Sharp trace-degree certificates for rank-two permanent-derivative compensation

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Luminary Oddities

Abstract

We study an explicitly defined homogeneous polynomial F_4 obtained from directional derivatives of permanents on centered Hermitian positive semidefinite matrices of order six. For every nonzero such matrix K of rank at most two, we prove $F_4(K) > 1000(\operatorname{tr} K)^4$, with arbitrary complex phases and unequal nonzero eigenvalues. In a two-column Gram factorization, we determine the least nonnegative integer power of the trace that makes this polynomial a sum of squared absolute values of holomorphic polynomials: the least exponent is five. The proof combines analytic coefficient identities and symmetry reduction with finite exact rational matrix calculations. An explicit negative coefficient direction proves minimality of the exponent; it is not a negative value of F_4 at a positive semidefinite matrix. We also prove a real-matrix lower bound and an inequality along imaginary directions with fixed real part. Together these extend the functional bound to every centered positive semidefinite matrix whose real part has rank at most three.

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1 The functional, the result and its context

All matrices below have order six. A matrix is *centered* when $K\mathbf{1} = 0$. Positive semidefinite matrices are Hermitian; a bar denotes entrywise conjugation. Let $J = \mathbf{1}\mathbf{1}^*$ and $W = 6I - J$. Define

$$f_K(v) = \operatorname{per}(K + vJ), \qquad \mathcal{P}(K) = f_K(1) + f_K'(1), \qquad D(K) = \mathcal{P}(K)^2 - \operatorname{per}|K + \sqrt{2}J|^2,$$

where the absolute square is entrywise. The directional derivative in the fixed direction W defines

$$E_j(K) = [x^j]\, \partial_v D(xK + vW)|_{v=0}\,, \qquad F_4(K) = E_4(K) + \frac{\operatorname{tr} K}{2} E_3(K). \tag{1}$$

In particular vW is not multiplied by x. The functional F_4 is homogeneous of degree four under real scaling. It combines two homogeneous coefficients of the directional derivative: the cubic coefficient is multiplied by half the trace so that both terms have degree four. The question addressed here is a lower bound for this particular polynomial on a specified PSD rank domain. Positivity of a complete permanent expression would not by itself give the sign of one of its homogeneous derivative coefficients.

For a Hermitian bihomogeneous polynomial $h(z, \bar{z})$, a Hermitian sum of squares means a finite expression $\sum_j |p_j(z)|^2$ with holomorphic homogeneous polynomials p_j. We write $h \in \mathrm{HSOS}$.

Theorem 1.1 (Unrestricted complex rank two)**.** *For every nonzero centered Hermitian positive semidefinite matrix K with* $\operatorname{rank} K \le 2$ *and* $r = \operatorname{tr} K$*,*

$$F_4(K) > 1000 r^4.$$

No condition on the phases, zero coordinates or the two positive eigenvalues is imposed. At $K = 0$ both sides vanish.

Here is the stronger coefficient statement and its sharp degree. Let H be the real 6×5 integer Helmert matrix, with columns

$$H_{ij} = \begin{cases} 1 & i \le j, \\ -(j+1) & i = j+1, \\ 0 & i > j+1, \end{cases} \qquad 0 \le j \le 4,$$

using row indices $0 \le i \le 5$. Thus

$$H^T H = \operatorname{diag}(2, 6, 12, 20, 30), \qquad \mathbf{1}^T H = 0. \tag{2}$$

For $Y = (u, v) \in \mathbb{C}^{5\times 2}$ put

$$K = HYY^* H^T, \quad h(Y, \bar{Y}) = F_4(K), \quad r(Y) = \sum_{i=1}^{5} d_i(|u_i|^2 + |v_i|^2), \quad d = (2, 6, 12, 20, 30).$$

Every centered PSD matrix of rank at most two has this form.

Theorem 1.2 (Sharp trace multiplier)**.** *The Hermitian coefficient matrix of*

$$r^5(h - 1000 r^4) \tag{3}$$

in all holomorphic monomials of degree nine is positive definite. Consequently $r^5 h \in$ HSOS. For every integer $0 \le j \le 4$, $r^j h \notin$ HSOS. Thus five is the least nonnegative integer trace exponent for this unrestricted two-column Hermitian SOS representation.

The dimension of the complete degree-nine holomorphic space is $\binom{18}{9} = 48620$. The certificate below avoids constructing its full matrix. Positive-definiteness in Theorem 1.2, rather than numerical sampling of Gram factors, proves Theorem 1.1.

Corollary 1.3 (Three-dimensional real support)**.** *Let $K = R + iS$ be nonzero, centered and positive semidefinite, with R real symmetric and S real antisymmetric. If* $\operatorname{rank} R \le 3$*, then $F_4(K) > 1000(\operatorname{tr} K)^4$. Every PSD-allowed phase and singular R is included.*

Context and proof strategy. The main result is a lower bound for a specific degree-four polynomial in a positive semidefinite matrix, together with the exact degree at which multiplying by powers of the trace makes its Hermitian coefficient matrix positive semidefinite. The coefficient obstruction is not a negative value of the polynomial at a matrix. These are different questions. A real-matrix lower bound and a reduction along imaginary directions, both proved below, extend the functional inequality to the stated real-part rank domain.

Border occupation identities for permanent polynomials are developed by Li, Zhao and Luo [6]. Here the target is the homogeneous coefficient of the additive-W derivative in (1). Its positivity and its first successful trace multiplier are established together.

The norm-multiplier hierarchy is classical. Quillen's theorem and the Catlin–D'Angelo stabilization theorem give eventual Hermitian SOS for strictly positive bihomogeneous polynomials; see [2]. Signature restrictions and positivity classes are studied in [3]. The spectral formulation and quantitative convergence of Hermitian hierarchies are treated in [4, 5]. The present certificate establishes positivity of this permanent derivative and determines the least successful trace multiplier in the stated hierarchy. Highest weights, Young symmetrizers and Schur functors are standard representation theory [7]. Exact arithmetic and symmetry compression are proof tools.

The proof begins by deriving the compensated functional from the additive permanent derivatives, then expresses it in the coefficient inner product of the two-column Gram coordinates. Column-unitary and row-permutation symmetries reduce the positive exponent-five certificate to the primitive forms checked below. A separate negative coefficient direction excludes exponent four; multiplication by the squared norm then excludes every smaller nonnegative trace exponent. Finally, the real-matrix estimate and the phase-boundary argument proved in Section 7 transfer functional positivity to real-part rank three. This does not enlarge the two-column domain of the coefficient certificate.

2 Original derivatives and the degree-four operator

We first specify the source of every coefficient. Work temporarily with 36 formal entries k_{ij}, using k_{ji} for the conjugate on the Hermitian specialization. Let a_k be the coefficient of x^k in $\operatorname{per}(J + xK)$, and let b_k be its additive-W derivative coefficient. Since $\mathcal{P}(xK) = \sum_k (7-k)a_k x^k$, its squared derivative has coefficient

$$O_j = 2\sum_{k=0}^{j} (7-k)(6-j+k)a_k b_{j-k}. \tag{4}$$

For the paired permanent, use the rescaled single-slot polynomials

$$2 + t(k_{ij} + k_{ji}) + \frac{t^2}{2} k_{ij}k_{ji}, \qquad 2W_{ij} + \frac{t}{2} W_{ij}(k_{ij} + k_{ji}) \tag{5}$$

for its value and derivative, and let p_j denote the resulting degree j derivative coefficient. Substitution $t = \sqrt{2}x$ and multiplication of the derivative by $\sqrt{2}$ recover the original absolute-square slots. Thus the actual degree-j paired derivative is $(\sqrt{2})^{j+1} p_j$. The two components of F_4 are exactly

$$F_4 = F_0 + \sqrt{2}F_1, \quad F_0 = O_4 + \frac{r}{2}(O_3 - 4p_3), \qquad F_1 = -4p_4. \tag{6}$$

The subset permanent recursion gives a short fully specified coefficient algorithm. For a column subset A, use row $i = |A|$, initialize the empty subset to value 1 and derivative 0, and for $j \notin A$ update

$$V_{A\cup\{j\}} \mathrel{+}= V_A s_{ij}, \qquad B_{A\cup\{j\}} \mathrel{+}= B_A s_{ij} + V_A \dot{s}_{ij}.$$

For ordinary slots $s = 1 + xk_{ij}$, $\dot{s} = W_{ij}$; for paired slots use (5). Truncation above degree four is exact for (6). This recurrence determines every quartic coefficient by rational arithmetic in the field $\mathbb{Q}(\sqrt{2})$. To impose centering one may use the complete 25-variable substitution

$$K = EXE^T, \qquad E = \begin{pmatrix} I_5 \\ -\mathbf{1}^T \end{pmatrix}, \qquad X \in \mathbb{C}^{5\times 5}. \tag{7}$$

Alternatively substitute the Helmert coordinates directly. Both procedures are polynomial substitutions into the displayed permanent recursion, so this directly specifies the polynomial used in the subsequent construction.

To turn a quartic monomial $\prod_{s=1}^{4} k_{i_s j_s}$ into an operator, place its coefficient in row $(j_1, \dots, j_4)$ and column $(i_1, \dots, i_4)$. Average over the 24 simultaneous slot permutations and over transpose. This preserves its pairing with $K^{\otimes 4}$. Contract the eight matrix indices with H. The result is

$$T = (T_0 + \sqrt{2}T_1)/48,$$

where T_0, T_1 are integer 625×625 matrices. The denominator 48 records the two averages; both components are determined by (6).

The column weight k records degree k in the second Gram column v and degree $4-k$ in the first column u. Assign the first $4-k$ slots to u and the last k to v. Let C_k be the zero–one matrix sending a row tuple to its two occupation multisets. The raw coefficient matrix at that weight is

$$A_{4,k} = \binom{4}{k} C_k^T T C_k. \tag{8}$$

The binomial factor counts the identical column assignments after slot symmetrization. There are at most 225 monomials in any such weight. Equations (4)–(8) specify every base coefficient by finite polynomial arithmetic, without a numerical tensor input or an external coefficient file.

3 The weighted Fischer inner product and trace multiplication

Write $z = (u_1, \dots, u_5, v_1, \dots, v_5)$ and repeat d across the two Gram columns. For a multi-index η, let $\eta! = \prod_i \eta_i!$ and $d^\eta = \prod_i d_i^{\eta_i}$. At degree D, the normalized monomials for the trace-weighted norm are

$$m_\eta(z) = s_\eta z^\eta, \qquad s_\eta = \sqrt{D!/\eta!}\, d^{\eta/2}.$$

If A is a raw Hermitian coefficient matrix, its normalized operator is $G = S^{-1} A S^{-1}$, $S = \mathrm{diag}(s_\eta)$. A test polynomial $q = \sum c_\eta z^\eta$ has normalized coefficient vector c_η / s_η. Consequently its operator form evaluates A on the dual vector

$$v_\eta = \frac{c_\eta \eta!}{D! d^\eta}. \tag{9}$$

Both divisions by the monomial normalization are required. Bare tableau coefficients or one factorial correction give a different form.

All computations are integral after using

$$e_\eta = c_\eta \eta! \prod_i (60/d_i)^{\eta_i}, \qquad v_\eta = e_\eta / (60^D D!). \tag{10}$$

The test polynomials, initially in the center chart (7), are first composed with the first five rows of H. Thus mixed row contents are transformed before the diagonal metric in (10) is applied.

For $D = 4 + m$, multiplication by r^m has exact convolution

$$A_D[\alpha + \gamma, \beta + \gamma] \mathrel{+}= \frac{m!}{\gamma!} d^\gamma A_4[\alpha, \beta], \qquad |\gamma| = m. \tag{11}$$

For a basis of test polynomials, put $(V_\gamma)_{\alpha,j} = e_{\alpha+\gamma,j}$. Up to the positive scalar $1/(48\, 60^{2D} (D!)^2)$, its form is

$$Q = \sum_{|\gamma|=m} \frac{m!}{\gamma!} d^\gamma V_\gamma^T (48 A_4) V_\gamma, \tag{12}$$

including every compatible column weight. The raw coefficient matrix of r^D is diagonal with entries $(D!/\eta!)d^\eta$. Independently form

$$N_{ij} = \sum_{|\eta|=D} e_{\eta,i} e_{\eta,j} \frac{D!}{\eta!} d^\eta. \tag{13}$$

The margin $1000r^4$ therefore subtracts exactly $48 \cdot 1000N$ from the rational component of (12). This identity fixes the normalization of all certificate matrices.

4 Complete highest-weight spaces and the critical types

Column-unitary transformations of Y leave YY^* and r fixed, so all coefficient operators commute with $U(2)$. The Cauchy decomposition of $\mathrm{Sym}^D(\mathbb{C}^5 \otimes \mathbb{C}^2)$ has column types $\lambda = (D-k, k)$, $0 \le k \le \lfloor D/2 \rfloor$, with highest-weight multiplicity space $\mathbb{S}_\lambda \mathbb{C}^5$. We describe these spaces explicitly to make the completeness test clear. For constructing the tableau bases and the row-permutation action, use the first five rows of HY as the two sets of five variables. These are center-chart coordinates. This invertible change of row coordinates does not change the column highest-weight equations. Compose back with the first five rows of H before applying the diagonal Helmert metric in (10); the tableau coefficients are not treated as Helmert coefficients before that change.

The adjoint identities below are written in Helmert coordinates; in the center chart the inner product is transported by the same invertible coordinate change. Let V_k be the polynomials of u-degree $D-k$ and v-degree k, and set $L = \sum_i u_i \partial_{v_i}$, $L^* = \sum_i v_i \partial_{u_i}$. They are adjoints for the weighted Fischer inner product because both Gram columns have the same d_i. Their commutator is $D - 2k$ on V_k. On V_{k-1},

$$\|L^* w\|^2 = \|Lw\|^2 + (D - 2k + 2)\|w\|^2.$$

Thus L^* is injective and $L : V_k \to V_{k-1}$ is onto. For $\lambda = (a, b)$, the highest-weight space has dimension

$$\binom{a+4}{4}\binom{b+4}{4} - \binom{a+5}{4}\binom{b+3}{4}. \tag{14}$$

The usual finite-dimensional $\mathfrak{sl}_2$ strings then exhaust all column types.

For a semistandard tableau with sorted top row $t_1, \dots, t_a$ and bottom row $s_1, \dots, s_b$, $t_j < s_j$ for $j \le b$, put

$$q_T = \prod_{j=1}^{b} (u_{t_j} v_{s_j} - u_{s_j} v_{t_j}) \prod_{j=b+1}^{a} u_{t_j}. \tag{15}$$

It is killed by L. Lexicographic leading monomials record the two sorted row multisets and are distinct. Enumeration gives exactly (14), so these polynomials form a complete highest-weight basis.

The subgroup S_5 permuting the first five matrix indices and fixing the sixth acts by ordinary permutations in the center chart. It commutes with the permanent functional, the trace and the weighted inner product. For each row type $\mu \vdash 5$, use its rowwise canonical Young symmetrizer

$$c_\mu = \left(\sum_{r \in \mathcal{R}_\mu} r \right) \left(\sum_{s \in \mathcal{C}_\mu} \mathrm{sgn}(s)\, s \right),$$

where the rightmost factor acts first. Its exact group-algebra identity is $c_\mu^2 = h_\mu c_\mu$, with h_μ the hook product. Its image is one Specht direction per multiplicity copy. A commuting selfadjoint

operator is the identity on that Specht factor tensor a multiplicity operator. Positivity on the image of c_μ therefore tests the entire isotypic component, even though the Young symmetrizer is not orthogonal.

A partition is a tuple of weakly decreasing positive integers. In the tables, commas and parentheses are suppressed: 61 means $(6,1)$ and 311 means $(3,1,1)$. The column partition λ and the row partition μ label the two commuting group actions.

$\lambda\backslash\mu$	5	41	32	311	221	2111	11111	dimension
61	14	39	35	36	22	12	1	720
52	16	43	43	40	28	14	1	840
53	18	59	57	66	42	28	3	1260
54	22	66	69	72	54	33	5	1470

Table 1: Primitive multiplicities. The Specht dimensions are respectively $1, 4, 5, 6, 5, 4, 1$; their weighted sums give the last column.

For completeness the character calculation uses no approximate kernel. If a permutation has cycle lengths ν, the complete characters h_j come from $\prod_{\ell\in\nu}(1-t^\ell)^{-1}$. Surjectivity of L gives the highest-weight character $h_a h_b - h_{a+1}h_{b-1}$. Taking its inner product with the exact S_5 characters gives Table 1. To construct a basis without any preselected tableau data, enumerate the semistandard tableaux in lexicographic order of the concatenated rows. Expand $c_\mu q_T$ in the ordinary monomial basis, discard zero vectors, and retain a vector precisely when it is linearly independent of those already retained. Exact rational Gaussian elimination specifies this rule unambiguously. The character dimensions above prove that the process ends at the multiplicity in Table 1. Any other complete basis of the same image gives a congruent matrix and therefore the same definiteness. The 28 dimensions sum to 938 and the largest is 72.

5 The positive certificate and propagation

Proposition 5.1 (Finite exact certificate)**.** *For the four types* $61, 52, 53, 54$ *at degrees* $7, 7, 8, 9$*, respectively, the forms* (12) *minus* $48000N$ *are positive definite on every primitive range in Table 1. The pure type* 7 *at degree seven is also positive definite.*

Proof. This is the finite exact-arithmetic part of the proof. The following procedure specifies its matrices without a stored matrix table.

1. Use the subset recurrence in Section 2 to expand F_0 and F_1 in the 36 formal matrix entries. Form T_0, T_1 by the slot symmetrization and Helmert contraction specified there, and form all $A_{4,k}$ by (8).

2. For every row of Table 1 and every partition μ of five, construct the exact highest-weight basis just described. Replace its center-chart variables by the first five rows of H; expand all coefficients as integers.

3. Form the integer dual columns by (10). Evaluate the full convolution (12), including all compatible color weights, and form N independently by (13). Subtract $48000N$ from the rational component. These operations use only finite sums and products of integers, factorials and tableau signs.

4. Evaluate the two rational endpoint matrices below. Check their symmetry and compute their characteristic polynomials and rational LDL^T decompositions without numerical eigenvalues.

The 28 resulting forms have dimensions given in Table 1; each endpoint characteristic polynomial has every coefficient positive, and every rational LDL^T pivot is positive. The total number of pivots is 938 at each endpoint. This calculation can be performed in any complete basis: changing the exact basis is an invertible congruence and preserves positive-definiteness. Thus no special choice of source tableaux is a mathematical hypothesis.

At both endpoints

$$\ell = 1414213562373095/10^{15}, \qquad u = 1414213562373096/10^{15}, \qquad \ell < \sqrt{2} < u,$$

the exact polynomial $\det(tI + G)$ of each shifted form has strictly positive coefficients. Since G is real symmetric, a nonpositive eigenvalue would give a nonnegative root, which those coefficients exclude. A second exact rational LDL computation checks positive pivots at both endpoints. Convexity gives positive-definiteness at $\sqrt{2}$.

For pure type 7, form its entire 330×330 raw coefficient matrix from $48A_{4,0}$ by (11) with $m = 3$ and one color. This is the coefficient matrix of $48r^3h$ in that color, so its corresponding norm subtraction is $48000r^7$. Subtract the explicit diagonal coefficient matrix of $48000r^7$. Both endpoint characteristic polynomials have all 331 coefficients positive, proving the required definiteness by the same argument. □

Let B_D be the normalized coefficient operator of $r^{D-4}(h - 1000r^4)$. Multiplication by r is a sum of creation-operator congruences. In coordinates $\zeta_i = \sqrt{d_i}z_i$, with normalized annihilators a_i acting by $\sqrt{\eta_i}$,

$$B_{D+1} = \frac{1}{D+1}\sum_{i=1}^{10} a_i^* B_D a_i, \qquad \sum_i a_i^* a_i = (D+1)I.$$

Equivalently, its value on a degree $D + 1$ vector is a positively weighted sum of the values of B_D on coordinate annihilations. All annihilations of a nonzero homogeneous vector cannot vanish. By the one-box branching rule, a child column type can receive only the predecessor types obtained by removing one box. Thus a child is positive definite when all its predecessors are.

Here are all steps, including the types whose sign is not needed.

degree	direct positive inputs	positive inherited types
7	$7, 61, 52$	type 43 need not be positive
8	53	$8 \leftarrow 7$, $71 \leftarrow 7, 61$, $62 \leftarrow 61, 52$
9	54	$9 \leftarrow 8$, $81 \leftarrow 8, 71$,
		$72 \leftarrow 71, 62$, $63 \leftarrow 62, 53$

At degree eight the untreated type is 44; the direct 54 certificate at degree nine handles the only child that could inherit that obstruction. The degree-nine list $9, 81, 72, 63, 54$ is exhaustive. Their highest-weight dimensions are $715, 1760, 2475, 2400, 1470$ and their column dimensions are $10, 8, 6, 4, 2$, summing to 48620. This proves positive-definiteness of (3). Its monomial evaluation vector is nonzero at every nonzero Y. Since $r > 0$, division by r^5 proves Theorem 1.1 and the positive assertion of Theorem 1.2.

6 An explicit trace-four obstruction

We now prove the lower trace-degree assertion. In the center chart use indices $1,\dots,5$ and let

$$\mathcal{A}q=\sum_{\sigma\in S_5}\operatorname{sgn}(\sigma)q(u_{\sigma(1)},\dots,v_{\sigma(5)}).$$

The permutation acts on the row indices of both Gram columns. Define three polynomials q_j by applying $\mathcal{A}$ to (15) with rows

$$\begin{array}{c|c}(1,1,1,2)&(2,2,3,4)\\(1,1,1,2)&(2,3,3,4)\\(1,1,2,2)&(3,4,4,5).\end{array}\tag{16}$$

These belong to the type 44 sign range. The explicit integer test polynomial is

$$q=-q_1+6q_2+2q_3.\tag{17}$$

Expanding the three alternating sums specifies all 1080 nonzero coefficients of this polynomial; no separate witness file is required.

After the Helmert coordinate change and separate positive dual-column reductions, the complete three-dimensional form of r^4h is a positive multiple of $A+\sqrt{2}B$, where

$$A=\begin{pmatrix}938640&2408820&-735090\\2408820&14844180&-6217680\\-735090&-6217680&2713755\end{pmatrix},\qquad B=\begin{pmatrix}-6308&-17644&5668\\-17644&-64748&23552\\5668&23552&-9086\end{pmatrix}.$$

For $w=(-1,1,2)^T$, exact multiplication gives

$$w^TAw=-110160,\qquad w^TBw=-576.\tag{18}$$

The small form is obtained directly from (12) by integer contraction; there are 663 nonzero γ contributions.

The dual-column gcds are $1296000000, 216000000, 1296000000$. Writing $L=1296000000$, undoing them gives precisely (17); its coefficient-form value is the negative expression in (18) multiplied by

$$\frac{3566440179302400000\,L^2}{48\,60^{16}(8!)^2}>0.$$

The sign requires no approximate eigenvector or radical estimate.

A Hermitian bihomogeneous polynomial is HSOS exactly when its unique raw Hermitian coefficient matrix is PSD. Hence $r^4h\notin\mathrm{HSOS}$. If $r^jh\in\mathrm{HSOS}$ for an integer $j\le 4$, multiplication by $r^{4-j}\in\mathrm{HSOS}$ would imply $r^4h\in\mathrm{HSOS}$, a contradiction. Conversely (3) and $1000r^9\in\mathrm{HSOS}$ imply $r^5h\in\mathrm{HSOS}$. This proves the sharp assertion in Theorem 1.2. The negative coefficient-test vector is a coefficient-operator direction, not an actual negative value at a PSD matrix. The lower obstruction is therefore compatible with Theorem 1.1.

7 Real matrices and interpolation along phase rays

We supply the real-matrix estimate and the imaginary-direction reduction needed for Corollary 1.3. No result about a different permanent inequality is a premise. For a centered matrix define

$$\alpha_j(K)=[x^j]\operatorname{per}(J+xK),\qquad \delta_j(K)=\partial_v\alpha_j(K+vW)|_{v=0}\,.$$

Lemma 7.1. *For every centered real $R \succeq 0$, with $r = \operatorname{tr} R$ and $\beta = \operatorname{tr}(R^2)$,*

$$F_4(R) \geq A\delta_5(R) + 14800r\delta_4(R) + C_0 r^2\beta, \quad A = 20160 - 464\sqrt{2}, \quad C_0 = 39744 - 10368\sqrt{2}.$$

The two δ quantities are nonnegative. In particular $F_4(R) > 5000r^4$ for $R \neq 0$.

Proof. Normalize $r = 1$. Write $t = \operatorname{diag} R$, $b = \operatorname{diag}(R^2)$, $e = \operatorname{diag}(R^3)$, $d_j = \sum_i t_i^j$, $\beta_j = \operatorname{tr}(R^j)$, $S_j = \sum_{i,k} R_{ik}^j$, and set

$$U = t^T b, \quad C = t^T(R \circ R \circ R)\mathbf{1}, \quad X = \operatorname{tr}(R^2(R \circ R)).$$

Take centered real Gram linear forms g_i for R and use the Fischer inner product $\|z^\eta\|^2 = \eta!$. Put $p_j = \sum_i g_i^j$ and

$$\xi_i = g_i^4 - \tfrac{1}{2}g_i^2 p_2 - \tfrac{1}{3}g_i p_3, \qquad \psi_i = g_i^3 - \tfrac{1}{2}g_i p_2.$$

Newton's identities with $p_1 = 0$ express the omitted fourth products as ξ_i plus a common polynomial and the omitted third products as $p_3/3 - \psi_i$. Differentiation along $W = 6I - J$ cancels common terms. The remaining constant-slot factors are $(6-5)! = 1$ and $(6-4)! = 2$; therefore

$$\delta_5 = 6\sum_i \|\xi_i\|^2 - \Big\|\sum_i \xi_i\Big\|^2 \geq 0,$$
$$\delta_4 = 2\left(6\sum_i \|\psi_i\|^2 - \Big\|\sum_i \psi_i\Big\|^2\right) \geq 0.$$

Fischer pairing expands a product of linear forms by all Gram matchings; this product inner product is classical [1]. It gives

$$\delta_4 = 72d_3 - 12S_3 - 72U + 6\beta_2 + 12\beta_3, \tag{19}$$
$$\begin{aligned}\delta_5 = {} & 144d_4 - 144(t^{\circ 2})^T b + 30\|b\|^2 + 6\beta_2 d_2 + 24t^T e + 24t^T Rb \\ & + 4S_3 + 36X - 96C - 24S_4 - 2\beta_2^2 - 4\beta_4.\end{aligned} \tag{20}$$

For example $\|\psi_i\|^2 = 6t_i^3 + t_i\beta_2/2 + e_i - 6t_i b_i$, $\sum_i \psi_i = p_3$ and $\|p_3\|^2 = 6S_3$. For the fourth products,

$$\begin{aligned}\sum_i \|\xi_i\|^2 = {} & 24d_4 + \|b\|^2 + 4t^T e + \beta_2 d_2 + \tfrac{2}{3}S_3 + 6X \\ & - 24(t^{\circ 2})^T b - 16C + 4t^T Rb,\end{aligned}$$
$$\Big\|\sum_i \xi_i\Big\|^2 = 24S_4 - 24\|b\|^2 + 2\beta_2^2 + 4\beta_4.$$

These identities prove (19)–(20) directly.

Let T_4 be the degree-four derivative coefficient of the entrywise absolute-square permanent, and put $P_4 = T_4/(8\sqrt{2})$. The explicit mixed-slot calculation in Appendix A gives

$$\begin{aligned}P_4 = {} & 8208d_4 - 8352(t^{\circ 2})^T b + 1848\|b\|^2 + 744\beta_2 d_2 + 960t^T e \\ & - 5568C + 1824t^T Rb + 496S_3 + 1440X - 1368S_4 - 200\beta_2^2 - 160\beta_4.\end{aligned}$$

Subtracting $58\delta_5$ cancels both signed contractions:

$$\begin{aligned}P_4 = {} & 58\delta_5 + R_{58}, \\ R_{58} = {} & -144d_4 + 108\|b\|^2 + 396\beta_2 d_2 - 432t^T e + 432t^T Rb \\ & + 264S_3 - 648X + 24S_4 - 84\beta_2^2 + 72\beta_4.\end{aligned}$$

At trace one, $0 \preceq R \preceq I$, $d_2 \le \beta_2 \le 1$ and $\|b\|^2 \le \beta_4 \le \beta_2^2 \le \beta_2$. Schur positivity gives $X, S_3 \ge 0$, and

$$S_3 \le \|R\|_{\mathrm{op}} d_2 \le \beta_2, \qquad S_4 \le d_2^2 \le \beta_2, \qquad |t^T R b| \le \|R\|_{\mathrm{op}} \sqrt{d_2} \|b\| \le \beta_2.$$

Also $t^T e \ge 0$. Discarding negative terms therefore gives $R_{58} \le (108+396+432+264+24+72)\beta_2 = 1296\beta_2$.

The ordinary derivative and the cubic part of the compensation have sum O, with $F_4 = O - 8\sqrt{2}P_4$. Direct coefficient extraction, detailed in Appendix A, gives

$$\begin{aligned} O &= 20160\delta_5 + 1067904 d_3 - 154944 S_3 - 1065600 U + 128544\beta_2 + 179904\beta_3 \\ &\quad + 5760\beta_2(6d_2 - \beta_2) \\ &= 20160\delta_5 + 14800\delta_4 + 2304 d_3 + 22656 S_3 + 39744\beta_2 + 2304\beta_3 + 5760\beta_2(6d_2 - \beta_2). \end{aligned}$$

All additional terms in the second line are nonnegative: $6d_2 \ge 1 \ge \beta_2$ and the other signs follow from PSD and Schur positivity. Combining the estimates proves the asserted reserve. Centering implies rank at most five, so $\beta_2 \ge 1/5$; $(39744 - 10368\sqrt{2})/5 > 5000$, for example using $\sqrt{2} < 71/50$. Scaling restores general trace and includes zero and singular matrices. □

Lemma 7.2. *For every centered real antisymmetric S, if $q = \|S\|_F^2$, the coefficient of x^4 in $F_4(R + ixS)$ is independent of R and is at most $30000q^2$.*

Proof. Conjugation invariance and degree imply $F_4(R + ixS) = a + bx^2 + c(S)x^4$. Set

$$B = S \circ S, \quad u = \|B\mathbf{1}\|^2, \quad v = \|B\|_F^2, \quad w = \mathrm{tr}(S^4), \quad X = \mathrm{tr}(S^2 B).$$

The original coefficient recursion gives

$$\delta_5(iS) = 30u - 24v - 2q^2 - 4w - 12X, \qquad T_4(iS)/(8\sqrt{2}) = 120u - 24v - 24q^2.$$

The ordinary quartic is $20160\delta_5(iS) - 5760q^2$, hence

$$\begin{aligned} c(S) = (604800 - 960\sqrt{2})u + (-483840 + 192\sqrt{2})v \\ + (-46080 + 192\sqrt{2})q^2 - 80640w - 241920X. \end{aligned}$$

The rational quartic

$$L = (-604800 + 1344)u + (483840 - 1920/7)v + (76080 - 1920/7)q^2 + 80640w + 241920X$$

has the explicit positive-definite Gram matrix specified in Appendix B; in particular $L \ge 0$. Since $7/5 < \sqrt{2} < 10/7$ and $u, v, q^2 \ge 0$,

$$30000q^2 - c(S) - L = 960(\sqrt{2} - 7/5)u + 192(10/7 - \sqrt{2})(v + q^2) \ge 0.$$

This proves the upper bound without assuming positivity of the generally indefinite matrix iS. □

Lemma 7.3. *Let $K = R + iS \succeq 0$ be centered, nonzero, with $r = \mathrm{tr}\, R$ and $\beta = \mathrm{tr}(R^2)$. Define*

$$M(R) = A\delta_5(R) + 14800 r \delta_4(R) + (24744 - 10368\sqrt{2})r^2\beta.$$

If $S \neq 0$, there are $0 < \kappa \le 1$ and $K_* = R + iS_* \succeq 0$ with $S = \kappa S_*$, $\mathrm{rank}\, K_* \le \mathrm{rank}\, R - 1$, and

$$F_4(K) \ge \kappa^2 F_4(K_*) + (1 - \kappa^2)M(R), \qquad M(R) > 2016 r^4.$$

Proof. If v is real and $Rv = 0$, then $v^*Kv = 0$ and PSD implies $Kv = 0$, so $Sv = 0$. Thus S acts on the support of R. Also, writing $q = \|S\|_F^2$, PSD of $K, \overline{K}$ gives $\beta - q = \operatorname{tr}(K\overline{K}) \geq 0$, while $\beta + q = \operatorname{tr}(K^2) \leq r^2$. Consequently $q \leq r^2/2$ and $q^2 \leq r^2\beta/2$. On $\operatorname{ran} R$, set $T = R^{-1/2}SR^{-1/2}$. Its imaginary Hermitian multiple has paired eigenvalues $\pm\sigma_j$. Since $I + iT \succeq 0$, $0 < \kappa = \|T\|_{\mathrm{op}} \leq 1$. Then $S_* = S/\kappa$ gives a PSD matrix K_* singular on this support, so its rank is at most $\operatorname{rank} R - 1$. No inverse is taken outside the support; trace and centering are unchanged. For $f(y) = F_4(R + i\sqrt{y}S_*) = a + by + cy^2$ one has

$$f(y) = yf(1) + (1 - y)(a - cy).$$

Lemma 7.2 and the real-matrix reserve give $a - cy \geq a - 30000\|S_*\|_F^4 \geq M(R)$ for $0 \leq y \leq 1$; if $c < 0$, the first inequality follows instead from $a - cy \geq a$. Here $R + iS_* \succeq 0$, so the same intrinsic estimate applies to S_*. Finally $\beta \geq r^2/5$ and $24744 - 10368\sqrt{2} > 10080$; the latter follows by squaring since $14664^2 - 2 \cdot 10368^2 = 42048 > 0$. Thus $M(R) > 2016r^4$. Take $y = \kappa^2$. □

Proof of Corollary 1.3. For $S = 0$, Lemma 7.1 gives the stronger bound $5000r^4$. Otherwise $\operatorname{rank} R \leq 3$ gives $\operatorname{rank} K_* \leq 2$. Theorem 1.1 applies to this endpoint with the same trace. Both terms in the interpolation bound are strictly greater than $1000r^4$, including the endpoint case $\kappa = 1$. □

The certificate degree remains a statement about the specified two-column Hermitian sum-of-squares hierarchy. The phase argument proves functional positivity on an additional domain, not a coefficient certificate there.

8 Exact arithmetic and scope

The proof combines the analytic decompositions above with finite exact integer and rational matrix calculations. The subset recursion, monomial orders, complete tableau bases, convolution and endpoint tests specify those calculations within this article. The trace-four obstruction is a coefficient direction, whereas the positive certificate controls actual matrix values. Neither numerical sampling nor an eventual stabilization theorem is used to infer positivity of the polynomial.

Assistance AI systems assisted with algebraic development, source comparison, exact-code checking and manuscript preparation. The author is responsible for the statements and attribution.

A Coefficient derivations for the real-matrix estimate

This appendix details the finite coefficient identities in Lemma 7.1. The variables used in this appendix are independent auxiliary matrix variables until their stated specialization. For general real centered symmetric A and symmetric B, put

$$b_B = B\mathbf{1}, \quad q = (A \circ A)\mathbf{1}, \quad c = (A \circ A \circ A)\mathbf{1}, \quad h = (A \circ B)\mathbf{1}, \quad s_j = \sum A_{ik}^j, \quad H = \sum A_{ik}B_{ik}.$$

The mixed coefficients needed here are

$$\begin{aligned} V_4 := [x^3y]\alpha_4(xA + yB) &= 12\sum A_{ik}^3B_{ik} - 8c^Tb_B - 12q^Th + \tfrac{4}{3}s_3\sum B_{ik} \\ &\quad + 4q^TAb_B + s_2H + 2\sum B_{ik}(A^3)_{ik}, \\ V_3 := [xy^2]\alpha_3(xA + yB) &= 12\sum A_{ik}B_{ik}^2 - 24h^Tb_B + 6H\sum B_{ik} + 6b_B^TAb_B. \end{aligned}$$

Here B need not be centered. The ordered residual-slot prefactor for α_j is $(6-j)!/j!$. Partition inclusion–exclusion has block weight $(-1)^{m-1}(m-1)!$ for a block of size m. A singleton A slot vanishes by centering. For A^3B, the surviving row partitions are a four-block U of weight -6, an AAA block and B singleton V of weight 2, and three pairings P of B with one A, of weight 1. Column partitions are identical. Their grouped contractions, before the factor $4 \cdot 2!/4! = 1/3$, are

(U,U)	$36 \sum A_{ik}^3 B_{ik}$
$(U,V)+(V,U)$	$-24 c^T b_B$
$(U,P)+(P,U)$	$-36 q^T h$
(V,V)	$4 s_3 \sum B_{ik}$
$(V,P)+(P,V)$	$12 q^T A b_B$
(P,P), same partner	$3 s_2 H$
(P,P), different partners	$6 \sum B_{ik}(A^3)_{ik}$.

For AB^2, the surviving triple block has weight 2 and the two pairings of A with B have weight -1. Their grouped contractions are $4 \sum A_{ik} B_{ik}^2$, $-8 h^T b_B$, $2H \sum B_{ik}$ and $2 b_B^T A b_B$; multiply by $3 \cdot 3!/3! = 3$.

For an arbitrary centered Hermitian K, take $A = K + \overline{K}$, $B = K \circ \overline{K}$. The degree-five homogeneous part of the paired permanent is $8\sqrt{2}[\alpha_5(A) + V_4 + V_3]$. Its slot patterns are five A, three A and one B, or one A and two B. Under $K + vW$,

$$\dot{A} = 12I - 2J, \qquad \dot{B} = 12 \operatorname{diag}(t) - A, \qquad P_4 = 2\delta_5(A) + \dot{V}_4 + \dot{V}_3.$$

Differentiation is performed before any real or null-locus specialization. At a real basepoint $A = 2K$, $B = K \circ K$, one has $2\delta_5(A) = 32\delta_5(K)$. The complete derivative table is

term	derivative at real K
$12 \sum A_{ik}^3 B_{ik}$	$2880 d_4 - 480 S_4$
$-8 c^T b_B$	$-1152 (t^{\circ 2})^T b + 192 \|b\|^2 - 768 C$
$-12 q^T h$	$-1728 (t^{\circ 2})^T b + 288 \|b\|^2 - 1152 C$
$\frac{4}{3} s_3 \sum B_{ik}$	$192 \beta_2 d_2 - 32 \beta_2^2 + 128 S_3$
$4 q^T A b_B$	$768 t^T K b + 192 \|b\|^2 - 32 \beta_2^2$
$s_2 H$	$96 S_3 + 144 \beta_2 d_2 - 24 \beta_2^2$
$2 \sum B_{ik}(A^3)_{ik}$	$192 t^T e - 32 \beta_4 + 288 X$
$12 \sum A_{ik} B_{ik}^2$	$720 d_4 - 120 S_4$
$-24 h^T b_B$	$-864 (t^{\circ 2})^T b + 144 \|b\|^2 - 576 C$
$6H \sum B_{ik}$	$216 \beta_2 d_2 - 36 \beta_2^2 + 144 S_3$
$6 b_B^T A b_B$	$288 t^T K b + 72 \|b\|^2 - 12 \beta_2^2$.

Summing these eleven rows with 32 times (20) gives the displayed formula for P_4 in Lemma 7.1, and hence its cancellation identity.

For the ordinary coefficient at a trace-one real base, direct expansion gives $\alpha_0 = 720$, $\alpha_1 = 0$, $\alpha_2 = 12\beta_2$ and $\alpha_3 = 4S_3$. More generally, for a centered real matrix,

$$\alpha_4 = 3 \sum_{i,k} R_{ik}^4 - 3 \sum_i \left(\sum_k R_{ik}^2 \right)^2 + \tfrac{1}{4} \beta_2^2 + \tfrac{1}{2} \beta_4.$$

Only partition types 4 and $2+2$ survive centering. Differentiating $\mathcal{P}(R)=\sum_j(7-j)\alpha_j(R)$ gives the ordinary quartic

$$R_4 = 20160\delta_5 + 23040S_3 + 5760\beta_2(6d_2-\beta_2).$$

The cubic compensated coefficient, obtained by the same slot product rule, is

$$E_3 = 2135808d_3 - 355968S_3 - 2131200U + 257088\beta_2 + 359808\beta_3.$$

For an explicit reconstruction of this last extraction, set $A=2R$, $B=R\circ R$, $q=(A\circ A)\mathbf{1}$ and $b_B=B\mathbf{1}$. Before differentiation the degree-four part of the paired permanent is

$$16\alpha_4(A) + 192\sum_{i,k}A_{ik}^2B_{ik} - 192q^Tb_B + 48\left(\sum_{i,k}A_{ik}^2\right)\left(\sum_{i,k}B_{ik}\right) + 192\left[\left(\sum_{i,k}B_{ik}\right)^2 - 2\|b_B\|^2 + \|B\|_F^2\right].$$

Differentiate with $\dot{A}=12I-2J$ and $\dot{B}=12\,\mathrm{diag}(t)-A$, and subtract from the ordinary cubic derivative given by (4). This yields the displayed E_3. Finally $O=R_4+E_3/2$ gives both expressions for O in Lemma 7.1. All identities can alternatively be expanded directly by the finite subset recursion of Section 2.

B An explicit rational Gram matrix for the phase bound

Use the ten variables $x_{ij}=S_{ij}$ for $1\le i<j\le 5$, in lexicographic pair order. Define $S_{ji}=-S_{ij}$, $S_{i6}=-\sum_{j\le 5}S_{ij}$ and $S_{6i}=-S_{i6}$; all diagonal entries are zero. This parametrizes every centered real skew matrix. List the 55 quadratic monomials $m_{ab}=x_ax_b$ for $0\le a\le b<10$ in lexicographic order, with the ten variables numbered from zero. The matrix G below satisfies $L=m^TGm$ for the rational polynomial L in Lemma 7.2.

To give every entry compactly, use the following ordered dictionary $D_0,\dots,D_{45}$. The alphabet `0123456789ABCDEFGHIJKLMNOPQRSTUVWXYZabcdefghij` represents indices $0,\dots,45$ in that order. Row i of the string table encodes $G_{i,i},G_{i,i+1},\dots,G_{i,54}$; reflect across the diagonal to complete the symmetric matrix. Thus this is an exact matrix specification, not a floating compressed approximation.

0–5	1291968, 861312, −861312, 0, 3429875/7, 6544133/14
6–11	975316/7, −1146080/7, −3766001/14, 3766001/14, 3198070/7, 1228879/14
12–17	−1228879/14, 10686362/7, 11324923/14, −1246036/7, −3430432/7, −4845583/14
18–23	1246036/7, 6292224/7, 95510, 1129265/21, −747106/21, 747106/21
24–29	−95510, 614651/7, −2218843/21, 1551232/21, −1551232/21, 4845583/14
30–35	3430432/7, −1129265/21, −614651/7, 2218843/21, −975316/7, −11324923/14
36–41	−6544133/14, −6292224/7, 7048084/7, −520015/14, 520015/14, 2278922/7
42–45	−1128229/21, 1128229/21, 1146080/7, −2278922/7

```
01112223334556778834576793847763994558834593844 99ABCABA
DEEFGGHH31EEIHHGG35J3KLKLM53LKLKN6KKOO39NPQR9QPS9NR9SB
DEGFGT3H5EJK3LOML1EHIHU3G5L3KVNK9KNWRQ6KK3O9XSP9SMBS9
DGGF3TT5JEKL3MOV5ELK3NVO1HHI3UU9NKRWX9KSXW63KKBSR9N9
DEEHH3YOOF33KK38MTOVWXS8TVOXWR2ZZUU3abOVNaVOM8MS8RC
DET3H8OMOTVPSXYO3F3O3K8VTOQRWaZbKNV2ZG3UaLMO8RNCR8
```

```
D3TT8MOOVTSPQ8OVOTRQPY33F3OOabZNKLaZMLK23GGCRS8M8
cde73L3fgTVhiVj3hTgL3hg3dQXi3VHLg7LHhV3eXQ333333
cd7L33gfVTg3Vh3gQdQijh3gTViV3LHh3LXeX7hHL333333
c3LV3hgXQd73g3fLgTihj3hVT3VLQXei3VhH7gLH333333
01113322345i936784i39769499aa3AB93CA39B45849A
DEUT3FGH1ETU3IHG5VVN3KK9PSOMX9SKXMB3SS6KO9P9
DU3TGFT5EVNVK3O1T3UHIU9SPMOQBSS3R9XKN9KW6K9
DTTZZ39NTPQKLR9TQPLKS1UUZZ39NKLR9LKS5JM5NB
cdT3eiVfTg3hL3gdQg3XiTV3Lh33He33hX37LV3Q3
c3Td3VgQd3gQifgTh3ViVTL3g33Xh33eH33LX7L3
DEHYO3O3F3K8LQRTOWaOMEJL8RGLNC3MR2ZUaO8
DT8OLRQOTPY33O3FOaMOJEVCRM3S8LGMaZK2G8
c73hLg3fTighVj3T3XQLVd33Vg33hL3i3H7H3
0131313243i9i96A9B8C34953aA3B949aA94
D3UTUTF13TUTUIBSSRR39PNQO9QNK9PO9K6
cTdHe33dgQhX33T3T3eiV3hL3g3X7LV3Q3
DTE3Z9QTPVSK9UNORL1UE3Z9VSK5NbBN5
c3dTigfTgV33V3Q3hiTVg33d3H3QV3L7
DTZ9XVSTPK9KSGMV5NEVbB3SN1UZ9K5
cTihgVfT33X3L3g3QVdV3g3LiT33H7
DY33O3OFCMRNS3aObVE8VMGaOE8G2
0331311AB9C83A9B384355A98454
cdTeH333T3Td3V3gQih3V3LX7L3
cTd3H33V3Qg3T3dTigV33QV3L7
D3EE9NUROV9USVO13EEBNM5J5
cTH33X3Lh3Q3gV3dVL3TTi37
DE9SKMGLBNS3M5VEJ9UO1E5
DBSNRM39KNLG5LJE9KG5E1
011223456784767945849A
DEFGH1EIHG53KK6KO9P9
DGFT5EK3O1HIU9KW6K9
DEHYOF3K8TOW2ZUaO8
DT8OOTPY3FOaZK2G8
c733fTij3Ti3H7H3
011324i9649aA94
DUTF1TUI9PO9K6
DTZ9TPK1UZ9K5
cTifT3iT33H7
DY3OFaOE8G2
0311A98454
cTH3TTi37
DE9UO1E5
D9KG5E1
012464
DF1I6
DYF2
014
D1
0
```

For clarity, the complete exact verification consists of these finite operations. Substitute the ten-variable parametrization into u, v, q, w, X and form L with rational coefficients. Expand $m^T G m$ by adding $G_{aa} m_a^2$ and $2G_{ab} m_a m_b$ for $a < b$; all coefficients agree, including every zero coefficient. The resulting polynomial has 555 nonzero coefficients. Positive-definiteness may be checked without radicals or numerical eigenvalues: $42G$ is integral. Starting with $B = 42G$ and $d = 1$, for $k = 0, \ldots, 54$ take $p = B_{kk}$ and replace all trailing entries simultaneously by

$$B_{ij} \longleftarrow \frac{pB_{ij} - B_{ik}B_{kj}}{d} \quad (i, j > k), \qquad d \longleftarrow p.$$

The divisions are exact. The 55 successive pivots are strictly positive; they are the leading principal determinants of $42G$. Sylvester's criterion therefore gives $G \succ 0$. Equivalently rational LDL^T elimination has positive pivots. These finite integer operations and the complete matrix specification above prove the required SOS bound.

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