The small-regularization limit of the Sinkhorn metric tensor on bounded domains

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Abstract

Lavenant, Luckhardt, Mordant, Schmitzer and Tamanini built a Riemannian metric on probability measures whose tensor is the Hessian of the Sinkhorn divergence. For the quadratic cost on a bounded convex domain they derived by an informal computation that this tensor converges to the Otto metric tensor of quadratic transport as the regularization parameter tends to zero. We prove this limit for every probability measure with a continuous, strictly positive density on the closure of a bounded connected Lipschitz domain, in particular on every bounded convex domain, in every dimension. The convergence holds for every tangent vector of finite Otto norm, that is, every balanced continuous linear functional on the Sobolev space H^1, uniformly on compact sets, and in operator norm on tangents with square-integrable density. The Sinkhorn tensor is also dominated by a fixed multiple of the Otto tensor for all small regularization parameters. The proof bounds the symmetric Schrödinger potential above and below up to the boundary by a maximum argument that uses only the measure density of the domain, and shows through a Liouville theorem for the whole-space self-transport equation that it converges in the interior to its leading Laplace profile. Legendre duality then expresses the tensor through a nonlocal Dirichlet form, which converges with compactness to the weighted Neumann form. At smooth boundary points of a convex domain the potential exceeds this profile by half the regularization parameter to leading order, as an exact virial identity for the boundary layer shows.

Keywords. Sinkhorn divergence; entropic optimal transport; Otto metric; Schrödinger potentials; Γ-convergence; nonlocal Dirichlet forms.
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1 Introduction

1.1 The question

Let X be a compact metric space, $c \in C(X \times X)$ a cost and $\varepsilon > 0$. Entropic optimal transport and the Sinkhorn divergence are

$$\begin{aligned} \mathrm{OT}_\varepsilon(\mu,\nu) &= \inf_{\pi \in \Pi(\mu,\nu)} \Big\{ \iint c \,\mathrm{d}\pi + \varepsilon \,\mathrm{KL}(\pi \,|\, \mu \otimes \nu) \Big\}, \\ S_\varepsilon(\mu,\nu) &= \mathrm{OT}_\varepsilon(\mu,\nu) - \tfrac{1}{2}\mathrm{OT}_\varepsilon(\mu,\mu) - \tfrac{1}{2}\mathrm{OT}_\varepsilon(\nu,\nu). \end{aligned} \tag{1.1}$$

Lavenant, Luckhardt, Mordant, Schmitzer and Tamanini [6] showed that S_ε has a second-order expansion on the diagonal whose quadratic part is a metric tensor, and built from it a geodesic distance on probability measures. With the symmetric Schrödinger potential v_ε of $\mathrm{OT}_\varepsilon(\mu,\mu)$, the self-transport kernel $k_\mu(x,y) = \exp((v_\varepsilon(x) + v_\varepsilon(y) - c(x,y))/\varepsilon)$ and the operators

$$K_\mu[\phi](y) = \int_X \phi(x) k_\mu(x,y) \,\mathrm{d}\mu(x), \qquad H_\mu[\nu](y) = \langle \nu, k_\mu(\cdot, y) \rangle,$$

their Theorem 3.4 and Definition 4.1 give the metric tensor

$$\mathbf{g}^\varepsilon_\mu(\dot\mu_1, \dot\mu_2) = \frac{\varepsilon}{2} \langle \dot\mu_1, (\mathrm{id} - K_\mu^2)^{-1} H_\mu[\dot\mu_2] \rangle, \qquad \mathbf{g}^\varepsilon_\mu(\dot\mu, \dot\mu) = \lim_{t \to 0} \frac{S_\varepsilon(\mu, \mu_t)}{t^2}, \tag{1.2}$$

for balanced tangent vectors $\dot\mu$, that is, distributions with $\langle \dot\mu, 1 \rangle = 0$, and for smooth perturbations μ_t of μ with velocity $\dot\mu$.

In [6, Section 4.4] the authors fix a bounded convex set $X \subset \mathbb{R}^d$ with nonempty interior and the quadratic cost $c(x,y) = |x-y|^2$, identify measures with their Lebesgue densities, and set $Lf = -\Delta f - \nabla f \cdot \nabla \log \mu$. They claim (formula (35) in the arXiv version [6])

$$\mathbf{g}^\varepsilon_\mu(\dot\mu, \dot\mu) \;\longrightarrow\; \Big\langle \frac{\dot\mu}{\mu}, L^{-1}\Big[\frac{\dot\mu}{\mu}\Big] \Big\rangle_{L^2(X,\mu)} \qquad (\varepsilon \to 0). \tag{1.3}$$

By their Remark 4.18, for $\dot\mu = -\operatorname{div}(\mu \nabla \psi)$ the right-hand side equals $\int_X |\nabla \psi|^2 \,\mathrm{d}\mu$, the metric tensor of quadratic optimal transport (the Otto metric). They derive (1.3) by an informal computation based on the expansion $K_\mu = \mathrm{id} - \frac{\varepsilon}{4} L + o(\varepsilon)$, which is known on smooth test functions in smooth settings [8, 2], and the convergence of the metric tensor as $\varepsilon \to 0$ is also among the directions listed in their Outlook [6, Section 8]. The difficulty is that (1.2) contains the inverse of $\mathrm{id} - K_\mu^2$, whose spectral gap closes as $\varepsilon \to 0$, so a first-order expansion of K_μ on smooth functions does not control the tensor, and on a domain the kernel k_μ also has a boundary layer.

The operator L in (1.3) is the weighted Laplacian with its natural (Neumann) boundary condition: this is the realization for which Remark 4.18 of [6] holds, because $\dot\mu = -\operatorname{div}(\mu\nabla\psi)$, read as a balanced distribution on X, acts on test functions ζ by $\langle\dot\mu,\zeta\rangle = \int_X \nabla\psi\cdot\nabla\zeta\,\mathrm{d}\mu$. Accordingly, for a balanced distribution $\nu \in C^1(X)^*$ we write

$$\|\nu\|^2_{-1,\mu} = \sup_{\psi\in C^1(X)} \Big\{ 2\langle\nu,\psi\rangle - \int_\Omega |\nabla\psi|^2\,\mathrm{d}\mu \Big\} \in [0,\infty], \tag{1.4}$$

where Ω is the interior of X. The supremum is finite exactly when ν extends continuously to the Sobolev space $H^1(\Omega)$, and then it may be taken over $\psi \in H^1(\Omega)$. When $\nu = \xi\mu$ with $\xi \in L^2(\mu)$, (1.4) is the right-hand side of (1.3), and for $\nu = -\operatorname{div}(\mu\nabla\psi)$ it is $\int|\nabla\psi|^2\,\mathrm{d}\mu$.

Question 1.1 ([6, Section 4.4, formula (35), and Section 8])**.** *For the quadratic cost on a bounded convex set $X \subset \mathbb{R}^d$ with nonempty interior, does the Sinkhorn metric tensor $\mathbf{g}^\varepsilon_\mu(\dot\mu,\dot\mu)$ converge to the Otto metric tensor, the right-hand side of* (1.3)*, as $\varepsilon \to 0$?*

The claim in [6, Section 4.4] is made for measures identified with their densities, without a specified class of densities or tangent vectors; Theorem 1.2 below states the classes for which we prove it.

1.2 Main results

Throughout, $\Omega \subset \mathbb{R}^d$ is a bounded connected open set with Lipschitz boundary and $X = \overline{\Omega}$. Every bounded open convex set is such a domain, and the closures of these sets are exactly the bounded convex sets with nonempty interior considered in [6, Section 4.4]. The cost is $c(x,y) = |x-y|^2$ and $\mu = f\,\mathrm{d}x$ with

$$f \in C(X), \qquad f_- := \min_X f > 0, \qquad f_+ := \max_X f, \qquad \int_\Omega f\,\mathrm{d}x = 1. \tag{1.5}$$

We write $L^2_0(\mu) = \{\xi \in L^2(\mu) : \int\xi\,\mathrm{d}\mu = 0\}$, $\mathcal{H}^*_0$ for the space of bounded linear functionals ν on $H^1(\Omega)$ with $\langle\nu,1\rangle = 0$, normed by the dual norm $\|\nu\|_*$, and L for the weighted Neumann Laplacian, the nonnegative selfadjoint operator on $L^2_0(\mu)$ with quadratic form $\int_\Omega|\nabla\psi|^2\,\mathrm{d}\mu$ on $H^1(\Omega)\cap L^2_0(\mu)$. Every element of $\mathcal{H}^*_0$ restricts to a balanced distribution in $C^1(X)^*$, so (1.2) applies to it; see Section 2.

Theorem 1.2 (The limit (1.3))**.** *Assume* (1.5)*. Then, as $\varepsilon \to 0$:*

(a) (operator norm on densities)

$$\sup_{\xi\in L^2_0(\mu),\ \|\xi\|_{L^2(\mu)}\le 1} \big|\mathbf{g}^\varepsilon_\mu(\xi\mu,\xi\mu) - \langle\xi, L^{-1}\xi\rangle_{L^2(\mu)}\big| \ \longrightarrow\ 0;$$

(b) (uniform domination) there are $\varepsilon_0 > 0$ and $C < \infty$, depending only on d, Ω, f_- and f_+, such that $\mathbf{g}^\varepsilon_\mu(\nu,\nu) \le C\|\nu\|^2_*$ for all $\nu \in \mathcal{H}^*_0$ and $0 < \varepsilon \le \varepsilon_0$;

(c) (every tangent of finite Otto norm) $\mathbf{g}^\varepsilon_\mu(\nu,\nu) \to \|\nu\|^2_{-1,\mu}$ for every $\nu \in \mathcal{H}^*_0$, uniformly on compact subsets of $\mathcal{H}^*_0$.

By polarization, $\mathbf{g}^\varepsilon_\mu(\nu_1,\nu_2)$ converges to the Otto inner product of $\nu_1,\nu_2 \in \mathcal{H}^*_0$.

For $\nu \in \mathcal{H}_0^*$ and $\psi \in H^1(\Omega)$, (1.4) gives $\langle \nu, \psi \rangle^2 \le \|\nu\|_{-1,\mu}^2 \int |\nabla \psi|^2 \,\mathrm{d}\mu \le f_+ \|\nu\|_{-1,\mu}^2 \|\psi\|_{H^1(\Omega)}^2$, so $\|\nu\|_*^2 \le f_+ \|\nu\|_{-1,\mu}^2$. Hence (b) gives $\mathbf{g}_\mu^\varepsilon(\nu,\nu) \le C f_+ \|\nu\|_{-1,\mu}^2$: for $\varepsilon \le \varepsilon_0$ the Sinkhorn tensor is dominated by a fixed multiple of the Otto tensor.

Theorem 1.2 answers Question 1.1 affirmatively for every measure with a continuous strictly positive density and for every tangent vector at which the right-hand side of (1.3) is finite, on every bounded convex domain and more generally on every bounded connected Lipschitz domain, in every dimension. It covers both kinds of perturbation in [6, Remark 3.2]. Vertical perturbations $\dot\mu = \xi\mu$ with bounded or square-integrable ξ are covered by (a). Horizontal perturbations $\dot\mu = -\operatorname{div}(\mu v)$ are covered by (c): this includes the velocities of the horizontal curves of [6, Remark 3.2], for which v is continuous and tangent to ∂X where ∂X is smooth, and extends to every $v \in L^2(\mu;\mathbb{R}^d)$. The limit is the Otto norm $\min\{\int |w|^2 \,\mathrm{d}\mu : \ -\operatorname{div}(\mu w) = \dot\mu\}$, which equals $\int |\nabla\psi|^2 \,\mathrm{d}\mu$ when $v = \nabla\psi$ (Remark 5.4).

The analytic core is a uniform description of the symmetric Schrödinger potential. In the normalization of [6], Section 4.4, the informal computation predicts $\exp(v_\varepsilon/\varepsilon) \approx (\pi\varepsilon)^{-d/4}\mu^{-1/2}$.

Theorem 1.3 (Schrödinger potential up to the boundary)**.** *Assume* (1.5) *and let* $\beta_\varepsilon(x) = v_\varepsilon(x)/\varepsilon + \frac{d}{4}\log(\pi\varepsilon) + \frac{1}{2}\log f(x)$. *There are* $\varepsilon_0 > 0$ *and* $B < \infty$*, depending only on* d*,* Ω*,* f_- *and* f_+*, with* $|\beta_\varepsilon| \le B$ *on* X *for* $0 < \varepsilon \le \varepsilon_0$*, and* $\beta_\varepsilon \to 0$ *uniformly on every compact subset of* Ω *as* $\varepsilon \to 0$.

The bound in Theorem 1.3 is explicit (Proposition 3.1), and its proof uses only the measure-density property (2.1) of the domain. At boundary points whose tangent cone is an orthant with k facets, the potential exceeds the interior profile by $k\varepsilon/2$ to leading order.

Theorem 1.4 (Boundary values of the potential)**.** *Let* Ω *be a bounded open convex set, assume* (1.5)*, and let* $x_* \in \partial\Omega$ *be a point at which the tangent cone* $\bigcup_{t>0}(\Omega - x_*)/t$ *is an orthant* $\{\zeta : \ \zeta\cdot n_i > 0, \ 1 \le i \le k\}$ *for some orthonormal* $n_1, \dots, n_k$*. Then* $\beta_\varepsilon(x_*) \to k/2$ *as* $\varepsilon \to 0$*, that is,*

$$v_\varepsilon(x_*) = -\frac{d\varepsilon}{4}\log(\pi\varepsilon) - \frac{\varepsilon}{2}\log f(x_*) + \frac{k\varepsilon}{2} + o(\varepsilon).$$

In particular $\beta_\varepsilon(x_*) \to \frac{1}{2}$ *at every point where* $\partial\Omega$ *has a unique supporting hyperplane, which is the case at* $\mathcal{H}^{d-1}$*-almost every boundary point, and* $\beta_\varepsilon \to \frac{d}{2}$ *at the vertices of a rectangular box.*

The constant comes from the boundary-layer profile m, the bounded positive solution of

$$m(s)\int_0^\infty g_1(s-t)\, m(t)\,\mathrm{d}t = 1 \qquad (s \ge 0),$$

where g_1 is the standard Gaussian density on $\mathbb{R}$; an exact virial identity gives $m(0) = e^{1/2}$ (Lemma 3.8). Thus, on convex domains, the convergence in Theorem 1.3 holds uniformly on compact subsets of Ω and never uniformly on X.

A concrete consequence concerns finite-dimensional statistical models. For a parametric family of densities, the Gram matrix of the score functions in the Otto metric is the Wasserstein information matrix of Li and Zhao [7], and the corresponding Gram matrix in the Sinkhorn tensor is its entropic counterpart.

Corollary 1.5 (Information matrices)**.** *Let* $\Theta \subset \mathbb{R}^p$ *be open,* $\theta \mapsto f_\theta$ *a map into densities satisfying* (1.5)*,* $\mu_\theta = f_\theta\,\mathrm{d}x$*, and* $\theta \in \Theta$ *a point at which the partial derivatives* $\partial_i f_\theta$ *exist in* $L^1(\Omega)$ *with* $\partial_i f_\theta / f_\theta \in L^2(\mu_\theta)$*. Then the matrices* $\big(\mathbf{g}_{\mu_\theta}^\varepsilon(\partial_i\mu_\theta, \partial_j\mu_\theta)\big)_{i,j}$ *converge as* $\varepsilon \to 0$ *to the Wasserstein information matrix* $\big(\langle \partial_i \log f_\theta, L_\theta^{-1}\partial_j \log f_\theta\rangle_{L^2(\mu_\theta)}\big)_{i,j}$*, where* L_θ *is the weighted Neumann Laplacian of* μ_θ.

Proof. The tangent $\partial_i \mu_\theta = \partial_i f_\theta \, \mathrm{d}x$ equals $\xi_i \mu_\theta$ with $\xi_i = \partial_i f_\theta / f_\theta$, and $\int \xi_i \, \mathrm{d}\mu_\theta = \partial_i \int f_\theta = 0$. Apply Theorem 1.2(a) to ξ_i, ξ_j and $\xi_i + \xi_j$, and polarize. □

The proof of Theorem 1.2(a) also shows that the inverse of the self-transport Laplacian $\frac{2}{h}(I - K_\mu)$ converges in norm to L^{-1} on $L^2_0(\mu)$. Hence its eigenvalues and spectral projections converge to those of the weighted Neumann Laplacian (Corollary 5.2). Marshall and Coifman [8, Theorem 1, Corollaries 1–2] identified the generator of the bistochastic normalization of the Gaussian kernel on compact manifolds, including manifolds with boundary, on finite spans of Neumann eigenfunctions and for C^3 densities; with their weight $w \equiv 1$ this is the expansion $K_\mu = \mathrm{id} - \frac{\varepsilon}{4} L + o(\varepsilon)$ quoted in [6, Section 4.4]. Corollary 5.2 upgrades this first-order expansion to norm convergence of the inverses, and hence to convergence of eigenvalues and spectral projections, for continuous densities on Lipschitz domains. For graph Laplacians built from samples, spectral convergence to weighted Laplacians with Neumann conditions was proved by García Trillos and Slepčev [5], consistency on manifolds with boundary was studied by Vaughn, Berry and Antil [17], and bistochastic normalizations were analysed on closed manifolds by Cheng and Landa [4] and on the torus by Wormell and Reich [18].

1.3 Relation to earlier work and route of the proof

The tensor (1.2), its regularity in ε, its large-ε limit and the informal derivation of (1.3) are due to [6]. Mordant and Xu [10] proved convergence of the tensor and of its derivatives as $\varepsilon \to 0$ for finitely supported measures and horizontal perturbations, and they explain why their method does not extend to measures with a density. Navas [11, Remark 4.10] records the formal limit $\frac{2}{\varepsilon}(K_\mu^{-1} - K_\mu) \to -\Delta_\mu$ on the torus and notes that the expansion of K_μ does not pass to K_μ^{-1}. For fully supported measures on $\mathbb{R}^d$, Agarwal, Harchaoui, Mulcahy and Pal [2] proved that the self-transport Markov operators have the Langevin generator as their derivative at $\varepsilon = 0$ on smooth test functions; the operator K_μ above is of this type. Symmetric (bistochastic) normalizations of Gaussian kernels were analysed by Marshall and Coifman [8] on compact manifolds, with or without boundary, and by Wormell and Reich [18] on the torus (see the discussion after Corollary 1.5), and Mordant [9] studied the self-transport problem as $\varepsilon \to 0$ with a view to score estimation. Nutz and Wiesel [12] proved convergence of entropic potentials to unregularized ones. For heat-kernel costs, which vary with the regularization, the author has studied the corresponding limits on tori [15] and on manifolds with boundary [16]; those records are cited for context only, and nothing below depends on them.

The proof works directly with the fixed quadratic cost, where the Gibbs kernel is the Gaussian kernel truncated to X, with no reflection symmetry at the boundary. It has four steps.

1. *Two-sided bounds* (Proposition 3.1). The scaling $\rho = e^{v_\varepsilon/\varepsilon}(\pi\varepsilon)^{d/4} f$ solves $\rho \, \mathcal{G}_h \rho = f$, where $\mathcal{G}_h$ is the Gaussian smoothing on Ω at variance $h = \varepsilon/2$. A short argument at a maximum point of ρ, using only the measure density of the domain, bounds ρ above and below uniformly in h.

2. *Interior convergence and boundary values* (Section 3). After blow-up at scale $\sqrt{h}$, an interior point sees the whole-space equation $\vartheta \, (g * \vartheta) = \text{const}$, and a boundary point with an orthant as tangent cone sees the same equation on the orthant. Bounded positive solutions of these equations are unique (Lemma 3.5); on $\mathbb{R}^d$ they are constant. Hence $\rho \to \sqrt{f}$ locally uniformly in Ω, and a virial identity for the half-line profile gives Theorem 1.4.

3. *Duality and nonlocal Dirichlet forms* (Section 4). Up to an error of operator norm $h/2$, the tensor is the Legendre dual of the nonlocal Dirichlet form $\mathcal{E}_h(\psi) = h^{-1} \iint (\psi(x) -$

$\psi(y))^2\rho(x)\rho(y)G_h(x-y)\,\mathrm{d}x\,\mathrm{d}y$. With the bounds of step 1 we prove compactness and a uniform Poincaré inequality, and with step 2 the liminf and limsup inequalities toward $\int|\nabla\psi|^2\,\mathrm{d}\mu$. This follows most closely the variational approach of García Trillos and Slepčev [5], who proved Γ-convergence with compactness for weighted nonlocal energies $\varepsilon^{-2}\iint\eta_\varepsilon(x-y)|u(x)-u(y)|^2\rho(x)\rho(y)\,\mathrm{d}x\,\mathrm{d}y$, with a fixed density ρ and rescaled kernels η_ε, on Lipschitz domains, building on the nonlocal characterizations of Sobolev spaces of Bourgain, Brezis and Mironescu [3] and Ponce [13, 14]. The new features here are the h-dependent weights ρ_h, controlled only through steps 1 and 2, and the truncated kernel at the boundary; we give complete proofs.

4. *Tangents in the dual of H^1* (Section 5). A gradient bound for K_μ and a smoothing bound for the kernel give the domination (b), and density of L^2 tangents in $\mathcal{H}_0^*$ gives (c).

2 Setting and the tensor on density tangents

2.1 The domain

We use the following standard properties of a bounded connected Lipschitz domain Ω; see [1, Chapters 4–6].

(D1) Measure density: there are $\kappa>0$ and $s_0>0$ with

$$|B(x,s)\cap\Omega|\ge\kappa s^d\qquad(x\in X,\ 0<s\le s_0).\tag{2.1}$$

(D2) Rellich: the embedding $H^1(\Omega)\hookrightarrow L^2(\Omega)$ is compact.

(D3) Poincaré–Wirtinger: $\|\zeta-\bar\zeta\|_{L^2(\Omega)}\le C_{\mathrm{PW}}\|\nabla\zeta\|_{L^2(\Omega)}$ for $\zeta\in H^1(\Omega)$, where $\bar\zeta$ is the mean of ζ over Ω.

(D4) Density: restrictions to Ω of functions in $C_c^\infty(\mathbb{R}^d)$ are dense in $H^1(\Omega)$.

For a bounded open convex Ω containing a ball $B(x_c,r_0)$ and of diameter D, (D1) holds with $\kappa=\omega_d(r_0/2D)^d$ and $s_0=2D$, where ω_d is the volume of the unit ball: for $x\in X$ and $t=s/(2D)\le1$, the ball $x+t(B(x_c,r_0)-x)$ of radius tr_0 lies in $\Omega\cap B(x,s)$ by convexity.

2.2 Normalization

Fix $\varepsilon>0$ and write $h=\varepsilon/2$. Let $G_h(z)=(2\pi h)^{-d/2}e^{-|z|^2/(2h)}$ and $g=G_1$. For a bounded measurable u on Ω put

$$\mathcal{G}_hu(x)=\int_\Omega G_h(x-y)u(y)\,\mathrm{d}y\qquad(x\in\mathbb{R}^d).$$

Let $v_\varepsilon\in C(X)$ be the symmetric Schrödinger potential of $\mathrm{OT}_\varepsilon(\mu,\mu)$ [6, Section 3], normalized by [6, (15)]:

$$\int_X k_\mu(x,y)\,\mathrm{d}\mu(x)=1\quad(y\in X),\qquad k_\mu(x,y)=\exp\bigl((v_\varepsilon(x)+v_\varepsilon(y)-|x-y|^2)/\varepsilon\bigr).\tag{2.2}$$

Set $a(x)=(\pi\varepsilon)^{d/4}e^{v_\varepsilon(x)/\varepsilon}$. Since $(\pi\varepsilon)^{-d/2}e^{-|z|^2/\varepsilon}=G_h(z)$, we have $k_\mu(x,y)=a(x)a(y)G_h(x-y)$.

Lemma 2.1 (Scaling equation)**.** *The function $\rho = af \in C(X)$ is positive and satisfies*

$$\rho(x)\,\mathcal{G}_h\rho(x) = f(x) \qquad (x \in X). \tag{2.3}$$

Moreover $a = 1/\mathcal{G}_h\rho$ on X, and $\beta_\varepsilon = \log(\rho/\sqrt{f})$ in Theorem 1.3.

Proof. By (2.2), $a(y)\int_\Omega G_h(x-y)a(x)f(x)\,\mathrm{d}x = 1$; multiply by $f(y)$. The last identity is

$$\log(af/\sqrt{f}) = \tfrac{d}{4}\log(\pi\varepsilon) + v_\varepsilon/\varepsilon + \tfrac{1}{2}\log f. \qquad \square$$

2.3 The self-transport operator

On $L^2(\mu)$ let $K = K_\mu$, so that

$$K\phi(x) = \int_\Omega k_\mu(x,y)\phi(y)f(y)\,\mathrm{d}y = \int_\Omega \tilde{k}(x,y)\phi(y)\,\mathrm{d}y, \qquad \tilde{k}(x,y) = \frac{G_h(x-y)\rho(y)}{\mathcal{G}_h\rho(x)}. \tag{2.4}$$

For every $x \in \mathbb{R}^d$, $\tilde{k}(x,\cdot)$ is a probability density on Ω, and $\tilde{k}(x,y)f(x) = p_h(x,y)$ for $x \in X$, where

$$p_h(x,y) = \rho(x)\rho(y)G_h(x-y) \tag{2.5}$$

is the density of the optimal self-transport plan $k_\mu\,\mu\otimes\mu$; its marginals are both μ.

Lemma 2.2. *K is a compact selfadjoint operator on $L^2(\mu)$ with $0 \le K \le I$, $K1 = 1$, and*

$$\langle\psi, (I-K)\psi\rangle_{L^2(\mu)} = \frac{1}{2}\iint_{\Omega\times\Omega}(\psi(x)-\psi(y))^2 p_h(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.6}$$

The eigenvalue 1 is simple, K maps $L^2_0(\mu)$ into itself, and $I - K \ge \gamma_h$ on $L^2_0(\mu)$ for some $\gamma_h > 0$.

Proof. The kernel k_μ is bounded and symmetric, so K is selfadjoint and Hilbert–Schmidt. (2.2) gives $K1 = 1$, and (2.6) follows by expanding the square and using the symmetry and both marginal identities. It gives $K \le I$. Further,

$$\langle\phi, K\phi\rangle_{L^2(\mu)} = \iint G_h(x-y)(a\phi f)(x)(a\phi f)(y)\,\mathrm{d}x\,\mathrm{d}y = \|G_{h/2} * (a\phi f\mathbf{1}_\Omega)\|^2_{L^2(\mathbb{R}^d)} \ge 0$$

by the semigroup property $G_h = G_{h/2} * G_{h/2}$. If $K\phi = \phi$, the right side of (2.6) vanishes, and since $p_h > 0$, ϕ is constant. Selfadjointness and $K1 = 1$ give $KL^2_0(\mu) \subset L^2_0(\mu)$, and compactness gives the gap γ_h. $\square$

2.4 The tensor and its dual form

Lemma 2.3 (Tensor on density tangents)**.** *For $\xi_1, \xi_2 \in L^2_0(\mu)$,*

$$\mathbf{g}^\varepsilon_\mu(\xi_1\mu, \xi_2\mu) = \langle\xi_1, B_h\xi_2\rangle_{L^2(\mu)}, \qquad B_h := hK(I-K^2)^{-1} \ \text{ on } L^2_0(\mu). \tag{2.7}$$

Moreover, with

$$\mathcal{E}_h(\psi) = \frac{1}{h}\iint_{\Omega\times\Omega}(\psi(x)-\psi(y))^2 p_h(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{2.8}$$

we have $B_h = D_h - \frac{h}{2}(I+K)^{-1}$ with $0 \le \frac{h}{2}(I+K)^{-1} \le \frac{h}{2}$ and

$$\langle\xi, D_h\xi\rangle_{L^2(\mu)} = \sup_{\psi\in L^2(\mu)}\{2\langle\xi,\psi\rangle_{L^2(\mu)} - \mathcal{E}_h(\psi)\} \qquad (\xi \in L^2_0(\mu)), \tag{2.9}$$

where $D_h := \frac{h}{2}(I-K)^{-1}$ on $L^2_0(\mu)$. The supremum is attained at $\psi = D_h\xi$, and its value equals $\mathcal{E}_h(D_h\xi) = \langle\xi, D_h\xi\rangle$.

Proof. For $\dot\mu = \xi\mu$ we have $H_\mu[\dot\mu] = K\xi$, a continuous function with μ-mean $\langle K\xi, 1\rangle = \langle \xi, K1\rangle = 0$. In [6] the operator $(\mathrm{id} - K_\mu^2)^{-1}$ acts on continuous (indeed C^m) functions modulo constants. If $F \in C(X) \cap L^2_0(\mu)$, then $\Phi = F + K^2(I-K^2)^{-1}F$, with the inverse taken on $L^2_0(\mu)$, is continuous because K^2 maps $L^2(\mu)$ into $C(X)$, and it solves $(I-K^2)\Phi = F$. A continuous solution of $(I-K^2)\Phi \equiv F$ modulo constants is unique modulo constants: integrating against μ shows the constant is zero, and Lemma 2.2 gives uniqueness in $L^2_0(\mu)$. Since $\dot\mu$ is balanced, the pairing does not see constants, so (1.2) with $\varepsilon = 2h$ gives $\mathbf{g}^\varepsilon_\mu = h\langle \xi_1, (I-K^2)^{-1}K\xi_2\rangle$, which is (2.7). The spectral identity $\lambda/(1-\lambda^2) = \frac12(1-\lambda)^{-1} - \frac12(1+\lambda)^{-1}$ gives the decomposition, and $0 \le K$ gives $0 \le (I+K)^{-1} \le I$. Finally $A_h = \frac{2}{h}(I-K)$ is positive and invertible on $L^2_0(\mu)$, with $\langle \psi, A_h\psi\rangle = \mathcal{E}_h(\psi)$ by (2.6); completing the square, $2\langle \xi, \psi\rangle - \langle \psi, A_h\psi\rangle = \langle \xi, A_h^{-1}\xi\rangle - \langle \psi - A_h^{-1}\xi, A_h(\psi - A_h^{-1}\xi)\rangle$ on $L^2_0(\mu)$. Both terms in (2.9) are unchanged when a constant is added to ψ, so the supremum over $L^2(\mu)$ is the one over $L^2_0(\mu)$. □

The next lemma places the tangents of Theorem 1.2(b),(c) in the framework of [6]. Let $C^1(X)$ carry its usual norm. Since $\|\zeta\|_{H^1(\Omega)} \le |\Omega|^{1/2}\|\zeta\|_{C^1(X)}$, every $\nu \in \mathcal{H}_0^*$ restricts to a balanced element of $C^1(X)^*$ with $\|\nu\|_{C^1(X)^*} \le |\Omega|^{1/2}\|\nu\|_*$, and by (D4) the restriction is injective.

Lemma 2.4 (Fixed ε)**.** *For fixed ε,* (1.2) *defines a bilinear form on the balanced elements of $C^1(X)^*$ with $|\mathbf{g}^\varepsilon_\mu(\nu_1,\nu_2)| \le C_\varepsilon\|\nu_1\|_{C^1(X)^*}\|\nu_2\|_{C^1(X)^*}$. The set $\{\xi\mu : \xi \in L^2_0(\mu)\}$ is dense in $\mathcal{H}_0^*$, and on $\mathcal{H}_0^*$ the form is symmetric and nonnegative.*

Proof. The function $a = 1/\mathcal{G}_h\rho$ is smooth on a neighbourhood of X, so k_μ is smooth on a neighbourhood of $X \times X$. Hence H_μ maps $C^1(X)^*$ boundedly into $C^1(X)$, K^2 maps $L^2(\mu)$ boundedly into $C^1(X)$, and as in the proof of Lemma 2.3, $(\mathrm{id}-K^2)^{-1}H_\mu\nu_2 = H_\mu\nu_2 + K^2(I-K^2)^{-1}H_\mu\nu_2$ is bounded by $C\|\nu_2\|_{C^1(X)^*}$ in $C^1(X)$ modulo constants; here $H_\mu\nu_2$ has μ-mean $\langle \nu_2, K1\rangle = 0$. Pairing with ν_1 gives the bound. For density, $H^1(\Omega)$ is reflexive; if $\zeta \in H^1(\Omega)$ annihilates every $\xi\mu$ with $\xi \in L^2_0(\mu)$, then ζ is constant, so it annihilates $\mathcal{H}_0^*$, and the Hahn–Banach theorem gives density. On the measures $\xi\mu$ the form is symmetric and nonnegative by Lemma 2.3. Since $\|\cdot\|_{C^1(X)^*} \le |\Omega|^{1/2}\|\cdot\|_*$, these measures are dense in $\mathcal{H}_0^*$ also for the norm of $C^1(X)^*$, and both properties pass to $\mathcal{H}_0^*$ by continuity. □

3 The symmetric scaling up to the boundary

3.1 Two-sided bounds

For $t \ge 0$ let $\tau_d(t)$ be the probability that a standard Gaussian vector in $\mathbb{R}^d$ has norm at least t. Chernoff's bound with $\mathbb{E}e^{|Z|^2/4} = 2^{d/2}$ gives

$$\tau_d(t) \le 2^{d/2}e^{-t^2/4}. \tag{3.1}$$

Proposition 3.1 (Two-sided bounds)**.** *Let Ω be a bounded open set satisfying* (2.1)*, let $f \in C(X)$ satisfy $0 < f_- \le f \le f_+$, let $0 < h \le s_0^2$, and let $\rho \in C(X)$ be positive with* (2.3)*. Put $A = (e+1)e^{1/2}(2\pi)^{d/2}f_+^2/(\kappa f_-)$ and*

$$\Lambda^2 = \max\Big\{2^{-d/2}f_+,\ \sup\big\{t>0:\ t \le A\big(1+2\sqrt{\log_+(2^{d/2}t/f_+)}\big)^d\big\}\Big\} < \infty.$$

Then

$$\frac{f_-}{\Lambda} \le \rho \le \Lambda \quad \textit{and} \quad \frac{f_-}{\Lambda} \le \mathcal{G}_h\rho \le \frac{f_+\Lambda}{f_-} \qquad \textit{on } X. \tag{3.2}$$

Proof. The supremum defining Λ^2 is finite because the right side of its inequality grows only like a power of $\sqrt{\log t}$. Let $M = \max_X \rho = \rho(x_0)$. If $M^2 \le 2^{-d/2} f_+$ there is nothing to prove for the upper bound, so assume $2^{d/2}M^2/f_+ > 1$, put $t_M = 2\sqrt{\log(2^{d/2}M^2/f_+)}$ and $R = 1 + t_M$, so that $R - 1/R \ge t_M$ and, by (3.1), $M\tau_d(R - 1/R) \le f_+/M$.

Step 1: ρ is large near x_0. Let $y \in X$ with $|y - x_0| \le \sqrt{h}/R$. For $z \in \Omega$ with $|z - x_0| \le R\sqrt{h}$,

$$|x_0 - z|^2 - |y - z|^2 = -|y - x_0|^2 - 2(y - x_0)\cdot(x_0 - z) \le 2|y - x_0|\,|x_0 - z| \le 2h,$$

so $G_h(y - z) \le e\, G_h(x_0 - z)$. For $|z - x_0| > R\sqrt{h}$ we have $|z - y| \ge (R - 1/R)\sqrt{h}$ and $\rho(z) \le M$. Hence, using (2.3) at x_0,

$$\mathcal{G}_h\rho(y) \le e\,\mathcal{G}_h\rho(x_0) + M\tau_d(R - 1/R) \le \frac{ef(x_0)}{M} + \frac{f_+}{M} \le \frac{(e+1)f_+}{M},$$

and (2.3) at y gives $\rho(y) \ge f_- M/((e+1)f_+)$.

Step 2: this forces M to be bounded. On $B(x_0, \sqrt{h}/R)$ we have $G_h(x_0 - y) \ge (2\pi h)^{-d/2} e^{-1/2}$, and (2.1) with $s = \sqrt{h}/R \le s_0$ gives $|B(x_0, \sqrt{h}/R) \cap \Omega| \ge \kappa h^{d/2} R^{-d}$. Therefore

$$\frac{f_+}{M} \ge \frac{f(x_0)}{M} = \mathcal{G}_h\rho(x_0) \ge \frac{f_- M}{(e+1)f_+}(2\pi)^{-d/2} e^{-1/2}\kappa R^{-d},$$

that is, $M^2 \le AR^d = A(1 + t_M)^d$. By the definition of Λ, $M \le \Lambda$.

Step 3: lower bounds. Since $\int_\Omega G_h(x - y)\,\mathrm{d}y \le 1$, we get $\mathcal{G}_h\rho \le \Lambda$ and $\rho = f/\mathcal{G}_h\rho \ge f_-/\Lambda$. Then $\mathcal{G}_h\rho \ge f_-/\Lambda$, and $\mathcal{G}_h\rho = f/\rho \le f_+\Lambda/f_-$. □

Corollary 3.2. *Under the hypotheses of Proposition 3.1, $c_1 G_h(x - y) \le p_h(x, y) \le c_2 G_h(x - y)$ on $X \times X$, with $c_1 = (f_-/\Lambda)^2$ and $c_2 = \Lambda^2$.*

3.2 Self-transport solutions on orthant-type sets

Blow-ups of (2.3) at scale $\sqrt{h}$ lead to the following limit problems.

Definition 3.3. An *orthant-type set* is $U = \{\zeta \in \mathbb{R}^d :\ \zeta \cdot n_i > a_i,\ 1 \le i \le k\}$ with $k \in \{0, \dots, d\}$, orthonormal vectors $n_1, \dots, n_k$ and $a_i \in \mathbb{R}$; for $k = 0$, $U = \mathbb{R}^d$. For bounded measurable ϑ on U put $(g *_U \vartheta)(\zeta) = \int_U g(\zeta - \eta)\vartheta(\eta)\,\mathrm{d}\eta$ for $\zeta \in \mathbb{R}^d$. A *self-transport solution of level $c > 0$ on U* is a continuous $\vartheta : \overline{U} \to (0, \infty)$ with $0 < \inf\vartheta \le \sup\vartheta < \infty$ and $\vartheta\,(g *_U \vartheta) = c$ on $\overline{U}$.

For $\zeta \in \overline{U}$ we have $\operatorname{dist}(\zeta, \partial U) = \min_i(\zeta \cdot n_i - a_i)$, and $\operatorname{dist}(\zeta, \overline{U})^2 = \sum_i \max\{0, a_i - \zeta \cdot n_i\}^2$ for every $\zeta \in \mathbb{R}^d$.

Lemma 3.4 (Translation compactness)**.** *Let ϑ be a self-transport solution of level c on an orthant-type set U, let $\Gamma = g *_U \vartheta$, and let $\zeta_n \in \overline{U}$. Then there are a subsequence, an orthant-type set U_∞ with $0 \in \overline{U_\infty}$, and a self-transport solution ϑ_∞ of level c on U_∞ with $\inf\vartheta \le \vartheta_\infty \le \sup\vartheta$, such that $\Gamma(\cdot + \zeta_n) \to g *_{U_\infty} \vartheta_\infty$ locally uniformly on $\mathbb{R}^d$ and $\vartheta(\zeta_n) \to \vartheta_\infty(0)$. If $\operatorname{dist}(\zeta_n, \partial U) \to \infty$, then $U_\infty = \mathbb{R}^d$.*

Proof. The function Γ is defined on $\mathbb{R}^d$, bounded by $\sup\vartheta$ and Lipschitz with constant $L = \sup\vartheta\,\|\nabla g\|_{L^1}$, and $\Gamma = c/\vartheta$ on $\overline{U}$. Put $\Gamma_n = \Gamma(\cdot + \zeta_n)$ and $U_n = U - \zeta_n = \{\zeta \cdot n_i > a_i - \zeta_n \cdot n_i\}$, where $a_i - \zeta_n \cdot n_i \le 0$. By the Arzelà–Ascoli theorem and a diagonal argument, along a subsequence $\Gamma_n \to \Gamma_\infty$ locally uniformly on $\mathbb{R}^d$ and $a_i - \zeta_n \cdot n_i \to -t_i \in [-\infty, 0]$ for each i. Let $U_\infty = \{\zeta \cdot n_i > -t_i$ for the i with $t_i < \infty\}$; then $0 \in \overline{U_\infty}$, and $U_\infty = \mathbb{R}^d$ if $\operatorname{dist}(\zeta_n, \partial U) \to \infty$. For $\zeta \in \overline{U_\infty}$ the

distance formula gives $\operatorname{dist}(\zeta, \overline{U_n}) \to 0$, so the Lipschitz bound yields $c/\sup\vartheta \le \Gamma_\infty(\zeta) \le c/\inf\vartheta$; put $\vartheta_\infty = c/\Gamma_\infty$ on $\overline{U_\infty}$. On U_n we have $\vartheta(\cdot + \zeta_n) = c/\Gamma_n$, hence

$$\Gamma_n(\zeta) = \int_{\mathbb{R}^d} g(\zeta - \eta)\mathbf{1}_{U_n}(\eta)\frac{c}{\Gamma_n(\eta)}\,\mathrm{d}\eta.$$

If $\eta \in U_\infty$, then $\eta \in U_n$ for large n and the integrand tends to $g(\zeta - \eta)c/\Gamma_\infty(\eta)$; if $\eta \notin \overline{U_\infty}$, then $\eta \notin U_n$ for large n; and ∂U_∞ is a null set. The integrand is at most $g(\zeta - \eta)\sup\vartheta$, so dominated convergence gives $\Gamma_\infty = g *_{U_\infty} \vartheta_\infty$, that is, ϑ_∞ is a self-transport solution of level c on U_∞. Finally $\vartheta(\zeta_n) = c/\Gamma_n(0) \to c/\Gamma_\infty(0) = \vartheta_\infty(0)$. □

Lemma 3.5 (Uniqueness)**.** *Two self-transport solutions of the same level on the same orthant-type set coincide.*

Proof. Let ϑ_1, ϑ_2 be solutions of level c on U, $\Gamma_j = g *_U \vartheta_j$, and $u = \log(\vartheta_1/\vartheta_2) = \log(\Gamma_2/\Gamma_1)$ on $\overline{U}$, a bounded continuous function. For $\zeta \in \overline{U}$,

$$e^{-u(\zeta)} = \frac{\Gamma_1(\zeta)}{\Gamma_2(\zeta)} = \int_U e^u\,\mathrm{d}\nu_\zeta, \qquad \mathrm{d}\nu_\zeta(\eta) = \frac{g(\zeta - \eta)\vartheta_2(\eta)}{\Gamma_2(\zeta)}\,\mathrm{d}\eta, \tag{3.3}$$

and ν_ζ is a probability measure with positive density on U. Let $S = \sup u$ and $s = \inf u$. By (3.3), $e^s \le e^{-u} \le e^S$ on $\overline{U}$, so $-s \le S$ and $-S \ge s$; hence $s = -S$. Choose $\zeta_n \in \overline{U}$ with $u(\zeta_n) \to S$ and apply Lemma 3.4 to ϑ_1 and ϑ_2 along a common subsequence; the limit set U_∞ depends only on (ζ_n). The limits $\vartheta_{j,\infty}$ satisfy (3.3) on $\overline{U_\infty}$ with $u_\infty = \log(\vartheta_{1,\infty}/\vartheta_{2,\infty}) = \lim \log(\Gamma_2/\Gamma_1)(\cdot + \zeta_n)$. Since $\log(\Gamma_2/\Gamma_1) = u$ takes values in $[s, S]$ on $\overline{U}$, the Γ_j are Lipschitz and bounded below near $\overline{U}$, and $\operatorname{dist}(\zeta + \zeta_n, \overline{U}) \to 0$ for $\zeta \in \overline{U_\infty}$, we get $s \le u_\infty \le S$ on $\overline{U_\infty}$, and $u_\infty(0) = \lim u(\zeta_n) = S$. At $\zeta = 0$, (3.3) gives $e^{-S} = \int e^{u_\infty}\,\mathrm{d}\nu_0^\infty \ge e^s = e^{-S}$, so $u_\infty = s$ almost everywhere on U_∞, hence everywhere on $\overline{U_\infty}$ by continuity, and $S = u_\infty(0) = s$. With $s = -S$ we get $u \equiv 0$. □

Corollary 3.6 (Liouville)**.** *Every self-transport solution of level λ^2 on $\mathbb{R}^d$ is the constant λ.*

Proof. The constant λ is a solution, because $g * \lambda = \lambda$; apply Lemma 3.5. □

3.3 Interior convergence

Proposition 3.7 (Interior convergence)**.** *Assume* (2.1) *and* (1.5)*, and for $0 < h \le s_0^2$ let $\rho_h \in C(X)$ be positive with $\rho_h \mathcal{G}_h \rho_h = f$. Then $\rho_h \to \sqrt{f}$ uniformly on every compact subset of Ω as $h \to 0$.*

Proof. Suppose not. Then there are a compact $C \subset \Omega$, $\eta > 0$, $h_n \to 0$ and $x_n \in C$ with $|\rho_{h_n}(x_n) - \sqrt{f(x_n)}| \ge \eta$; passing to a subsequence, $x_n \to x_* \in C$. Let $\delta = \operatorname{dist}(C, \partial\Omega)$, $\Omega_n = (\Omega - x_n)/\sqrt{h_n} \supset B(0, \delta/\sqrt{h_n})$ and $\vartheta_n(\zeta) = \rho_{h_n}(x_n + \sqrt{h_n}\zeta)$ for $\zeta \in \overline{\Omega_n}$. The change of variables $y = x_n + \sqrt{h_n}\zeta'$ turns (2.3) into

$$\vartheta_n(\zeta)\,\Gamma_n(\zeta) = f(x_n + \sqrt{h_n}\zeta), \qquad \Gamma_n(\zeta) = \int_{\Omega_n} g(\zeta - \zeta')\vartheta_n(\zeta')\,\mathrm{d}\zeta'. \tag{3.4}$$

By Proposition 3.1, $f_-/\Lambda \le \vartheta_n \le \Lambda$ and $\Gamma_n \ge f_-/\Lambda$ on $\overline{\Omega_n}$, and Γ_n is Lipschitz on $\mathbb{R}^d$ with constant $\Lambda\|\nabla g\|_{L^1}$. Since f is uniformly continuous on X, (3.4) shows that (ϑ_n) is equicontinuous on each ball $B(0, j)$ once $\delta/\sqrt{h_n} > j$. By the Arzelà–Ascoli theorem and a diagonal argument, a subsequence converges locally uniformly on $\mathbb{R}^d$ to a continuous ϑ with $f_-/\Lambda \le \vartheta \le \Lambda$. For fixed ζ, the integrands of $\Gamma_n(\zeta)$ converge pointwise to $g(\zeta - \cdot)\vartheta$ and are dominated by $\Lambda g(\zeta - \cdot)$, so $\Gamma_n(\zeta) \to (g * \vartheta)(\zeta)$, while $f(x_n + \sqrt{h_n}\zeta) \to f(x_*)$. Thus ϑ is a self-transport solution of level $f(x_*)$ on $\mathbb{R}^d$, and Corollary 3.6 gives $\vartheta \equiv \sqrt{f(x_*)}$. But $\vartheta(0) = \lim \rho_{h_n}(x_n)$ and $\sqrt{f(x_n)} \to \sqrt{f(x_*)}$, contradicting $|\rho_{h_n}(x_n) - \sqrt{f(x_n)}| \ge \eta$. □

Proof of Theorem 1.3. By Lemma 2.1, $\beta_\varepsilon = \log(\rho/\sqrt{f})$ with ρ solving (2.3) at $h = \varepsilon/2$. For $\varepsilon \le \varepsilon_0 := 2s_0^2$, Proposition 3.1 gives $f_-/(\Lambda\sqrt{f_+}) \le \rho/\sqrt{f} \le \Lambda/\sqrt{f_-}$, hence $|\beta_\varepsilon| \le B := \max\{|\log(\Lambda/\sqrt{f_-})|, |\log(\Lambda\sqrt{f_+}/f_-)|\}$, and Proposition 3.7 gives local uniform convergence of $\rho/\sqrt{f}$ to 1 in Ω, since $f \ge f_- > 0$. □

3.4 The boundary profile

Lemma 3.8 (Half-line profile)**.** *There is exactly one self-transport solution m of level 1 on the half-line $(0,\infty) \subset \mathbb{R}$. It satisfies $m(s) \to 1$ as $s \to \infty$ and*

$$m(0) = e^{1/2}. \tag{3.5}$$

Proof. Existence. Take $d = 1$, $\Omega = (0,1)$, which satisfies (2.1) with $\kappa = s_0 = 1$, and $f \equiv 1$. Let ρ_h be the corresponding scalings (Lemma 2.1) and $\vartheta_n(\zeta) = \rho_{h_n}(\sqrt{h_n}\zeta)$ for $\zeta \in [0, h_n^{-1/2}]$ and some $h_n \to 0$. As in the proof of Proposition 3.7, Proposition 3.1 and the Arzelà–Ascoli theorem give a subsequence converging locally uniformly on $[0,\infty)$ to a continuous m with $1/\Lambda \le m \le \Lambda$, and dominated convergence, with $\mathbf{1}_{(0,h_n^{-1/2})} \to \mathbf{1}_{(0,\infty)}$, shows that m is a self-transport solution of level 1 on $(0,\infty)$.

Uniqueness is Lemma 3.5.

Behaviour at infinity. Let $R_n \to \infty$. By Lemma 3.4, a subsequence of $m(\cdot + R_n)$ converges, locally uniformly on $\mathbb{R}$, to a self-transport solution of level 1 on $\mathbb{R}$, which is 1 by Corollary 3.6; the local uniform convergence follows from that of $\Gamma(\cdot + R_n)$, where $\Gamma = g *_{(0,\infty)} m = 1/m$ on $[0,\infty)$. Since every sequence $R_n \to \infty$ has such a subsequence, $m(R + \cdot) \to 1$ locally uniformly as $R \to \infty$.

The value at the boundary. Let $\Gamma = g *_{(0,\infty)} m$. Since $g'(z) = -zg(z)$, $\Gamma'(s) = \int_0^\infty (t-s)g(t-s)m(t)\,\mathrm{d}t$, and $\Gamma = 1/m$ on $[0,\infty)$ gives $m\Gamma' = \Gamma'/\Gamma = -(\log m)'$. For $R > 0$,

$$\log m(0) - \log m(R) = \int_0^R m(s)\Gamma'(s)\,\mathrm{d}s = \int_0^R\int_0^\infty m(s)m(t)(t-s)g(t-s)\,\mathrm{d}t\,\mathrm{d}s.$$

The part with $t \in [0,R]$ vanishes, because its integrand is antisymmetric in (s,t) and integrable on $[0,R]^2$. Substituting $s = R + u$ and $t = R + v$ in the remaining part,

$$\log m(0) - \log m(R) = \int_{-R}^0\int_0^\infty m(R+u)m(R+v)(v-u)g(v-u)\,\mathrm{d}v\,\mathrm{d}u.$$

As $R \to \infty$, the integrand tends to $(v-u)g(v-u)$ pointwise and is dominated by $\Lambda^2(v-u)g(v-u)$, whose integral over $\{u < 0 < v\}$ is $\int_0^\infty w \cdot wg(w)\,\mathrm{d}w = \frac{1}{2}$. Since $\log m(R) \to 0$, we obtain $\log m(0) = \frac{1}{2}$. □

Proposition 3.9 (Boundary blow-up)**.** *Let Ω be a bounded open convex set, assume (1.5), and let $x_* \in \partial\Omega$. Suppose the tangent cone $T = \bigcup_{t>0}(\Omega - x_*)/t$ is an orthant-type set $\{\zeta \cdot n_i > 0,\ 1 \le i \le k\}$. Then, for every compact $Q \subset \mathbb{R}^d$, as $h \to 0$,*

$$\sup\Big\{\Big|\frac{\rho_h(x_* + \sqrt{h}\,\zeta)}{\sqrt{f(x_*)}} - \prod_{i=1}^k m(\zeta \cdot n_i)\Big| :\ \zeta \in Q,\ x_* + \sqrt{h}\,\zeta \in X\Big\} \ \longrightarrow\ 0,$$

where m is the half-line profile of Lemma 3.8. In particular $\rho_h(x_*)/\sqrt{f(x_*)} \to e^{k/2}$.

Proof. Let $\Omega_t = (\Omega - x_*)/t$. For $0 < t' < t$, convexity gives $x_* + (t'/t)(y - x_*) \in \Omega$ for $y \in \Omega$, that is, $\Omega_t \subset \Omega_{t'}$; moreover $\Omega_t \subset T$ and $\bigcup_t \Omega_t = T$; hence $\mathbf{1}_{\Omega_t} \to \mathbf{1}_T$ off the null set ∂T, and $\operatorname{dist}(\zeta, \overline{\Omega_t}) \to 0$

for each $\zeta \in \overline{T}$. Let $h_n \to 0$, $\vartheta_n(\zeta) = \rho_{h_n}(x_* + \sqrt{h_n}\zeta)$ on $\overline{\Omega_{\sqrt{h_n}}}$, and let Γ_n be as in (3.4) with $x_n = x_*$. As in the proof of Lemma 3.4, a subsequence of Γ_n converges locally uniformly on $\mathbb{R}^d$ to some Γ_∞ with $f_-/\Lambda \le \Gamma_\infty \le f_+\Lambda/f_-$ on $\overline{T}$, and dominated convergence in

$$\Gamma_n(\zeta) = \int_{\mathbb{R}^d} g(\zeta - \eta)\mathbf{1}_{\Omega_{\sqrt{h_n}}}(\eta)\frac{f(x_* + \sqrt{h_n}\eta)}{\Gamma_n(\eta)}\,\mathrm{d}\eta$$

shows that $\vartheta = f(x_*)/\Gamma_\infty$ is a self-transport solution of level $f(x_*)$ on T. The function $\zeta \mapsto \sqrt{f(x_*)}\prod_{i\le k} m(\zeta\cdot n_i)$ is another one: completing $n_1,\dots,n_k$ to an orthonormal basis, g factorizes into one-dimensional standard Gaussians g_1, so that for $\zeta \in \overline{T}$

$$\Big(g *_T \prod_{i\le k} m(\cdot\, n_i)\Big)(\zeta) = \prod_{i\le k}(g_1 *_{(0,\infty)} m)(\zeta\cdot n_i) = \prod_{i\le k}\frac{1}{m(\zeta\cdot n_i)}.$$

By Lemma 3.5 the two coincide. On $\overline{\Omega_{\sqrt{h_n}}} \subset \overline{T}$ we have $\vartheta_n = f(x_* + \sqrt{h_n}\,\cdot)/\Gamma_n$ and $\Gamma_n \ge f_-/\Lambda$, while $\vartheta = f(x_*)/\Gamma_\infty$ on $\overline{T}$; since $f(x_* + \sqrt{h_n}\zeta) \to f(x_*)$ and $\Gamma_n \to \Gamma_\infty$ uniformly on Q, we get $\sup\{|\vartheta_n - \vartheta| :\ \zeta \in Q \cap \overline{\Omega_{\sqrt{h_n}}}\} \to 0$. The limit is independent of the subsequence, so the whole family converges, and (3.5) gives the value at $\zeta = 0$, which lies in $\overline{\Omega_t}$ for every t. □

Proof of Theorem 1.4. By Lemma 2.1, $\beta_\varepsilon(x_*) = \log(\rho_h(x_*)/\sqrt{f(x_*)})$ with $h = \varepsilon/2$, and Proposition 3.9 gives $\beta_\varepsilon(x_*) \to \log e^{k/2} = k/2$. If $\partial\Omega$ has a unique supporting hyperplane at x_*, with inner unit normal n, then the open convex cone T has closure $\{\zeta\cdot n \ge 0\}$, so $T = \{\zeta\cdot n > 0\}$ and $k = 1$. Near each boundary point, $\partial\Omega$ is the graph of a convex, hence locally Lipschitz, function, which is differentiable almost everywhere by Rademacher's theorem; at points of differentiability the supporting hyperplane is unique, which gives the $\mathcal{H}^{d-1}$-almost-everywhere statement. At a vertex of a box, T is an orthant with $k = d$. □

4 Nonlocal Dirichlet forms

The arguments of this section follow the variational approach of García Trillos and Slepčev [5] for weighted nonlocal energies, which builds on [3, 13, 14]; here the weights ρ_h depend on h, and they are controlled by Section 3. In this section Ω is a bounded connected Lipschitz domain, (1.5) holds, $0 < h \le s_0^2$, and $\rho = \rho_h$, p_h, $\mathcal{E}_h$ are as in Section 2. Besides $\mathcal{E}_h$ we use the Gaussian form and the limit form

$$\mathcal{E}_h^G(\psi) = \frac{1}{h}\iint_{\Omega\times\Omega}(\psi(x)-\psi(y))^2 G_h(x-y)\,\mathrm{d}x\,\mathrm{d}y, \qquad \mathcal{E}_0(\psi) = \int_\Omega |\nabla\psi|^2 f\,\mathrm{d}x \quad (\psi \in H^1(\Omega)),$$

with $\mathcal{E}_0 = +\infty$ off $H^1(\Omega)$. By Corollary 3.2, $c_1\mathcal{E}_h^G \le \mathcal{E}_h \le c_2\mathcal{E}_h^G$. Put $\mathcal{M}_h(x) = \int_\Omega G_h(x-y)\,\mathrm{d}y$; by (2.1) with $s = \sqrt{h}$,

$$c_0 := (2\pi)^{-d/2}e^{-1/2}\kappa \le \mathcal{M}_h(x) \le 1 \qquad (x \in X). \tag{4.1}$$

The norms of $L^2(\Omega)$ and $L^2(\mu)$ are equivalent, with constants $f_\pm^{1/2}$.

Lemma 4.1 (Compactness)**.** *Let $h_n \to 0$ and $\psi_n \in L^2(\Omega)$ with $\sup_n(\|\psi_n\|_{L^2(\Omega)} + \mathcal{E}_{h_n}^G(\psi_n)) < \infty$. Then a subsequence of (ψ_n) converges in $L^2(\Omega)$ to a limit in $H^1(\Omega)$.*

Proof. For $\psi \in L^2(\Omega)$ and h fixed define the smoothed function $\bar\psi(x) = \mathcal{M}_h(x)^{-1}\int_\Omega G_h(x-y)\psi(y)\,\mathrm{d}y$, which is smooth on $\mathbb{R}^d$. Jensen's inequality for the probability density $G_h(x-\cdot)/\mathcal{M}_h(x)$ and (4.1) give

$$\|\psi - \bar\psi\|_{L^2(\Omega)}^2 \le \frac{1}{c_0}\iint(\psi(x)-\psi(y))^2 G_h(x-y)\,\mathrm{d}y\,\mathrm{d}x = \frac{h}{c_0}\mathcal{E}_h^G(\psi). \tag{4.2}$$

Since $\nabla \mathcal{M}_h(x) = \int_\Omega \nabla_x G_h(x-y)\,\mathrm{d}y$, the quotient rule gives

$$\nabla \bar\psi(x) = \frac{1}{\mathcal{M}_h(x)} \int_\Omega (\psi(y) - \bar\psi(x)) \nabla_x G_h(x-y)\,\mathrm{d}y, \qquad |\nabla_x G_h(x-y)| = \frac{|x-y|}{h} G_h(x-y).$$

By the Cauchy–Schwarz inequality and $\int |z|^2 G_h(z)\,\mathrm{d}z = dh$,

$$|\nabla \bar\psi(x)|^2 \le \frac{d}{c_0^2 h} \int_\Omega (\psi(y) - \bar\psi(x))^2 G_h(x-y)\,\mathrm{d}y.$$

Using $(\psi(y) - \bar\psi(x))^2 \le 2(\psi(y) - \psi(x))^2 + 2(\psi(x) - \bar\psi(x))^2$, integrating over $x \in \Omega$ and using (4.2),

$$\|\nabla \bar\psi\|_{L^2(\Omega)}^2 \le \frac{2d}{c_0^2}\Big(1 + \frac{1}{c_0}\Big) \mathcal{E}_h^G(\psi). \tag{4.3}$$

Applied to ψ_n at h_n, (4.2) gives $\|\psi_n - \bar\psi_n\| \to 0$ and (4.3) with $\|\bar\psi_n\| \le \|\psi_n\| + \|\psi_n - \bar\psi_n\|$ shows that $(\bar\psi_n)$ is bounded in $H^1(\Omega)$. By (D2) and weak compactness, a subsequence of $(\bar\psi_n)$ converges in $L^2(\Omega)$ and weakly in $H^1(\Omega)$ to some $\psi \in H^1(\Omega)$, and the same subsequence of (ψ_n) converges to ψ in $L^2(\Omega)$. □

Lemma 4.2 (Local consistency)**.** *Let $C \subset \mathbb{R}^d$ be compact, $\delta > 0$, $C_\delta = C + \overline{B(0,\delta)}$, and let $u_n, u \in C^1$ on a neighbourhood of C_δ with $u_n \to u$ and $\nabla u_n \to \nabla u$ uniformly on C_δ. If $h_n \to 0$, then*

$$\frac{1}{h_n} \int_{B(x,\delta)} (u_n(x) - u_n(y))^2 G_{h_n}(x-y)\,\mathrm{d}y \;\longrightarrow\; |\nabla u(x)|^2 \qquad \text{uniformly in } x \in C.$$

Proof. Let ω be a modulus of continuity of ∇u on C_δ and $\eta_n = \sup_{C_\delta} |\nabla u_n - \nabla u|$. For $x \in C$ and $|z| < \delta$, $u_n(x+z) - u_n(x) = \nabla u_n(x) \cdot z + r_n$ with $|r_n| \le (2\eta_n + \omega(|z|))|z|$. Since $h^{-1} \int_{B(0,\delta)} (e \cdot z)^2 G_h(z)\,\mathrm{d}z \to |e|^2$ uniformly for bounded e, and $h^{-1} \int |z|^2 (2\eta_n + \omega(|z|))^j G_h(z)\,\mathrm{d}z = \int |\zeta|^2 (2\eta_n + \omega(\sqrt{h}|\zeta|))^j g(\zeta)\,\mathrm{d}\zeta \to 0$ for $j = 1, 2$ by dominated convergence, expanding the square proves the claim, with $|\nabla u_n(x)|^2 \to |\nabla u(x)|^2$ uniformly. □

Lemma 4.3 (Liminf inequality)**.** *Let $h_n \to 0$, $\psi_n \to \psi$ in $L^2(\Omega)$, and let $w_n \ge 0$ and w be functions on Ω with w continuous and positive and $w_n \to w$ uniformly on compact subsets of Ω. Then*

$$\liminf_{n\to\infty} \frac{1}{h_n} \iint_{\Omega\times\Omega} (\psi_n(x) - \psi_n(y))^2 w_n(x) w_n(y) G_{h_n}(x-y)\,\mathrm{d}x\,\mathrm{d}y \;\ge\; \int_\Omega w^2 |\nabla \psi|^2\,\mathrm{d}x,$$

in the sense that if the left side is finite, then $\psi \in H^1_{\mathrm{loc}}(\Omega)$ and the inequality holds.

Proof. Let I be the left side, assumed finite, and pass to a subsequence realizing it.

Constant weights. Let U be a bounded open set with $\overline{U} \subset \Omega$, and write $U^t = \{x \in U : \operatorname{dist}(x, \partial U) > t\}$. We claim

$$\liminf_{n} \frac{1}{h_n} \iint_{U\times U} (\psi_n(x) - \psi_n(y))^2 G_{h_n}(x-y)\,\mathrm{d}x\,\mathrm{d}y \;\ge\; \int_U |\nabla \psi|^2\,\mathrm{d}x, \tag{4.4}$$

in the same extended sense. Let $\delta > 0$ and let $\eta_\delta \ge 0$ be a smooth mollifier supported in $B(0,\delta)$ with integral one. The functions $u_n = \psi_n * \eta_\delta$ and $u = \psi * \eta_\delta$ are defined and smooth on U^δ, and $u_n \to u$ in C^1 on compact subsets of U^δ, since $\psi_n \to \psi$ in $L^2(U)$. For $x, y \in U^\delta$, Jensen's inequality gives $(u_n(x) - u_n(y))^2 \le \int \eta_\delta(z)(\psi_n(x-z) - \psi_n(y-z))^2\,\mathrm{d}z$. Let $C = \overline{U^{3\delta}}$; then $B(x,\delta) \subset U^\delta$ for $x \in C$, and the substitution $(x, y) \mapsto (x - z, y - z)$, which preserves $x - y$, gives

$$\frac{1}{h_n} \int_C \int_{B(x,\delta)} (u_n(x) - u_n(y))^2 G_{h_n}(x-y)\,\mathrm{d}y\,\mathrm{d}x \le \frac{1}{h_n} \iint_{U\times U} (\psi_n(x) - \psi_n(y))^2 G_{h_n}(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

By Lemma 4.2 the left side tends to $\int_C |\nabla(\psi*\eta_\delta)|^2 \ge \int_{U^{3\delta}} |\nabla(\psi*\eta_\delta)|^2$. If J denotes the left side of (4.4), assumed finite, then for each open U' with $\overline{U'} \subset U$ the family $\psi*\eta_\delta$ (δ small) is bounded in $H^1(U')$ with $\|\nabla(\psi*\eta_\delta)\|^2_{L^2(U')} \le J$; as $\psi*\eta_\delta \to \psi$ in $L^2(U')$, weak lower semicontinuity gives $\psi \in H^1(U')$ and $\int_{U'} |\nabla\psi|^2 \le J$. Letting $U' \uparrow U$ proves (4.4).

Variable weights. For every U as above, $\min_{\overline{U}} w > 0$ and $w_n \to w$ uniformly on $\overline{U}$, so (4.4) and $I < \infty$ give $\psi \in H^1_{\mathrm{loc}}(\Omega)$. Let $\mathcal{Q}_k$ be the finite family of open dyadic cubes of side 2^{-k} whose closures lie in Ω, and for $Q \in \mathcal{Q}_k$ let $w_Q = \min_{\overline{Q}} w$. Fix $\theta \in (0,1)$. For n large, $w_n(x)w_n(y) \ge \theta w_Q^2$ for $x, y \in Q$ and all $Q \in \mathcal{Q}_k$. Discarding pairs (x,y) in different cubes and applying (4.4) on each cube,

$$I \ge \theta \sum_{Q\in\mathcal{Q}_k} w_Q^2 \int_Q |\nabla\psi|^2 \,\mathrm{d}x = \theta \int_\Omega W_k |\nabla\psi|^2 \,\mathrm{d}x, \qquad W_k = \sum_{Q\in\mathcal{Q}_k} w_Q^2 \mathbf{1}_Q.$$

The dyadic children of a cube in $\mathcal{Q}_k$ belong to $\mathcal{Q}_{k+1}$ and have larger or equal minima, so W_k is nondecreasing in k; by continuity of w, $W_k \to w^2$ at every point of Ω off the dyadic grid. Monotone convergence and $\theta \uparrow 1$ finish the proof. □

Lemma 4.4 (Uniform Poincaré inequality)**.** *There are $h_1 \in (0, s_0^2]$ and $c_P > 0$, depending only on Ω, such that $\mathcal{E}_h^G(\psi) \ge c_P \|\psi - \bar\psi_\Omega\|^2_{L^2(\Omega)}$ for all $\psi \in L^2(\Omega)$ and $0 < h \le h_1$, where $\bar\psi_\Omega$ is the mean of ψ over Ω. Consequently, for $h \le h_1$ and $\psi \in L^2_0(\mu)$,*

$$\mathcal{E}_h(\psi) \ge \alpha \|\psi\|^2_{L^2(\mu)}, \qquad \alpha = c_1 c_P / f_+. \tag{4.5}$$

Proof. If the first claim fails, there are $h_n \to 0$ and ψ_n with mean zero, $\|\psi_n\|_{L^2(\Omega)} = 1$ and $\mathcal{E}^G_{h_n}(\psi_n) \to 0$. By Lemma 4.1, a subsequence converges in $L^2(\Omega)$ to some $\psi \in H^1(\Omega)$ with mean zero and norm one, and Lemma 4.3 with $w_n = w = 1$ gives $\int |\nabla\psi|^2 = 0$. As Ω is connected, ψ is constant, hence zero, a contradiction. For (4.5), the μ-mean minimizes $c \mapsto \|\psi - c\|_{L^2(\mu)}$, so $\|\psi\|^2_{L^2(\mu)} \le \|\psi - \bar\psi_\Omega\|^2_{L^2(\mu)} \le f_+ \|\psi - \bar\psi_\Omega\|^2_{L^2(\Omega)}$, and $\mathcal{E}_h \ge c_1 \mathcal{E}^G_h$. □

Lemma 4.5 (Convergence on smooth functions)**.** *For every $\psi \in C^1_c(\mathbb{R}^d)$, $\mathcal{E}_h(\psi) \to \mathcal{E}_0(\psi)$ as $h \to 0$.*

Proof. Let $\delta > 0$ and $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge \delta\}$, and write $\ell = \|\nabla\psi\|_\infty$. By Corollary 3.2 and $|\psi(x) - \psi(y)| \le \ell|x-y|$, the pairs with $x \in \Omega \setminus \Omega_\delta$ contribute at most $c_2 \ell^2 h^{-1} \int_{\Omega\setminus\Omega_\delta} \int |x-y|^2 G_h(x-y) \,\mathrm{d}y \,\mathrm{d}x = c_2 \ell^2 d\, |\Omega \setminus \Omega_\delta|$, and the pairs with $x \in \Omega_\delta$, $|y - x| \ge \delta/2$ contribute at most $c_2 \ell^2 |\Omega| \int_{|\zeta| \ge \delta/(2\sqrt{h})} |\zeta|^2 g(\zeta) \,\mathrm{d}\zeta \to 0$. For $x \in \Omega_\delta$ and $|y - x| < \delta/2$ we have $y \in \Omega_{\delta/2}$, and by Proposition 3.7 and the uniform continuity of f,

$$|p_h(x,y) - f(x) G_h(x-y)| \le (\sigma_h + \sqrt{f_+}\, \omega(|x-y|)) G_h(x-y),$$

where $\sigma_h = \sup_{x,y\in\Omega_{\delta/2}} |\rho_h(x)\rho_h(y) - \sqrt{f(x)f(y)}| \to 0$ and ω is a modulus of continuity of $\sqrt{f}$. The corresponding error is at most $\ell^2 |\Omega| \int |\zeta|^2 (\sigma_h + \sqrt{f_+}\omega(\sqrt{h}|\zeta|)) g(\zeta) \,\mathrm{d}\zeta \to 0$. Finally, by Lemma 4.2 with $u_n = u = \psi$ and $C = \Omega_\delta$,

$$\frac{1}{h} \int_{\Omega_\delta} f(x) \int_{B(x,\delta/2)} (\psi(x) - \psi(y))^2 G_h(x-y) \,\mathrm{d}y \,\mathrm{d}x \to \int_{\Omega_\delta} f |\nabla\psi|^2.$$

Hence $\limsup_h |\mathcal{E}_h(\psi) - \mathcal{E}_0(\psi)| \le (c_2 d + f_+) \ell^2 |\Omega \setminus \Omega_\delta|$, which tends to zero as $\delta \to 0$. □

5 Proofs of the main results

Let $B_0 = L^{-1}$ on $L^2_0(\mu)$. By (D2) and (D3), L has compact resolvent and $L \geq \lambda_1 > 0$ on $L^2_0(\mu)$, so B_0 is compact, and

$$\langle \xi, B_0 \xi \rangle_{L^2(\mu)} = \sup_{\psi \in H^1(\Omega)} \{ 2\langle \xi, \psi \rangle_{L^2(\mu)} - \mathcal{E}_0(\psi) \} = \|\xi\mu\|^2_{-1,\mu} \qquad (\xi \in L^2_0(\mu)). \tag{5.1}$$

By (D4) and continuity of the bracket on $H^1(\Omega)$, the supremum may be restricted to $\psi \in C^1_c(\mathbb{R}^d)$.

Proposition 5.1. *If $h_n \to 0$ and $\xi_n \rightharpoonup \xi$ weakly in $L^2_0(\mu)$, then $\langle \xi_n, B_{h_n} \xi_n \rangle \to \langle \xi, B_0 \xi \rangle$.*

Proof. By Lemma 2.3, $|\langle \xi_n, (B_{h_n} - D_{h_n})\xi_n \rangle| \leq \frac{h_n}{2} \|\xi_n\|^2 \to 0$, so it suffices to treat D_{h_n}; we write $\mathcal{E}_n = \mathcal{E}_{h_n}$ and take n so large that $h_n \leq h_1$.

Lower bound. For $\psi \in C^1_c(\mathbb{R}^d)$, (2.9) and Lemma 4.5 give $\liminf_n \langle \xi_n, D_{h_n} \xi_n \rangle \geq \lim_n \{ 2\langle \xi_n, \psi \rangle - \mathcal{E}_n(\psi) \} = 2\langle \xi, \psi \rangle - \mathcal{E}_0(\psi)$. Taking the supremum over ψ and using (5.1) gives $\liminf \geq \langle \xi, B_0 \xi \rangle$.

Upper bound. Let $\psi_n = D_{h_n} \xi_n$, so $\langle \xi_n, D_{h_n} \xi_n \rangle = \mathcal{E}_n(\psi_n) = \langle \xi_n, \psi_n \rangle$ by Lemma 2.3. By (4.5), $\alpha \|\psi_n\|^2 \leq \langle \xi_n, \psi_n \rangle \leq \|\xi_n\| \|\psi_n\|$, so $\|\psi_n\|_{L^2(\mu)} \leq \sup_m \|\xi_m\|/\alpha$, and $\mathcal{E}^G_{h_n}(\psi_n) \leq \mathcal{E}_n(\psi_n)/c_1$ is bounded. Given any subsequence, Lemma 4.1 yields a further subsequence with $\psi_n \to \psi_*$ in $L^2(\mu)$ and $\psi_* \in H^1(\Omega)$. Along it, $\langle \xi_n, \psi_n \rangle \to \langle \xi, \psi_* \rangle$, and Lemma 4.3 with $w_n = \rho_{h_n} \to w = \sqrt{f}$ (Proposition 3.7) gives $\liminf \mathcal{E}_n(\psi_n) \geq \mathcal{E}_0(\psi_*)$. Hence

$$\limsup \langle \xi_n, D_{h_n} \xi_n \rangle = \limsup \{ 2\langle \xi_n, \psi_n \rangle - \mathcal{E}_n(\psi_n) \} \leq 2\langle \xi, \psi_* \rangle - \mathcal{E}_0(\psi_*) \leq \langle \xi, B_0 \xi \rangle$$

along the further subsequence. As the subsequence was arbitrary, the upper bound holds for the whole sequence. □

Proof of Theorem 1.2(a). By Lemma 2.3, the supremum in (a) is $\|B_h - B_0\|$, the norm on $L^2_0(\mu)$ of a selfadjoint operator. If it does not tend to zero, there are $h_n \to 0$, $\eta > 0$ and unit vectors $\xi_n \in L^2_0(\mu)$ with $|\langle \xi_n, (B_{h_n} - B_0)\xi_n \rangle| \geq \eta$. Pass to a weakly convergent subsequence $\xi_n \rightharpoonup \xi$. Proposition 5.1 gives $\langle \xi_n, B_{h_n} \xi_n \rangle \to \langle \xi, B_0 \xi \rangle$, and compactness of B_0 gives $\langle \xi_n, B_0 \xi_n \rangle \to \langle \xi, B_0 \xi \rangle$, a contradiction. □

Since $\|B_h - D_h\| \leq h/2$, the proof also gives $\|D_h - L^{-1}\| \to 0$ on $L^2_0(\mu)$, where $D_h^{-1} = \frac{2}{h}(I - K_\mu)$ is the self-transport Laplacian. Norm convergence of selfadjoint operators moves each eigenvalue by at most the norm of the difference (min-max principle), which gives the following spectral statement.

Corollary 5.2 (Spectral convergence)**.** *Assume* (1.5)*, and let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of L on $L^2_0(\mu)$, repeated with multiplicity. For each j, the j-th smallest eigenvalue of $\frac{2}{h}(I - K_\mu)$ on $L^2_0(\mu)$ converges to λ_j as $h = \varepsilon/2 \to 0$, and the spectral projections of $\frac{2}{h}(I - K_\mu)$ onto eigenvalues in any interval (a, b) with $0 < a < b < \infty$ and $a, b \notin \{\lambda_j\}$ converge in norm to those of L.*

Proof. The eigenvalues of D_h above $h/2$ are the reciprocals of the eigenvalues of $\frac{2}{h}(I - K_\mu)$ below $2/h$, ordered decreasingly, and the j-th largest eigenvalue of D_h differs from $1/\lambda_j$ by at most $\|D_h - L^{-1}\|$. For the projections, write them as Riesz integrals of the resolvents of D_h and L^{-1} over a contour around $(1/b, 1/a)$ that avoids the spectrum of L^{-1}; the resolvents converge in norm uniformly on the contour. □

For part (b) we need a gradient bound for K.

Lemma 5.3 (Gradient bound)**.** *For $0 < h \leq s_0^2$ and $\psi \in L^2(\mu)$, $K\psi$ is smooth and $\|\nabla K\psi\|^2_{L^2(\Omega)} \leq \Lambda^2 d f_-^{-2} \mathcal{E}_h(\psi)$.*

Proof. Differentiating (2.4) gives

$$\nabla_x \tilde{k}(x,y) = \frac{\tilde{k}(x,y)(y-\bar{y}_x)}{h}, \qquad \bar{y}_x = \int y\, \tilde{k}(x,y)\,\mathrm{d}y, \qquad \int \nabla_x \tilde{k}(x,y)\,\mathrm{d}y = 0.$$

Hence

$$\nabla K\psi(x) = \frac{1}{h}\int \tilde{k}(x,y)(y-\bar{y}_x)(\psi(y)-\psi(x))\,\mathrm{d}y,$$

and by the Cauchy–Schwarz inequality and $\int \tilde{k}(x,y)|y-\bar{y}_x|^2\,\mathrm{d}y \le \int \tilde{k}(x,y)|y-x|^2\,\mathrm{d}y$,

$$|\nabla K\psi(x)|^2 \le \frac{1}{h^2}\int \tilde{k}(x,y)|y-x|^2\,\mathrm{d}y \int \tilde{k}(x,y)(\psi(y)-\psi(x))^2\,\mathrm{d}y.$$

By (3.2), $\int \tilde{k}(x,y)|y-x|^2\,\mathrm{d}y \le \Lambda\, dh/\mathcal{G}_h\rho(x) \le \Lambda^2 dh/f_-$, and $\tilde{k}(x,y) = p_h(x,y)/f(x) \le p_h(x,y)/f_-$. Integrating over $x \in \Omega$ gives the claim. □

Proof of Theorem 1.2(b). Let $\varepsilon_0 = 2s_0^2$, $h = \varepsilon/2 \le s_0^2$, and first let $\nu = \xi\mu$ with $\xi \in L_0^2(\mu)$. For $\lambda \in [0,1)$ we have $\lambda/(1-\lambda^2) \le \lambda/(1-\lambda) = \lambda + \lambda^2/(1-\lambda)$, so by the spectral theorem on $L_0^2(\mu)$,

$$\mathbf{g}_\mu^\varepsilon(\nu,\nu) \le h\langle \xi, K\xi\rangle + h\langle K\xi, (I-K)^{-1}K\xi\rangle. \tag{5.2}$$

Second term. As in (2.9), $h\langle \chi, (I-K)^{-1}\chi\rangle = \sup_\psi\{2\langle\chi,\psi\rangle - \frac{1}{2}\mathcal{E}_h(\psi)\}$ for $\chi \in L_0^2(\mu)$. With $\chi = K\xi$, $\langle K\xi,\psi\rangle_{L^2(\mu)} = \langle \xi, K\psi\rangle_{L^2(\mu)} = \langle \nu, K\psi - c\rangle$ for every constant c. Choosing c as the mean of $K\psi$ over Ω, (D3) and Lemma 5.3 give $\langle K\xi,\psi\rangle \le \|\nu\|_*(1+C_{\mathrm{PW}})\|\nabla K\psi\| \le C_3\|\nu\|_*\mathcal{E}_h(\psi)^{1/2}$ with $C_3 = (1+C_{\mathrm{PW}})\Lambda\sqrt{d}/f_-$. Maximizing $2C_3\|\nu\|_* t - t^2/2$ over $t \ge 0$ bounds the second term by $2C_3^2\|\nu\|_*^2$.

First term. As in the proof of Lemma 2.2, $h\langle \xi, K\xi\rangle = h\|G_{h/2} * (a\xi f \mathbf{1}_\Omega)\|_{L^2(\mathbb{R}^d)}^2$. For $\zeta \in L^2(\mathbb{R}^d)$ with $\|\zeta\| \le 1$,

$$\int_{\mathbb{R}^d} (G_{h/2} * (a\xi f\mathbf{1}_\Omega))\zeta = \int_\Omega a\,(G_{h/2}*\zeta)\,\xi f\,\mathrm{d}x = \langle \nu, a\,(G_{h/2}*\zeta) - c\rangle$$

for every constant c. By (3.2),

$$0 < a = \frac{1}{\mathcal{G}_h\rho} \le \frac{\Lambda}{f_-}, \qquad |\nabla a| = \frac{|\nabla \mathcal{G}_h\rho|}{(\mathcal{G}_h\rho)^2} \le \frac{\Lambda^3\|\nabla G_h\|_{L^1}}{f_-^2}.$$

Since $\|\nabla G_t\|_{L^1} \le \sqrt{d/t}$, $\|G_{h/2}*\zeta\| \le 1$ and $\|\nabla(G_{h/2}*\zeta)\| \le \sqrt{2d/h}$, we get $\|\nabla(a\,G_{h/2}*\zeta)\|_{L^2(\Omega)} \le C_4 h^{-1/2}$ with $C_4 = \sqrt{d}(\Lambda^3 f_-^{-2} + \sqrt{2}\Lambda f_-^{-1})$. With (D3) this gives $h\langle\xi, K\xi\rangle \le (1+C_{\mathrm{PW}})^2 C_4^2\|\nu\|_*^2$. Hence (b) holds on $\{\xi\mu\}$ with $C = (1+C_{\mathrm{PW}})^2C_4^2 + 2C_3^2$.

For general $\nu \in \mathcal{H}_0^*$, choose $\xi_j\mu \to \nu$ in $\mathcal{H}_0^*$ (Lemma 2.4). Then $\xi_j\mu \to \nu$ in $C^1(X)^*$, so $\mathbf{g}_\mu^\varepsilon(\xi_j\mu,\xi_j\mu) \to \mathbf{g}_\mu^\varepsilon(\nu,\nu)$ by Lemma 2.4, and the bound passes to the limit. □

Proof of Theorem 1.2(c). Write $Q_\varepsilon(\nu) = \mathbf{g}_\mu^\varepsilon(\nu,\nu)$ and $Q_0(\nu) = \|\nu\|_{-1,\mu}^2$. Both are nonnegative quadratic forms on $\mathcal{H}_0^*$: for Q_ε this is Lemma 2.4, and $Q_0(\nu) = \int|\nabla\psi_\nu|^2\,\mathrm{d}\mu$ where $\psi_\nu \in H^1(\Omega)$ solves $\int \nabla\psi_\nu\cdot\nabla\zeta\,\mathrm{d}\mu = \langle\nu,\zeta\rangle$ for all $\zeta \in H^1(\Omega)$. By (b), and by $Q_0(\nu) \le (1+C_{\mathrm{PW}})^2\|\nu\|_*^2/f_-$, which follows from (1.4) and (D3), both are bounded by $C'\|\nu\|_*^2$ for $\varepsilon \le \varepsilon_0$. The Cauchy–Schwarz inequality for nonnegative forms then gives, for $Q \in \{Q_\varepsilon, Q_0\}$,

$$|Q(\nu_1) - Q(\nu_2)| \le Q(\nu_1-\nu_2)^{1/2}Q(\nu_1+\nu_2)^{1/2} \le C'\|\nu_1-\nu_2\|_*\|\nu_1+\nu_2\|_*,$$

so the family $\{Q_\varepsilon\}_{\varepsilon\le\varepsilon_0} \cup \{Q_0\}$ is equi-Lipschitz on bounded sets. By (a), $Q_\varepsilon \to Q_0$ on the dense set $\{\xi\mu : \xi \in L_0^2(\mu)\}$ (Lemma 2.4; by (5.1), $Q_0(\xi\mu) = \langle\xi, L^{-1}\xi\rangle$). Equicontinuity gives convergence on all of $\mathcal{H}_0^*$, uniformly on compact sets. □

Remark 5.4 (Horizontal perturbations and the Otto metric). Let $v \in L^2(\mu;\mathbb{R}^d)$ and let $\dot\mu = -\operatorname{div}(\mu v)$ act by $\langle \dot\mu, \zeta\rangle = \int v\cdot\nabla\zeta\,\mathrm{d}\mu$. The horizontal curves $\mu_t = \Psi(t,\cdot)_\#\mu$ of [6, Remark 3.2] with $\Psi(0,\cdot)=\mathrm{id}$ have velocities of this form, with $v = \partial_t\Psi(0,\cdot)$ continuous and, since $\Psi(t,\cdot)$ maps X into X for t of both signs, tangent to ∂X wherever ∂X is smooth; the class of all $v\in L^2(\mu;\mathbb{R}^d)$ is larger. Since $|\langle\dot\mu,\zeta\rangle| \le \|v\|_{L^2(\mu)} f_+^{1/2}\|\zeta\|_{H^1(\Omega)}$, $\dot\mu\in\mathcal{H}_0^*$, and (1.4) gives $\|\dot\mu\|_{-1,\mu}^2 = \|Pv\|_{L^2(\mu)}^2$, where P is the orthogonal projection of $L^2(\mu;\mathbb{R}^d)$ onto the closure of $\{\nabla\psi:\psi\in H^1(\Omega)\}$. This is the Otto norm $\min\{\int|w|^2\,\mathrm{d}\mu : -\operatorname{div}(\mu w)=\dot\mu\}$, and for $v=\nabla\psi$ it equals $\int|\nabla\psi|^2\,\mathrm{d}\mu = \mathbf{g}_\mu^0(\dot\mu,\dot\mu)$, as in [6, Remark 4.18]. Theorem 1.2(c) shows that $\mathbf{g}_\mu^\varepsilon(\dot\mu,\dot\mu)$ converges to this value.

Remark 5.5 (Scope of the hypotheses). The proofs use the domain only through (D1)–(D4) and connectedness. The constants in Proposition 3.1, Lemma 5.3 and Theorem 1.2(b) depend on f only through f_- and f_+; the modulus of continuity of f enters through Proposition 3.7, which identifies the limit of the nonlocal forms.

6 Numerical illustration

The scripts `v1_interval.py` and `v2_planar.py`, included with this article, discretize (2.3) and (2.7) by a partition of the domain into cells with one node per cell, solve (2.3) by Newton's method, and evaluate (2.7) by diagonalization or by linear solves. These computations illustrate the theorems; no proof uses them.

On $X=[0,1]$ the limit is explicit: $\langle\xi, L^{-1}\xi\rangle_{L^2(\mu)} = \int_0^1 F^2/f$ with $F(x)=\int_0^x f\xi$, and $\|\delta_s-\delta_t\|_{-1,\mu}^2 = \int_s^t \mathrm{d}x/f$. Table 1 lists the ratio of the Sinkhorn tensor to this limit for a smooth tangent, a discontinuous tangent and a difference of point masses, which lies in $\mathcal{H}_0^*$ but not in $L^2(\mu)$, together with the relative operator-norm error of Theorem 1.2(a) and two values of $\exp(\beta_\varepsilon)=\rho/\sqrt{f}$. In these runs the deficit $1-\text{ratio}$ decreases roughly like $h^{3/2}$ for the smooth tangent and like $h^{1/2}$ for the point masses, and the operator-norm error roughly like h.

The last column shows the boundary layer of Theorem 1.4: the first node lies at distance $s\sqrt{h}$ from the endpoint with $s=1/60$ in the last three rows of each block, and the half-line profile computed by `v3_halfline.py` has $m(1/60)=1.62932$, while $m(0)=e^{1/2}=1.64872$ to twelve digits. In dimension two, Table 2 shows the same behaviour on a square, on a right isosceles triangle and on a disk, at uniform and nonuniform densities, for the tangent $\xi=\cos(\pi x_1)(1+x_2)+x_1x_2^2$ minus its mean; the limit is computed by a Legendre–Galerkin method on the exact domain. At the corner node of the square, at distance $s\sqrt{h}$ from both sides, the values 2.2604, 2.2565, 2.2067 agree within 0.4% with $m(s)^2 = 2.2536$, 2.2496, 2.1985 from `v3_halfline.py`, as the product profile of Proposition 3.9 predicts; at the corner itself that profile equals e. On the disk the reported node lies at $s=0.126$, 0.127, 0.142 from the circle, where $m(s)=1.515$, 1.514, 1.499; the tabulated values exceed $m(s)$ by 0.062, 0.043, 0.030, that is, by about $0.69\sqrt{h}$ in each case (`v5_table_notes.py`). This is consistent with Proposition 3.9 at the curved boundary, with an excess of order $\sqrt{h}$, and the disk entries in the column "max" likewise decrease as h decreases.

Finally, `v4_spectrum.py` illustrates Corollary 5.2: on $[0,1]$ with $f\equiv1$, where $\lambda_j=(\pi j)^2$, the five smallest nonzero eigenvalues of $\frac{2}{h}(I-K_\mu)$ have relative errors between $-8.6\cdot10^{-3}$ and -0.20 at $h=4\cdot10^{-3}$, and between $-6.0\cdot10^{-4}$ and $-1.5\cdot10^{-2}$ at $h=2.5\cdot10^{-4}$.

density	$h = \varepsilon/2$	N	smooth ξ	step ξ	$\delta_s - \delta_t$	$\|B_h - B_0\|/\|B_0\|$	$\min \rho/\sqrt{f}$	$\rho/\sqrt{f}$ at x_1
$f \equiv 1$	$4 \cdot 10^{-3}$	600	0.998867	0.998175	0.952015	$5.2 \cdot 10^{-3}$	0.9914	1.6334
	$1 \cdot 10^{-3}$	948	0.999869	0.998164	0.975984	$1.3 \cdot 10^{-3}$	0.9914	1.6294
	$2.5 \cdot 10^{-4}$	1897	0.999984	0.999771	0.988000	$3.3 \cdot 10^{-4}$	0.9914	1.6294
	$1 \cdot 10^{-4}$	3000	0.999996	0.999993	0.992416	$1.3 \cdot 10^{-4}$	0.9914	1.6294
f_{s}	$4 \cdot 10^{-3}$	600	0.998829	0.998615	0.964491	$3.3 \cdot 10^{-3}$	0.9850	1.6359
	$1 \cdot 10^{-3}$	948	0.999884	0.998452	0.982418	$8.1 \cdot 10^{-4}$	0.9925	1.6287
	$2.5 \cdot 10^{-4}$	1897	0.999987	0.999808	0.991228	$2.0 \cdot 10^{-4}$	0.9917	1.6286
	$1 \cdot 10^{-4}$	3000	0.999997	0.999996	0.994459	$8.0 \cdot 10^{-5}$	0.9915	1.6288
f_{r}	$4 \cdot 10^{-3}$	600	0.998622	0.998207	0.955141	$4.9 \cdot 10^{-3}$	0.9576	1.6424
	$1 \cdot 10^{-3}$	948	0.999805	0.998178	0.977529	$1.2 \cdot 10^{-3}$	0.9685	1.6338
	$2.5 \cdot 10^{-4}$	1897	0.999967	0.999762	0.988788	$3.0 \cdot 10^{-4}$	0.9774	1.6315
	$1 \cdot 10^{-4}$	3000	0.999988	0.999985	0.992918	$1.2 \cdot 10^{-4}$	0.9827	1.6307

Table 1: The interval $[0, 1]$ with N equal cells; $x_1 = 1/(2N)$ is the first node. Columns 4–6: the ratio $\mathbf{g}^{\varepsilon}_{\mu}(\dot\mu, \dot\mu)/\|\dot\mu\|^2_{-1,\mu}$ for $\dot\mu = \xi\mu$ with $\xi = \cos \pi x + x^2$ and with $\xi = \mathbf{1}_{[0,0.3)} - \frac{1}{2}\mathbf{1}_{[0.3,1]}$ (both centred), and for $\dot\mu = \delta_s - \delta_t$ at the nodes nearest 0.25 and 0.7. Densities: $f \equiv 1$; $f_{\mathrm{s}} \propto 1 + \frac{1}{2}\cos 2\pi x + 0.3x$; $f_{\mathrm{r}} \propto 1 + 0.6|x - 0.4|^{1/2}$, which is continuous but not Lipschitz. The step tangent is sampled at the nodes: for $N = 948$ and $N = 1897$ the jump at 0.3 falls inside a cell, which by itself multiplies the limit by 0.99839 and 0.99980 when $f \equiv 1$ (`v5_table_notes.py`), so the deficits in that column at those N come mostly from this sampling. Output of `v1_interval.py`.

domain, density	h	N	ratio	min $\frac{\rho}{\sqrt{f}}$	max $\frac{\rho}{\sqrt{f}}$	inner $\lvert\frac{\rho}{\sqrt{f}} - 1\rvert$	$\frac{\rho}{\sqrt{f}}$ at node
square, uniform	$8 \cdot 10^{-3}$	1600	0.996339	0.9825	2.2604	$1.4 \cdot 10^{-2}$	2.2604
	$4 \cdot 10^{-3}$	3136	0.998729	0.9826	2.2565	$8.1 \cdot 10^{-4}$	2.2565
	$2 \cdot 10^{-3}$	4900	0.999562	0.9823	2.2067	$2.2 \cdot 10^{-4}$	2.2067
square, nonuniform	$8 \cdot 10^{-3}$	1600	0.996396	0.9774	2.2773	$1.8 \cdot 10^{-2}$	2.2622
	$4 \cdot 10^{-3}$	3136	0.998756	0.9793	2.2665	$3.4 \cdot 10^{-3}$	2.2556
	$2 \cdot 10^{-3}$	4900	0.999578	0.9803	2.2126	$1.2 \cdot 10^{-3}$	2.2051
triangle, uniform	$8 \cdot 10^{-3}$	2304	0.994985	0.9827	3.4744	$6.3 \cdot 10^{-2}$	2.4429
	$4 \cdot 10^{-3}$	4096	0.998126	0.9826	3.4409	$1.7 \cdot 10^{-2}$	2.4278
	$2 \cdot 10^{-3}$	6400	0.999297	0.9826	3.3661	$1.5 \cdot 10^{-2}$	2.3937
triangle, nonuniform	$8 \cdot 10^{-3}$	2304	0.994794	0.9822	3.4825	$6.5 \cdot 10^{-2}$	2.4432
	$4 \cdot 10^{-3}$	4096	0.998046	0.9812	3.4511	$1.9 \cdot 10^{-2}$	2.4258
	$2 \cdot 10^{-3}$	6400	0.999267	0.9814	3.3750	$1.7 \cdot 10^{-2}$	2.3913
disk, uniform	$8 \cdot 10^{-3}$	1600	0.994894	0.9877	1.7257	$1.1 \cdot 10^{-2}$	1.5764
	$4 \cdot 10^{-3}$	3120	0.998224	0.9890	1.6931	$8.5 \cdot 10^{-4}$	1.5572
	$2 \cdot 10^{-3}$	4920	0.999321	0.9897	1.6865	$2.2 \cdot 10^{-4}$	1.5291
disk, nonuniform	$8 \cdot 10^{-3}$	1600	0.995013	0.9857	1.7536	$1.2 \cdot 10^{-2}$	1.5837
	$4 \cdot 10^{-3}$	3120	0.998273	0.9877	1.7108	$2.0 \cdot 10^{-3}$	1.5622
	$2 \cdot 10^{-3}$	4920	0.999346	0.9889	1.6995	$8.8 \cdot 10^{-4}$	1.5325

Table 2: Planar domains: the unit square, the triangle with vertices $(0,0), (1,0), (0,1)$, and the disk of radius $\frac{1}{2}$ centred at $(\frac{1}{2}, \frac{1}{2})$, each partitioned into N cells (for the disk, the cells of a square grid cut by the disk). The column "ratio" is $\mathbf{g}^{\varepsilon}_{\mu}(\xi\mu, \xi\mu)/\langle \xi, L^{-1}\xi\rangle_{L^2(\mu)}$; "inner" means distance at least 0.2 (square, disk) or 0.1 (triangle) from the boundary. The last column is the value at the node nearest the corner $(0,0)$ (square, triangle) or the boundary point $(1, \frac{1}{2})$ (disk). Nonuniform density: $f \propto 1 + 0.4\cos(\pi x_1)\cos(\pi x_2) + 0.3x_1$. Output of `v2_planar.py`.

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