Single-time boundary heat-flux rigidity for smooth domains

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Abstract

Cavallina and Pinamonti (2026) studied domains on which the Dirichlet heat flow with initial temperature one has constant boundary flux at a discrete set of times, and conjectured that complements of closed balls are the only such exterior domains. Their exterior problem also asks the temperature to vanish at infinity uniformly in time, which initial temperature one excludes on every unbounded domain; the natural reading selects the bounded solution, whose deficit, one minus the temperature, vanishes at infinity uniformly in time in dimension three and higher and on bounded time intervals in the plane. We prove the conjecture in this reading from a single observation time: if the bounded solution on an exterior domain with $C^{2,\alpha}$ boundary has constant boundary flux at one positive time, the domain is the complement of a closed ball. The obstacle may be nonconvex and disconnected. In the moving-plane argument, comparison at every admissible plane follows from the uniform initial datum; the single time enters only at the critical plane, through a Hopf contradiction and a parabolic corner estimate. The same argument shows that a bounded connected $C^{2,\alpha}$ domain with constant flux at one time is a ball, answering Cavallina and Pinamonti's finite-time question in this class and excluding their conjectured nonball examples from it. Near a ball, at fixed volume and centroid, the flux defect controls the deformation in Hölder norms, uniformly at late times for the mean-normalized flux; a moving proper observation cap admits infinitely many nonball shapes with the ball's flux.

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1 Statement and context

Let $\Omega \subset \mathbb{R}^N$ be bounded and connected with $C^{2,\alpha}$ boundary, where $N \geq 2$ and $0 < \alpha < 1$. Write ν for its outward unit normal and Δ_D for the Dirichlet Laplacian. We study

$$u_t = \Delta u \quad \text{in } \Omega \times (0,\infty), \qquad u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \qquad u(\cdot,t) \longrightarrow 1 \quad \text{in } L^2(\Omega) \quad (t \downarrow 0). \tag{1}$$

Thus $u(\cdot,t) = e^{t\Delta_D}1$. The initial temperature is prescribed in L^2, and the zero boundary value is imposed for positive times. The signed normal derivative is $\partial_\nu u = \nabla u \cdot \nu$.

Theorem 1.1. *Suppose that, for some $T > 0$ and $b \in \mathbb{R}$,*

$$\partial_\nu u(x,T) = b \qquad \text{for every } x \in \partial\Omega. \tag{2}$$

Then Ω is a ball. Conversely, on a ball condition (2) holds at every positive time.

The same scalar b is required on every boundary component. The domain may be nonconvex and its boundary may be disconnected. The observation is the complete spatial boundary flux at a single instant. Theorem 1.5 below proves the same rigidity for exterior domains, which answers the exterior conjecture of Cavallina and Pinamonti for exterior domains with $C^{2,\alpha}$ boundary, in the reading described there. *This edition* adds the exterior theorem (Theorem 1.5, Corollary 1.6 and Section 11) and states Theorem 1.1 for $C^{2,\alpha}$ boundaries, where edition 5 assumed C^∞ boundaries.

Cavallina–Pinamonti [1] consider precisely the initial datum and flux in (1)–(2), and the condition

$$\partial_\nu u(\cdot, t_n) \equiv b_n \quad \text{on } \partial\Omega \qquad (n \in \mathbb{N}) \tag{CP}$$

for sequences $(t_n) \subset (0,\infty)$ and $(b_n) \subset \mathbb{R}$ [1, (1.4)]. Their Theorem I proves that a bounded Lipschitz domain satisfying (CP) is a ball when (t_n) has an accumulation point in $(0,\infty]$, and their Theorem II proves the same for bounded domains with C^∞ boundary when (t_n) accumulates at 0. Their Section 7 poses three questions. *Problem 3* asks for which bounded domains (CP) holds for finitely many times $t_1, \dots, t_m$, and *Conjecture 7.4* predicts nonball solutions of Problem 3 for every m. Theorem 1.1 shows that one time already forces a ball: in the class of bounded connected domains with $C^{2,\alpha}$ boundary it answers Problem 3 (the solutions are exactly the balls) and shows that Conjecture 7.4 fails in this class: the boundary of every nonball solution of Problem 3 is not of class $C^{2,\alpha}$. *Problem 2* asks for Theorem II under milder regularity. Zeraoulia [2] proves Theorem II for C^2 boundaries by a uniform short-time flux expansion along sequences tending to zero; Theorem 1.1 gives the conclusion of Theorem II for $C^{2,\alpha}$ boundaries from any single positive time. *Problem 1* concerns unbounded domains; its exterior case and *Conjecture 7.1* are the subject of Theorem 1.5.

The proof belongs to the classical moving-plane tradition of Serrin [3] and the parabolic symmetry literature of Alessandrini–Garofalo [4], Alessandrini [5, 6], and Garofalo–Sartori [7]. In particular, reflected parabolic comparison, tangency Hopf contradictions and orthogonal-contact arguments already occur in that literature. The boundary-point principle used here is classical; see Friedman [8] and Li–Nirenberg [9, Section 3]. Coordinate-times-exponential parabolic corner constructions also have antecedents; compare Attouchi–Souplet [10, Appendix 2]. Our proof gives directly the corner estimate for the parabolic evolution in (1). The quantitative results and the exterior theorem below extend this classical symmetry framework. The role of the one-time condition is explicit: it enters only in the final tangency or corner contradiction, whereas comparison uses the evolution from its uniform initial datum.

The stationary-level-surface results of Alessandrini [5, 6] are described and developed by Magnanini–Marini [11, Introduction and Theorem 1.1]. That framework assumes temporal invariance of interior level surfaces and leads to isoparametric or space-time splitting alternatives. These results provide further classical context for the global argument.

For approximate observations we establish the following separate local statement. Let B be the unit ball in $\mathbb{R}^N$, let $X = C^{m,\alpha}(\mathbb{S}^{N-1})$ and $Y = C^{m-1,\alpha}(\mathbb{S}^{N-1})$, and define

$$\Omega_f = \{r\theta : 0 \le r < 1 + f(\theta),\ \theta \in \mathbb{S}^{N-1}\}.$$

Pull its signed flux back to the sphere by the radial graph:

$$\Phi_T(f)(\theta) = \partial_{\nu_f} u_f((1+f(\theta))\theta, T), \qquad u_f(t) = e^{t\Delta_{\Omega_f, D}} 1.$$

Theorem 1.2 (Local quantitative estimate)**.** *Fix $N \ge 2$, integer $m \ge 3$, $0 < \alpha < 1$ and $0 < T_0 \le T_1 < \infty$. There are $\varepsilon, C > 0$ such that, if $f \in X$, $\|f\|_X < \varepsilon$,*

$$|\Omega_f| = |B|, \qquad \int_{\Omega_f} x\, dx = 0,$$

then for every $T \in [T_0, T_1]$,

$$\|f\|_{C^{m,\alpha}(\mathbb{S}^{N-1})} \le C \inf_{b\in\mathbb{R}} \|\Phi_T(f) - b\|_{C^{m-1,\alpha}(\mathbb{S}^{N-1})}. \tag{3}$$

The constants are uniform on that time interval. The same statement holds in the corresponding little Hölder spaces.

Quantitative elliptic Serrin estimates provide a natural comparison at the limiting first eigenfunction. Ciraolo, Magnanini and Vespri [12, Theorems 4.2 and 5.3] obtain linear annular control for positive semilinear Dirichlet solutions and identify the role of a lower bound on the boundary derivative. The estimate below concerns the mean-normalized heat-flux family in a fixed near-ball Hölder chart, with a common inverse bound and a common shape neighborhood for all $T \ge T_0$. Its proof controls the shape derivative of the spectral remainder and the limiting reduced boundary operator uniformly.

The main quantitative strengthening concerns late-time conditioning. Use the physical surface mean of the signed flux to form a dimensionless observation. The following estimate has a common neighborhood even as the observation time tends to infinity.

Theorem 1.3 (Relative-flux stability uniformly at late times)**.** *Fix* $N \ge 2$, *integer* $m \ge 3$, $0 < \alpha < 1$ *and* $T_0 > 0$. *For a radial graph* f, *let*

$$\overline{\Phi}_T(f) = \frac{1}{|\partial\Omega_f|}\int_{\partial\Omega_f} \partial_{\nu_f} u_f(x,T)\, dS_x < 0, \qquad \mathcal{R}_T(f) = \Phi_T(f)/\overline{\Phi}_T(f).$$

There exist $\varepsilon, C > 0$ *such that every volume- and centroid-normalized* $f \in C^{m,\alpha}$ *with* $\|f\|_{C^{m,\alpha}} < \varepsilon$ *satisfies*

$$\|f\|_{C^{m,\alpha}} \le C\|\mathcal{R}_T(f) - 1\|_{C^{m-1,\alpha}} = C\frac{\|\Phi_T(f) - \overline{\Phi}_T(f)\|_{C^{m-1,\alpha}}}{|\overline{\Phi}_T(f)|} \qquad (T \ge T_0).$$

Both the neighborhood and the constant are independent of T. *The result also holds in the corresponding little Hölder spaces.*

The raw-flux estimate concerns the entire spatial boundary trace at one instant, in a near-ball chart with fixed size and position. Its constants are uniform on each compact subinterval of $(0,\infty)$. The nondegeneracy behind it is proved directly by comparison of spherical heat modes. The subsequent material-coordinate, Schauder, Fredholm and inverse-function arguments use established methods. Elliptic local Serrin stability via shape linearization and a modified implicit function theorem is treated by Gilsbach–Onodera [13]; analytic dependence of elliptic Dirichlet–Neumann maps by Hu–Nicholls [14]; and smooth shape dependence of heat layers in sharp parabolic Hölder spaces by Dalla Riva–Luzzini [15]. For the same constant initial temperature, Savo [16] derives a shape variation of heat content involving a time convolution of fluxes, rather than the raw flux at one instant. Kenioua [17] obtains short-time quantitative geometric estimates from a flux defect plus a positive time-error term. The local estimate (3) controls the deformation directly by the flux defect in the stated near-ball geometry and strong boundary norms. For partial observations, Corollary 1.4 constructs arbitrarily small normalized nonball perturbations with constant flux on a proper cap transported by the radial graph. This is a submersion consequence of the full-boundary isomorphism. Partially overdetermined elliptic counterexamples have classical antecedents [18]; the present statement concerns the datum-one heat equation at one time and a moving observation cap.

The partial-observation contrast. Write $-a(T)$ for the constant signed flux of the unit ball at time T. The radial-graph observation map $\Phi_T(f)$ is defined above. The next conclusion concerns a cap transported with that graph.

Corollary 1.4. *Fix $N \geq 2$, integer $m \geq 3$, $0 < \alpha < 1$ and $T > 0$. Let $U \subset \mathbb{S}^{N-1}$ be a nonempty open spherical cap with smooth boundary whose complement contains a nonempty open cap. There are arbitrarily small nonzero $f \in C^{m,\alpha}$ satisfying $|\Omega_f| = |B|$ and $\int_{\Omega_f} x\,dx = 0$ for which*

$$\Phi_T(f)(\theta) = -a(T) \qquad (\theta \in \overline{U}). \tag{4}$$

They form a local infinite-dimensional C^1 submanifold of the normalized near-ball shapes. Every nonzero member is a nonball. The observed physical cap is the moving radial graph over $\overline{U}$.

Exterior domains. An *exterior domain* is a connected set $\Omega = \mathbb{R}^N \setminus \overline{K}$, where K is a nonempty bounded open set; K may be nonconvex and disconnected. Its outward unit normal ν points into K. Cavallina and Pinamonti [1, Problem 1] ask for which unbounded domains Ω the solution of

$$u_t = \Delta u \text{ in } \Omega \times (0,\infty), \qquad u = 1 \text{ on } \Omega \times \{0\},$$
$$u = 0 \text{ on } \partial\Omega \times (0,\infty), \qquad u(x,t) \to 0 \text{ uniformly in } t \text{ as } |x| \to \infty,$$

satisfies (CP), and they conjecture [1, Conjecture 7.1] that "complements of closed balls are the only solutions to Problem 1 among exterior domains." The last condition of this system cannot hold together with the initial value one on any unbounded domain (Proposition 11.1): far from the boundary the solution stays close to one for small times. We take the solution to be the bounded solution (Definition 11.2). After the substitution $v = 1 - u$ this is the normalization of Garofalo and Sartori [7] with boundary datum $f \equiv 1$, to which [1, Remark 7.2] compares Conjecture 7.1: initial value zero, boundary value one and decay at infinity, on a bounded time interval. The deficit $1 - u$ of the bounded solution tends to zero at infinity, uniformly on bounded time intervals in every dimension and uniformly for all $t > 0$ when $N \geq 3$ (Lemma 11.4), so we read the decay condition as the decay (27) of the deficit. The substitution changes (CP) only by the signs of the b_n.

Theorem 1.5 (Exterior domains)**.** *Let $N \geq 2$, $0 < \alpha < 1$, and let $\Omega = \mathbb{R}^N \setminus \overline{K}$ be an exterior domain, where K has $C^{2,\alpha}$ boundary. Let u be the bounded solution of the exterior problem (Definition 11.2). Suppose that, for some $T > 0$ and $b \in \mathbb{R}$,*

$$\partial_\nu u(x,T) = b \qquad \text{for every } x \in \partial\Omega. \tag{2$'$}$$

Then K is a ball. Conversely, if K is a ball, then $\partial_\nu u(\cdot,t)$ is constant on $\partial\Omega$ for every $t > 0$.

Corollary 1.6 (Conjecture 7.1 of Cavallina–Pinamonti)**.** *Let Ω be an exterior domain with $C^{2,\alpha}$ boundary and let u be its bounded solution, which satisfies (27). If $\partial_\nu u(\cdot,t)$ is constant on $\partial\Omega$ for every t in a nonempty set of positive times, finite or infinite and with or without accumulation points (in particular, if (CP) holds), then Ω is the complement of a closed ball. Conversely, complements of closed balls satisfy (CP) for every sequence of times. Thus, with the decay condition read as (27), the complements of closed balls are exactly the solutions of [1, Problem 1] among exterior domains with $C^{2,\alpha}$ boundary; this proves [1, Conjecture 7.1] in that class, in the stronger form with a single observation time.*

Remark 7.2 of [1] points out that the moving planes of Garofalo–Sartori [7] use the overdetermined condition on a whole time interval, and that eigenfunction expansions are

not available on unbounded domains. Garofalo and Sartori treat quasilinear equations outside simply connected bounded sets. For the heat equation, Theorem 1.5 uses the flux at one time, only at the critical plane, and allows obstacles that are disconnected or not simply connected; comparison for all admissible planes comes from the uniform initial datum through a comparison principle for bounded functions (Lemma 11.3).

Plan of the proof. Section 2 establishes reflected comparison from the L^2 initial datum and reconstructs symmetry from a zero cap component. The corner estimate in Section 3 completes the moving-plane argument in Section 4. Sections 6–8 establish the linearized nondegeneracy, shape regularity and local stability. Section 9 proves uniform late-time relative-flux stability. Section 10 proves the moving-cap contrast. Section 11 treats exterior domains: the decay condition, the bounded solution, comparison on unbounded sets, and the exterior moving-plane argument for Theorem 1.5 and Corollary 1.6.

2 Reflected comparison and zero components

Positive-time parabolic regularity gives $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [\delta, M])$ for every $0 < \delta < M < \infty$, and $u \in C^\infty(\overline{\Omega} \times [\delta, M])$ when $\partial\Omega$ is C^∞. The semigroup maximum principle and connectedness give

$$0 < u(x,t) \le 1 \qquad (x \in \Omega,\ t > 0). \tag{5}$$

For a C^∞ boundary, smoothing follows by writing $A = -\Delta_D$ and using $e^{-tA}1 \in D(A^m)$ for every m, then iterating elliptic regularity and Sobolev embedding. For a $C^{2,\alpha}$ boundary, let $\theta \in C^\infty(\mathbb{R})$ vanish on $(-\infty, \delta/2]$ and equal one on $[\delta, \infty)$. The function $\theta(t)u$ has zero initial and boundary values and solves the heat equation with the bounded source $\theta'(t)u$; the parabolic L^p estimates and then the Schauder estimates up to the boundary [19], applied with two nested choices of θ, give the stated regularity. In both cases $e^{-tA}1 \in D(A^m)$ for every m, so the analytic semigroup and interior elliptic regularity show that $t \mapsto u(x,t)$ is real analytic for each fixed interior point x and $t > 0$.

Fix a unit vector e, a plane $\Pi_\lambda = \{x : x \cdot e = \lambda\}$, and reflection

$$R_\lambda x = x - 2(x \cdot e - \lambda)e.$$

Put $\Sigma_\lambda = \Omega \cap \{x \cdot e > \lambda\}$. Whenever $R_\lambda \Sigma_\lambda \subset \Omega$, define

$$w(x,t) = u(R_\lambda x, t) - u(x,t), \qquad x \in \Sigma_\lambda. \tag{6}$$

The cap may have more than one component.

Lemma 2.1. *For every admissible plane, $w \ge 0$ on $\Sigma_\lambda \times (0,\infty)$. On each connected component C of the cap, either $w > 0$ for all positive times, or $w \equiv 0$ for all positive times.*

Proof. The difference is caloric. On the plane face it vanishes; on the original boundary face it has nonnegative values by (5). These are legitimate continuous positive-time boundary values since reflection of the closed cap lies in $\overline{\Omega}$.

We justify comparison without making a time-zero boundary continuity assumption. On any compact positive-time interval $[s,t]$, w is uniformly Lipschitz on the closed cap, by C^1 spatial extension of u from $\overline{\Omega}$. Its negative part $v = w_- := \max\{-w, 0\}$ vanishes continuously on the cap boundary. In particular $v(x,\tau) \le L \operatorname{dist}(x, \partial\Sigma_\lambda)$. Cutting v off in a boundary layer of width ε produces functions converging to v in H^1. Indeed the extra gradient is bounded on this shrinking layer, whose volume tends to zero; the cap boundary lies in the $C^{2,\alpha}$ hypersurface $\partial\Omega$ together with a plane and has zero N-dimensional measure. Interior

smoothing of the cutoffs shows $v \in H_0^1(\Sigma_\lambda)$. The same cutoff argument applies at the critical cap.

Testing the heat equation with the negative part gives the standard energy inequality

$$\|w_-(\cdot,t)\|^2_{L^2(\Sigma_\lambda)} + 2\int_s^t \|\nabla w_-(\cdot,\tau)\|^2_{L^2(\Sigma_\lambda)}\,d\tau \le \|w_-(\cdot,s)\|^2_{L^2(\Sigma_\lambda)}. \tag{7}$$

Reflection preserves volume and maps the cap into Ω. Hence

$$\|w(\cdot,s)\|_{L^2(\Sigma_\lambda)} \le 2\|u(\cdot,s)-1\|_{L^2(\Omega)} \longrightarrow 0.$$

Letting $s \downarrow 0$ in (7) proves nonnegativity.

If w has an interior zero in C at a positive time, the strong parabolic maximum principle makes it zero throughout C at every earlier positive time. For each fixed $x \in C$, analyticity of $w(x,\cdot)$ extends this interval of zeros to all positive times. Otherwise it is strictly positive everywhere on that component. □

Lemma 2.2. *If $w \equiv 0$ on one cap component C for all positive times, then Ω is symmetric about Π_λ.*

Proof. Abbreviate $R = R_\lambda$, $\Pi = \Pi_\lambda$, and set

$$D = C \cup R(C) \cup (\Omega \cap \Pi \cap \overline{C}).$$

This is a nonempty relatively open subset of Ω. At a plane point in the third set, choose a small ball lying inside Ω. Its positive half is connected and meets C, so belongs to C; its negative half then belongs to $R(C)$.

The set D is also relatively closed. A positive-side limit in Ω belongs to C, since a component is closed relative to the cap. If $y_j = Rx_j \in R(C)$ tends to a negative-side point $y \in \Omega$, then $x = Ry \in \overline{C}$. If $x \in \partial\Omega$, the identity $u(x_j,t) = u(y_j,t)$ and continuity would give $u(y,t) = 0$, contradicting (5). Thus $x \in \Omega$, hence $x \in C$, and $y \in R(C)$. Limits on the plane lie in the third set. Connectedness now gives $D = \Omega$, which proves reflection symmetry. □

3 A parabolic corner estimate

The following local estimate isolates the only corner conclusion needed in the proof. Denote by e_1 the first coordinate vector.

Lemma 3.1. *Let $n \perp e_1$ be a unit vector, let $c = Rn$ with $R > 0$, and put $B^+ = B_R(c) \cap \{x_1 > 0\}$. Suppose w is caloric, positive in $B^+ \times (0,\infty)$, nonnegative on its lateral boundary, and vanishes on the plane face $x_1 = 0$. Suppose it is smooth up to that face near the closed inner half-sphere $|x-c| = R/2$. Then, for every $T > 0$,*

$$\liminf_{\rho \downarrow 0} \frac{w(\rho(e_1+n),T)}{\rho^2} > 0. \tag{8}$$

Here continuity up to the outer spherical boundary is assumed wherever the comparison below uses its values.

Proof. Choose

$$\beta > \frac{3R^2}{4T}, \qquad s = T - \frac{3R^2}{4\beta} > 0, \qquad \alpha > \frac{\beta + 2(N+2)}{R^2}.$$

With $r = |x - c|$ and $q = r^2 + \beta(T - t)$ consider

$$E = \{(x,t) : x_1 > 0,\ r > R/2,\ q < R^2,\ s < t \le T\}.$$

It is contained in $B^+ \times (s,T]$ and is empty at $t = s$. The positive function

$$h(x,t) = x_1\big(e^{-\alpha q} - e^{-\alpha R^2}\big) \tag{9}$$

satisfies

$$(\partial_t - \Delta)h = \alpha x_1 e^{-\alpha q}\big[\beta + 2(N+2) - 4\alpha r^2\big] < 0 \quad \text{in } E. \tag{10}$$

In this calculation $c_1 = 0$; the cross derivatives of $x_1 e^{-\alpha q}$ account for the $N + 2$ term. The chosen α and $r > R/2$ give the strict sign.

On the plane and the moving face $q = R^2$, $h = 0$. On the remaining face $r = R/2$, the quotient w/x_1 extends continuously to its plane rim, with value $\partial_1 w$. The ordinary parabolic Hopf principle on that flat face gives $\partial_1 w > 0$ at each rim point and every $s \le t \le T$. The relevant earlier half-cylinders lie inside B^+ near the rim. Positivity away from the rim and compactness therefore give

$$\inf_{\substack{x_1 \ge 0,\ |x-c| = R/2 \\ s \le t \le T}} \frac{w(x,t)}{x_1} > 0,$$

where the quotient has its continuous rim interpretation. Since h/x_1 is bounded there, a fixed $\varepsilon > 0$ gives $w \ge \varepsilon h$ on that face.

Comparison yields $w \ge \varepsilon h$ throughout E. To see that the moving boundary causes no extra hypothesis, apply the minimum argument to $w - \varepsilon h + \delta(t - s)$ on the closure through any time $t_0 \le T$. It is nonnegative on each lateral face and at the birth set $t = s$, which lies on $r = R/2$. A negative minimum at an interior spatial point would have nonpositive backward time derivative and nonnegative Laplacian, contradicting its strictly positive heat operator. Let $\delta \downarrow 0$.

At $t = T$, the point $x = \rho(e_1 + n)$ lies in E for small $\rho > 0$, and

$$x_1 = \rho, \qquad r^2 = R^2 - 2R\rho + 2\rho^2, \qquad h(x,T) = 2\alpha R e^{-\alpha R^2}\rho^2 + O(\rho^3).$$

This proves (8). □

4 Moving planes and proof of the theorem

Proof of Theorem 1.1. Fix e. Start the plane Π_λ near the support value $\max_{\overline{\Omega}} x \cdot e$ and move it inward, maintaining $R_\lambda \Sigma_\lambda \subset \Omega$. The classical geometric moving-plane lemma for bounded C^2 domains gives a first limiting plane. At that plane either the reflected original boundary is tangent to the original boundary away from the plane, or the plane meets $\partial\Omega$ orthogonally; see Serrin [3, Sections 1–2]. Reflection of the open cap remains in Ω at this position. This geometric lemma applies to the full cap and does not require it to be connected.

Use (6) at the limiting position. By Lemmas 2.1–2.2, either the domain is already symmetric about this plane or $w > 0$ on every cap component at all positive times. Suppose the latter.

In the off-plane tangency case, let P be the original cap boundary point whose reflection also lies on $\partial\Omega$. Write S for the linear part of R_λ. Tangency gives $\nu(R_\lambda P) = S\nu(P)$. At time T, the boundary condition and (2) give

$$\nabla w(P,T) = S\nabla u(R_\lambda P, T) - \nabla u(P,T) = bS\nu(R_\lambda P) - b\nu(P) = 0.$$

But $w(P,T) = 0$ and w is strictly positive in the adjacent cap component at preceding positive times. The ordinary parabolic Hopf principle at this smooth face gives a strictly positive inward derivative, a contradiction.

In the orthogonal-contact case translate the contact point to $Q = 0$ and rotate coordinates so the plane is $x_1 = 0$, with the cap on $x_1 > 0$. Let n be the inward unit normal at Q. Orthogonality means $n \perp e_1$. Choose a sufficiently small interior tangent ball $B_R(Rn) \subset \Omega$ whose closure meets $\partial\Omega$ only at Q. Such a ball exists by smoothness, after decreasing its radius if necessary. The ball is invariant under reflection in $x_1 = 0$; its positive half lies in one cap component. Lemma 3.1 therefore gives

$$\liminf_{\rho \downarrow 0} \rho^{-2} w(\rho(e_1 + n), T) > 0. \tag{11}$$

The inner sphere and its plane rim lie strictly inside Ω, so all flat-face regularity and Hopf hypotheses in that lemma hold.

Set $f = u(\cdot, T)$. On $\partial\Omega$,

$$\nabla f = b\nu. \tag{12}$$

Since e_1 is tangent at Q, differentiate (12) along a boundary curve with velocity e_1 and dot with ν. The scalar b is constant and $(D_{e_1}\nu) \cdot \nu = 0$, hence

$$D^2 f(Q)[e_1, \nu] = 0, \qquad D^2 f(Q)[e_1, n] = 0. \tag{13}$$

Reflection fixes Q, n and reverses e_1. Consequently $w(Q,T) = 0$ and $\nabla w(Q,T) = 0$. Its Hessian on the span of e_1, n has vanishing diagonal terms by reflection, and vanishing mixed term by (13). Taylor's formula gives

$$w(\rho(e_1 + n), T) = o(\rho^2),$$

contradicting (11). Thus the first limiting plane is a symmetry plane in every direction.

Let z be the centroid of Ω. Reflection invariance forces the plane offset to equal $z \cdot e$. Translating z to zero, the domain is invariant under every origin hyperplane reflection, hence under every orthogonal transformation. Its radii form an interval by connectedness. A bounded open radial connected set is a ball or a shell, with the punctured-ball possibility excluded by smooth boundary when $N \geq 2$.

For completeness radial symmetry alone does not exclude a shell. If $\Omega = \{a < |x| < R_0\}$ with $a > 0$, the first limiting plane in direction e has offset

$$\lambda_* = (R_0 + a)/2 > 0.$$

Indeed, for $\lambda > \lambda_*$ the full cap reflects inside the outer ball and has reflected axial coordinate greater than $2\lambda - R_0 > a$, so avoids the inner ball. At λ_* the outer pole $R_0 e$ reflects to the inner pole ae, giving the first off-plane tangency. This limiting plane cannot be a symmetry plane: the centered shell has centroid zero. This contradicts what was proved for every first limiting plane. Thus Ω is a ball.

Conversely the datum, equation and boundary value on a ball are invariant under rotations about its center. Uniqueness makes u radial, and its normal derivative is constant on the boundary at every positive time. □

5 Consequences and the role of the initial temperature

Theorem 1.1 also applies whenever a nonempty set of positive times, finite or infinite, has constant whole-boundary signed flux: apply the theorem at any one of them. This rules

out the connected nonball examples proposed in [1, Problem 3 and Conjecture 7.4] in the $C^{2,\alpha}$ class. It also gives the conclusion of [1, Theorem II] for bounded connected domains with $C^{2,\alpha}$ boundary from a single observation time, in place of a sequence of times tending to zero; this complements, within [1, Problem 2], Zeraoulia's theorem [2] for C^2 boundaries along such sequences. Corollary 1.6 gives the corresponding statement for exterior domains.

For a general initial temperature g, the reflected comparison starts from $g(R_\lambda x) - g(x)$, whose sign need not be nonnegative. Thus positivity of g cannot replace the datum $g \equiv 1$ in the argument. This explains the compatibility with the tailored-data shell results in [20]: on a fixed spherical shell, nonuniform positive preparations can produce a common signed outward flux on both boundary components at suitable positive times. The exterior comparison of Lemma 11.6 starts in the same way, so Theorem 1.5 also relies on the constant datum.

Both Theorems 1.1 and 1.2 use the prescribed constant initial temperature. The first is a global rigidity statement for an exact whole-boundary observation; the second gives quantitative control in a normalized near-ball chart with Hölder observation norms.

6 The spherical flux derivative

Throughout the quantitative argument the reference domain is the unit ball $B \subset \mathbb{R}^N$. Write $u(t,r) = e^{t\Delta_D}1$ and $a(t) = -u_r(t,1) > 0$. The radial heat-content expansion and the identity $H_B'(t) = -|\partial B| a(t)$ give

$$a(t) = 2\sum_{n\geq 1} e^{-j_{N/2-1,n}^2 t}, \tag{14}$$

where $j_{\nu,n}$ are the positive zeros of J_ν. One obtains the heat-content formula by expanding the constant function in radial Dirichlet eigenfunctions and integrating their Bessel expressions: $H_B(t)/|B| = 2N\sum_{n\geq 1} j_{N/2-1,n}^{-2} e^{-j_{N/2-1,n}^2 t}$. The series and all its time derivatives converge locally normally for $t > 0$. In particular $a'(t) < 0$ and $a(t) = O(t^{-1/2})$ at zero, by the usual linear growth of the Bessel zeros. Thus a is locally integrable even though the initial boundary data are incompatible.

For spherical degree $\ell \geq 1$, put $\lambda_\ell = \ell(\ell + N - 2)$ and

$$L_\ell = \partial_{rr} + \frac{N-1}{r}\partial_r - \frac{\lambda_\ell}{r^2}.$$

Let w_ℓ be the positive heat Poisson solution of

$$(\partial_t - L_\ell)w_\ell = 0, \quad w_\ell(t,1) = a(t), \quad w_\ell(t,r) = O(r^\ell)\ (r \downarrow 0), \tag{15}$$

with zero interior initial value. More precisely it is the increasing limit of solutions with boundary input $a_\epsilon(t) = a(t)\mathbf{1}_{t\geq\epsilon}$ and zero initial value. Local integrability of the input and off-boundary decay of the heat Poisson kernel make the limit finite and smooth on compact positive-time sets. The omitted input is separated from any fixed observation time, so convergence also holds for the normal trace there. This specifies the solution class; pointwise zero interior initial values alone would not specify an arbitrary singular-boundary solution.

Lemma 6.1. *The derivative $G_T = D\Phi_T(0)$ has multiplier*

$$\gamma_\ell(T) = \partial_r w_\ell(T,1) + (N-1)a(T) \tag{16}$$

on degree-ℓ spherical harmonics. For all $T > 0$,

$$\gamma_1(T) = 0 < \gamma_2(T) < \gamma_3(T) < \cdots. \tag{17}$$

Uniformly for $T \in [T_0, T_1]$,

$$\gamma_\ell(T) = a(T)(\ell + N - 1) + O_{N,T_0,T_1}(1). \tag{18}$$

Consequently, for $\ell \geq 2$, there are positive constants independent of ℓ, T in this range such that

$$c(1+\ell) \leq \gamma_\ell(T) \leq C(1+\ell). \tag{19}$$

Proof. For a smooth normal displacement f, the Eulerian heat derivative u' satisfies the heat equation with boundary input $-fu_r = a(t)f$. Rotational equivariance gives $u' = w_\ell(t,r)Y_\ell(\theta)$ for a degree-ℓ harmonic. Differentiating the normal flux at the radial graph yields

$$G_T f = \partial_r u'(T,1) + f u_{rr}(T,1).$$

The normal change is tangential and has zero dot product with the radial reference gradient. At positive times $u_t = 0$ on the boundary, so $u_{rr}(T,1) = -(N-1)u_r(T,1) = (N-1)a(T)$, proving (16). Section 7 supplies the full positive-time Banach differentiability used here.

A translation by ϵe has displacement $e\cdot\theta$ and Eulerian derivative $-e\cdot\nabla u$. Thus $w_1 = -u_r$ and $\gamma_1 = 0$. To verify that this is the specified Poisson solution despite the corner singularity, take Laplace transforms in time. For $s > 0$, $\nu = N/2 - 1$, define

$$F_\ell(r,s) = r^{-\nu}\frac{I_{\ell+\nu}(\sqrt{s}\,r)}{I_{\ell+\nu}(\sqrt{s})}, \qquad D_\ell(s) = \partial_r F_\ell(1,s).$$

The ball resolvent is $U(s,r) = (1 - F_0(r,s))/s$ and $A(s) = \mathcal{L}[a](s) = D_0(s)/s$. The Poisson solution has transform $A(s)F_\ell(r,s)$. The Bessel recurrence gives $-\partial_r U = AF_1$, proving the desired equality by ordinary Laplace uniqueness at each interior point.

For truncated inputs, if $k > \ell \geq 1$ then $z = w_\ell - w_k$ satisfies

$$(\partial_t - L_k)z = (\lambda_k - \lambda_\ell)w_\ell/r^2 \geq 0,$$

with zero initial and boundary values. Comparison with the nonnegative radial killing potential gives $z \geq 0$. A negative minimum cannot occur at $r = 0$, where both modes vanish, so the singular center causes no obstruction to the annular limiting argument. Passing to the Poisson limits gives $w_\ell \geq w_k \geq 0$. The input is positive at every positive time, so the strong principle gives $z > 0$ for $0 < r < 1, t > 0$. On a small positive-time cylinder at $r = 1$, the coefficients are smooth and the ordinary parabolic Hopf principle gives $z_r(T,1) < 0$. This proves the strict inequalities in (17).

For the high-degree estimate set $\tau = T_0/2$ and $v_\ell(t,r) = w_\ell(t,r) - a(t)r^\ell$ for $t \geq \tau$. Then

$$(\partial_t - L_\ell)v_\ell = -a'(t)r^\ell, \qquad v_\ell(t,1) = 0.$$

The comparison $w_\ell \leq w_1$ gives a common initial bound $|v_\ell(\tau,r)| \leq C_0$, with $C_0 = \sup_r w_1(\tau,r) + a(\tau)$. Put $C_1 = \sup_{\tau\leq t\leq T_1}|a'(t)|$ and

$$V(t,r) = C_0 u(t-\tau,r) + C_1\int_0^{t-\tau} u(s,r)\,ds.$$

It is nonnegative and $(\partial_t - L_\ell)V = C_1 + \lambda_\ell V/r^2 \geq C_1$. Comparison gives $-V \leq v_\ell \leq V$. Differentiation at their common zero boundary therefore yields

$$|\partial_r v_\ell(T,1)| \leq C_0 a(T-\tau) + C_1\int_0^{T-\tau} a(s)\,ds.$$

This is bounded uniformly for $T \in [T_0, T_1]$, proving (18). Since $\min_{[T_0,T_1]} a > 0$, sufficiently high degrees satisfy (19); the remaining finite degrees do so by strict positivity and continuity in time. □

The preceding computation also gives an exact ordinary Laplace formula for the multipliers:

$$\mathcal{L}[\gamma_\ell](s) = A(s)(D_\ell(s) - D_1(s)), \qquad s > 0. \tag{20}$$

Here the difference is important. The Bessel recurrence gives $A(s)(D_1(s) + N - 1) = 1$; thus $A(s)(D_\ell(s) + N - 1)$ would contain an additional impulse at time zero. To justify (20) without interchanging singular boundary derivatives, let $z_\epsilon = w_{1,\epsilon} - w_{\ell,\epsilon}$. Its nonnegative source increases as $\epsilon \downarrow 0$, so z_ϵ and its inward normal trace increase. Therefore $-\partial_r z_\epsilon(1)$ increases to γ_ℓ, and its transform is $A_\epsilon(s)(D_\ell - D_1)$. Monotone convergence proves (20) and ordinary local integrability of γ_ℓ from the finite right hand side.

As an immediate spectral consequence, on the subspace of degrees at least two, G_T is an isomorphism from $H^s(\mathbb{S}^{N-1})$ to $H^{s-1}(\mathbb{S}^{N-1})$ for every real s, uniformly in the stated time interval. This follows by squaring harmonic coefficients in (19). The Hölder-space inverse needed for Theorem 1.2 is justified separately below; a Sobolev multiplier bound alone does not supply it.

7 Positive-time material regularity

We give a fixed-domain argument to address the incompatible initial temperature directly. We use the standard sectorial and analytic-semigroup theory for Dirichlet elliptic operators in [21], and the elliptic Schauder and oblique-derivative theory in [22, Chapter 6]. Choose a smooth cutoff χ, zero for $r \le 1/2$ and one near $r = 1$, and put

$$F_f(r\theta) = (r + \chi(r) f(\theta))\theta.$$

For small $\|f\|_X$ this is a $C^{m,\alpha}$ diffeomorphism from B to Ω_f. Define $v_f(t) = u_f(t) \circ F_f$. With $J_f = DF_f$ and $K_f = J_f^{-1}$, the pulled-back Laplacian is

$$A_f v = a_f^{kl} \partial_{kl} v + b_f^l \partial_l v, \quad a_f^{kl} = \sum_i (K_f)_{ki}(K_f)_{li}, \quad b_f^l = \sum_{k,i} (K_f)_{ki} \partial_k (K_f)_{li}. \tag{21}$$

The inverse here is a pointwise matrix inverse at a fixed reference point. This fixed-coordinate formula allows Banach-algebra calculus to give analytic coefficient maps

$$a_f \in C^{m-1,\alpha}(\overline{B}), \qquad b_f \in C^{m-2,\alpha}(\overline{B}).$$

Lemma 7.1. *The maps $\Phi_T : X \to Y$ are continuously Fréchet differentiable on a common neighborhood of zero, with the derivative jointly continuous in (T, f) for $T \in [T_0, T_1]$.*

Proof. Fix finite p sufficiently large that $W^{2,p}(B)$ embeds in $C^{1,\beta}(\overline{B})$ for some $\beta > \alpha$. On $L^p(B)$, all A_f have the same Dirichlet domain $D_p = W^{2,p}(B) \cap W_0^{1,p}(B)$ and $f \mapsto A_f$ is C^1 into $\mathcal{L}(D_p, L^p)$. Uniform ellipticity holds in a small X ball. A fixed shifted sectorial contour for the ball Dirichlet Laplacian remains a resolvent contour for all these operators: the usual bounds $L^p \to L^p$ of order $|z|^{-1}$ and $L^p \to D_p$ of order one make $(A_f - A_0)(z - A_0)^{-1}$ uniformly small there. A Neumann series gives a common sectorial family and equivalent graph norms. The resolvent derivative is

$$D_f(z - A_f)^{-1}[h] = (z - A_f)^{-1} A_f'[h] (z - A_f)^{-1}. \tag{22}$$

The Dunford formula

$$v_f(t) = \frac{1}{2\pi i} \int_\Gamma e^{tz} (z - A_f)^{-1} 1 \, dz$$

and (22) show that, for every fixed integer $j \geq 0$, $f \mapsto \partial_t^j v_f(t)$ is C^1 into D_p, jointly continuously for positive times. The contour products have at most polynomial growth; exponential decay along its rays dominates them uniformly for $t \geq T_0/2$. Resolvent difference identities also control the Fréchet remainders. The initial value is the constant one in $L^p(B)$; all the asserted boundary regularity is at positive times.

We detail the spatial bootstrap. For every j and positive time,

$$A_f \partial_t^j v_f = \partial_t^{j+1} v_f, \qquad \partial_t^j v_f|_{\partial B} = 0.$$

The D_p result begins with C^1 dependence in $C^{0,\alpha}$. Uniform elliptic Schauder estimates on the fixed smooth ball then upgrade it successively to $C^{k,\alpha}$, $2 \leq k \leq m$. The induction is simultaneous over time derivatives: to treat a fixed (k, j) only finitely many higher time derivatives at lower spatial orders are needed, all supplied by the contour argument. The available coefficient regularity in (21) suffices.

For clarity, state continuity follows by subtracting the elliptic equations and applying Schauder with the uniform solution bounds. The parameter derivative solves

$$A_f \dot{v}_f[h] = \partial_t \dot{v}_f[h] - A_f'[h] v_f, \qquad \dot{v}_f[h]|_{\partial B} = 0.$$

Subtracting this from the exact increment equation leaves a time-derivative remainder already $o(\|h\|_X)$ two spatial orders lower, a C^1 coefficient remainder times a bounded state, and a coefficient increment times an $o(1)$ state difference. Schauder makes the state remainder $o(\|h\|_X)$ in $C^{k,\alpha}$. The same difference estimates give continuity of the derivative. Thus $f \mapsto v_f(T)$ is C^1 into $C^{m,\alpha}$.

Finally

$$\Phi_T(f) = (\nu_f \circ F_f) \cdot (K_f^\top \nabla v_f(T)) \quad \text{on } \partial B.$$

The geometric factors are algebraic C^1 functions of DF_f in $C^{m-1,\alpha}$; for example the pulled normal is the normalized $K_f^\top \nu_0$. This proves the claimed $X \to Y$ regularity. It also validates the Eulerian differentiation used in Lemma 6.1. Restriction to little Hölder spaces follows by smooth approximation and the proved map and derivative continuity. □

This argument is a positive-time material-coordinate application of classical operator perturbation and Schauder theory. The regularity conclusion applies to the combined Eulerian expression. The derivative-loss discussion for an elliptic parametrization in [13] motivates checking this point; the fixed-coordinate matrix formula (21) supplies the needed cancellation directly. The heat-layer shape regularity in [15] is related general ancestry. Here the semigroup argument starts from the prescribed L^p datum and establishes regularity at positive times.

8 Hölder isomorphism and local stability

Let $\mathcal{H}f$ be harmonic extension into B and let $\Lambda f = \partial_r \mathcal{H}f|_{r=1}$ be the spherical Dirichlet–Neumann operator. Its degree-ℓ multiplier is ℓ. Put $\tau = T_0/2$. For the Eulerian heat derivative with boundary input $a(t)f$, Duhamel gives, for $T \geq T_0$,

$$w_f(T) = a(T)\mathcal{H}f + e^{(T-\tau)\Delta_D}[w_f(\tau) - a(\tau)\mathcal{H}f] - \int_0^{T-\tau} e^{s\Delta_D} \mathcal{H}f \, a'(T-s)\, ds. \tag{23}$$

The middle term has a fixed positive smoothing interval and its normal trace is compact from X to Y. The low-norm bound on $f \mapsto w_f(\tau)$ follows from Lemma 7.1 or the heat Poisson representation; subsequent heat smoothing supplies any fixed boundary regularity.

The integral term also gives a compact trace operator $X \to Y$. Indeed set $B_s f = \partial_\nu e^{s\Delta_D} \mathcal{H} f$. The standard positive-time Dirichlet boundary-gradient estimate (see [21]) gives

$$\|B_s f\|_\infty \le C s^{-1/2} \|f\|_\infty, \qquad 0 < s \le T_1,$$

since harmonic extension contracts the supremum norm. Rotations commute with harmonic extension, the ball heat semigroup and normal trace. Apply the same estimate to all ordered rotational derivatives of order at most $m-1$. The top-order Hölder seminorm is equivalently measured by differences under small rotations; applying the same commutation estimate to those differences yields

$$\|B_s f\|_Y \le C s^{-1/2} \|f\|_Y. \tag{24}$$

Rotational vector fields span the tangent bundle, so these are equivalent standard sphere norms. The half-power is integrable and all derivatives of a are bounded on $[\tau, T_1]$. Thus the integral trace in (23) is bounded $Y \to Y$, and its restriction $X \to Y$ is compact by the compact embedding. Small-s domination in (24) and positive-s smoothing also give operator-norm continuity in T.

It follows that

$$G_T = a(T)\Lambda + (N-1)a(T)I + K_T, \tag{25}$$

where $K_T : X \to Y$ is compact and norm-continuous in time. These operators preserve harmonic degrees. On mean-zero Hölder spaces Λ is an isomorphism by harmonic Neumann solvability and Schauder estimates; removing its finite degree-one sector leaves an isomorphism $X_{\ge 2} \to Y_{\ge 2}$. The restriction of G_T to these subspaces is therefore Fredholm of index zero. It has zero kernel by (17), since every harmonic coefficient of a kernel vector vanishes. Completeness in L^2 suffices here; density in a large Hölder norm is not required. Fredholm index zero and the bounded inverse theorem give an isomorphism

$$G_T : X_{\ge 2} \longrightarrow Y_{\ge 2}. \tag{26}$$

Continuity in T and compactness of $[T_0, T_1]$ give a common bound for the inverse norms.

Proof of Theorem 1.2. Let P project onto harmonic degrees at least two. Write $f = h + c + b\cdot\theta$ with $h \in X_{\ge 2}$. The volume and unnormalized first-moment maps are

$$V(f) = \frac{1}{N} \int_{\mathbb{S}^{N-1}} (1+f)^N, \qquad M(f) = \frac{1}{N+1} \int_{\mathbb{S}^{N-1}} \theta (1+f)^{N+1}.$$

These are smooth polynomial Banach maps. Their derivative in (c, b) at zero consists of nonzero scalar and vector diagonal blocks, with zero cross terms by spherical symmetry. The parameter-dependent finite-dimensional implicit function theorem gives smooth $c(h), b(h)$ imposing $V(f) = |B|$ and $M(f) = 0$. Since h has no degrees zero or one, their first derivatives at zero vanish. In particular

$$f(h) = h + O(\|h\|_X^2),$$

and the two norms are comparable locally. Every sufficiently small constrained f is uniquely represented this way.

Set $Q_T(h) = P\Phi_T(f(h))$. It vanishes at zero, because the ball flux is constant. Lemma 7.1 gives C^1 regularity and its derivative at zero is (26). By joint continuity of the derivatives on the compact set $[T_0, T_1] \times \{0\}$ and the uniform inverse bound, there is a common neighborhood where

$$\|Q_T(h) - G_T h\|_Y \le \eta \|h\|_X$$

with η less than half the reciprocal of that bound. Hence

$$\|h\|_X \le C \|Q_T(h)\|_Y \le C' \inf_{b \in \mathbb{R}} \|\Phi_T(f(h)) - b\|_Y.$$

The last inequality uses boundedness of P and its annihilation of constants. Norm comparability proves (3). □

The positive spectrum is the specific nondegeneracy input; the final local estimate is a standard consequence of that isomorphism and the proved C^1 material map. The observation norm is one derivative weaker than the shape norm but still strong, and the theorem requires a priori near-sphericity, with constants uniform on compact positive-time intervals. A proper moving observation cap has a different behavior, as Corollary 1.4 shows (Section 10).

9 Relative flux and uniform late-time stability

We prove Theorem 1.3. The normalization removes the decaying amplitude through the first eigenspace, while its shape derivative retains an invertible order-one boundary operator.

Proof. Use the material coordinates, spaces X, Y, and normalized chart $f(h)$ of the preceding sections. The surface Jacobian j_f is a C^1 map into Y, so the displayed average and quotient are C^1 whenever the denominator is nonzero. Hopf's principle makes the physical average negative for each positive time. On each compact time interval the previous proof therefore applies equally to the relative map. We establish uniformity as $T \to \infty$ separately.

Write A_f for the pulled-back Dirichlet generator on $L^p(B)$ with its fixed domain D_p, and let $-\lambda_f$ be its largest eigenvalue. The unit-ball first eigenvalue is simple and isolated. On a sufficiently small common X neighborhood, the resolvent Neumann-series argument already used for A_f gives a C^1 Riesz projection P_f, a C^1 real simple eigenvalue λ_f, and a uniform gap to the remaining spectrum. The material realization is selfadjoint on the weighted Hilbert space $L^2(B, \det DF_f\, dx)$, by change of variables from Ω_f. Its compact resolvent and elliptic regularity identify its eigenvalues and the isolated spectral projections with those of the L^p realization; no selfadjointness on L^p is asserted. Let $v_f = P_f 1$. It is a positive multiple of the first eigenfunction, and v_f is C^1 into $C^{m,\alpha}$ by the fixed-domain elliptic bootstrap. Its pulled normal trace g_f is C^1 into Y; its physical surface average $\bar{g}_f$ stays bounded away from zero on a smaller neighborhood, because $\bar{g}_0 < 0$.

For completeness, the uniform remainder assertion needed here is

$$e^{\lambda_f T}\Phi_T(f) = g_f + E_T(f).$$

$$\sup_{\|f\|_X<\varepsilon} \big(\|E_T(f)\|_Y + \|D_f E_T(f)\|_{\mathcal{L}(X,Y)}\big) \le C(1+T)e^{-\delta T} \quad (T \ge 2). \tag{*}$$

We give the operator-norm details. Write $Z = L^p(B)$ and $D = D_p$. Choose a fixed real number β_* strictly between $\sup_f \lambda_f$ and $\inf_f \lambda_2(f)$, reducing the shape ball if necessary. A fixed complementary sectorial contour has its finite right portion on $\Re z = -\beta_*$ and rays extending to the left inside the common sectorial resolvent region. Its Dunford integral is $S_f(t) = e^{tA_f}(I - P_f)$. Differentiation on this same fixed contour gives the integrand $(z-A_f)^{-1}A_f'[h](z-A_f)^{-1}$; it already includes the derivative of the complementary projection. The uniform graph-norm bounds give, for $t \ge 1$,

$$\|S_f(t)\|_{\mathcal{L}(Z,Z)} + \|D_f S_f(t)\|_{\mathcal{L}(X,\mathcal{L}(Z,Z))} \le Ce^{-\beta_* t},$$

with a slightly smaller β_* if needed to absorb contour prefactors, still larger than $\sup_f \lambda_f$. Choose $\delta > 0$ with $\beta_* - \sup_f \lambda_f \ge 2\delta$. Multiplying by $e^{\lambda_f t}$ and differentiating adds only $t\lambda_f'[h]S_f(t)$, which is bounded by $C(1+t)e^{-2\delta t}\|h\|_X$.

We also need the positive-time smoothing assertion in operator norm:

$$f \longmapsto L_f = e^{A_f} \quad \text{is } C^1 \text{ into } \mathcal{L}(Z, C^{m,\alpha}(\overline{B})).$$

Indeed the fixed-domain resolvent argument first gives C^1 operator dependence of $\partial_t^j e^{tA_f}$: $Z \to D$ for each fixed j, uniformly on a positive-time window around time one. Every estimate is linear in the initial norm and uniform over its unit ball. In $A_f \partial_t^j v = \partial_t^{j+1} v$, the coefficient regularities $a_f \in C^{m-1,\alpha}$ and $b_f \in C^{m-2,\alpha}$ allow simultaneous elliptic Schauder bootstrap through $C^{m,\alpha}$. At step k, the differentiated forcing $A_f'[h]v$ in $C^{k-2,\alpha}$ is bounded by $C\|h\|_X\|z\|_Z$ for initial vector z. Subtracting the exact increment and linearized equations gives an $o(\|h\|_X)\|z\|_Z$ remainder uniformly for $\|z\|_Z \le 1$. Only finitely many higher time derivatives are needed for this induction, all supplied by the contour argument. Thus this is Fréchet differentiability in the displayed operator space, not merely differentiability for initial datum one. No derivative of the shape beyond m or compatible boundary trace of z is used. The eigenvalue equation and the same differentiated bootstrap likewise give $v_f = P_f 1$ as a C^1 map into $C^{m,\alpha}$.

For $T \ge 2$ write the complementary state remainder as $L_f S_f(T-1)1$. Its derivative is $(D_f L_f)S_f(T-1)1 + L_f(D_f S_f(T-1))1$. The preceding estimates, followed by multiplication by $e^{\lambda_f T}$ and its derivative, give (*) in the state $C^{m,\alpha}$ norm. Finally the material normal-trace map is C^1 into $\mathcal{L}(C^{m,\alpha}, Y)$, by the geometric coefficient formula in Lemma 7.1, proving (*) in the required boundary norm. Taking the physical average in (*) preserves its C^1 remainder bound, since j_f and its derivative are uniformly bounded. The factor $e^{-\lambda_f T}$ cancels exactly in the quotient:

$$\mathcal{R}_T(f) = \frac{g_f + E_T(f)}{\bar{g}_f + \overline{E}_T(f)} \longrightarrow \mathcal{R}_\infty(f) := g_f/\bar{g}_f \quad \text{in } C^1(X, Y),$$

uniformly on that neighborhood. This is the reason the shape-dependent exponential decay does not spoil the estimate.

We verify the limiting derivative's nondegeneracy rather than infer it from finite-time positivity. Put $\nu = N/2 - 1$, $j = j_{\nu,1}$ and $\lambda = j^2$. On degree $\ell \ge 1$, the limiting normalized positive multiplier is

$$\eta_\ell = N - 1 + d_\ell, \qquad d_\ell = \partial_r \left[r^{-\nu} \frac{J_{\ell+\nu}(jr)}{J_{\ell+\nu}(j)} \right]\Big|_{r=1}.$$

In fact, a mean-zero shape variation has $\lambda_0'[f] = 0$ by the first-eigenvalue variation formula, and the Eulerian eigenfunction derivative solves $(\Delta + \lambda)v' = 0$ with boundary data $-f\partial_\nu v$. Its degree-ℓ radial solution is the displayed Bessel quotient. Differentiating the normal trace contributes the curvature term $N - 1$. Normalization divides by the negative reference flux and kills any scalar eigenfunction normalization derivative. Thus $D\mathcal{R}_\infty(0)$ has multiplier $-\eta_\ell$ on degrees at least2. Translation invariance gives $\eta_1 = 0$ and $d_1 = -(N-1)$.

For $k > \ell \ge 1$, compare the positive regular radial solutions of $(L_\ell + \lambda)F_\ell = 0$ with boundary value1. The first Dirichlet eigenvalue in sector k exceeds the full first eigenvalue λ, so $-L_k - \lambda$ has a positive inverse. Consequently $F_\ell - F_k > 0$ in $(0, 1)$, as its source under this operator is $(\lambda_k - \lambda_\ell)F_\ell/r^2 > 0$. The ordinary boundary Hopf inequality gives $d_k > d_\ell$. Moreover the fixed-argument large-order Bessel series gives $d_\ell = \ell - j^2/(2\ell + N) + O(\ell^{-2})$. Thus $\eta_\ell > 0$ for $\ell \ge 2$ and $\eta_\ell \asymp 1 + \ell$. All Helmholtz inverses and boundary operators in this argument are restricted to harmonic sectors of degree at least two. The full Dirichlet operator at λ is resonant and is not being inverted. On the reduced sectors the limiting boundary operator is the Helmholtz Dirichlet–Neumann operator plus $(N-1)I$; relative to the harmonic Dirichlet–Neumann operator it is compact $X_{\ge 2} \to Y_{\ge 2}$, by elliptic Schauder estimates on the zero-boundary difference. Fredholm index zero and the just-proved harmonic-mode kernel exclusion therefore make it an isomorphism in the stated Hölder spaces.

Now compactify the parameter interval to $[T_0, \infty]$. The proved C^1 convergence and finite-time regularity give joint continuity at infinity. For finite T, the derivative on degrees

at least2 is $-G_T/a(T)$, an isomorphism by the preceding theorem; the limit is also an isomorphism. Continuity and compactness give a uniform bound on the inverses and a common neighborhood on which the nonlinear remainder is less than half that inverse bound. Apply the same normalized-chart argument to $P_{\geq 2}(\mathcal{R}_T(f(h)) - 1)$. It gives the displayed inequality uniformly for all $T \geq T_0$. Restriction to little Hölder spaces follows by the same smoothing and continuity argument as before. □

Remark 9.1 (Why the relative normalization is necessary). The preceding theorem removes the vanishing signal amplitude, not the strong full-boundary observation requirement. The ball flux satisfies $a(T) \sim 2e^{-\lambda T}$, $\lambda = j^2_{N/2-1,1}$. On any fixed degree-two harmonic the raw derivative is $\gamma_2(T) = a(T)(\eta_2 + o(1))$, with $\eta_2 > 0$. Consequently any local raw-flux estimate in these norms has a constant bounded below by $ce^{\lambda T}$ as $T \to \infty$: insert the normalized chart curve $f(\epsilon Y_2)$ and let $\epsilon \to 0$. Thus no time-uniform raw-flux Lipschitz constant exists, even infinitesimally near the ball. The relative theorem uses the experimentally different dimensionless defect divided by the measured nonzero mean flux, so its constants are relative to the size of the signal at the observation time.

10 Nonuniqueness for a moving observation cap

Corollary 1.4, stated with the main results, follows from the normalized full-boundary isomorphism.

Proof. Use the smooth volume/centroid chart $f(h)$ from the preceding proof, with $h \in X_{\geq 2}$. Let $Y_U = C^{m-1,\alpha}(\overline{U})$ and define

$$\mathcal{F}(h) = (\Phi_T(f(h)) + a(T))|_{\overline{U}}.$$

This is a C^1 map to Y_U, vanishes at zero, and has derivative $D\mathcal{F}(0) = R_U G_T$, where R_U is restriction.

We construct a bounded right inverse for $R_U : Y_{\geq 2} \to Y_U$. Take a bounded Hölder extension $E : Y_U \to Y$, using the smooth cap boundary and local charts. Choose a nonzero nonnegative $\beta \in C_c^\infty(V)$, where V is an unobserved open cap disjoint from $\overline{U}$. Put $v(\theta) = (1, \theta_1, \ldots, \theta_N)^\top$ and

$$M = \int_{\mathbb{S}^{N-1}} \beta(\theta) v(\theta) v(\theta)^\top \, d\theta.$$

This matrix is positive definite: a nonzero affine function cannot vanish on a nonempty open portion of the sphere. The corrected extension

$$Jg = Eg - \beta v^\top M^{-1} \int_{\mathbb{S}^{N-1}} v(Eg)$$

has zero constant and degree-one moments and agrees with g on $\overline{U}$. Hence $J : Y_U \to Y_{\geq 2}$ is a bounded right inverse for restriction. By (26), $\mathcal{R} = G_T^{-1} J$ is a bounded right inverse for $D\mathcal{F}(0)$.

The kernel $\mathcal{K}$ of this derivative is infinite-dimensional: smooth functions supported in V with zero constant and coordinate moments form an infinite-dimensional subspace of $Y_{\geq 2}$, and their G_T^{-1} images lie in $\mathcal{K}$. The projection $\mathcal{R}D\mathcal{F}(0)$ splits $X_{\geq 2}$ as $\mathcal{K} \oplus \operatorname{range} \mathcal{R}$. In the coordinates $h = k + \mathcal{R}z$, the z derivative of $\mathcal{F}$ at zero is the identity on Y_U. The parameter-dependent Banach implicit function theorem therefore gives all nearby zeros in the form

$$h = k + \mathcal{R}z(k), \qquad z(0) = 0, \quad Dz(0) = 0,$$

for k near zero in $\mathcal{K}$. The kernel projection of h is k, so a nonzero k gives $h \neq 0$, and consequently $f(h) \neq 0$. An exactly volume- and centroid-normalized ball is the centered unit ball. Thus these nonzero shapes are nonballs and satisfy (4). The chart transports the split level set to the claimed local submanifold of normalized shapes. □

The corollary concerns $C^{m,\alpha}$ boundaries and observations on a moving cap at a single positive time. Its argument is the standard submersion deduction from the full-boundary isomorphism. Classical partially overdetermined elliptic counterexamples, such as those in [18], provide context; they are not used to infer this heat statement.

11 Exterior domains

Throughout this section $N \geq 2$, $0 < \alpha < 1$, $K \subset \mathbb{R}^N$ is a nonempty bounded open set with $C^{2,\alpha}$ boundary, and $\Omega = \mathbb{R}^N \setminus \overline{K}$ is connected. The boundary $\partial\Omega = \partial K$ is compact, K lies locally on one side of it, and ν is the outward unit normal of Ω, which points into K. The set K may have several components.

11.1 The decay condition

The system in [1, Problem 1] prescribes the initial value one and asks the solution to tend to zero at infinity uniformly in time. The first requirement determines the behavior far from the boundary at small times, and the two are incompatible on every unbounded set.

Proposition 11.1. *Let $U \subset \mathbb{R}^N$ be an unbounded open set and let $u \in C(U \times (0,\infty))$ satisfy*

$$\int_U u(x,t)\varphi(x)\,dx \longrightarrow \int_U \varphi\,dx \qquad (t \downarrow 0)$$

for every $\varphi \in C_c^\infty(U)$. Then, for every $R > 0$,

$$\sup\{|u(x,t)| : x \in U,\ |x| > R,\ t > 0\} \geq 1.$$

In particular no unbounded domain carries a solution of the system in [1, Problem 1].

Proof. Since U is open and unbounded, it contains a ball $B \subset \{|x| > R\}$. Choose $\varphi \in C_c^\infty(B)$ with $\varphi \geq 0$ and $\int \varphi = 1$. If $|u| \leq m$ on $(U \cap \{|x| > R\}) \times (0,\infty)$, then $|\int u(\cdot,t)\varphi| \leq m$ for every $t > 0$, and the limit $t \downarrow 0$ gives $m \geq 1$. □

The hypothesis of Proposition 11.1 holds for every usual meaning of the initial condition: locally uniform, L^1_{loc} or L^2_{loc} convergence, or pointwise convergence with local bounds. We take the solution in [1, Problem 1] to be the bounded solution of Definition 11.2 below; after the substitution $v = 1 - u$ this is the normalization of Garofalo and Sartori [7] with boundary datum $f \equiv 1$, to which [1, Remark 7.2] compares Conjecture 7.1. Its deficit $1 - u$ tends to zero at infinity (Lemma 11.4), so we read the decay condition of [1, Problem 1] as

$$1 - u(x,t) \longrightarrow 0 \quad \text{as } |x| \to \infty, \text{ uniformly for } t \text{ in bounded subsets of } (0,\infty). \tag{27}$$

For $N \geq 3$ the convergence in (27) is uniform for all $t > 0$ (Lemma 11.4(v)); in the plane it is uniform only on bounded time intervals (Remark 11.5). The substitution $v = 1 - u$ turns the system into $v_t = \Delta v$, $v = 0$ at $t = 0$, $v = 1$ on $\partial\Omega$, with $v \to 0$ at infinity, and changes the flux condition (CP) only by the sign of each b_n.

Definition 11.2. A *bounded solution* of the exterior problem is a function $u \in C^{2,1}(\Omega \times (0,\infty)) \cap C(\overline{\Omega} \times (0,\infty))$ such that $u_t = \Delta u$ in $\Omega \times (0,\infty)$, $u = 0$ on $\partial\Omega \times (0,\infty)$, u is bounded on $\Omega \times (0,T_1)$ for every $T_1 > 0$, and $u(\cdot,t) \to 1$ in $L^1_{\mathrm{loc}}(\Omega)$ as $t \downarrow 0$.

The boundedness requirement is the one that selects the solution: the uniqueness proof in Lemma 11.4(ii) uses it on $\Omega \times (0,T_1)$, in particular near $t=0$ and $\partial\Omega$, where the decay (27) gives no control.

11.2 Comparison for bounded functions

Write $\langle x\rangle = (1+|x|^2)^{1/2}$ and $\psi(x) = e^{-\langle x\rangle}$, so that $\psi \in L^1(\mathbb{R}^N)$ and $|\nabla\psi| \le \psi$. For a function w write $w_- = \max\{-w, 0\}$.

Lemma 11.3. *Let $U \subset \mathbb{R}^N$ be open, let $T_1 > 0$, and let $w \in C^{2,1}(U\times(0,T_1]) \cap C(\overline{U}\times(0,T_1])$ be bounded with $w_t = \Delta w$ in $U\times(0,T_1]$. Assume:*

(a) for each $s \in (0,T_1)$ there are constants L_s, C_s such that, for $t \in [s,T_1]$, $|w(x,t) - w(y,t)| \le L_s|x-y|$ for $x, y \in \overline{U}$ and $|w_t(x,t)| \le C_s$ for $x \in U$;

(b) $w \ge 0$ on $\partial U \times (0,T_1]$;

(c) $w_-(\cdot,t) \to 0$ in $L^1_{\mathrm{loc}}(U)$ as $t \downarrow 0$.

Then $w \ge 0$ in $U\times(0,T_1]$.

Proof. Let $M = \sup|w|$, fix $0 < s < T_1$ and $\varepsilon > 0$, and set $\zeta = (w+\varepsilon)_-$ and

$$g(t) = \int_U \zeta(x,t)^2 \psi(x)\,dx \qquad (s \le t \le T_1).$$

If $w(x,t) < -\varepsilon$ and $y \in \partial U$, then $w(y,t) \ge 0$ by (b), so (a) gives $|x-y| > \varepsilon/L_s$. Thus, for every $t \in [s,T_1]$, the support of $\zeta(\cdot,t)$ lies in the closed set $F = \{x \in U : \operatorname{dist}(x,\partial U) \ge \varepsilon/L_s\}$.

The map $r \mapsto (r+\varepsilon)_-^2$ is C^1, and $|\partial_t \zeta^2| = 2\zeta|w_t| \le 2(M+\varepsilon)C_s$. Dominated convergence shows that $g \in C^1([s,T_1])$ with $g' = -2\int_U \zeta w_t \psi = -2\int_U \zeta\psi\Delta w$. Take $\chi \in C_c^\infty(B_2)$ with $\chi = 1$ on B_1 and put $\chi_R = \chi(\cdot/R)$. The function $\zeta\psi\chi_R$ is Lipschitz with compact support in $F \cap \overline{B_{2R}} \subset U$, and $w(\cdot,t) \in C^2(U)$, so

$$\int_U \zeta\psi\chi_R\,\Delta w = -\int_U \nabla(\zeta\psi\chi_R)\cdot\nabla w.$$

By (a), $|\nabla w| \le L_s$ and $|\Delta w| = |w_t| \le C_s$; also $|\nabla(\zeta\psi)| \le (L_s + M + \varepsilon)\psi$ and $\zeta\psi|\nabla\chi_R| \le C(M+\varepsilon)\psi/R$. Letting $R \to \infty$ by dominated convergence gives $g' = 2\int_U \nabla(\zeta\psi)\cdot\nabla w$. Almost everywhere, $\nabla\zeta = -\nabla w$ on $\{\zeta > 0\}$ and $\nabla\zeta = 0$ on $\{\zeta = 0\}$. Hence

$$g' = -2\int_U \psi|\nabla\zeta|^2 - 2\int_U \zeta\nabla\psi\cdot\nabla\zeta \le -2\int_U \psi|\nabla\zeta|^2 + 2\int_U \psi\zeta|\nabla\zeta| \le \frac{1}{2}\,g,$$

where we used $|\nabla\psi| \le \psi$ and $2ab \le 2a^2 + b^2/2$. Gronwall's inequality and $\zeta \le w_-$ give, for $s \le t \le T_1$,

$$g(t) \le e^{T_1}\int_U w_-(x,s)^2\psi(x)\,dx \le e^{T_1}\Big(M\int_{U\cap B_\rho} w_-(x,s)\,dx + M^2\int_{|x|>\rho}\psi\,dx\Big)$$

for every $\rho > 0$. For $\delta > 0$ let $A_\delta = \{x \in U\cap\overline{B_\rho} : \operatorname{dist}(x,\partial U) < \delta\}$. The set $(U\cap\overline{B_\rho})\setminus A_\delta$ is a compact subset of U, so (c) and $w_- \le M$ give

$$\limsup_{s\downarrow 0}\int_{U\cap B_\rho} w_-(x,s)\,dx \le M|A_\delta|.$$

The sets A_δ lie in $\overline{B_\rho}$ and decrease to the empty set as $\delta \downarrow 0$, so $|A_\delta| \to 0$. Fix t, let first $s \downarrow 0$, then $\delta \downarrow 0$, and then $\rho \to \infty$. Thus $g(t) = 0$, that is, $w(\cdot,t) \ge -\varepsilon$. Since $\varepsilon > 0$ and t are arbitrary, $w \ge 0$. □

11.3 The bounded solution

Let $\operatorname{erfc}(r) = 2\pi^{-1/2}\int_r^\infty e^{-\sigma^2}d\sigma$. For each $a \in \mathbb{R}$ the function $(y,t) \mapsto \operatorname{erfc}\big((y-a)/(2\sqrt{t})\big)$ solves the one-dimensional heat equation, takes the value one at $y = a$, and tends to zero as $t \downarrow 0$, locally uniformly in $y > a$.

Lemma 11.4. *(i) There is a bounded solution u. For $0 < \delta < M < \infty$ it belongs to $C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [\delta, M])$, and u, ∇u and u_t are bounded on $\Omega \times [\delta, M]$.*

(ii) There is only one bounded solution.

(iii) $0 < u < 1$ in $\Omega \times (0,\infty)$; $u(\cdot,t) \to 1$ uniformly on compact subsets of Ω as $t \downarrow 0$; and, for $t \in [\delta, M]$, $u(\cdot,t)$ is Lipschitz on $\overline{\Omega}$ with a constant independent of t.

(iv) If $a > 0$ and $K \subset \{|x_i| < a\}$ for every coordinate x_i, then $1 - u(x,t) \le \operatorname{erfc}\big((|x_i| - a)/(2\sqrt{t})\big)$ whenever $|x_i| > a$. Consequently (27) holds.

(v) If $N \ge 3$, $z \in K$ and $\rho = \max_{y\in\overline{K}} |y-z|$, then $1-u(x,t) \le (\rho/|x-z|)^{N-2}$ in $\Omega\times(0,\infty)$, so the convergence in (27) is uniform for all $t > 0$.

(vi) If K is a ball with centre z, then $u(\cdot,t)$ is radial about z for every $t > 0$.

Proof. (i) Fix $\eta \in C^\infty(\mathbb{R}^N)$ with $0 \le \eta \le 1$, $\eta = 0$ near $\overline{K}$ and $\eta = 1$ outside a ball, and let $A = -\Delta_D$ be the Dirichlet Laplacian on $L^2(\Omega)$. Since $1 - \eta \in L^2(\Omega)$ and $\Delta\eta \in C_c^\infty(\Omega) \subset D(A^m)$ for every m, the function

$$z(t) = e^{-tA}(1-\eta) + \int_0^t e^{-\sigma A}\Delta\eta\, d\sigma$$

belongs to $C([0,\infty); L^2(\Omega)) \cap C^\infty((0,\infty); D(A^m))$ for every m, with $z(0) = 1-\eta$ and $z' = -Az + \Delta\eta$. Put $u = \eta + z$. Then $u_t = \Delta u$ in the sense of distributions, interior elliptic regularity makes u smooth in $\Omega \times (0,\infty)$, $u(\cdot,t) - \eta \in H_0^1(\Omega)$, and $u(\cdot,t) \to 1$ in $L^2_{\mathrm{loc}}(\Omega)$. With $C = \sup|\Delta\eta|$, the functions $(z - 1 - Ct)_+$ and $(z + Ct)_-$ belong to $H_0^1(\Omega)$ and tend to zero in $L^2(\Omega)$ as $t \downarrow 0$; testing $z' = -Az + \Delta\eta$ with them shows that their L^2 norms do not increase, so $-Ct \le z \le 1 + Ct$ and u is bounded on $\Omega \times (0, T_1)$. The parabolic regularity theory up to the boundary [19], applied as in Section 2 near the compact boundary, gives $u \in C^{2+\alpha,1+\alpha/2}$ up to $\partial\Omega$ on $[\delta, M]$ and $u = 0$ on $\partial\Omega$. Interior estimates for bounded caloric functions bound ∇u and u_t on $\{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \ge 1\} \times [\delta, M]$.

(ii) Let u_1, u_2 be bounded solutions, fix $T_1 > 0$, let C_0 bound $|u_1| + |u_2|$ on $\Omega \times (0, T_1)$, and put $w = u_1 - u_2$. Write $\Gamma(t,x) = (4\pi t)^{-N/2} e^{-|x|^2/(4t)}$. Fix $0 < \tau < T_1$. The function $w(\cdot,\tau)$ is continuous on $\overline{\Omega}$ and vanishes on $\partial\Omega$, so its extension by zero is continuous and bounded on $\mathbb{R}^N$, and

$$Z_\tau(x,t) = \int_\Omega \Gamma(t-\tau, x-y)\,|w(y,\tau)|\,dy \qquad (t > \tau)$$

is caloric on $\mathbb{R}^N \times (\tau,\infty)$, satisfies $0 \le Z_\tau \le C_0$, and extends continuously to $t = \tau$ with value $|w(\cdot,\tau)|$. The function $d = w - Z_\tau$ is therefore bounded, caloric in $\Omega \times (\tau, T_1]$, continuous on $\overline{\Omega} \times [\tau, T_1]$, and nonpositive on $\partial\Omega \times [\tau, T_1]$ and on $\overline{\Omega} \times \{\tau\}$. For $\varepsilon > 0$ and $\varepsilon R^2 \ge C_0$, the caloric function $d - \varepsilon(|x|^2 + 2Nt)$ is nonpositive on the parabolic boundary of $(\Omega \cap B_R) \times (\tau, T_1]$, so the weak maximum principle on this bounded set, followed by $R \to \infty$ and $\varepsilon \to 0$, gives $w \le Z_\tau$ on $\Omega \times [\tau, T_1]$.

Now fix $x \in \Omega$ and $t \in (0, T_1]$, and let $\tau \downarrow 0$ with $\tau < t/2$, so that $\Gamma(t-\tau,\cdot) \le (2\pi t)^{-N/2}$. Given $\rho, \kappa > 0$, split the integral defining $Z_\tau(x,t)$ over the compact set $F = \{y \in \Omega \cap \overline{B_\rho} :$

$\operatorname{dist}(y,\partial\Omega)\ge\kappa\}$, over $(\Omega\cap\overline{B_\rho})\setminus F$, and over $\Omega\setminus\overline{B_\rho}$. The first part tends to zero as $\tau\downarrow 0$ because $w(\cdot,\tau)\to 0$ in $L^1(F)$; the second is at most $C_0(2\pi t)^{-N/2}|(\Omega\cap\overline{B_\rho})\setminus F|$, which tends to zero as $\kappa\downarrow 0$; the third is at most $C_0\sup_{t/2\le\sigma\le t}\int_{|y|>\rho}\Gamma(\sigma,x-y)\,dy$, which tends to zero as $\rho\to\infty$. Hence $Z_\tau(x,t)\to 0$ and $w(x,t)\le 0$. Exchanging u_1 and u_2 gives $w=0$.

(iii) For $t\in[\delta,M]$, part (i) bounds $|\nabla u|$ on $\Omega\times[\delta,M]$ by some L. A segment $[x,y]$ with endpoints in $\overline{\Omega}$ either lies in $\overline{\Omega}$, or meets $\partial\Omega$ at a first point x' and a last point y'; in the second case $u(x')=u(y')=0$ and $|u(x)-u(y)|\le L(|x-x'|+|y'-y|)\le L|x-y|$. This gives the Lipschitz statement. Apply Lemma 11.3 on $U=\Omega$ to $w=u$ and to $w=1-u$: the boundary values are 0 and 1, and $u_-\le|1-u|$, $(1-u)_-\le|1-u|$ tend to zero in L^1_{loc}. Thus $0\le u\le 1$. If $u(x_0,t_0)=0$ or $u(x_0,t_0)=1$ at an interior point, the strong maximum principle on the connected set Ω makes u constant on $\Omega\times(0,t_0]$, which contradicts the initial or the boundary values. For the convergence, let $Q=\prod_i(a_i,b_i)$ be a cube with $\overline{Q}\subset\Omega$ and

$$S(x,t)=\sum_{i=1}^{N}\Big[\operatorname{erfc}\Big(\frac{x_i-a_i}{2\sqrt{t}}\Big)+\operatorname{erfc}\Big(\frac{b_i-x_i}{2\sqrt{t}}\Big)\Big].$$

It is caloric, $S\ge 1\ge 1-u$ on ∂Q, and $S(\cdot,t)\to 0$ uniformly on compact subsets of Q. Since $(S-(1-u))_-\le 1-u$, Lemma 11.3 on $U=Q$ gives $0\le 1-u\le S$ in Q. Finitely many such cubes cover a compact subset of Ω.

(iv) Fix i and a sign $\sigma\in\{1,-1\}$. The half-space $U=\{\sigma x_i>a\}$ lies in Ω, and $S=\operatorname{erfc}((\sigma x_i-a)/(2\sqrt{t}))$ is caloric and equals one on ∂U. Lemma 11.3 applied to $S-(1-u)$ on U gives the inequality. If $t\le T_1$ and $|x|\ge 2\sqrt{N}\,a$, some coordinate satisfies $|x_i|\ge|x|/\sqrt{N}\ge 2a$, and $1-u(x,t)\le\operatorname{erfc}\big((|x|/\sqrt{N}-a)/(2\sqrt{T_1})\big)$, which tends to zero as $|x|\to\infty$.

(v) Let $d=\operatorname{dist}(z,\Omega)>0$ and $h(x)=(\rho/|x-z|)^{N-2}$. The function h is harmonic and bounded on $\{|x-z|\ge d\}\supset\Omega$, Lipschitz there, and $h\ge 1$ on $\partial\Omega$. Lemma 11.3 applied to $h-(1-u)$ on $U=\Omega$ gives the bound.

(vi) Every orthogonal map fixing z maps a bounded solution to a bounded solution, so (ii) gives invariance. □

Remark 11.5. In the plane the deficit $v=1-u$ tends to one at every point as $t\to\infty$. Indeed, Lemma 11.3 applied to $u(\cdot,t)-u(\cdot,t+\tau)$ shows that v is nondecreasing in t. By interior estimates its limit $v_\infty\le 1$ is harmonic in Ω, and $v_\infty\ge v(\cdot,1)$ gives $v_\infty=1$ on $\partial\Omega$. If $B_{r_0}(z)\subset K$, the maximum principle on $\Omega\cap B_R$ with $R\to\infty$ gives $1-v_\infty\le\varepsilon\log(|x-z|/r_0)$ for every $\varepsilon>0$, so $v_\infty\equiv 1$. Thus for $N=2$ the convergence in (27) is uniform on bounded time intervals and not on $(0,\infty)$, while for $N\ge 3$ Lemma 11.4(v) gives uniformity on $(0,\infty)$.

11.4 Reflection

Fix a unit vector e and $\lambda\in\mathbb{R}$, and keep the notation Π_λ and R_λ of Section 2. Write $H_\lambda=\{x: x\cdot e>\lambda\}$ and let S be the linear part of R_λ. Call λ *admissible* if

$$R_\lambda\big(\overline{K}\cap\overline{H_\lambda}\big)\subset\overline{K}. \tag{28}$$

For admissible λ put

$$\Sigma_\lambda^{\mathrm{ext}}=H_\lambda\setminus R_\lambda(\overline{K}),\qquad W(x,t)=u(x,t)-u(R_\lambda x,t).$$

The set $\Sigma_\lambda^{\mathrm{ext}}$ is open and contains all points of H_λ far from K.

Lemma 11.6. *Let λ be admissible. Then $\Sigma_\lambda^{\mathrm{ext}}\subset\Omega$ and $R_\lambda\Sigma_\lambda^{\mathrm{ext}}\subset\Omega$, and $W\ge 0$ on $\Sigma_\lambda^{\mathrm{ext}}\times(0,\infty)$. On each connected component C of $\Sigma_\lambda^{\mathrm{ext}}$, either $W>0$ on $C\times(0,\infty)$, or $W(\cdot,t_0)\equiv 0$ on C for some $t_0>0$.*

Proof. If $x \in \Sigma_\lambda^{\mathrm{ext}}$, then $R_\lambda x \notin \overline{K}$, so $R_\lambda x \in \Omega$. If moreover $x \in \overline{K}$, then $x \in \overline{K} \cap H_\lambda$ and (28) gives $R_\lambda x \in \overline{K}$, which is impossible; hence $x \in \Omega$. Consequently W is caloric in $\Sigma_\lambda^{\mathrm{ext}}$ and continuous on its closure, which lies in $\overline{\Omega} \cap R_\lambda \overline{\Omega}$.

Let y be a boundary point of $\Sigma_\lambda^{\mathrm{ext}}$. If $y \in \Pi_\lambda$, then $W(y,t) = 0$. Otherwise $y \in H_\lambda$ and $y \notin \Sigma_\lambda^{\mathrm{ext}}$, so $R_\lambda y \in \overline{K}$; as $R_\lambda y$ is also a limit of points of Ω, it lies on $\partial\Omega$, and $W(y,t) = u(y,t) \geq 0$. By Lemma 11.4(i),(iii), $W(\cdot,t)$ is Lipschitz on the closure with a constant uniform for $t \in [s, T_1]$, and W_t is bounded there. If $F \subset \Sigma_\lambda^{\mathrm{ext}}$ is compact, then F and $R_\lambda F$ are compact subsets of Ω, so $W(\cdot,t) \to 0$ uniformly on F by Lemma 11.4(iii). Lemma 11.3 on $U = \Sigma_\lambda^{\mathrm{ext}}$ and every $T_1 > 0$ gives $W \geq 0$.

If $W(x_0, t_0) = 0$ for some $x_0 \in C$ and $t_0 > 0$, the strong maximum principle gives $W \equiv 0$ on $C \times (0, t_0]$, in particular at time t_0. □

Lemma 11.7. *Let λ be admissible. If $W(\cdot, t_0) \equiv 0$ on a component C of $\Sigma_\lambda^{\mathrm{ext}}$ for some $t_0 > 0$, then $R_\lambda \overline{K} = \overline{K}$.*

Proof. Abbreviate $R = R_\lambda$, $\Pi = \Pi_\lambda$, $H = H_\lambda$, $\Sigma = \Sigma_\lambda^{\mathrm{ext}}$ and $f = u(\cdot, t_0)$, and set

$$D = C \cup R(C) \cup (\Omega \cap \Pi \cap \overline{C}).$$

Recall that $f > 0$ in Ω, $f = 0$ on $\partial\Omega$, and $f = f \circ R$ on C. We show that D is open and closed in Ω.

Openness. The sets C and $R(C)$ are open. Let $p \in \Omega \cap \Pi \cap \overline{C}$ and let $B \subset \Omega$ be a ball centred at p. Then $RB = B$, so $B \cap H \subset \Sigma$. This half ball is connected and contains points of C, hence lies in C; therefore $B \cap R(H) \subset R(C)$ and $B \cap \Pi \subset \overline{C}$, and $B \subset D$.

Closedness. Let $x_j \in D$ converge to $x \in \Omega$. If infinitely many x_j lie in $\Omega \cap \Pi \cap \overline{C}$, then so does x. If infinitely many lie in C, then $x \in \overline{H}$. When $x \in \Pi$, we get $x \in \Omega \cap \Pi \cap \overline{C}$. When $x \in H$, the points Rx_j lie in Ω and converge to $Rx \in \overline{\Omega}$; if $Rx \in \partial\Omega$, then $f(x) = \lim f(x_j) = \lim f(Rx_j) = f(Rx) = 0$, contradicting $x \in \Omega$. Hence $Rx \notin \overline{K}$, so $x \in \Sigma \cap \overline{C} = C$. If infinitely many $x_j = Ry_j$ with $y_j \in C$, then $y_j \to y = Rx$ and $y \in \overline{\Omega} \cap \overline{H}$. If $y \in \partial\Omega$, then $f(x) = \lim f(Ry_j) = \lim f(y_j) = 0$, a contradiction; so $y \in \Omega$. When $y \in \Pi$, $x = y \in \Omega \cap \Pi \cap \overline{C}$. When $y \in H$, we have $Ry = x \in \Omega$, so $y \in \Sigma \cap \overline{C} = C$ and $x \in R(C)$.

The set D is nonempty, and Ω is connected, so $D = \Omega$. Since $R(D) = D$, the domain Ω is symmetric about Π, and so is $\overline{K} = \mathbb{R}^N \setminus \Omega$. □

11.5 Proof of Theorem 1.5

Proof of Theorem 1.5. Fix a unit vector e and let $\lambda_0 = \max_{\overline{K}} x \cdot e$. The geometric moving-plane lemma used in Section 4, applied to the bounded C^2 open set K, gives $\lambda_* < \lambda_0$ such that every $\lambda \in [\lambda_*, \lambda_0]$ is admissible and, at λ_*, one of the following holds [3, Sections 1–2]:

(T) there is $P \in \partial K \cap H_{\lambda_*}$ with $R_{\lambda_*} P \in \partial K$ and $\nu(R_{\lambda_*} P) = S\nu(P)$ (internal tangency of the reflected cap of K);

(O) there is $Q \in \partial K \cap \Pi_{\lambda_*}$ with $e \cdot \nu(Q) = 0$ (orthogonal contact).

Serrin's argument is local near $\partial K \cap \Pi_\lambda$ and near reflected boundary points, and it uses only the compactness and C^2 regularity of ∂K, not the connectedness of K. When the plane reaches the top of a further component of K, the new small cap lies near a point where the outward normal of K equals e, and it reflects into that component by the same local argument. Let $\lambda = \lambda_*$. By Lemmas 11.6 and 11.7, either $\overline{K}$ is symmetric about Π_λ, or $W > 0$ on $\Sigma_\lambda^{\mathrm{ext}} \times (0, \infty)$. Suppose the latter. Since $u = 0$ on $\partial\Omega$, condition (2′) of Theorem 1.5 means

$$\nabla u(y, T) = b\, \nu(y) \qquad (y \in \partial\Omega). \tag{29}$$

Case (T). Since $R_\lambda P \in \partial K$, near P the set $R_\lambda \overline{K}$ is the closure of one side of the $C^{2,\alpha}$ hypersurface $R_\lambda(\partial K)$, so near P the set $\Sigma_\lambda^{\mathrm{ext}}$ is the other side; it satisfies an interior sphere condition at P, and it lies in Ω. The function W is C^1 up to $R_\lambda(\partial K)$ near P, positive in $\Sigma_\lambda^{\mathrm{ext}} \times (0,T]$, and $W(P,T) = 0$. The parabolic Hopf principle [8] gives a strictly positive derivative of $W(\cdot,T)$ at P in the direction pointing into $\Sigma_\lambda^{\mathrm{ext}}$. But (29) and $S\nu(R_\lambda P) = \nu(P)$ give

$$\nabla W(P,T) = \nabla u(P,T) - S\nabla u(R_\lambda P,T) = b\nu(P) - bS\nu(R_\lambda P) = 0,$$

a contradiction.

Case (O). Translate Q to the origin and rotate so that $e = e_1$ and $\Pi_\lambda = \{x_1 = 0\}$. Let $n = -\nu(Q)$, the unit normal pointing into Ω; then $n \perp e_1$. Since ∂K is C^2, there is $r > 0$ such that the ball $B = B_r(rn)$ lies in Ω and $\overline{B} \cap \partial\Omega = \{Q\}$. Its centre lies on Π_λ, so $R_\lambda B = B \subset \Omega$; hence $B \cap H_\lambda \subset \Sigma_\lambda^{\mathrm{ext}}$, and $W = u - u \circ R_\lambda$ is caloric and smooth in $B \times (0,\infty)$, vanishes on $B \cap \Pi_\lambda$, is positive on $(B \cap H_\lambda) \times (0,\infty)$, and is continuous and nonnegative on $(\partial B \cap \overline{H_\lambda}) \times (0,\infty)$. Lemma 3.1, with $R = r$ and $c = rn$, gives

$$\liminf_{\rho \downarrow 0} \rho^{-2} W(\rho(e_1 + n), T) > 0. \tag{30}$$

On the other hand $f = u(\cdot,T) \in C^{2,\alpha}(\overline{\Omega})$ near Q, and $\nabla f = b\nu$ on $\partial\Omega$ by (29). As in the proof of Theorem 1.1, differentiating this identity along a boundary curve with velocity e_1 at Q and taking the inner product with $\nu(Q)$ gives $D^2 f(Q)[e_1, n] = 0$. Reflection fixes Q and n and reverses e_1, so $W(Q,T) = 0$, $\nabla W(Q,T) = 2(\nabla f(Q) \cdot e_1)e_1 = 2b(\nu(Q) \cdot e_1)e_1 = 0$, and the Hessian of $W(\cdot,T)$ at Q restricted to the span of e_1, n has diagonal entries 0 and off-diagonal entry $2D^2 f(Q)[e_1, n] = 0$. Taylor's formula, applied to a C^2 extension of f across $\partial\Omega$, gives $W(\rho(e_1 + n), T) = o(\rho^2)$, contradicting (30).

Hence $\overline{K}$ is symmetric about $\Pi_{\lambda_*(e)}$ for every unit vector e. Each such plane contains the centroid z of K, so $\overline{K}$ is invariant under every reflection in a hyperplane through z, hence under every orthogonal map fixing z. Let $\rho = \max_{\overline{K}} |x - z|$; the sphere $\{|x - z| = \rho\}$ lies in $\overline{K}$. If some point $x \in \Omega$ had $|x - z| < \rho$, every path in $\mathbb{R}^N$ from x to a point with $|x - z| > \rho$ would meet this sphere, and Ω would be disconnected. Hence $\Omega = \{|x - z| > \rho\}$ and $K = B_\rho(z)$, because K is the interior of $\overline{K}$.

Conversely, if $K = B_\rho(z)$, Lemma 11.4(vi) makes $u(\cdot,t)$ radial about z, so its normal derivative is constant on $\partial\Omega$ for every $t > 0$. □

Proof of Corollary 1.6. Lemma 11.4 gives the bounded solution and the decay (27). If $\partial_\nu u(\cdot,t)$ is constant on $\partial\Omega$ for every t in a nonempty set of positive times, Theorem 1.5 applied at any one of them shows that K is a ball. Conversely, for a ball the flux is constant at every positive time. □

Remark 11.8. Exterior overdetermined elliptic problems have been treated by the moving-plane method by Reichel [23], Aftalion and Busca [24] and Sirakov [25]. In the heat problem above, comparison at each admissible plane comes directly from the uniform initial datum (Lemma 11.6), exactly as for bounded domains, and the admissible planes are determined by K alone. The single observation time enters only at the critical plane, through the tangency and corner contradictions.

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Data and code availability. No datasets were generated or analyzed. The arguments are analytic, and no proof relies on computation. Three supplementary scripts accompany the source files: exact symbolic checks of the barrier and Taylor identities used in Sections 3 and 11 and of the linearized flux multipliers of the ball complement in $\mathbb{R}^3$; a numerical evaluation of the linearized single-time flux of perturbed ball complements; and a finite-difference test of the exterior reflected comparison.

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