Single-time boundary heat-flux rigidity for smooth domains

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Abstract

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded connected domain with smooth boundary. Consider the Dirichlet heat evolution with initial temperature identically one. We prove that if its signed outward normal derivative is spatially constant on the entire boundary at one positive time, then Ω is a ball. No connectedness assumption is imposed on the boundary. The argument uses the classical method of moving planes, with an L^2 comparison at the incompatible initial corner, a component reconstruction lemma, and an explicit parabolic corner barrier. In particular, the result gives a negative answer in the smooth connected Euclidean class to the finite-observation nonball scenario in Cavallina–Pinamonti's Problem 3 and Conjecture 7.4. The initial datum is essential: the statement does not apply to tailored nonuniform positive temperatures on a fixed shell.

1 Statement and context

Let $\Omega \subset \mathbb{R}^N$ be bounded and connected with C^∞ boundary, where $N \geq 2$. Write ν for its outward unit normal and Δ_D for the Dirichlet Laplacian. We study

$$u_t = \Delta u \quad \text{in } \Omega \times (0,\infty), \qquad u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \qquad u(\cdot,t) \longrightarrow 1 \quad \text{in } L^2(\Omega) \quad (t \downarrow 0). \tag{1}$$

Thus $u(\cdot,t) = e^{t\Delta_D}1$. The initial condition is understood in L^2; it is not compatible with the lateral boundary value at time zero. The signed normal derivative is $\partial_\nu u = \nabla u \cdot \nu$.

Theorem 1.1. *Suppose that, for some $T > 0$ and $b \in \mathbb{R}$,*

$$\partial_\nu u(x,T) = b \qquad \text{for every } x \in \partial\Omega. \tag{2}$$

Then Ω is a ball. Conversely, on a ball condition (2) holds at every positive time.

The same scalar b is required on every boundary component. The theorem does not assume convexity, simple connectivity or connected boundary. It concerns the complete spatial boundary flux at a single instant, rather than a finite collection of point measurements.

Cavallina–Pinamonti [5] consider precisely the initial datum and flux in (1)–(2). Their Theorems I–II establish ball rigidity from infinite sequences of observation times in several time-accumulation and boundary-regularity regimes. Their Problem 3 asks about finitely many times, and Conjecture 7.4 proposes nonball examples for finite prescribed observation counts. Theorem 1.1 rules out even one such time in the bounded connected C^∞ Euclidean class.

The proof belongs to the classical moving-plane tradition of Serrin [10] and the parabolic symmetry literature of Alessandrini–Garofalo [1], Alessandrini [2, 3], and Garofalo–Sartori [7]. In particular, reflected parabolic comparison, tangency Hopf contradictions and orthogonal-contact arguments already occur in that literature. The boundary-point principle used here is classical; see Friedman [6] and Li–Nirenberg [8, Section 3]. Coordinate-times-exponential

parabolic corner constructions also have antecedents; compare Attouchi–Souplet [4, Appendix 2]. Our proof gives directly the corner estimate needed for (1); it does not apply an elliptic corner lemma to a frozen heat profile. The role of the one-time condition is explicit: it enters only in the final tangency or corner contradiction, whereas comparison uses the evolution from its uniform initial datum.

2 Reflected comparison and zero components

Positive-time parabolic and elliptic regularity give $u \in C^\infty(\overline{\Omega} \times [\delta, M])$ for every $0 < \delta < M < \infty$. The semigroup maximum principle and connectedness give

$$0 < u(x,t) \le 1 \qquad (x \in \Omega,\ t > 0). \tag{3}$$

For example, smoothing follows by writing $A = -\Delta_D$ and using $e^{-tA}1 \in D(A^m)$ for every m, then iterating elliptic regularity and Sobolev embedding. The analytic semigroup and the same smoothing show that $t \mapsto u(x,t)$ is real analytic for each fixed interior point x and $t > 0$.

Fix a unit vector e, a plane $\Pi_\lambda = \{x : x \cdot e = \lambda\}$, and reflection

$$R_\lambda x = x - 2(x \cdot e - \lambda)e.$$

Put $\Sigma_\lambda = \Omega \cap \{x \cdot e > \lambda\}$. Whenever $R_\lambda \Sigma_\lambda \subset \Omega$, define

$$w(x,t) = u(R_\lambda x, t) - u(x,t), \qquad x \in \Sigma_\lambda. \tag{4}$$

The cap may have more than one component.

Lemma 2.1. *For every admissible plane, $w \ge 0$ on $\Sigma_\lambda \times (0,\infty)$. On each connected component C of the cap, either $w > 0$ for all positive times, or $w \equiv 0$ for all positive times.*

Proof. The difference is caloric. On the plane face it vanishes; on the original boundary face it has nonnegative values by (3). These are legitimate continuous positive-time boundary values since reflection of the closed cap lies in $\overline{\Omega}$.

We justify comparison without making a time-zero boundary continuity assumption. On any compact positive-time interval $[s,t]$, w is uniformly Lipschitz on the closed cap, by smooth spatial extension of u from $\overline{\Omega}$. Its negative part $v = w_- := \max\{-w, 0\}$ vanishes continuously on the cap boundary. In particular $v(x,\tau) \le L \operatorname{dist}(x, \partial\Sigma_\lambda)$. Cutting v off in a boundary layer of width ε produces functions converging to v in H^1. Indeed the extra gradient is bounded on this shrinking layer, whose volume tends to zero; the cap boundary lies in a smooth hypersurface together with a plane and has zero N-dimensional measure. Interior smoothing of the cutoffs shows $v \in H^1_0(\Sigma_\lambda)$. This argument uses the critical cap through the bounds proved above.

Testing the heat equation with the negative part gives the standard energy inequality

$$\|w_-(\cdot,t)\|^2_{L^2(\Sigma_\lambda)} + 2\int_s^t \|\nabla w_-(\cdot,\tau)\|^2_{L^2(\Sigma_\lambda)}\, d\tau \le \|w_-(\cdot,s)\|^2_{L^2(\Sigma_\lambda)}. \tag{5}$$

Reflection preserves volume and maps the cap into Ω. Hence

$$\|w(\cdot,s)\|_{L^2(\Sigma_\lambda)} \le 2\|u(\cdot,s) - 1\|_{L^2(\Omega)} \longrightarrow 0.$$

Letting $s \downarrow 0$ in (5) proves nonnegativity.

If w has an interior zero in C at a positive time, the strong parabolic maximum principle makes it zero throughout C at every earlier positive time. For each fixed $x \in C$, analyticity of $w(x,\cdot)$ extends this interval of zeros to all positive times. Otherwise it is strictly positive everywhere on that component. Analyticity is used on an interval of zeros, not at a single zero. □

Lemma 2.2. *If $w \equiv 0$ on one cap component C for all positive times, then Ω is symmetric about Π_λ.*

Proof. Abbreviate $R = R_\lambda$, $\Pi = \Pi_\lambda$, and set

$$D = C \cup R(C) \cup (\Omega \cap \Pi \cap \overline{C}).$$

This is a nonempty relatively open subset of Ω. At a plane point in the third set, choose a small ball lying inside Ω. Its positive half is connected and meets C, so belongs to C; its negative half then belongs to $R(C)$.

The set D is also relatively closed. A positive-side limit in Ω belongs to C, since a component is closed relative to the cap. If $y_j = Rx_j \in R(C)$ tends to a negative-side point $y \in \Omega$, then $x = Ry \in \overline{C}$. If $x \in \partial\Omega$, the identity $u(x_j, t) = u(y_j, t)$ and continuity would give $u(y, t) = 0$, contradicting (3). Thus $x \in \Omega$, hence $x \in C$, and $y \in R(C)$. Limits on the plane lie in the third set. Connectedness now gives $D = \Omega$, which proves reflection symmetry. □

3 A parabolic corner estimate

The following local estimate isolates the only corner conclusion needed in the proof. Denote by e_1 the first coordinate vector.

Lemma 3.1. *Let $n \perp e_1$ be a unit vector, let $c = Rn$ with $R > 0$, and put $B^+ = B_R(c) \cap \{x_1 > 0\}$. Suppose w is caloric, positive in $B^+ \times (0, \infty)$, nonnegative on its lateral boundary, and vanishes on the plane face $x_1 = 0$. Suppose it is smooth up to that face near the closed inner half-sphere $|x - c| = R/2$. Then, for every $T > 0$,*

$$\liminf_{\rho \downarrow 0} \frac{w(\rho(e_1 + n), T)}{\rho^2} > 0. \tag{6}$$

Here continuity up to the outer spherical boundary is assumed wherever the comparison below uses its values.

Proof. Choose

$$\beta > \frac{3R^2}{4T}, \qquad s = T - \frac{3R^2}{4\beta} > 0, \qquad \alpha > \frac{\beta + 2(N+2)}{R^2}.$$

With $r = |x - c|$ and $q = r^2 + \beta(T - t)$ consider

$$E = \{(x, t) : x_1 > 0,\ r > R/2,\ q < R^2,\ s < t \le T\}.$$

It is contained in $B^+ \times (s, T]$ and is empty at $t = s$. The positive function

$$h(x, t) = x_1 \big(e^{-\alpha q} - e^{-\alpha R^2}\big) \tag{7}$$

satisfies

$$(\partial_t - \Delta)h = \alpha x_1 e^{-\alpha q}\big[\beta + 2(N+2) - 4\alpha r^2\big] < 0 \quad \text{in } E. \tag{8}$$

In this calculation $c_1 = 0$; the cross derivatives of $x_1 e^{-\alpha q}$ account for the $N + 2$ term. The chosen α and $r > R/2$ give the strict sign.

On the plane and the moving face $q = R^2$, $h = 0$. On the remaining face $r = R/2$, the quotient w/x_1 extends continuously to its plane rim, with value $\partial_1 w$. The ordinary parabolic Hopf principle on that flat face gives $\partial_1 w > 0$ at each rim point and every $s \le t \le T$. The

relevant earlier half-cylinders lie inside B^+ near the rim. Positivity away from the rim and compactness therefore give

$$\inf_{\substack{x_1\geq 0,\ |x-c|=R/2\\ s\leq t\leq T}} \frac{w(x,t)}{x_1} > 0,$$

where the quotient has its continuous rim interpretation. Since h/x_1 is bounded there, a fixed $\varepsilon > 0$ gives $w \geq \varepsilon h$ on that face.

Comparison yields $w \geq \varepsilon h$ throughout E. To see that the moving boundary causes no extra hypothesis, apply the minimum argument to $w - \varepsilon h + \delta(t-s)$ on the closure through any time $t_0 \leq T$. It is nonnegative on each lateral face and at the birth set $t = s$, which lies on $r = R/2$. A negative minimum at an interior spatial point would have nonpositive backward time derivative and nonnegative Laplacian, contradicting its strictly positive heat operator. Let $\delta \downarrow 0$.

At $t = T$, the point $x = \rho(e_1 + n)$ lies in E for small $\rho > 0$, and

$$x_1 = \rho, \qquad r^2 = R^2 - 2R\rho + 2\rho^2, \qquad h(x,T) = 2\alpha R e^{-\alpha R^2}\rho^2 + O(\rho^3).$$

This proves (6). □

4 Moving planes and proof of the theorem

Proof of Theorem 1.1. Fix e. Start the plane Π_λ near the support value $\max_{\overline{\Omega}} x\cdot e$ and move it inward, maintaining $R_\lambda\Sigma_\lambda \subset \Omega$. The classical geometric moving-plane lemma for bounded C^2 domains gives a first limiting plane. At that plane either the reflected original boundary is tangent to the original boundary away from the plane, or the plane meets $\partial\Omega$ orthogonally; see Serrin [10, Sections 1–2]. Reflection of the open cap remains in Ω at this position. This geometric lemma applies to the full cap and does not require it to be connected.

Use (4) at the limiting position. By Lemmas 2.1–2.2, either the domain is already symmetric about this plane or $w > 0$ on every cap component at all positive times. Suppose the latter.

In the off-plane tangency case, let P be the original cap boundary point whose reflection also lies on $\partial\Omega$. Write S for the linear part of R_λ. Tangency gives $\nu(R_\lambda P) = S\nu(P)$. At time T, the boundary condition and (2) give

$$\nabla w(P,T) = S\nabla u(R_\lambda P, T) - \nabla u(P,T) = bS\nu(R_\lambda P) - b\nu(P) = 0.$$

But $w(P,T) = 0$ and w is strictly positive in the adjacent cap component at preceding positive times. The ordinary parabolic Hopf principle at this smooth face gives a strictly positive inward derivative, a contradiction.

In the orthogonal-contact case translate the contact point to $Q = 0$ and rotate coordinates so the plane is $x_1 = 0$, with the cap on $x_1 > 0$. Let n be the inward unit normal at Q. Orthogonality means $n \perp e_1$. Choose a sufficiently small interior tangent ball $B_R(Rn) \subset \Omega$ whose closure meets $\partial\Omega$ only at Q. Such a ball exists by smoothness, after decreasing its radius if necessary. The ball is invariant under reflection in $x_1 = 0$; its positive half lies in one cap component. Lemma 3.1 therefore gives

$$\liminf_{\rho\downarrow 0} \rho^{-2} w(\rho(e_1+n), T) > 0. \tag{9}$$

The inner sphere and its plane rim lie strictly inside Ω, so all flat-face regularity and Hopf hypotheses in that lemma hold.

Set $f = u(\cdot, T)$. On $\partial\Omega$,

$$\nabla f = b\nu. \tag{10}$$

Since e_1 is tangent at Q, differentiate (10) along a boundary curve with velocity e_1 and dot with ν. The scalar b is constant and $(D_{e_1}\nu)\cdot\nu = 0$, hence

$$D^2 f(Q)[e_1, \nu] = 0, \qquad D^2 f(Q)[e_1, n] = 0. \tag{11}$$

Reflection fixes Q, n and reverses e_1. Consequently $w(Q,T) = 0$ and $\nabla w(Q,T) = 0$. Its Hessian on the span of e_1, n has vanishing diagonal terms by reflection, and vanishing mixed term by (11). Taylor's formula gives

$$w(\rho(e_1 + n), T) = o(\rho^2),$$

contradicting (9). Thus the first limiting plane is a symmetry plane in every direction.

Let z be the centroid of Ω. Reflection invariance forces the plane offset to equal $z \cdot e$. Translating z to zero, the domain is invariant under every origin hyperplane reflection, hence under every orthogonal transformation. Its radii form an interval by connectedness. A bounded open radial connected set is a ball or a shell, with the punctured-ball possibility excluded by smooth boundary when $N \geq 2$.

For completeness radial symmetry alone does not exclude a shell. If $\Omega = \{a < |x| < R_0\}$ with $a > 0$, the first limiting plane in direction e has offset

$$\lambda_* = (R_0 + a)/2 > 0.$$

Indeed, for $\lambda > \lambda_*$ the full cap reflects inside the outer ball and has reflected axial coordinate greater than $2\lambda - R_0 > a$, so avoids the inner ball. At λ_* the outer pole $R_0 e$ reflects to the inner pole ae, giving the first off-plane tangency. This limiting plane cannot be a symmetry plane: the centered shell has centroid zero. This contradicts what was proved for every first limiting plane. Thus Ω is a ball.

Conversely the datum, equation and boundary value on a ball are invariant under rotations about its center. Uniqueness makes u radial, and its normal derivative is constant on the boundary at every positive time. □

5 Consequences and the role of the initial temperature

Corollary 5.1. *Let Ω satisfy the hypotheses of Theorem 1.1. If there is a nonempty finite set of positive times at each of which the solution of (1) has spatially constant whole-boundary signed flux, then Ω is a ball. In particular no smooth connected nonball example can realize even one of the observations proposed in [5, Problem 3 and Conjecture 7.4].*

Proof. Apply Theorem 1.1 at any one time in the set. □

For a general initial temperature g, the reflected comparison starts from $g(R_\lambda x) - g(x)$, whose sign need not be nonnegative. Thus positivity of g cannot replace the datum $g \equiv 1$ in the argument. This explains the compatibility with the tailored-data shell results in [9]: on a fixed spherical shell, nonuniform positive preparations can produce a common signed outward flux on both boundary components at suitable positive times. They do not provide nonball examples for (1).

The theorem is an exact qualitative statement.

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