Single-time boundary heat-flux rigidity for smooth domains

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Abstract

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded connected domain with smooth boundary. Consider the Dirichlet heat evolution with initial temperature identically one. We prove that if its signed outward normal derivative is spatially constant on the entire boundary at one positive time, then Ω is a ball. The same flux value is imposed on all boundary components. The proof uses moving planes, an L^2 initial-time comparison, and a parabolic corner barrier. We also establish local quantitative stability near a ball: after fixing volume and centroid, the full boundary flux defect in $C^{m-1,\alpha}$ controls the radial deformation in $C^{m,\alpha}$, for $m \geq 3$ and $0 < \alpha < 1$, uniformly on compact positive-time intervals. The key input is positivity and order-one growth of the linearized spherical multipliers above the translation modes. By contrast, observation on a moving proper boundary cap admits an infinite-dimensional family of nearby normalized nonballs with exactly the ball flux at the prescribed time.

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1 Statement and context

Let $\Omega \subset \mathbb{R}^N$ be bounded and connected with C^∞ boundary, where $N \geq 2$. Write ν for its outward unit normal and Δ_D for the Dirichlet Laplacian. We study

$$u_t = \Delta u \quad \text{in } \Omega \times (0,\infty), \qquad u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \qquad u(\cdot,t) \longrightarrow 1 \quad \text{in } L^2(\Omega) \quad (t \downarrow 0). \tag{1}$$

Thus $u(\cdot,t) = e^{t\Delta_D}1$. The initial temperature is prescribed in L^2, and the zero boundary value is imposed for positive times. The signed normal derivative is $\partial_\nu u = \nabla u \cdot \nu$.

Theorem 1.1. *Suppose that, for some $T > 0$ and $b \in \mathbb{R}$,*

$$\partial_\nu u(x,T) = b \qquad \text{for every } x \in \partial\Omega. \tag{2}$$

Then Ω is a ball. Conversely, on a ball condition (2) holds at every positive time.

The same scalar b is required on every boundary component. The domain may be nonconvex and its boundary may be disconnected. The observation is the complete spatial boundary flux at a single instant.

Cavallina–Pinamonti [1] consider precisely the initial datum and flux in (1)–(2). Their Theorems I–II establish ball rigidity from infinite sequences of observation times in several time-accumulation and boundary-regularity regimes. Their Problem 3 asks about finitely many times, and Conjecture 7.4 proposes nonball examples for finite prescribed observation counts. Theorem 1.1 rules out even one such time in the bounded connected C^∞ Euclidean class. Zeraoulia [2, Theorem 1.1] proves the short-time sequence result for C^2 boundaries

by a uniform two-term flux expansion. The two results address different aspects of the observation: that theorem allows lower boundary regularity along a sequence tending to zero, while Theorem 1.1 uses one positive time on a smooth domain.

The proof belongs to the classical moving-plane tradition of Serrin [3] and the parabolic symmetry literature of Alessandrini–Garofalo [4], Alessandrini [5, 6], and Garofalo–Sartori [7]. In particular, reflected parabolic comparison, tangency Hopf contradictions and orthogonal-contact arguments already occur in that literature. The boundary-point principle used here is classical; see Friedman [8] and Li–Nirenberg [9, Section 3]. Coordinate-times-exponential parabolic corner constructions also have antecedents; compare Attouchi–Souplet [10, Appendix 2]. Our proof gives directly the corner estimate for the parabolic evolution in (1). The role of the one-time condition is explicit: it enters only in the final tangency or corner contradiction, whereas comparison uses the evolution from its uniform initial datum.

For approximate observations we establish the following separate local statement. Let B be the unit ball in $\mathbb{R}^N$, let $X = C^{m,\alpha}(\mathbb{S}^{N-1})$ and $Y = C^{m-1,\alpha}(\mathbb{S}^{N-1})$, and define

$$\Omega_f = \{r\theta : 0 \le r < 1 + f(\theta),\ \theta \in \mathbb{S}^{N-1}\}.$$

Pull its signed flux back to the sphere by the radial graph:

$$\Phi_T(f)(\theta) = \partial_{\nu_f} u_f((1 + f(\theta))\theta, T), \qquad u_f(t) = e^{t\Delta_{\Omega_f, D}} 1.$$

Theorem 1.2 (Local quantitative estimate)**.** *Fix $N \ge 2$, integer $m \ge 3$, $0 < \alpha < 1$ and $0 < T_0 \le T_1 < \infty$. There are $\varepsilon, C > 0$ such that, if $f \in X$, $\|f\|_X < \varepsilon$,*

$$|\Omega_f| = |B|, \qquad \int_{\Omega_f} x\, dx = 0,$$

then for every $T \in [T_0, T_1]$,

$$\|f\|_{C^{m,\alpha}(\mathbb{S}^{N-1})} \le C \inf_{b \in \mathbb{R}} \|\Phi_T(f) - b\|_{C^{m-1,\alpha}(\mathbb{S}^{N-1})}. \tag{3}$$

The constants are uniform on that time interval. The same statement holds in the corresponding little Hölder spaces.

The estimate concerns the entire spatial boundary trace at one instant, in a near-ball chart with fixed size and position. Its constants are uniform on each compact subinterval of $(0, \infty)$. The nondegeneracy behind it is proved directly by comparison of spherical heat modes. The subsequent material-coordinate, Schauder, Fredholm and inverse-function arguments use established methods. Elliptic local Serrin stability via shape linearization and a modified implicit function theorem is treated by Gilsbach–Onodera [11]; analytic dependence of elliptic Dirichlet–Neumann maps by Hu–Nicholls [12]; and smooth shape dependence of heat layers in sharp parabolic Hölder spaces by Dalla Riva–Luzzini [13]. For the same constant initial temperature, Savo [14] derives a shape variation of heat content involving a time convolution of fluxes, rather than the raw flux at one instant. Kenioua [15] obtains short-time quantitative geometric estimates from a flux defect plus a positive time-error term. The local estimate (3) controls the deformation directly by the flux defect in the stated near-ball geometry and strong boundary norms. For partial observations, Corollary 1.3 constructs arbitrarily small normalized nonball perturbations with constant flux on a proper cap transported by the radial graph. This is a submersion consequence of the full-boundary isomorphism. Partially overdetermined elliptic counterexamples have classical antecedents [16]; the present statement concerns the datum-one heat equation at one time and a moving observation cap.

The partial-observation contrast. Write $-a(T)$ for the constant signed flux of the unit ball at time T. The radial-graph observation map $\Phi_T(f)$ is defined above. The next conclusion concerns a cap transported with that graph.

Corollary 1.3. *Fix $N \geq 2$, integer $m \geq 3$, $0 < \alpha < 1$ and $T > 0$. Let $U \subset \mathbb{S}^{N-1}$ be a nonempty open spherical cap with smooth boundary whose complement contains a nonempty open cap. There are arbitrarily small nonzero $f \in C^{m,\alpha}$ satisfying $|\Omega_f| = |B|$ and $\int_{\Omega_f} x\,dx = 0$ for which*

$$\Phi_T(f)(\theta) = -a(T) \qquad (\theta \in \overline{U}). \tag{4}$$

They form a local infinite-dimensional C^1 submanifold of the normalized near-ball shapes. Every nonzero member is a nonball. The observed physical cap is the moving radial graph over $\overline{U}$.

Plan of the proof. Section 2 establishes reflected comparison from the L^2 initial datum and reconstructs symmetry from a zero cap component. The corner estimate in Section 3 completes the moving-plane argument in Section 4. Sections 6–8 establish the linearized nondegeneracy, shape regularity and local stability. Section 9 proves the moving-cap contrast.

2 Reflected comparison and zero components

Positive-time parabolic and elliptic regularity give $u \in C^\infty(\overline{\Omega} \times [\delta, M])$ for every $0 < \delta < M < \infty$. The semigroup maximum principle and connectedness give

$$0 < u(x,t) \leq 1 \qquad (x \in \Omega,\ t > 0). \tag{5}$$

For example, smoothing follows by writing $A = -\Delta_D$ and using $e^{-tA}1 \in D(A^m)$ for every m, then iterating elliptic regularity and Sobolev embedding. The analytic semigroup and the same smoothing show that $t \mapsto u(x,t)$ is real analytic for each fixed interior point x and $t > 0$.

Fix a unit vector e, a plane $\Pi_\lambda = \{x : x \cdot e = \lambda\}$, and reflection

$$R_\lambda x = x - 2(x \cdot e - \lambda)e.$$

Put $\Sigma_\lambda = \Omega \cap \{x \cdot e > \lambda\}$. Whenever $R_\lambda \Sigma_\lambda \subset \Omega$, define

$$w(x,t) = u(R_\lambda x, t) - u(x,t), \qquad x \in \Sigma_\lambda. \tag{6}$$

The cap may have more than one component.

Lemma 2.1. *For every admissible plane, $w \geq 0$ on $\Sigma_\lambda \times (0,\infty)$. On each connected component C of the cap, either $w > 0$ for all positive times, or $w \equiv 0$ for all positive times.*

Proof. The difference is caloric. On the plane face it vanishes; on the original boundary face it has nonnegative values by (5). These are legitimate continuous positive-time boundary values since reflection of the closed cap lies in $\overline{\Omega}$.

We justify comparison without making a time-zero boundary continuity assumption. On any compact positive-time interval $[s,t]$, w is uniformly Lipschitz on the closed cap, by smooth spatial extension of u from $\overline{\Omega}$. Its negative part $v = w_- := \max\{-w, 0\}$ vanishes continuously on the cap boundary. In particular $v(x,\tau) \leq L\,\mathrm{dist}(x, \partial\Sigma_\lambda)$. Cutting v off in a boundary layer of width ε produces functions converging to v in H^1. Indeed the extra gradient is bounded on this shrinking layer, whose volume tends to zero; the cap boundary lies in a smooth hypersurface together with a plane and has zero N-dimensional measure.

Interior smoothing of the cutoffs shows $v \in H_0^1(\Sigma_\lambda)$. The same cutoff argument applies at the critical cap.

Testing the heat equation with the negative part gives the standard energy inequality

$$\|w_-(\cdot,t)\|^2_{L^2(\Sigma_\lambda)} + 2\int_s^t \|\nabla w_-(\cdot,\tau)\|^2_{L^2(\Sigma_\lambda)}\,d\tau \le \|w_-(\cdot,s)\|^2_{L^2(\Sigma_\lambda)}. \tag{7}$$

Reflection preserves volume and maps the cap into Ω. Hence

$$\|w(\cdot,s)\|_{L^2(\Sigma_\lambda)} \le 2\|u(\cdot,s)-1\|_{L^2(\Omega)} \longrightarrow 0.$$

Letting $s \downarrow 0$ in (7) proves nonnegativity.

If w has an interior zero in C at a positive time, the strong parabolic maximum principle makes it zero throughout C at every earlier positive time. For each fixed $x \in C$, analyticity of $w(x,\cdot)$ extends this interval of zeros to all positive times. Otherwise it is strictly positive everywhere on that component. □

Lemma 2.2. *If $w \equiv 0$ on one cap component C for all positive times, then Ω is symmetric about Π_λ.*

Proof. Abbreviate $R = R_\lambda$, $\Pi = \Pi_\lambda$, and set

$$D = C \cup R(C) \cup (\Omega \cap \Pi \cap \overline{C}).$$

This is a nonempty relatively open subset of Ω. At a plane point in the third set, choose a small ball lying inside Ω. Its positive half is connected and meets C, so belongs to C; its negative half then belongs to $R(C)$.

The set D is also relatively closed. A positive-side limit in Ω belongs to C, since a component is closed relative to the cap. If $y_j = Rx_j \in R(C)$ tends to a negative-side point $y \in \Omega$, then $x = Ry \in \overline{C}$. If $x \in \partial\Omega$, the identity $u(x_j,t) = u(y_j,t)$ and continuity would give $u(y,t) = 0$, contradicting (5). Thus $x \in \Omega$, hence $x \in C$, and $y \in R(C)$. Limits on the plane lie in the third set. Connectedness now gives $D = \Omega$, which proves reflection symmetry. □

3 A parabolic corner estimate

The following local estimate isolates the only corner conclusion needed in the proof. Denote by e_1 the first coordinate vector.

Lemma 3.1. *Let $n \perp e_1$ be a unit vector, let $c = Rn$ with $R > 0$, and put $B^+ = B_R(c) \cap \{x_1 > 0\}$. Suppose w is caloric, positive in $B^+ \times (0,\infty)$, nonnegative on its lateral boundary, and vanishes on the plane face $x_1 = 0$. Suppose it is smooth up to that face near the closed inner half-sphere $|x-c| = R/2$. Then, for every $T > 0$,*

$$\liminf_{\rho \downarrow 0} \frac{w(\rho(e_1+n),T)}{\rho^2} > 0. \tag{8}$$

Here continuity up to the outer spherical boundary is assumed wherever the comparison below uses its values.

Proof. Choose

$$\beta > \frac{3R^2}{4T}, \qquad s = T - \frac{3R^2}{4\beta} > 0, \qquad \alpha > \frac{\beta + 2(N+2)}{R^2}.$$

With $r = |x - c|$ and $q = r^2 + \beta(T - t)$ consider

$$E = \{(x,t) : x_1 > 0,\ r > R/2,\ q < R^2,\ s < t \le T\}.$$

It is contained in $B^+ \times (s,T]$ and is empty at $t = s$. The positive function

$$h(x,t) = x_1\big(e^{-\alpha q} - e^{-\alpha R^2}\big) \tag{9}$$

satisfies

$$(\partial_t - \Delta)h = \alpha x_1 e^{-\alpha q}\big[\beta + 2(N+2) - 4\alpha r^2\big] < 0 \quad \text{in } E. \tag{10}$$

In this calculation $c_1 = 0$; the cross derivatives of $x_1 e^{-\alpha q}$ account for the $N + 2$ term. The chosen α and $r > R/2$ give the strict sign.

On the plane and the moving face $q = R^2$, $h = 0$. On the remaining face $r = R/2$, the quotient w/x_1 extends continuously to its plane rim, with value $\partial_1 w$. The ordinary parabolic Hopf principle on that flat face gives $\partial_1 w > 0$ at each rim point and every $s \le t \le T$. The relevant earlier half-cylinders lie inside B^+ near the rim. Positivity away from the rim and compactness therefore give

$$\inf_{\substack{x_1 \ge 0,\ |x-c| = R/2 \\ s \le t \le T}} \frac{w(x,t)}{x_1} > 0,$$

where the quotient has its continuous rim interpretation. Since h/x_1 is bounded there, a fixed $\varepsilon > 0$ gives $w \ge \varepsilon h$ on that face.

Comparison yields $w \ge \varepsilon h$ throughout E. To see that the moving boundary causes no extra hypothesis, apply the minimum argument to $w - \varepsilon h + \delta(t - s)$ on the closure through any time $t_0 \le T$. It is nonnegative on each lateral face and at the birth set $t = s$, which lies on $r = R/2$. A negative minimum at an interior spatial point would have nonpositive backward time derivative and nonnegative Laplacian, contradicting its strictly positive heat operator. Let $\delta \downarrow 0$.

At $t = T$, the point $x = \rho(e_1 + n)$ lies in E for small $\rho > 0$, and

$$x_1 = \rho, \qquad r^2 = R^2 - 2R\rho + 2\rho^2, \qquad h(x,T) = 2\alpha R e^{-\alpha R^2}\rho^2 + O(\rho^3).$$

This proves (8). □

4 Moving planes and proof of the theorem

Proof of Theorem 1.1. Fix e. Start the plane Π_λ near the support value $\max_{\overline{\Omega}} x \cdot e$ and move it inward, maintaining $R_\lambda \Sigma_\lambda \subset \Omega$. The classical geometric moving-plane lemma for bounded C^2 domains gives a first limiting plane. At that plane either the reflected original boundary is tangent to the original boundary away from the plane, or the plane meets $\partial\Omega$ orthogonally; see Serrin [3, Sections 1–2]. Reflection of the open cap remains in Ω at this position. This geometric lemma applies to the full cap and does not require it to be connected.

Use (6) at the limiting position. By Lemmas 2.1–2.2, either the domain is already symmetric about this plane or $w > 0$ on every cap component at all positive times. Suppose the latter.

In the off-plane tangency case, let P be the original cap boundary point whose reflection also lies on $\partial\Omega$. Write S for the linear part of R_λ. Tangency gives $\nu(R_\lambda P) = S\nu(P)$. At time T, the boundary condition and (2) give

$$\nabla w(P,T) = S\nabla u(R_\lambda P, T) - \nabla u(P,T) = bS\nu(R_\lambda P) - b\nu(P) = 0.$$

But $w(P,T) = 0$ and w is strictly positive in the adjacent cap component at preceding positive times. The ordinary parabolic Hopf principle at this smooth face gives a strictly positive inward derivative, a contradiction.

In the orthogonal-contact case translate the contact point to $Q = 0$ and rotate coordinates so the plane is $x_1 = 0$, with the cap on $x_1 > 0$. Let n be the inward unit normal at Q. Orthogonality means $n \perp e_1$. Choose a sufficiently small interior tangent ball $B_R(Rn) \subset \Omega$ whose closure meets $\partial\Omega$ only at Q. Such a ball exists by smoothness, after decreasing its radius if necessary. The ball is invariant under reflection in $x_1 = 0$; its positive half lies in one cap component. Lemma 3.1 therefore gives

$$\liminf_{\rho \downarrow 0} \rho^{-2} w(\rho(e_1 + n), T) > 0. \tag{11}$$

The inner sphere and its plane rim lie strictly inside Ω, so all flat-face regularity and Hopf hypotheses in that lemma hold.

Set $f = u(\cdot, T)$. On $\partial\Omega$,

$$\nabla f = b\nu. \tag{12}$$

Since e_1 is tangent at Q, differentiate (12) along a boundary curve with velocity e_1 and dot with ν. The scalar b is constant and $(D_{e_1}\nu) \cdot \nu = 0$, hence

$$D^2 f(Q)[e_1, \nu] = 0, \qquad D^2 f(Q)[e_1, n] = 0. \tag{13}$$

Reflection fixes Q, n and reverses e_1. Consequently $w(Q,T) = 0$ and $\nabla w(Q,T) = 0$. Its Hessian on the span of e_1, n has vanishing diagonal terms by reflection, and vanishing mixed term by (13). Taylor's formula gives

$$w(\rho(e_1 + n), T) = o(\rho^2),$$

contradicting (11). Thus the first limiting plane is a symmetry plane in every direction.

Let z be the centroid of Ω. Reflection invariance forces the plane offset to equal $z \cdot e$. Translating z to zero, the domain is invariant under every origin hyperplane reflection, hence under every orthogonal transformation. Its radii form an interval by connectedness. A bounded open radial connected set is a ball or a shell, with the punctured-ball possibility excluded by smooth boundary when $N \geq 2$.

For completeness radial symmetry alone does not exclude a shell. If $\Omega = \{a < |x| < R_0\}$ with $a > 0$, the first limiting plane in direction e has offset

$$\lambda_* = (R_0 + a)/2 > 0.$$

Indeed, for $\lambda > \lambda_*$ the full cap reflects inside the outer ball and has reflected axial coordinate greater than $2\lambda - R_0 > a$, so avoids the inner ball. At λ_* the outer pole $R_0 e$ reflects to the inner pole ae, giving the first off-plane tangency. This limiting plane cannot be a symmetry plane: the centered shell has centroid zero. This contradicts what was proved for every first limiting plane. Thus Ω is a ball.

Conversely the datum, equation and boundary value on a ball are invariant under rotations about its center. Uniqueness makes u radial, and its normal derivative is constant on the boundary at every positive time. □

5 Consequences and the role of the initial temperature

Corollary 5.1. *Let Ω satisfy the hypotheses of Theorem 1.1. If there is a nonempty finite set of positive times at each of which the solution of (1) has spatially constant whole-boundary signed flux, then Ω is a ball. In particular no smooth connected nonball example can realize even one of the observations proposed in [1, Problem 3 and Conjecture 7.4].*

Proof. Apply Theorem 1.1 at any one time in the set. □

For a general initial temperature g, the reflected comparison starts from $g(R_\lambda x) - g(x)$, whose sign need not be nonnegative. Thus positivity of g cannot replace the datum $g \equiv 1$ in the argument. This explains the compatibility with the tailored-data shell results in [17]: on a fixed spherical shell, nonuniform positive preparations can produce a common signed outward flux on both boundary components at suitable positive times.

Both Theorems 1.1 and 1.2 use the prescribed constant initial temperature. The first is a global rigidity statement for an exact whole-boundary observation; the second gives quantitative control in a normalized near-ball chart with Hölder observation norms.

6 The spherical flux derivative

Throughout the quantitative argument the reference domain is the unit ball $B \subset \mathbb{R}^N$. Write $u(t,r) = e^{t\Delta_D}1$ and $a(t) = -u_r(t,1) > 0$. The radial heat-content expansion and the identity $H_B'(t) = -|\partial B| a(t)$ give

$$a(t) = 2\sum_{n\geq 1} e^{-j_{N/2-1,n}^2 t}, \tag{14}$$

where $j_{\nu,n}$ are the positive zeros of J_ν. One obtains the heat-content formula by expanding the constant function in radial Dirichlet eigenfunctions and integrating their Bessel expressions: $H_B(t)/|B| = 2N\sum_{n\geq 1} j_{N/2-1,n}^{-2} e^{-j_{N/2-1,n}^2 t}$. The series and all its time derivatives converge locally normally for $t > 0$. In particular $a'(t) < 0$ and $a(t) = O(t^{-1/2})$ at zero, by the usual linear growth of the Bessel zeros. Thus a is locally integrable even though the initial boundary data are incompatible.

For spherical degree $\ell \geq 1$, put $\lambda_\ell = \ell(\ell+N-2)$ and

$$L_\ell = \partial_{rr} + \frac{N-1}{r}\partial_r - \frac{\lambda_\ell}{r^2}.$$

Let w_ℓ be the positive heat Poisson solution of

$$(\partial_t - L_\ell)w_\ell = 0, \quad w_\ell(t,1) = a(t), \quad w_\ell(t,r) = O(r^\ell)\ (r \downarrow 0), \tag{15}$$

with zero interior initial value. More precisely it is the increasing limit of solutions with boundary input $a_\epsilon(t) = a(t)\mathbf{1}_{t\geq\epsilon}$ and zero initial value. Local integrability of the input and off-boundary decay of the heat Poisson kernel make the limit finite and smooth on compact positive-time sets. The omitted input is separated from any fixed observation time, so convergence also holds for the normal trace there. This specifies the solution class; pointwise zero interior initial values alone would not specify an arbitrary singular-boundary solution.

Lemma 6.1. *The derivative $G_T = D\Phi_T(0)$ has multiplier*

$$\gamma_\ell(T) = \partial_r w_\ell(T,1) + (N-1)a(T) \tag{16}$$

on degree-ℓ spherical harmonics. For all $T > 0$,

$$\gamma_1(T) = 0 < \gamma_2(T) < \gamma_3(T) < \cdots. \tag{17}$$

Uniformly for $T \in [T_0, T_1]$,

$$\gamma_\ell(T) = a(T)(\ell+N-1) + O_{N,T_0,T_1}(1). \tag{18}$$

Consequently, for $\ell \geq 2$, there are positive constants independent of ℓ, T in this range such that

$$c(1+\ell) \leq \gamma_\ell(T) \leq C(1+\ell). \tag{19}$$

Proof. For a smooth normal displacement f, the Eulerian heat derivative u' satisfies the heat equation with boundary input $-fu_r = a(t)f$. Rotational equivariance gives $u' = w_\ell(t,r)Y_\ell(\theta)$ for a degree-ℓ harmonic. Differentiating the normal flux at the radial graph yields

$$G_T f = \partial_r u'(T,1) + f u_{rr}(T,1).$$

The normal change is tangential and has zero dot product with the radial reference gradient. At positive times $u_t = 0$ on the boundary, so $u_{rr}(T,1) = -(N-1)u_r(T,1) = (N-1)a(T)$, proving (16). Section 7 supplies the full positive-time Banach differentiability used here.

A translation by ϵe has displacement $e\cdot\theta$ and Eulerian derivative $-e\cdot\nabla u$. Thus $w_1 = -u_r$ and $\gamma_1 = 0$. To verify that this is the specified Poisson solution despite the corner singularity, take Laplace transforms in time. For $s > 0$, $\nu = N/2 - 1$, define

$$F_\ell(r,s) = r^{-\nu}\frac{I_{\ell+\nu}(\sqrt{s}\,r)}{I_{\ell+\nu}(\sqrt{s})}, \qquad D_\ell(s) = \partial_r F_\ell(1,s).$$

The ball resolvent is $U(s,r) = (1 - F_0(r,s))/s$ and $A(s) = \mathcal{L}[a](s) = D_0(s)/s$. The Poisson solution has transform $A(s)F_\ell(r,s)$. The Bessel recurrence gives $-\partial_r U = AF_1$, proving the desired equality by ordinary Laplace uniqueness at each interior point.

For truncated inputs, if $k > \ell \geq 1$ then $z = w_\ell - w_k$ satisfies

$$(\partial_t - L_k)z = (\lambda_k - \lambda_\ell)w_\ell/r^2 \geq 0,$$

with zero initial and boundary values. Comparison with the nonnegative radial killing potential gives $z \geq 0$. A negative minimum cannot occur at $r = 0$, where both modes vanish, so the singular center causes no obstruction to the annular limiting argument. Passing to the Poisson limits gives $w_\ell \geq w_k \geq 0$. The input is positive at every positive time, so the strong principle gives $z > 0$ for $0 < r < 1, t > 0$. On a small positive-time cylinder at $r = 1$, the coefficients are smooth and the ordinary parabolic Hopf principle gives $z_r(T,1) < 0$. This proves the strict inequalities in (17).

For the high-degree estimate set $\tau = T_0/2$ and $v_\ell(t,r) = w_\ell(t,r) - a(t)r^\ell$ for $t \geq \tau$. Then

$$(\partial_t - L_\ell)v_\ell = -a'(t)r^\ell, \qquad v_\ell(t,1) = 0.$$

The comparison $w_\ell \leq w_1$ gives a common initial bound $|v_\ell(\tau,r)| \leq C_0$, with $C_0 = \sup_r w_1(\tau,r) + a(\tau)$. Put $C_1 = \sup_{\tau\leq t\leq T_1}|a'(t)|$ and

$$V(t,r) = C_0 u(t-\tau,r) + C_1\int_0^{t-\tau} u(s,r)\,ds.$$

It is nonnegative and $(\partial_t - L_\ell)V = C_1 + \lambda_\ell V/r^2 \geq C_1$. Comparison gives $-V \leq v_\ell \leq V$. Differentiation at their common zero boundary therefore yields

$$|\partial_r v_\ell(T,1)| \leq C_0 a(T-\tau) + C_1\int_0^{T-\tau} a(s)\,ds.$$

This is bounded uniformly for $T \in [T_0, T_1]$, proving (18). Since $\min_{[T_0,T_1]} a > 0$, sufficiently high degrees satisfy (19); the remaining finite degrees do so by strict positivity and continuity in time. □

The preceding computation also gives an exact ordinary Laplace formula for the multipliers:

$$\mathcal{L}[\gamma_\ell](s) = A(s)(D_\ell(s) - D_1(s)), \qquad s > 0. \tag{20}$$

Here the difference is important. The Bessel recurrence gives $A(s)(D_1(s)+N-1)=1$; thus $A(s)(D_\ell(s)+N-1)$ would contain an additional impulse at time zero. To justify (20) without interchanging singular boundary derivatives, let $z_\epsilon = w_{1,\epsilon} - w_{\ell,\epsilon}$. Its nonnegative source increases as $\epsilon \downarrow 0$, so z_ϵ and its inward normal trace increase. Therefore $-\partial_r z_\epsilon(1)$ increases to γ_ℓ, and its transform is $A_\epsilon(s)(D_\ell - D_1)$. Monotone convergence proves (20) and ordinary local integrability of γ_ℓ from the finite right hand side.

As an immediate spectral consequence, on the subspace of degrees at least two, G_T is an isomorphism from $H^s(\mathbb{S}^{N-1})$ to $H^{s-1}(\mathbb{S}^{N-1})$ for every real s, uniformly in the stated time interval. This follows by squaring harmonic coefficients in (19). The Hölder-space inverse needed for Theorem 1.2 is justified separately below; a Sobolev multiplier bound alone does not supply it.

7 Positive-time material regularity

We give a fixed-domain argument to address the incompatible initial temperature directly. We use the standard sectorial and analytic-semigroup theory for Dirichlet elliptic operators in [18], and the elliptic Schauder and oblique-derivative theory in [19, Chapter 6]. Choose a smooth cutoff χ, zero for $r \le 1/2$ and one near $r=1$, and put

$$F_f(r\theta) = (r + \chi(r) f(\theta))\theta.$$

For small $\|f\|_X$ this is a $C^{m,\alpha}$ diffeomorphism from B to Ω_f. Define $v_f(t) = u_f(t) \circ F_f$. With $J_f = DF_f$ and $K_f = J_f^{-1}$, the pulled-back Laplacian is

$$A_f v = a_f^{kl}\partial_{kl} v + b_f^l \partial_l v, \quad a_f^{kl} = \sum_i (K_f)_{ki}(K_f)_{li}, \quad b_f^l = \sum_{k,i} (K_f)_{ki}\partial_k (K_f)_{li}. \tag{21}$$

The inverse here is a pointwise matrix inverse at a fixed reference point. This fixed-coordinate formula allows Banach-algebra calculus to give analytic coefficient maps

$$a_f \in C^{m-1,\alpha}(\overline{B}), \qquad b_f \in C^{m-2,\alpha}(\overline{B}).$$

Lemma 7.1. *The maps $\Phi_T : X \to Y$ are continuously Fréchet differentiable on a common neighborhood of zero, with the derivative jointly continuous in (T, f) for $T \in [T_0, T_1]$.*

Proof. Fix finite p sufficiently large that $W^{2,p}(B)$ embeds in $C^{1,\beta}(\overline{B})$ for some $\beta > \alpha$. On $L^p(B)$, all A_f have the same Dirichlet domain $D_p = W^{2,p}(B) \cap W_0^{1,p}(B)$ and $f \mapsto A_f$ is C^1 into $\mathcal{L}(D_p, L^p)$. Uniform ellipticity holds in a small X ball. A fixed shifted sectorial contour for the ball Dirichlet Laplacian remains a resolvent contour for all these operators: the usual bounds $L^p \to L^p$ of order $|z|^{-1}$ and $L^p \to D_p$ of order one make $(A_f - A_0)(z - A_0)^{-1}$ uniformly small there. A Neumann series gives a common sectorial family and equivalent graph norms. The resolvent derivative is

$$D_f (z - A_f)^{-1}[h] = (z - A_f)^{-1} A_f'[h] (z - A_f)^{-1}. \tag{22}$$

The Dunford formula

$$v_f(t) = \frac{1}{2\pi i} \int_\Gamma e^{tz} (z - A_f)^{-1} 1 \, dz$$

and (22) show that, for every fixed integer $j \ge 0$, $f \mapsto \partial_t^j v_f(t)$ is C^1 into D_p, jointly continuously for positive times. The contour products have at most polynomial growth; exponential decay along its rays dominates them uniformly for $t \ge T_0/2$. Resolvent difference

identities also control the Fréchet remainders. The initial value is the constant one in $L^p(B)$; all the asserted boundary regularity is at positive times.

We detail the spatial bootstrap. For every j and positive time,

$$A_f \partial_t^j v_f = \partial_t^{j+1} v_f, \qquad \partial_t^j v_f|_{\partial B} = 0.$$

The D_p result begins with C^1 dependence in $C^{0,\alpha}$. Uniform elliptic Schauder estimates on the fixed smooth ball then upgrade it successively to $C^{k,\alpha}$, $2 \le k \le m$. The induction is simultaneous over time derivatives: to treat a fixed (k, j) only finitely many higher time derivatives at lower spatial orders are needed, all supplied by the contour argument. The available coefficient regularity in (21) suffices.

For clarity, state continuity follows by subtracting the elliptic equations and applying Schauder with the uniform solution bounds. The parameter derivative solves

$$A_f \dot{v}_f[h] = \partial_t \dot{v}_f[h] - A_f'[h] v_f, \qquad \dot{v}_f[h]|_{\partial B} = 0.$$

Subtracting this from the exact increment equation leaves a time-derivative remainder already $o(\|h\|_X)$ two spatial orders lower, a C^1 coefficient remainder times a bounded state, and a coefficient increment times an $o(1)$ state difference. Schauder makes the state remainder $o(\|h\|_X)$ in $C^{k,\alpha}$. The same difference estimates give continuity of the derivative. Thus $f \mapsto v_f(T)$ is C^1 into $C^{m,\alpha}$.

Finally

$$\Phi_T(f) = (\nu_f \circ F_f) \cdot (K_f^\top \nabla v_f(T)) \quad \text{on } \partial B.$$

The geometric factors are algebraic C^1 functions of DF_f in $C^{m-1,\alpha}$; for example the pulled normal is the normalized $K_f^\top \nu_0$. This proves the claimed $X \to Y$ regularity. It also validates the Eulerian differentiation used in Lemma 6.1. Restriction to little Hölder spaces follows by smooth approximation and the proved map and derivative continuity. □

This argument is a positive-time material-coordinate application of classical operator perturbation and Schauder theory. The regularity conclusion applies to the combined Eulerian expression. The derivative-loss discussion for an elliptic parametrization in [11] motivates checking this point; the fixed-coordinate matrix formula (21) supplies the needed cancellation directly. The heat-layer shape regularity in [13] is related general ancestry. Here the semigroup argument starts from the prescribed L^p datum and establishes regularity at positive times.

8 Hölder isomorphism and local stability

Let $\mathcal{H}f$ be harmonic extension into B and let $\Lambda f = \partial_r \mathcal{H}f|_{r=1}$ be the spherical Dirichlet–Neumann operator. Its degree-ℓ multiplier is ℓ. Put $\tau = T_0/2$. For the Eulerian heat derivative with boundary input $a(t)f$, Duhamel gives, for $T \ge T_0$,

$$\begin{aligned} w_f(T) = a(T)\mathcal{H}f + e^{(T-\tau)\Delta_D}[w_f(\tau) - a(\tau)\mathcal{H}f] \\ - \int_0^{T-\tau} e^{s\Delta_D} \mathcal{H}f \, a'(T-s)\, ds. \end{aligned} \tag{23}$$

The middle term has a fixed positive smoothing interval and its normal trace is compact from X to Y. The low-norm bound on $f \mapsto w_f(\tau)$ follows from Lemma 7.1 or the heat Poisson representation; subsequent heat smoothing supplies any fixed boundary regularity.

The integral term also gives a compact trace operator $X \to Y$. Indeed set $B_s f = \partial_\nu e^{s\Delta_D} \mathcal{H}f$. The standard positive-time Dirichlet boundary-gradient estimate (see [18]) gives

$$\|B_s f\|_\infty \le C s^{-1/2} \|f\|_\infty, \qquad 0 < s \le T_1,$$

since harmonic extension contracts the supremum norm. Rotations commute with harmonic extension, the ball heat semigroup and normal trace. Apply the same estimate to all ordered rotational derivatives of order at most $m-1$. The top-order Hölder seminorm is equivalently measured by differences under small rotations; applying the same commutation estimate to those differences yields

$$\|B_s f\|_Y \le C s^{-1/2} \|f\|_Y. \tag{24}$$

Rotational vector fields span the tangent bundle, so these are equivalent standard sphere norms. The half-power is integrable and all derivatives of a are bounded on $[\tau, T_1]$. Thus the integral trace in (23) is bounded $Y \to Y$, and its restriction $X \to Y$ is compact by the compact embedding. Small-s domination in (24) and positive-s smoothing also give operator-norm continuity in T.

It follows that

$$G_T = a(T)\Lambda + (N-1)a(T)I + K_T, \tag{25}$$

where $K_T : X \to Y$ is compact and norm-continuous in time. These operators preserve harmonic degrees. On mean-zero Hölder spaces Λ is an isomorphism by harmonic Neumann solvability and Schauder estimates; removing its finite degree-one sector leaves an isomorphism $X_{\ge 2} \to Y_{\ge 2}$. The restriction of G_T to these subspaces is therefore Fredholm of index zero. It has zero kernel by (17), since every harmonic coefficient of a kernel vector vanishes. Completeness in L^2 suffices here; density in a large Hölder norm is not required. Fredholm index zero and the bounded inverse theorem give an isomorphism

$$G_T : X_{\ge 2} \longrightarrow Y_{\ge 2}. \tag{26}$$

Continuity in T and compactness of $[T_0, T_1]$ give a common bound for the inverse norms.

Proof of Theorem 1.2. Let P project onto harmonic degrees at least two. Write $f = h + c + b\cdot\theta$ with $h \in X_{\ge 2}$. The volume and unnormalized first-moment maps are

$$V(f) = \frac{1}{N}\int_{\mathbb{S}^{N-1}} (1+f)^N, \qquad M(f) = \frac{1}{N+1}\int_{\mathbb{S}^{N-1}} \theta (1+f)^{N+1}.$$

These are smooth polynomial Banach maps. Their derivative in (c, b) at zero consists of nonzero scalar and vector diagonal blocks, with zero cross terms by spherical symmetry. The parameter-dependent finite-dimensional implicit function theorem gives smooth $c(h), b(h)$ imposing $V(f) = |B|$ and $M(f) = 0$. Since h has no degrees zero or one, their first derivatives at zero vanish. In particular

$$f(h) = h + O(\|h\|_X^2),$$

and the two norms are comparable locally. Every sufficiently small constrained f is uniquely represented this way.

Set $Q_T(h) = P\Phi_T(f(h))$. It vanishes at zero, because the ball flux is constant. Lemma 7.1 gives C^1 regularity and its derivative at zero is (26). By joint continuity of the derivatives on the compact set $[T_0, T_1]\times\{0\}$ and the uniform inverse bound, there is a common neighborhood where

$$\|Q_T(h) - G_T h\|_Y \le \eta \|h\|_X$$

with η less than half the reciprocal of that bound. Hence

$$\|h\|_X \le C\|Q_T(h)\|_Y \le C' \inf_{b\in\mathbb{R}} \|\Phi_T(f(h)) - b\|_Y.$$

The last inequality uses boundedness of P and its annihilation of constants. Norm comparability proves (3). □

The positive spectrum is the specific nondegeneracy input; the final local estimate is a standard consequence of that isomorphism and the proved C^1 material map. The observation norm is one derivative weaker than the shape norm but still strong, and the theorem requires a priori near-sphericity, with constants uniform on compact positive-time intervals. A proper moving observation cap has a different behavior, as the following corollary shows.

9 Nonuniqueness for a moving observation cap

Corollary 1.3, stated with the main results, follows from the normalized full-boundary isomorphism.

Proof. Use the smooth volume/centroid chart $f(h)$ from the preceding proof, with $h \in X_{\geq 2}$. Let $Y_U = C^{m-1,\alpha}(\overline{U})$ and define

$$\mathcal{F}(h) = (\Phi_T(f(h)) + a(T))|_{\overline{U}}\,.$$

This is a C^1 map to Y_U, vanishes at zero, and has derivative $D\mathcal{F}(0) = R_U G_T$, where R_U is restriction.

We construct a bounded right inverse for $R_U : Y_{\geq 2} \to Y_U$. Take a bounded Hölder extension $E : Y_U \to Y$, using the smooth cap boundary and local charts. Choose a nonzero nonnegative $\beta \in C_c^\infty(V)$, where V is an unobserved open cap disjoint from $\overline{U}$. Put $v(\theta) = (1, \theta_1, \dots, \theta_N)^\top$ and

$$M = \int_{\mathbb{S}^{N-1}} \beta(\theta) v(\theta) v(\theta)^\top \, d\theta.$$

This matrix is positive definite: a nonzero affine function cannot vanish on a nonempty open portion of the sphere. The corrected extension

$$Jg = Eg - \beta v^\top M^{-1} \int_{\mathbb{S}^{N-1}} v(Eg)$$

has zero constant and degree-one moments and agrees with g on $\overline{U}$. Hence $J : Y_U \to Y_{\geq 2}$ is a bounded right inverse for restriction. By (26), $\mathcal{R} = G_T^{-1} J$ is a bounded right inverse for $D\mathcal{F}(0)$.

The kernel $\mathcal{K}$ of this derivative is infinite-dimensional: smooth functions supported in V with zero constant and coordinate moments form an infinite-dimensional subspace of $Y_{\geq 2}$, and their G_T^{-1} images lie in $\mathcal{K}$. The projection $\mathcal{R}D\mathcal{F}(0)$ splits $X_{\geq 2}$ as $\mathcal{K} \oplus \operatorname{range} \mathcal{R}$. In the coordinates $h = k + \mathcal{R}z$, the z derivative of $\mathcal{F}$ at zero is the identity on Y_U. The parameter-dependent Banach implicit function theorem therefore gives all nearby zeros in the form

$$h = k + \mathcal{R}z(k), \qquad z(0) = 0, \quad Dz(0) = 0,$$

for k near zero in $\mathcal{K}$. The kernel projection of h is k, so a nonzero k gives $h \neq 0$, and consequently $f(h) \neq 0$. An exactly volume- and centroid-normalized ball is the centered unit ball. Thus these nonzero shapes are nonballs and satisfy (4). The chart transports the split level set to the claimed local submanifold of normalized shapes. □

The corollary concerns $C^{m,\alpha}$ boundaries and observations on a moving cap at a single positive time. Its argument is the standard submersion deduction from the full-boundary isomorphism. Classical partially overdetermined elliptic counterexamples, such as those in [16], provide context; they are not used to infer this heat statement.

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