1-(h/N)^r)_+$ for integers $h\ge 0$, and put

$$\alpha_m=(m+1)(w_m-2w_{m+1}+w_{m+2}),\qquad 0\le m\le N-1.$$

The decreasing sequence w_h is convex: its second derivative on $(0,N)$ is $r(1-r)N^{-r}h^{r-2}>0$, and its slope jumps upward to zero at N. Thus $\alpha_m\ge 0$. With $d_m=w_m-w_{m+1}$, telescoping gives

$$\sum_{m=0}^{N-1}\alpha_m=\sum_{m=0}^{N-1}(m+1)(d_m-d_{m+1})=1,$$
$$\sum_{m=h}^{N-1}\alpha_m\left(1-\frac{h}{m+1}\right)=\sum_{m=h}^{N-1}(m+1-h)(d_m-d_{m+1})=w_h.$$

Consequently $F_{N,r} = \sum_{m=0}^{N-1} \alpha_m F_m$ is a nonnegative integral-one kernel with Fourier coefficients $w_{|h|}$.

For $0 < \gamma < r$, these mixture coefficients satisfy

$$\sum_{m=0}^{N-1} \alpha_m (m+1)^{-\gamma} \le \left(3 + \frac{2r(1-r)}{r-\gamma}\right) N^{-\gamma}. \tag{8.4}$$

Indeed $\alpha_0 = (2-2^r)N^{-r} \le 2N^{-\gamma}$, while the last coefficient contributes at most $N^{-\gamma}$ since $\alpha_{N-1} \le 1$. For $1 \le m \le N-2$, integrating the second derivative of x^r gives

$$2(m+1)^r - m^r - (m+2)^r = r(1-r)\int_0^1 \int_0^1 (m+s+t)^{r-2}\, ds\, dt \le r(1-r)m^{r-2}.$$

It follows that $\alpha_m (m+1)^{-\gamma} \le 2r(1-r)N^{-r}m^{r-\gamma-1}$. Since $r-\gamma-1 \in (-1,0)$, $\sum_{m=1}^{N-2} m^{r-\gamma-1} \le N^{r-\gamma}/(r-\gamma)$, which proves (8.4).

Take $\gamma = 1/2$ and write $A_{N,r}u = F_{N,r} * u$, where $u = g - at$. The ordinary pinned Fejér error proved in Section 4 is at most $14\sqrt{2}(L+2)/(3\sqrt{m+1})$ for $m \ge 1$. For $m = 0$ its pinned mean is zero and $\|u\|_\infty \le L+2$, so that bound is still valid. Mixing the errors by (8.4) yields

$$\|A_{N,r}u - A_{N,r}u(0) - u\|_\infty \le D_r(L+2)N^{-1/2}.$$

Choose $N = \lceil 64 D_r^2 (L+2)^2 \epsilon^{-2} \rceil$, so this error is at most $\epsilon/8$ and $N \le Q_r(L+2)^2\epsilon^{-2}$. For $0 < |h| \le N-1$ put

$$x_h = (|h|/N)^r, \qquad b_h = (1-x_h)\beta_h, \qquad r_h = \min(1/2, x_h).$$

Keep $b_0 = (1-\epsilon/8)a$ and $r_0 = \epsilon/8$. The pinned polynomial $q = b_0 t + A_{N,r}u - A_{N,r}u(0) = \sum_{|h|\le N-1} b_h v_h$ has $\|q-g\|_\infty \le 3\epsilon/8$.

Radial monotonicity makes Ψ nonincreasing; it is nonnegative by definition. On the hth coefficient box, $|b_h + z| \le 2 - x_h$; hence $f(b_h+z) \ge \exp[-\Psi(x_h)]$. The local factor against the centered uniform density is at least $x_h \exp[-\Psi(x_h)]$, since $2r_h \ge x_h$. The decreasing function $t \mapsto \Psi(t^r)$ obeys

$$\sum_{h=1}^{N-1} \Psi((h/N)^r) \le N \int_0^{(N-1)/N} \Psi(t^r)\, dt \le N I_r.$$

Together with the factorial estimate used in (4.7), the two-sign product is at least

$$\left(\frac{(N-1)!}{N^{N-1}}\right)^{2r} \exp\left[-2\sum_{h=1}^{N-1} \Psi((h/N)^r)\right] \ge \exp[-(2r+2I_r)N].$$

The endpoint factor is at least $(\epsilon/4)\exp[-\Psi(\epsilon/8)]$. Moreover (8.1) implies

$$I_r \ge \int_0^{s^{1/r}} \Psi(t^r)\, dt \ge s^{1/r}\Psi(s), \qquad \Psi(\epsilon/8) \le 8^{1/r} I_r \epsilon^{-1/r}.$$

The bounded symmetric coefficients satisfy the same endpoint-chord exponential bound as in Lemma 2.1, so K_f converges uniformly by that proof. Lemma 2.3 gives the centered finite-sum comparison. The Gaussian reference and the limiting tail argument therefore retain the bounds (3.1) and (3.3). The auxiliary uniform variables in finite density matching are themselves log-concave, so

that step is unchanged. Finite density matching, followed by the centered comparison, now bounds the low-frequency event of radius $\epsilon/4$ below by

$$\frac{\epsilon}{4}\exp[-\Psi(\epsilon/8)-(2r+2I_r)N-1024\epsilon^{-2}].$$

The independent tail event of radius $\epsilon/4$ contributes $\exp(-4096\epsilon^{-2})$. Their intersection has norm at most $7\epsilon/8<\epsilon$. Since $1/r<2$ and $\epsilon\le 1$, $\epsilon^{-1/r}\le\epsilon^{-2}$. Inserting the bound on N and $\log(4/\epsilon)\le(\log 4+1)\epsilon^{-2}$ proves (8.2) with the stated constant.

For the upper bound and an arbitrary continuous complex center, define a,u,β_h as in the proof of Theorem 1.1. The endpoint/Parseval inequality (5.1) is unchanged. Every interval of length 2η has mass at most $2B\eta=1/2$. Thus the indicators $\mathbf{1}_{\{|X_h-\operatorname{Re}\beta_h|\ge\eta\}}$ have independent success probabilities at least p_0. Their sum on the small-ball event is at most $16\pi^2\eta^{-2}N^2\epsilon^2$. With $N=\lfloor\delta_0\epsilon^{-2}\rfloor$, this is at most $p_0N/2$. The explicit exponential-moment argument in the main proof bounds its probability by $\mathrm{e}^{-p_0N/8}$. For $\epsilon^2\le\delta_0/2$ this is at most $\exp[-(p_0\delta_0/16)\epsilon^{-2}]$, as asserted. □

An unweighted integrable boundary logarithm is sufficient for (8.1): $t^r\ge t$ and monotonicity give $\Psi(t^r)\le\Psi(t)$. In particular every polynomial bound $f(x)\ge d(2-|x|)^\kappa$, $d>0$, $\kappa\ge 0$, works, since $\Psi(s)\le\max(0,-\log d)+\kappa\log(1/s)$. This includes the symmetric Jacobi densities

$$f_\kappa(x)=\frac{\Gamma(\kappa+3/2)}{2^{2\kappa+1}\sqrt{\pi}\,\Gamma(\kappa+1)}(4-x^2)^\kappa\mathbf{1}_{[-2,2]}(x),\qquad \kappa\ge 0.$$

Their normalization follows from the beta integral, and concavity of $\log(4-x^2)$ proves log-concavity. The cases $\kappa=0$ and $\kappa=1/2$ are uniform and Sato–Tate, respectively.

The density hypotheses also include laws that are not log-concave. For example,

$$f(x)=\begin{cases}1/3, & |x|<1,\\ 1/6, & 1\le|x|<2,\\ 0, & |x|\ge 2,\end{cases}$$

is the mixture of a uniform law on $[-1,1]$ of weight $1/3$ and a uniform law on $[-2,2]$ of weight $2/3$. It has an interior jump and therefore is not log-concave, but its boundary function satisfies $\Psi(s)=\log 6$ for $0<s\le 1$, so the weighted condition holds for every permitted r.

There is also an exact threshold for the uniform lower order in a faster-vanishing family. For $A>0$ and $a>0$, put

$$f_{A,a}(x)=Z_{A,a}^{-1}\exp[-A(2-|x|)^{-a}]\mathbf{1}_{(-2,2)}(x). \tag{8.5}$$

This density is even, positive in the interior, bounded and log-concave: $2-|x|$ is concave, and $y\mapsto -Ay^{-a}$ is increasing and concave on positive y. Its boundary function is at most $|\log Z_{A,a}|+As^{-a}$. If $a<2$, choose $1/2<r<\min(1,1/a)$; then $ar<1$ and (8.1) holds. Proposition 8.1 proves the uniform ϵ^{-2} logarithmic order throughout this range.

For $a\ge 2$, Proposition 9.1 supplies a fixed-center obstruction in every prescribed $\mathcal{H}_L$, including the critical case $a=2$. Thus $a<2$ is the exact threshold for the existence of a uniform lower bound of order $\exp(-C\epsilon^{-2})$ in this family. The obstruction does not provide a matching lower bound or an exact critical small-deviation asymptotic.

Corollary 8.2. *Fix $0<\gamma<1/2$ and $L\ge 1$. Let f satisfy the density hypotheses of Proposition 8.1, but suppose more generally that (8.1) holds for some $\gamma<r<1$. Then*

$$\inf_{g\in\mathcal{H}_{\gamma,L}}\mathbb{P}(\|K_f-g\|_\infty<\epsilon)$$

has negative logarithm of order $\epsilon^{-1/\gamma}$, for all sufficiently small ϵ, with constants depending on the fixed parameters and law. For the family (8.5), this conclusion holds whenever $a < 1/\gamma$. Theorem 10.1 proves failure at and beyond equality in every prescribed $\mathcal{H}_{\gamma,L}$.

Proof. The mixture and its moment estimate (8.4) hold for every $0 < \gamma < r < 1$, including $r \le 1/2$. With D_γ from Section 7 and $T_{\gamma,r} = 3 + 2r(1-r)/(r-\gamma)$, the mixed pinned error is at most $D_\gamma T_{\gamma,r}(L+2)N^{-\gamma}$. For the degree-zero component the same estimate follows from $\|u\|_\infty \le L+2$, so no mixture term is omitted. Choose

$$N = \left\lceil [8D_\gamma T_{\gamma,r}(L+2)]^{1/\gamma}\epsilon^{-1/\gamma} \right\rceil .$$

The density matching proof gives the same nonzero-coefficient factor $\exp[-(2r+2I_r)N]$ and endpoint factor $(\epsilon/4)\exp[-\Psi(\epsilon/8)]$. Now $\Psi(\epsilon/8) \le 8^{1/r} I_r \epsilon^{-1/r}$ is absorbed by $\epsilon^{-1/\gamma}$ since $r > \gamma$; the Gaussian and tail cost $5120\epsilon^{-2}$ is absorbed since $\gamma < 1/2$. For example a lower-bound constant is

$$(2r+2I_r)\{[8D_\gamma T_{\gamma,r}(L+2)]^{1/\gamma} + 1\} + 5120 + \log 4 + 1 + 8^{1/r} I_r.$$

The signed-spike upper proof in Section 7 uses only independent centered coefficients in $[-2,2]$. It therefore applies to f without change, with the same c_γ and small-radius threshold. This proves matching worst-case order.

For (8.5) and $a < 1/\gamma$, choose $\gamma < r < \min(1, 1/a)$; then $ar < 1$ makes the weighted integral finite. The separate negative assertion is Theorem 10.1. □

9 Flat-edge threshold and critical-rate bounds

The sufficient condition in Section 8 excludes the critical case $a = 2$ of (8.5). In that family the exclusion reflects an actual obstruction. Infinitely many boundary coefficients can force an additional harmonic-sum cost while classical Rudin–Shapiro cancellation keeps their center 1/2-Hölder. Rapidly vanishing origin distributions and weighted-sum small deviations belong to classical Laplace-transform theory; see, for example, Rozovsky [19]. The elementary estimate constructs a fixed admissible harmonic boundary center.

Proposition 9.1. *For every $L \ge 1$, there is a fixed $g_L \in \mathcal{H}_L$ such that, for the coefficient law (8.5) with $A > 0$ and $a \ge 2$, some $c_{A,a,L} > 0$ and $\epsilon_{A,a,L} > 0$ satisfy*

$$\mathbb{P}(\|K_f - g_L\|_\infty < \epsilon) \le \exp[-c_{A,a,L}\epsilon^{-2}\log^2(1/\epsilon)], \qquad 0 < \epsilon \le \epsilon_{A,a,L}. \tag{9.1}$$

In particular no lower bound $\exp(-C\epsilon^{-2})$ with finite C holds even at this one center for all sufficiently small radii. Consequently, within (8.5), a uniform lower bound of that order on $\mathcal{H}_L$ holds if and only if $a < 2$, for every fixed $L \ge 1$.

Proof. Use the polynomials P_m, Q_m in the proof of (7.3). The first 2^m coefficients of P_{m+1} equal those of P_m, so they define infinite signs $s_0, s_1, \ldots$; put $r_h = s_{h-1}$. A prefix of P_m or Q_m decomposes recursively into signed, shifted complete P_j or Q_j blocks, with at most one block of each dyadic size. An entire block is kept intact. Otherwise a prefix in the first half is a prefix of P_{m-1}; a longer one consists of that whole first block and a signed shifted prefix of Q_{m-1}. Iteration follows its binary length expansion. Each complete block of length 2^j has norm at most $\sqrt{2\cdot 2^j}$ by the complementary identity. Thus, uniformly on $|z| = 1$,

$$\left|\sum_{h=1}^{n} r_h z^h\right| \le C_{\mathrm{RS}}\sqrt{n}, \qquad C_{\mathrm{RS}} = \frac{\sqrt{2}}{1-2^{-1/2}} < 5. \tag{9.2}$$

The shift from index zero multiplies by z and preserves the norm.

Consider the deterministic center

$$g(t) = \frac{2}{\pi} \sum_{h \geq 1} \frac{r_h \sin(2\pi h t)}{h}.$$

Its uniform convergence and regularity can be checked directly. For $S_n(t) = \sum_{h=1}^{n} r_h \mathrm{e}^{2\pi i h t}$, summation by parts gives

$$\sum_{h=M+1}^{H} \frac{r_h \mathrm{e}^{2\pi i h t}}{h} = \frac{S_H(t)}{H} - \frac{S_M(t)}{M} + \sum_{h=M}^{H-1} S_h(t) \left(\frac{1}{h} - \frac{1}{h+1} \right).$$

The first term tends uniformly to zero as $H \to \infty$, and $\sum_{h=M}^{\infty} \sqrt{h}/(h(h+1)) \leq 3M^{-1/2}$. The complex-series tail is therefore at most $4C_{\mathrm{RS}} M^{-1/2}$, and the sine-series tail at most $8C_{\mathrm{RS}}/(\pi\sqrt{M})$. The center is consequently continuous, real, zero at both endpoints, and has normalized coefficients $\beta_h = \beta_{-h} = 2r_h$ for $h \geq 1$.

Its degree-M partial sum has derivative $4\sum_{h=1}^{M} r_h \cos(2\pi h t)$, of norm at most $4C_{\mathrm{RS}}\sqrt{M}$. For $\delta = |t - s| \in (0, 1]$, choose $M = \lfloor \delta^{-1} \rfloor$. The partial sum and its two tails give

$$|g(t) - g(s)| \leq 4C_{\mathrm{RS}}\sqrt{M}\,\delta + \frac{16C_{\mathrm{RS}}}{\pi\sqrt{M}} \leq \left(4 + \frac{16\sqrt{2}}{\pi} \right) C_{\mathrm{RS}}\sqrt{\delta} < 80\sqrt{\delta}.$$

Thus we may fix $L_0 = 80$.

Extend g periodically. Its zero endpoints give circular 1/2-Hölder constant at most $\sqrt{2}L_0$, by the same wrapped-pair argument as in (4.1). For a positive integer q, put $g_q(t) = q^{-1} g(qt)$. This has Hölder constant at most $\sqrt{2}L_0/\sqrt{q}$, zero endpoints, and normalized coefficients

$$\beta_{qk} = \beta_{-qk} = 2r_k \quad (k \geq 1), \qquad \beta_h = 0 \quad (q \nmid h).$$

The formula follows directly from its sine series, or by partitioning the Fourier integral into q periods. Choose the fixed integer $q = \max\{1, \lceil 2L_0^2/L^2 \rceil\}$, and let $g_L = g_q$. It belongs to $\mathcal{H}_L$ and is independent of the radius.

On $\|K_f - g_q\|_\infty < \epsilon$, the endpoint identity gives $|X_0| < \epsilon$. Thus $D = K_f - tX_0 - g_q$ has norm less than 2ϵ. Parseval, retaining positive frequencies qk for $1 \leq k \leq M$, gives

$$\sum_{k=1}^{M} \frac{(X_{qk} - 2r_k)^2}{(qk)^2} \leq 16\pi^2\epsilon^2.$$

Write $d_k = 2 - r_k X_{qk} = |X_{qk} - 2r_k| \in (0, 4)$ almost surely, and $H_M = \sum_{k=1}^{M} 1/k$. Cauchy's inequality implies

$$\left(\frac{H_M}{q} \right)^2 \leq \left(\sum_{k=1}^{M} \frac{d_k^2}{(qk)^2} \right) \left(\sum_{k=1}^{M} d_k^{-2} \right), \qquad \sum_{k=1}^{M} d_k^{-2} \geq \frac{H_M^2}{16\pi^2 q^2 \epsilon^2}.$$

For $a \geq 2$ and $\delta_x = 2 - |x| \in (0, 2]$, $\delta_x^{-a} \geq 2^{2-a}\delta_x^{-2}$. Also $\delta_{X_{qk}} \leq d_k$ for either sign r_k. With $A_* = A2^{2-a}$ and $Z = Z_{A,a}$, the marginal density is at most $Z^{-1}\exp(-A_*/d_k^2)$. Independence of these M coefficients, integration over their finite necessary set, and the full box volume 4^M therefore give

$$\mathbb{P}(\|K_f - g_q\|_\infty < \epsilon) \leq (4/Z)^M \exp\left[-\frac{A_* H_M^2}{16\pi^2 q^2 \epsilon^2} \right]. \tag{9.3}$$

All unselected coefficients were discarded through a necessary-event upper bound; no infinite density product is used.

Take $M = \lfloor \epsilon^{-2} \rfloor$ for $\epsilon \le 1/4$. Then $H_M \ge \log(M+1) \ge 2\log(1/\epsilon)$ and $M \le \epsilon^{-2}$. Put $b = \log(4/Z) > 0$. The right side of (9.3) is at most

$$\exp\left\{\epsilon^{-2}\left[b - \frac{A_*}{4\pi^2 q^2}\log^2(1/\epsilon)\right]\right\}.$$

For $\epsilon \le \min\{1/4, \exp[-\sqrt{8\pi^2 q^2 b/A_*}]\}$, this proves (9.1) with $c_{A,a,L} = A_*/(8\pi^2 q^2)$. The extra unbounded factor $\log^2(1/\epsilon)$ excludes every finite-C lower bound of order $\exp(-C\epsilon^{-2})$ at g_L. For $a < 2$, Proposition 8.1 already proves the positive assertion, completing the stated threshold. □

9.1 A lower bound at the critical law

For $a = 2$, a borderline positive taper supplies a complementary uniform lower bound. Write K_A for the process with density $f_{A,2}$, and $Z_A = Z_{A,2}$.

Theorem 9.2. *For every fixed $A > 0$ and $L \ge 1$, there is an explicitly finite $C_{A,L}$ such that*

$$\mathbb{P}(\|K_A - g\|_\infty < \epsilon) \ge \exp[-C_{A,L}\epsilon^{-2}\log^3(\mathrm{e}/\epsilon)], \qquad g \in \mathcal{H}_L, \quad 0 < \epsilon \le 1/4. \tag{9.4}$$

For the single fixed center g_L of Proposition 9.1, therefore,

$$c_{A,L}\epsilon^{-2}\log^2(1/\epsilon) \le -\log \mathbb{P}(\|K_A - g_L\|_\infty < \epsilon) \le C_{A,L}\epsilon^{-2}\log^3(\mathrm{e}/\epsilon) \tag{9.5}$$

at every sufficiently small radius. The upper and lower estimates have the displayed logarithmic powers.

Proof. Use the mixture from Section 8 with $w_h = (1 - \sqrt{h/N})_+$, $N \ge 2$, and $\alpha_m = (m+1)(w_m - 2w_{m+1} + w_{m+2})$. Its positivity, unit sum and exact Fejér representation hold also at this borderline exponent. For $1 \le m \le N-2$, the second-difference integral gives

$$\alpha_m(m+1)^{-1/2} \le \tfrac{1}{4}N^{-1/2}(m+1)^{1/2}m^{-3/2} \le \tfrac{1}{2}N^{-1/2}/m.$$

The $m = 0$ and $m = N-1$ contributions are at most $2N^{-1/2}$ and $N^{-1/2}$, respectively. Hence, with $H_0 = 0$,

$$\sum_{m=0}^{N-1} \alpha_m(m+1)^{-1/2} \le N^{-1/2}[3 + \tfrac{1}{2}H_{N-2}] \le N^{-1/2}[4 + \tfrac{1}{2}\log N]. \tag{9.6}$$

Put

$$D = \frac{14\sqrt{2}}{3}, \quad S = 8D(L+2), \quad B = 256S^2, \quad \ell = \log(\mathrm{e}/\epsilon), \quad N = \lceil B^2\epsilon^{-2}\ell^2 \rceil.$$

These constants satisfy

$$\sqrt{N} \ge B\epsilon^{-1}\ell, \quad N \le 2B^2\epsilon^{-2}\ell^2, \quad \log N \le \log 2 + 2\log B + 4\ell.$$

In particular $4 + \frac{1}{2}\log N \le (7 + \log B)\ell$ and $B \ge S(7 + \log B)$: use $S \ge 1$, $7 + \log 256 + 2\log S \le 13S$ and $B/S = 256S$. For $u = g - g(1)t$, the ordinary pinned Fejér error is at most $D(L+2)/\sqrt{m+1}$, including $m = 0$. Its mixture and (9.6) consequently give error at most $\epsilon/8$.

For $0 < |h| \le N-1$, put

$$x_h = \sqrt{|h|/N}, \quad b_h = (1 - x_h)\beta_h, \quad r_h = \min(1/2, x_h),$$

and put $b_0 = (1-\epsilon/8)g(1)$, $r_0 = \epsilon/8$. The pinned polynomial $q = \sum_{|h|\le N-1} b_h v_h$ has $\|q-g\|_\infty \le 3\epsilon/8$. For $|z| \le r_h$, $h \ne 0$, $|b_h + z| \le 2 - x_h$, so comparison with the centered uniform law on $[-r_h, r_h]$ pays at least

$$x_h \exp[-AN/|h| - z_A], \qquad z_A = \max(0, \log Z_A).$$

The factorial estimate, over both frequency signs, gives the nonzero-frequency product at least

$$\exp[-(1+2z_A)N - 2ANH_{N-1}].$$

For $h = 0$ the factor is at least $(\epsilon/4)\exp[-64A\epsilon^{-2} - z_A]$.

The centered uniform acceptance event of radius $\epsilon/4$ has probability at least $\exp(-1024\epsilon^{-2})$ by the peakedness and Gaussian lemmas. The independent tail of K_A has acceptance probability at least $\exp(-4096\epsilon^{-2})$. Indeed the even log-concave density $f_{A,2}$ dominates $N(0,4)$ in central peakedness, exactly as in Section 8. The deterministic and two random errors total $7\epsilon/8$. Thus the strict-ball probability is at least

$$\frac{\epsilon}{4}\exp[-z_A - (1+2z_A)N - 2ANH_{N-1} - (5120+64A)\epsilon^{-2}].$$

Since $H_{N-1} \le (6 + 2\log B)\ell$, one admissible constant is

$$C_{A,L} = 2B^2[1 + 2z_A + 2A(6 + 2\log B)] + 5120 + 64A + \log 4 + 1 + z_A.$$

This proves (9.4); combining it with (9.1) gives (9.5). □

The extra harmonic sum in (9.6) explains the loss in this lower-bound proof. Determining the exact logarithmic order at the fixed boundary center remains a separate problem.

9.2 A limitation of the positive-mixture error payment

The unresolved logarithm cannot be removed merely by optimizing the positive Fejér mixture while retaining its present termwise Hölder error bound. This statement concerns a proof certificate, not the true small-ball probability.

Proposition 9.3. *Let $\lambda_0, \dots, \lambda_N \ge 0$ sum to one, and define*

$$E = \sum_{m=0}^{N} \frac{\lambda_m}{\sqrt{m+1}}, \qquad x_h = \sum_{m=0}^{N} \lambda_m \min\{1, h/(m+1)\} \quad (h \ge 1).$$

For $0 < \tau \le 1/4$, suppose $E \le \tau$, and put

$$J = \lfloor \log_2(\tau^{-2}) \rfloor - 1, \qquad C_* = \frac{2}{1 - 2^{-1/2}}.$$

Then $2^J \le N$ and

$$\sum_{h=1}^{2^J} x_h^{-2} \ge \frac{J^3}{2C_*^2 E^2} \ge \frac{J^3}{2C_*^2\tau^2}. \tag{9.7}$$

Proof. Since $E \ge (N+1)^{-1/2}$, we have $N+1 \ge \tau^{-2}$ and $2^J \le \tau^{-2}/2 \le N$. Set $y_i = x_{2^i}/\sqrt{2^i}$. With $a = m+1$ and $e_m = \lambda_m/\sqrt{a}$,

$$y_i = \sum_m e_m \min\{\sqrt{2^i/a}, \sqrt{a/2^i}\}.$$

Choose an integer k with $2^k \le a < 2^{k+1}$. The terms for $i \le k$ are bounded by $2^{(i-k)/2}$, and those for $i \ge k+1$ by $2^{(k+1-i)/2}$. Their sum over all integer i is at most C_*. Nonnegativity therefore gives $\sum_{i=1}^{J} y_i \le C_* E$. On $2^{i-1} < h \le 2^i$, monotonicity gives $x_h \le x_{2^i}$, so

$$\sum_{h=1}^{2^J} x_h^{-2} \ge \tfrac{1}{2} \sum_{i=1}^{J} y_i^{-2} \ge \frac{J^3}{2(\sum_{i=1}^{J} y_i)^2}.$$

The last step is Jensen for the convex function t^{-2}. This proves the claim. □

For this mixture the inward multiplier margin is exactly x_h, and the pinned Hölder approximation estimate used above pays error at most $D(L+2)E$, where $D = 14\sqrt{2}/3$. To certify error $\epsilon/8$ by that estimate requires $E \le \tau = \epsilon/[8D(L+2)]$. Equation (9.7) then forces an inverse-square margin payment of order at least $\epsilon^{-2}\log^3(1/\epsilon)$, as appears in the current shifted finite-box density comparison. The actual approximation error of a particular center may be smaller than this paid bound. A different center-dependent approximation or density integral can escape the proposition; it neither proves log-cubed small-ball sharpness nor excludes a log-squared probability lower bound.

10 The exact joint roughness–density threshold

Corollary 8.2 gives a sufficient range for rough centers. The equality case in the flat-edge family can also be decided. The following construction uses classical Rudin–Shapiro cancellation and positive Fejér tests to force a separate density penalty at every geometric frequency scale.

Theorem 10.1. *Fix $0 < \gamma < 1/2$, $L \ge 1$, $A > 0$, and put $a = 1/\gamma$. There is one fixed real zero-endpoint $g_{\gamma,L} \in \mathcal{H}_{\gamma,L}$ such that, for the law $f_{A,a}$ in (8.5),*

$$\mathbb{P}(\|K_f - g_{\gamma,L}\|_\infty < \epsilon) \le \exp[-c_{A,\gamma,L}\epsilon^{-1/\gamma}\log(1/\epsilon)] \tag{10.1}$$

for every sufficiently small ϵ, with $c_{A,\gamma,L} > 0$. The same center excludes a bound of the form

$$\mathbb{P}(\|K_f - g_{\gamma,L}\|_\infty < \epsilon) \ge \exp(-C\epsilon^{-1/\gamma})$$

with finite C for all sufficiently small radii, for every law $f_{A,a'}$ with $a' \ge 1/\gamma$. Consequently, within this family a uniform bound

$$\inf_{g \in \mathcal{H}_{\gamma,L}} \mathbb{P}(\|K_f - g\|_\infty < \epsilon) \ge \exp(-C\epsilon^{-1/\gamma})$$

with a finite constant and a sufficiently small radius exists exactly when $a' < 1/\gamma$.

Proof. For $j \ge 1$, put

$$N_j = 4^j, \quad J_j = 2^{\lfloor 2\gamma \log_2 N_j \rfloor}, \quad m_j = N_j/J_j, \quad B_j = 4N_j.$$

Here J_j, m_j are positive integers, $\frac{1}{2}N_j^{2\gamma} < J_j \le N_j^{2\gamma}$, and $J_j m_j = N_j$. Choose independently for each band a deterministic Rudin–Shapiro polynomial $R_{J_j}(z) = \sum_{k=0}^{J_j-1} \sigma_k z^k$ of norm at most $\sqrt{2J_j}$, using the complete recursion in (7.3). No consistency between different finite sign blocks is needed.

In band j, assign $r_h = \sigma_k$ at

$$h = B_j + 2m_j k + \ell, \qquad 0 \le k < J_j, \quad |\ell| \le m_j - 1.$$

Subblocks leave a one-frequency gap; all band frequencies lie in $[3N_j, 6N_j]$, while the next band lies in $[12N_j, 24N_j]$. Thus every assignment is unambiguous. With $D_{m-1}(t) = \sum_{\ell=-m+1}^{m-1} \mathrm{e}^{2\pi i \ell t}$,

$$T_j(t) := \sum_{h \text{ in band } j} r_h \mathrm{e}^{2\pi i h t} = \mathrm{e}^{2\pi i B_j t} D_{m_j-1}(t) R_{J_j}(\mathrm{e}^{2\pi i 2m_j t}), \qquad \|T_j\|_\infty \le 2\sqrt{2} m_j \sqrt{J_j}.$$

Define

$$g_j(t) = \frac{2}{\pi} \sum_{h \text{ in band } j} \frac{r_h \sin(2\pi h t)}{h}, \qquad g(t) = \sum_{j \ge 1} g_j(t).$$

If $b_j \ge 3N_j$ is the smallest band frequency, write $T_j(z) = z^{b_j} Q_j(z)$. Maximum modulus gives $|Q_j(z)| \le \|T_j\|_\infty$ for $|z| \le 1$, and

$$\sum_{h \text{ in band } j} \frac{r_h z^h}{h} = z^{b_j} \int_0^1 t^{b_j - 1} Q_j(tz)\, dt.$$

Therefore

$$\|g_j\|_\infty \le \frac{4\sqrt{2}}{3\pi\sqrt{J_j}} \le \frac{8}{3\pi} N_j^{-\gamma} < N_j^{-\gamma}, \qquad \left\|g_j'\right\|_\infty \le 16 N_j^{1-\gamma}.$$

This proves uniform convergence, zero endpoints, reality, and exact normalized coefficients $\beta_h = \beta_{-h} = 2r_h$ on the bands, zero elsewhere. No prefix estimate for a truncated band is assumed.

Splitting bands at $N_k \le |t-s|^{-1} < N_{k+1}$ proves a Hölder-γ bound with constant

$$L_\gamma^* = \frac{16}{1 - 4^{-(1-\gamma)}} + \frac{2}{1 - 4^{-\gamma}}.$$

Indeed low-band increments are bounded by $16|t-s| \sum_{j \le k} N_j^{1-\gamma}$, and the two tails by $2 \sum_{j>k} N_j^{-\gamma}$; their geometric sums give the two displayed terms. For $|t-s| \ge 1/4$, the full tail bound gives the second term directly. Extend g periodically. Its circular constant is at most $2^{1-\gamma} L_\gamma^*$ by splitting wrapped pairs at the common zero endpoint. Fix an integer

$$q \ge \max\left\{1, (2^{1-\gamma} L_\gamma^* / L)^{1/(1-\gamma)}\right\}$$

and set $g_{\gamma,L}(t) = q^{-1} g(qt)$. Its Hölder constant is at most $2^{1-\gamma} L_\gamma^* q^{\gamma-1} \le L$, and its normalized boundary coefficients at qh are still $2r_h$. Thus the center is fixed in the required class.

For a band, suppress indices and put $N = Jm$, $B = 4N$. The analytic test

$$P_j(t) = \mathrm{e}^{2\pi i B t} F_{m-1}(t) R_J(\mathrm{e}^{2\pi i 2mt})$$

has coefficient $r_h w_h$ on the band, with $w_h = 1 - |\ell|/m \in (0, 1]$, and

$$W_j := \sum_h w_h = N, \qquad \|P_j\|_1 \le \sqrt{2J}.$$

The derivative satisfies

$$\|P_j'\|_1 \le DN\sqrt{J}, \qquad D = 4\pi(3 + \mathrm{e})\sqrt{2}. \tag{10.2}$$

For completeness, $F_{m-1} = |D_m^+|^2/m$ with $D_m^+ = \sum_{k=0}^{m-1} \mathrm{e}^{2\pi i k t}$; Cauchy–Schwarz gives $\|F_{m-1}'\|_1 \le 4\pi m$. An analytic polynomial R of degree $d \ge 1$ obeys $\|R'\|_{|z|=1} \le \mathrm{e} d \|R\|_{|z|=1}$: reverse the polynomial to bound its modulus outside the unit disk by $|z|^d \|R\|$, and use Cauchy's formula on radius $1/d$. For degree zero the derivative vanishes. Differentiating the three factors now bounds $\|P_j'\|_1$ by

$$2\pi B\sqrt{2J} + 4\pi m\sqrt{2J} + 4\pi \mathrm{e} m(J-1)\sqrt{2J},$$

which proves (10.2), including $m = 1$ or $J = 1$.

On the small-ball event, $|X_0| < \epsilon$ and $U = K_f - tX_0 - g_{\gamma,L}$ has norm less than 2ϵ. For a selected positive frequency define $d_h = 2 - r_h X_{qh} > 0$. Fourier multiplication gives the exact nonnegative identity

$$S_j := \sum_{h \text{ in band } j} w_h d_h = \int_0^1 U(t) \overline{\frac{d}{dt} P_j(qt)}\, dt \le 2qD\epsilon N_j \sqrt{J_j}.$$

In fact $\widehat{U}(qh) = (X_{qh} - 2r_h)/(2\pi i q h)$, and the conjugate derivative coefficient is $-2\pi i q h\, r_h w_h$, giving precisely $w_h d_h$. The L^1 derivative norm after integer dilation is $q\|P_j'\|_1$, so no further factor appears.

For $a = 1/\gamma$, weighted Jensen gives

$$\sum_{h \text{ in band } j} d_h^{-a} \ge \sum_h w_h d_h^{-a} \ge \frac{W_j^{a+1}}{S_j^a} \ge \frac{N_j}{(2qD\epsilon\sqrt{J_j})^a} \ge c_* \epsilon^{-a}, \qquad c_* = (2qD)^{-a}.$$

The last inequality uses $J_j^{a/2} \le N_j$. The actual distance to the coefficient boundary is at most d_h, so its density is at most $Z_{A,a}^{-1} \exp(-A d_h^{-a})$.

Select the first K bands, containing

$$V_K = \sum_{j=1}^K J_j(2m_j - 1) \le 2 \sum_{j=1}^K N_j \le \tfrac{8}{3} N_K$$

independent positive-frequency coefficients. Integrate just these finite marginals over their necessary set; the enclosing box has volume at most 4^{V_K}. It follows that

$$\mathbb{P}(\|K_f - g_{\gamma,L}\|_\infty < \epsilon) \le (4/Z_{A,a})^{V_K} \exp[-A c_* K \epsilon^{-a}].$$

This uses independence of scalar coefficients, not of process increments.

Put

$$b = \log(4/Z_{A,a}) > 0, \qquad K = \lfloor \log_4 \epsilon^{-a} \rfloor, \qquad K_0 = \max(2, \lceil 16b/(3Ac_*) \rceil).$$

For $0 < \epsilon \le \epsilon_0$, where $\epsilon_0 = \min(1/4, 4^{-\gamma(K_0+1)})$, the conditions are

$$N_K \le \epsilon^{-a}, \qquad K \ge K_0, \qquad K \ge \frac{a \log(1/\epsilon)}{2 \log 4}.$$

Absorbing the finite volume term therefore gives

$$\mathbb{P}(\|K_f - g_{\gamma,L}\|_\infty < \epsilon) \le \exp\left[-\frac{Aac_*}{4 \log 4} \epsilon^{-a} \log(1/\epsilon) \right].$$

For $a' \ge a$, the inequality $\delta^{-a'} \ge 2^{a-a'} \delta^{-a}$ on $(0, 2]$ applies the same proof with A replaced by $A 2^{a-a'}$ and with the actual normalization constant. The center depends only on γ, L, not on the law parameters or the radius. The positive assertion for $a' < 1/\gamma$ is Corollary 8.2; the extra logarithm excludes every finite-constant lower bound at and beyond equality. □

Together with Proposition 9.1, this determines the flat-edge lower-order property for all $0 < \gamma \le 1/2$: its threshold is $a < 1/\gamma$. The theorem supplies a one-sided critical obstruction below $\gamma = 1/2$; it does not identify the exact critical logarithmic order for those rough centers.

11 Supercritical laws and smoother centers

The same positive taper also covers the range beyond the joint threshold. Its approximation order becomes slower when its exponent is below the Hölder exponent, but the resulting density cost has exactly the required power. This also distinguishes the critical half-Hölder case from smoother centers.

Theorem 11.1. *Fix $0 < \gamma < 1$, $L \geq 1$, $A > 0$, and $a > 1/\gamma$. For the law $f_{A,a}$ of (8.5), there are explicitly finite constants C_a, C_2 such that*

$$\mathbb{P}(\|K_f - g\|_\infty < \epsilon) \geq \exp[-C_a\epsilon^{-a} - C_2\epsilon^{-2}], \qquad g \in \mathcal{H}_{\gamma,L}, \quad 0 < \epsilon \leq 1/4. \tag{11.1}$$

If $a > \max(2, 1/\gamma)$, a single fixed smooth center in $\mathcal{H}_{\gamma,L}$ gives a matching upper bound $\exp(-c\epsilon^{-a})$ at all sufficiently small radii. Consequently the worst-case negative logarithm has exact order ϵ^{-a} in this range. For $1/2 < \gamma < 1$ and $a = 2$, it instead has exact order ϵ^{-2}.

Proof. Choose $1/a < r < \gamma$. Use w_h and α_m from Section 8. The positive-kernel representation is unchanged, but the mixture moment is now

$$\sum_{m=0}^{N-1} \alpha_m (m+1)^{-\gamma} \leq TN^{-r}, \qquad T = 3 + 2r(1-r)\left(1 + \frac{1}{\gamma - r}\right). \tag{11.2}$$

Indeed the initial term is at most $2N^{-r}$, and the last term is at most $N^{-\gamma} \leq N^{-r}$. For $1 \leq m \leq N-2$, the same second-difference integral gives

$$\alpha_m (m+1)^{-\gamma} \leq 2r(1-r)N^{-r}m^{r-\gamma-1}.$$

The sum of $m^{-1-(\gamma-r)}$ is at most $1 + 1/(\gamma - r)$, which proves (11.2).

Put

$$E_\gamma = \frac{2}{\gamma+1} + \frac{1}{2(1-\gamma)}, \qquad D_\gamma = 2^{2-\gamma}E_\gamma, \qquad U = 8D_\gamma T(L+2), \qquad N = \left\lceil (U/\epsilon)^{1/r} \right\rceil.$$

The elementary Fejér moment calculation in Section 7 is valid for every $0 < \gamma < 1$. Thus the ordinary pinned approximation to $u = g - g(1)t$ has error at most $D_\gamma(L+2)(m+1)^{-\gamma}$, including $m = 0$. Mixing by (11.2) gives error at most $\epsilon/8$.

As before set, for $0 < |h| < N$,

$$x_h = (|h|/N)^r, \qquad b_h = (1 - x_h)\beta_h, \qquad \rho_h = \min(1/2, x_h),$$

and set $b_0 = (1 - \epsilon/8)g(1)$, $\rho_0 = \epsilon/8$. The pinned polynomial $q = \sum_{|h|<N} b_h v_h$ has $\|q - g\|_\infty \leq 3\epsilon/8$. Writing $z = \max(0, \log Z_{A,a})$, the local density factor against the centered uniform comparison is at least

$$x_h \exp[-AN^{ar}|h|^{-ar} - z] \quad (h \neq 0).$$

Now $ar > 1$, so

$$\sum_{h=1}^{N-1} h^{-ar} \leq S := 1 + \frac{1}{ar - 1}.$$

The factorial estimate and both frequency signs give a joint factor at least

$$\exp[-2rN - 2zN - 2ASN^{ar}] \geq \exp[-(2r + 2z + 2AS)N^{ar}].$$

The endpoint factor is at least $(\epsilon/4)\exp[-A8^a\epsilon^{-a}-z]$. The unchanged centered acceptance and independent tail cost at most $5120\epsilon^{-2}$ in the negative logarithm. Their two errors are $\epsilon/4$ each, so the total error is $7\epsilon/8<\epsilon$. Since $N\le 2(U/\epsilon)^{1/r}$, one possible choice in (11.1) is

$$C_a=(2r+2z+2AS)2^{ar}U^a+A8^a,\qquad C_2=5120+z+\log 4+1.$$

For the opposite estimate, choose a fixed positive integer q and put

$$g_q(t)=\frac{2}{\pi q}\sin(2\pi qt).$$

This center is real and has zero endpoints, with normalized coefficients $\beta_q=\beta_{-q}=2$ and all others zero. Its increments satisfy

$$|g_q(t)-g_q(s)|\le\min(4|t-s|,4/(\pi q))\le 4\pi^{\gamma-1}q^{\gamma-1}|t-s|^\gamma.$$

Taking $q\ge\max\{1,(4\pi^{\gamma-1}/L)^{1/(1-\gamma)}\}$ places it in $\mathcal{H}_{\gamma,L}$. On the small-ball event, $|X_0|<\epsilon$, and the qth Fourier coefficient of $K_f-tX_0-g_q$ gives $|X_q-2|\le 4\pi q\epsilon=\delta$. For $0<\delta\le 1$, this necessary event has probability at most

$$\frac{\delta}{Z_{A,a}}\exp(-A\delta^{-a})\le\exp\left[\max(0,-\log Z_{A,a})-A(4\pi q)^{-a}\epsilon^{-a}\right].$$

At sufficiently small radii this is bounded by $\exp[-\frac{A}{2}(4\pi q)^{-a}\epsilon^{-a}]$. Combining with (11.1) proves the supercritical assertion. For $\gamma>1/2, a=2$, both powers in (11.1) equal two, and the same witness supplies the reverse bound. □

For clarity, set

$$F^{A,a}_{\gamma,L}(\epsilon)=\inf_{g\in\mathcal{H}_{\gamma,L}}\mathbb{P}(\|K_f-g\|_\infty<\epsilon).$$

Theorem 11.1, the earlier subcritical results, and the inclusion $\mathcal{H}_{\gamma,L}\subseteq\mathcal{H}_{1/2,L}$ for $\gamma>1/2$ give the following complete noncritical power orders. All parameters in the table are fixed, with $A>0, L\ge 1$.

Hölder exponent	Density exponent	Order of $-\log F^{A,a}_{\gamma,L}$
$0<\gamma\le 1/2$	$a<1/\gamma$	$\epsilon^{-1/\gamma}$
$0<\gamma\le 1/2$	$a>1/\gamma$	ϵ^{-a}
$1/2<\gamma<1$	every $a>0$	$\epsilon^{-\max(2,a)}$

The excluded equality line $a=1/\gamma$ for $0<\gamma\le 1/2$ retains its proved obstruction and unresolved logarithmic order. In particular, the fixed half-Hölder RS center still has the bracket (9.5) at $a=2$. For that same fixed center, however, the new theorem and any one of its saturated coefficients give exact order ϵ^{-a} for every $a>2$.

A Why continuity alone does not give the lower order

Proposition A.1. *There is a continuous real g with $g(0)=g(1)=0$ and every normalized Fourier coefficient real in $[-1,1]$ for which no finite C satisfies $\mathbb{P}(\|K-g\|_\infty<\epsilon)\ge\exp(-C\epsilon^{-2})$ for all sufficiently small ϵ. The center is not $1/2$-Hölder.*

Proof. Put $a_j = 16^{-j}$, $M_j = a_j^{-8} = 16^{8j}$, and

$$g_j(t) = a_j \sum_{h=M_j}^{2M_j} \frac{\sin(2\pi h t)}{\pi h}, \qquad g(t) = \sum_{j\ge 1} g_j(t).$$

Because $\|g_j\|_\infty \le 2a_j/\pi$, this is a uniformly convergent continuous real series, zero at both endpoints. The frequency blocks are disjoint. Its normalized Fourier coefficients at both signs of a frequency in the jth block are a_j, and all others are zero, hence are strictly inside $[-1, 1]$.

At $t_j = 1/(8M_j)$, all sines of the current block are at least $\sqrt{2}/2$, and those of earlier blocks are nonnegative. The later blocks have total norm at most $2a_j/(15\pi)$. Therefore

$$g(t_j) \ge \left(\frac{\sqrt{2}}{4\pi} - \frac{2}{15\pi}\right) a_j \ge a_j/16. \tag{A.1}$$

For the real part of $K(t)$, the real scalar coefficients are $c_0 = t$ and $c_h = \sin(2\pi h t)/(2\pi h)$ for $h \ne 0$. If $0 < t \le 1/2$, identity (3.4) gives

$$\sum_h c_h^2 = t^2 + \frac{1}{2\pi^2} \sum_{h\ge 1} \frac{\sin^2(2\pi h t)}{h^2} = t/2.$$

Hoeffding's estimate, independence, and Fatou through the convergent symmetric sums yield $\mathbb{E}\exp(\lambda \operatorname{Re} K(t)) \le \exp(\lambda^2 t)$. Optimizing exponential Markov inequality gives

$$\mathbb{P}(\operatorname{Re} K(t) \ge A) \le \exp(-A^2/(4t)), \quad A > 0.$$

For $\epsilon_j = a_j/64$, equations (A.1) and the small-ball event imply $\operatorname{Re} K(t_j) > 3a_j/64$. Consequently

$$\mathbb{P}(\|K - g\|_\infty < \epsilon_j) \le \exp\{-\tfrac{9}{2048} a_j^2 M_j\} = \exp\{-\tfrac{9}{2048} a_j^{-6}\}.$$

Since $\epsilon_j^{-2} = 4096 a_j^{-2}$, this contradicts the proposed lower bound for every fixed C when j is sufficiently large. Finally, $g(t_j)/\sqrt{t_j} \ge (\sqrt{8}/16) a_j^{-3} \to \infty$, so this center is not $1/2$-Hölder. □

The proposition shows that the regularity hypothesis cannot be replaced by continuity alone, even strictly inside the permitted Fourier interval. Theorem 7.1 determines the corresponding uniform worst-case scale within the Hölder hierarchy.

B Center-dependent concentration obstructions

The next consequences expose the bounded-coefficient concentration mechanism behind the spike and continuity examples. They use classical exponential estimates and are not separate comparison principles.

Proposition B.1 (An increment-energy obstruction)**.** *Let $g \in C([0,1], \mathbb{C})$, let $\epsilon > 0$, and let $[s_j, t_j] \subset [0, 1/2]$ be finitely many intervals with disjoint interiors and positive lengths $\ell_j = t_j - s_j$. Then*

$$\mathbb{P}(\|K - g\|_\infty < \epsilon) \le \exp\left\{-\frac{1}{4} \sum_j \frac{(|g(t_j) - g(s_j)| - 2\epsilon)_+^2}{\ell_j}\right\}.$$

In particular, the same bound holds with the supremum of the displayed sum over all such finite collections; an infinite supremum gives probability zero.

Proof. Write $v_h = c_h + id_h$, so

$$c_0(t) = t, \quad d_0(t) = 0, \qquad c_h(t) = \frac{\sin(2\pi ht)}{2\pi h}, \quad d_h(t) = \frac{1-\cos(2\pi ht)}{2\pi h} \quad (h \neq 0).$$

For $0 \le s \le t \le 1/2$, the cosine-series identity (3.4) gives

$$\sum_h c_h(s)c_h(t) = \sum_h d_h(s)d_h(t) = s/2, \qquad \sum_h c_h(s)d_h(t) = 0.$$

The first identity for c_h was used in (7.6). For d_h, write the product of $1-\cos(2\pi hs)$ and $1-\cos(2\pi ht)$ using the cosine series at $0, s, t, t-s, t+s$; its value is $s/2$. The cross sum vanishes by pairing h and $-h$. Consequently, for arbitrary complex weights z_j, the real scalar variable

$$S = \sum_j \mathrm{Re}\{\overline{z_j}(K(t_j) - K(s_j))\}$$

has a coefficient vector of squared norm $\frac{1}{2}\sum_j |z_j|^2\ell_j$. Independence and the bounded-variable exponential estimate from Section 2 therefore give

$$\mathbb{E}\mathrm{e}^{uS} \le \exp\{u^2 \sum_j |z_j|^2\ell_j\}.$$

For the infinite series this follows by Fatou from the symmetric partial sums, which converge almost surely at all the finitely many endpoints.

Put $\Delta_j = g(t_j) - g(s_j)$ and $\eta_j = (|\Delta_j| - 2\epsilon)_+$. Take $z_j = 0$ if $\eta_j = 0$, and otherwise take $z_j = \eta_j\Delta_j/(\ell_j|\Delta_j|)$. If $E = \sum_j \eta_j^2/\ell_j > 0$, the strict small-ball event implies $S > E$, whereas $\sum_j |z_j|^2\ell_j = E$. Exponential Markov inequality at $u = 1/2$ yields $\mathbb{P}(S > E) \le \mathrm{e}^{-E/4}$. If $E = 0$ the conclusion is trivial. Taking the infimum over these bounds proves the final assertion. □

Corollary B.2 (A necessary regularity condition at a fixed center)**.** *Suppose $g \in C([0,1], \mathbb{C})$ and there exist $C, \epsilon_* > 0$ such that*

$$\mathbb{P}(\|K - g\|_\infty < \epsilon) \ge \exp(-C\epsilon^{-2}), \qquad 0 < \epsilon \le \epsilon_*.$$

Then g is $1/4$-Hölder on $[0,1]$. This is a necessary condition, not a sufficient condition or a characterization of centers with the displayed lower order.

Proof. For $s < t$ in $[0, 1/2]$, put $d = |g(t) - g(s)|$. When $0 < d \le 4\epsilon_*$, apply the preceding proposition to the single interval and take $\epsilon = d/4$. The assumed lower bound and the resulting upper bound imply

$$\exp(-16C/d^2) \le \mathbb{P}(\|K - g\|_\infty < d/4) \le \exp\{-d^2/[16(t-s)]\}, \qquad d^4 \le 256C(t-s).$$

By uniform continuity, the restriction $d \le 4\epsilon_*$ holds for all sufficiently short intervals. The longer intervals are controlled using the boundedness of g. Thus g is $1/4$-Hölder on the first half interval.

Every sample path satisfies $K(t) + \overline{K(1-t)} = K(1)$: the identity holds termwise for the symmetric partial sums and passes to their uniform limit. The assumed positive small-ball probabilities for arbitrarily small radii force the same identity for g. Indeed on such an event the uniform defect in that identity is less than 3ϵ. Reflection therefore gives the same Hölder bound on $[1/2, 1]$. Splitting an interval at $1/2$ and using $x^{1/4} + y^{1/4} \le 2^{3/4}(x+y)^{1/4}$ completes the proof. □

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