Sharp trace-degree certificates for rank-two permanent-derivative compensation

Scott Schubr
Luminary Oddities

25 October 2026

Abstract

We prove a uniform lower bound for a compensated homogeneous permanent derivative on every centered complex positive semidefinite matrix of order six and rank at most two. The bound is $F_4(K) > 1000(\operatorname{tr} K)^4$ for nonzero K, with arbitrary complex phases, unequal spectra and singular limits. In a two-column Gram factorization, the fifth trace multiplier gives a positive-definite Hermitian sum-of-squares certificate. An explicit degree-eight receiving polynomial has strictly negative coefficient-form value $-110160 - 576\sqrt{2}$, proving that five is the least nonnegative integer trace exponent in this multiplier hierarchy. Column-unitary symmetry and row permutation symmetry reduce the positive certificate to 28 primitive forms of dimensions at most 72, with exact coefficient reconstruction and rational positive-definiteness checks. The previously established paid phase-boundary reduction then gives the same uniform bound on every feasible phase fiber whose real part has rank at most three.

1 The functional, the result and its context

All matrices below have order six. A matrix is *centered* when $K\mathbf{1} = 0$. Positive semidefinite matrices are Hermitian; a bar denotes entrywise conjugation. Let $J = \mathbf{1}\mathbf{1}^*$ and $W = 6I - J$. Define

$$f_K(v) = \operatorname{per}(K + vJ), \qquad \mathcal{P}(K) = f_K(1) + f_K'(1), \qquad D(K) = \mathcal{P}(K)^2 - \operatorname{per}|K + \sqrt{2}J|^2,$$

where the absolute square is entrywise. The actual additive derivative is

$$E_j(K) = [x^j]\, \partial_v D(xK + vW)|_{v=0}\,, \qquad F_4(K) = E_4(K) + \frac{\operatorname{tr} K}{2} E_3(K). \tag{1}$$

In particular vW is not multiplied by x. The functional F_4 is homogeneous of degree four under real scaling. It is a particular compensation term arising in a permanent-inequality approach; positivity of this term is not the entire Chollet conjecture.

For a Hermitian bihomogeneous polynomial $h(z, \bar{z})$, a Hermitian sum of squares means a finite expression $\sum_j |p_j(z)|^2$ with holomorphic homogeneous polynomials p_j. We write $h \in \mathrm{HSOS}$.

Theorem 1.1 (Unrestricted complex rank two)**.** *For every nonzero centered Hermitian positive semidefinite matrix K with* $\operatorname{rank} K \le 2$ *and* $r = \operatorname{tr} K$*,*

$$F_4(K) > 1000 r^4.$$

No condition on the phases, zero coordinates or the two positive eigenvalues is imposed. At $K = 0$ both sides vanish.

Here is the stronger coefficient statement and its sharp degree. Let H be the real 6×5 integer Helmert matrix, with columns

$$H_{ij} = \begin{cases} 1 & i \le j, \\ -(j+1) & i = j+1, \\ 0 & i > j+1, \end{cases} \qquad 0 \le j \le 4,$$

using row indices $0 \le i \le 5$. Thus

$$H^T H = \operatorname{diag}(2, 6, 12, 20, 30), \qquad \mathbf{1}^T H = 0. \tag{2}$$

For $Y = (u, v) \in \mathbb{C}^{5\times 2}$ put

$$K = HYY^* H^T, \quad h(Y, \bar{Y}) = F_4(K), \quad r(Y) = \sum_{i=1}^{5} d_i(|u_i|^2 + |v_i|^2), \quad d = (2, 6, 12, 20, 30).$$

Every centered PSD matrix of rank at most two has this form.

Theorem 1.2 (Sharp trace multiplier)**.** *The Hermitian coefficient matrix of*

$$r^5(h - 1000r^4) \tag{3}$$

in all holomorphic monomials of degree nine is positive definite. Consequently $r^5 h \in$ HSOS*. For every integer* $0 \le j \le 4$*,* $r^j h \notin$ HSOS*. Thus five is the least nonnegative integer trace exponent for this unrestricted two-column Hermitian SOS representation.*

The dimension of the complete degree-nine holomorphic space is $\binom{18}{9} = 48620$. The certificate below avoids constructing its full matrix. Positive-definiteness in Theorem 1.2, rather than numerical sampling of Gram factors, proves Theorem 1.1.

Corollary 1.3 (Three-dimensional real support)**.** *Let* $K = R + iS$ *be nonzero, centered and positive semidefinite, with* R *real symmetric and* S *real antisymmetric. If* $\operatorname{rank} R \le 3$*, then* $F_4(K) > 1000(\operatorname{tr} K)^4$*. Every PSD-allowed phase and singular* R *is included.*

Relation to previous work. The real-section and phase-boundary papers [2, 1] prove, respectively, positivity on the entire real section and an actual-function chord inequality along feasible phase rays. The latter also proves unrestricted complex rank-one compensation with constant 8000 and all real-support-two fibers with constant 5040. Those stronger constants on the earlier domains remain available. The present increment is unrestricted complex rank two, all real-support-three phase fibers, and the exact trace degree of a finite certificate.

The norm-multiplier hierarchy is classical. Quillen's theorem and the Catlin–D'Angelo stabilization theorem give eventual Hermitian SOS for strictly positive bihomogeneous polynomials; see [3]. Signature restrictions and positivity classes are studied in [4]. The spectral formulation and quantitative convergence of Hermitian hierarchies are treated in [5, 6]. The present certificate supplies both for this permanent derivative. Highestweights, Young symmetrizers and Schur functors are standard representation theory [7]. Exact arithmetic and symmetry compression are proof tools.

2 Original derivatives and the degree-four operator

We first specify the source of every coefficient. Work temporarily with 36 formal entries k_{ij}, using k_{ji} for the conjugate on the Hermitian specialization. Let a_k be the coefficient of x^k in $\mathrm{per}(J + xK)$, and let b_k be its additive-W derivative coefficient. Since $\mathcal{P}(xK) = \sum_k (7-k)a_k x^k$, its squared derivative has coefficient

$$O_j = 2\sum_{k=0}^{j} (7-k)(6-j+k)a_k b_{j-k}. \tag{4}$$

For the paired permanent, use the rescaled single-slot polynomials

$$2 + t(k_{ij} + k_{ji}) + \frac{t^2}{2} k_{ij}k_{ji}, \qquad 2W_{ij} + \frac{t}{2} W_{ij}(k_{ij} + k_{ji}) \tag{5}$$

for its value and derivative, and let p_j denote the resulting degree j derivative coefficient. Substitution $t = \sqrt{2}x$ and multiplication of the derivative by $\sqrt{2}$ recover the original absolute-square slots. Thus the actual degree-j paired derivative is $(\sqrt{2})^{j+1} p_j$. The two components of F_4 are exactly

$$F_4 = F_0 + \sqrt{2}F_1, \quad F_0 = O_4 + \frac{r}{2}(O_3 - 4p_3), \qquad F_1 = -4p_4. \tag{6}$$

The subset permanent recursion gives a short fully specified coefficient algorithm. For a column subset A, use row $i = |A|$, initialize the empty subset to value 1 and derivative 0, and for $j \notin A$ update

$$V_{A\cup\{j\}} \mathrel{+}= V_A s_{ij}, \qquad B_{A\cup\{j\}} \mathrel{+}= B_A s_{ij} + V_A \dot{s}_{ij}.$$

For ordinary slots $s = 1 + xk_{ij}$, $\dot{s} = W_{ij}$; for paired slots use (5). Truncation above degree four is exact for (6). The complete implementation is `original.py`. It is independent of the contraction constructor `contractions.py`: the two formulas agree after the complete 25-variable substitution

$$K = EXE^T, \qquad E = \begin{pmatrix} I_5 \\ -\mathbf{1}^T \end{pmatrix}, \qquad X \in \mathbb{C}^{5\times 5}. \tag{7}$$

The ordinary and radical original polynomials have respectively 66795 and 25620 terms before this substitution. Their differences from the contraction formulas vanish identically in all 25 entries of X. This is a finite exact polynomial identity check, not a rank-one polarization or finite PSD test. Specializing X to Hermitian includes every centered Hermitian matrix.

To turn a quartic monomial $\prod_{s=1}^{4} k_{i_s j_s}$ into an operator, place its coefficient in row $(j_1, \ldots, j_4)$ and column $(i_1, \ldots, i_4)$. Average over the 24 simultaneous slot permutations and over transpose. This preserves its pairing with $K^{\otimes 4}$. Contract the eight physical indices with H. The result is

$$T = (T_0 + \sqrt{2}T_1)/48,$$

where T_0, T_1 are integer 625×625 matrices. The denominator 48 records the two averages; the radical component already includes the factor -8 in the contraction convention.

For color weight k, assign the first $4-k$ slots to u and the last k to v. Let C_k be the zero–one matrix sending a row tuple to its two occupation multisets. The raw coefficient matrix at that weight is

$$A_{4,k} = \binom{4}{k} C_k^T T C_k. \tag{8}$$

The binomial factor counts the identical color assignments after slot symmetrization. There are at most 225 monomials in any such weight. Equations (4)–(8), with the supplied constructor and independent polynomial check, specify every base coefficient without a numerical tensor input.

3 The physical apolar metric and trace multiplication

Write $z = (u_1, \dots, u_5, v_1, \dots, v_5)$ and repeat d across the two colors. For a multi-index η, let $\eta! = \prod_i \eta_i!$ and $d^\eta = \prod_i d_i^{\eta_i}$. At degree D, the normalized monomials for the physical trace norm are

$$m_\eta(z) = s_\eta z^\eta, \qquad s_\eta = \sqrt{D!/\eta!}\, d^{\eta/2}.$$

If A is a raw Hermitian coefficient matrix, its normalized operator is $G = S^{-1}AS^{-1}$, $S = \mathrm{diag}(s_\eta)$. A receiving polynomial $q = \sum c_\eta z^\eta$ has normalized coefficient vector c_η/s_η. Consequently its operator form evaluates A on the dual vector

$$v_\eta = \frac{c_\eta \eta!}{D! d^\eta}. \tag{9}$$

Both divisions by the monomial normalization are required. Bare tableau coefficients or one factorial correction give a different form.

All computations are integral after using

$$e_\eta = c_\eta \eta! \prod_i (60/d_i)^{\eta_i}, \qquad v_\eta = e_\eta/(60^D D!). \tag{10}$$

The receiving polynomials, initially in the center chart (7), are first composed with the first five rows of H. Thus mixed row contents are transformed before the diagonal metric in (10) is applied.

For $D = 4 + m$, multiplication by r^m has exact convolution

$$A_D[\alpha + \gamma, \beta + \gamma] \mathrel{+}= \frac{m!}{\gamma!} d^\gamma A_4[\alpha, \beta], \qquad |\gamma| = m. \tag{11}$$

For a basis of receiving polynomials, put $(V_\gamma)_{\alpha,j} = e_{\alpha+\gamma,j}$. Up to the positive scalar $1/(48\, 60^{2D} (D!)^2)$, its form is

$$Q = \sum_{|\gamma|=m} \frac{m!}{\gamma!} d^\gamma V_\gamma^T (48 A_4) V_\gamma, \tag{12}$$

including every compatible color weight. The raw coefficient matrix of r^D is diagonal with entries $(D!/\eta!)d^\eta$. Independently form

$$N_{ij} = \sum_{|\eta|=D} e_{\eta,i} e_{\eta,j} \frac{D!}{\eta!} d^\eta. \tag{13}$$

The margin $1000r^4$ therefore subtracts exactly $48 \cdot 1000N$ from the rational component of (12). This identity fixes the normalization of all supplied matrices.

4 Complete receiving spaces and the critical types

Column-unitary transformations of Y leave YY^* and r fixed, so all coefficient operators commute with $U(2)$. The Cauchy decomposition of $\mathrm{Sym}^D(\mathbb{C}^5 \otimes \mathbb{C}^2)$ has column types $\lambda = (D-k, k)$, $0 \le k \le \lfloor D/2 \rfloor$, with receiving multiplicity space $\mathbb{S}_\lambda \mathbb{C}^5$. We describe these spaces explicitly to make the completeness test clear.

Let V_k be the polynomials of u-degree $D-k$ and v-degree k, and set $L = \sum_i u_i \partial_{v_i}$, $L^* = \sum_i v_i \partial_{u_i}$. They are adjoints for the physical Fischer metric because both colors have the same d_i. Their commutator is $D - 2k$ on V_k. On V_{k-1},

$$\|L^* w\|^2 = \|Lw\|^2 + (D - 2k + 2)\|w\|^2.$$

Thus L^* is injective and $L : V_k \to V_{k-1}$ is onto. For $\lambda = (a, b)$, the highestweight space has dimension

$$\binom{a+4}{4}\binom{b+4}{4} - \binom{a+5}{4}\binom{b+3}{4}. \tag{14}$$

The usual finite-dimensional $\mathfrak{sl}_2$ strings then exhaust all column types.

For a semistandard tableau with sorted top row $t_1, \ldots, t_a$ and bottom row $s_1, \ldots, s_b$, $t_j < s_j$ for $j \le b$, put

$$q_T = \prod_{j=1}^{b} (u_{t_j} v_{s_j} - u_{s_j} v_{t_j}) \prod_{j=b+1}^{a} u_{t_j}. \tag{15}$$

It is killed by L. Lexicographic leading monomials record the two sorted row multisets and are distinct. Enumeration gives exactly (14), so these polynomials form a complete highestweight basis.

The subgroup S_5 permuting the first five physical sites and fixing the sixth acts by ordinary permutations in the center chart. It commutes with the permanent functional, the trace and the physical metric. For each row type $\mu \vdash 5$, use its rowwise canonical Young symmetrizer

$$c_\mu = \left(\sum_{r \in \mathcal{R}_\mu} r \right) \left(\sum_{s \in \mathcal{C}_\mu} \operatorname{sgn}(s) s \right),$$

where the rightmost factor acts first. Its exact group-algebra identity is $c_\mu^2 = h_\mu c_\mu$, with h_μ the hook product. Its image is one Specht direction per multiplicity copy. A commuting selfadjoint operator is the identity on that Specht factor tensor a multiplicity operator. Positivity on the image of c_μ therefore tests the entire isotypic component, even though the Young symmetrizer is not orthogonal.

$\lambda\backslash\mu$	5	41	32	311	221	2111	11111	dimension
61	14	39	35	36	22	12	1	720
52	16	43	43	40	28	14	1	840
53	18	59	57	66	42	28	3	1260
54	22	66	69	72	54	33	5	1470

Table 1: Primitive multiplicities. The Specht dimensions are respectively $1, 4, 5, 6, 5, 4, 1$; their weighted sums give the last column.

For completeness the character calculation uses no approximate kernel. If a permutation has cycle lengths ν, the complete characters h_j come from $\prod_{\ell \in \nu} (1 - t^\ell)^{-1}$. Surjectivity of L gives the highestweight character $h_a h_b - h_{a+1} h_{b-1}$. Taking its inner product with the exact S_5 characters gives Table 1. The supplied source tableaux generate $c_\mu q_T$; the verifier checks raising annihilation and independence by coefficient elimination modulo the prime 1000003. Independence modulo that prime implies independence over $\mathbb{Q}$. The character dimensions then prove full primitive range coverage. The 28 dimensions sum to 938 and the largest is 72.

5 The positive certificate and propagation

Proposition 5.1 (Finite exact certificate)**.** *For the four types* $61, 52, 53, 54$ *at degrees* $7, 7, 8, 9$*, respectively, the forms* (12) *minus* $48000N$ *are positive definite on every primitive range in Table 1. The pure type* 7 *at degree seven is also positive definite.*

Proof. This is the computer-assisted part of the proof. The complete finite data are `certificates.json`; the source tableaux generate every basis by (15) and the explicitly generated Young element. Each record contains the two shifted integer component matrices and a positive common factor, not an eigenvalue estimate.

For exact reconstruction, divide each integer dual column by the positive gcd of its coefficients. Check divisibility of all supplied form entries by the corresponding products of gcds; this is a positive diagonal congruence. Compute a conservative entry bound from (12): for each γ, sum its positive weight times the base dimension squared, the maximum absolute base coefficient and the maximum absolute receiving coefficient squared. Check enough pairwise coprime moduli that their product exceeds twice that bound. Every native integer residue is recomputed, together with a distinct coprime redundant modulus. The target lies inside the symmetric bound, so these equalities uniquely fix every entry.

All modular dot products sum at most 225 products of residues below p, and the verifier requires $225(p-1)^2 < 2^{63}$. Reduction occurs between the two products in V^TAV. Base aggregation has its own checked integer bound using the occupation multiplicities. No floating-point arithmetic is used for the reconstruction. The norm subtraction (13) is formed independently.

At both endpoints

$$\ell = 1414213562373095/10^{15}, \qquad u = 1414213562373096/10^{15}, \qquad \ell < \sqrt{2} < u,$$

the exact polynomial $\det(tI + G)$ of each supplied shifted form has strictly positive coefficients. Since G is real symmetric, a nonpositive eigenvalue would give a nonnegative root, which those coefficients exclude. A second exact rational LDL computation checks positive pivots at both endpoints. Convexity gives positive-definiteness at $\sqrt{2}$.

For pure type 7, `pure_type.py` regenerates its entire 330×330 raw coefficient matrix from $A_{4,0}$ by (11) with $m = 3$ and one color. It subtracts the explicit diagonal coefficient matrix of $48000r^7$ and checks both endpoint characteristic polynomials. Thus no earlier private rank-one certificate is a premise of the supplied reproduction. □

Let B_D be the normalized coefficient operator of $r^{D-4}(h - 1000r^4)$. Multiplication by r is a sum of creation-operator congruences. In coordinates $\zeta_i = \sqrt{d_i}z_i$, with normalized annihilators a_i acting by $\sqrt{\eta_i}$,

$$B_{D+1} = \frac{1}{D+1}\sum_{i=1}^{10} a_i^* B_D a_i, \qquad \sum_i a_i^* a_i = (D+1)I.$$

Equivalently, its value on a degree $D+1$ vector is a positively weighted sum of the values of B_D on coordinate annihilations. All annihilations of a nonzero homogeneous vector cannot vanish. By the one-box branching rule, a child column type can receive only the predecessor types obtained by removing one box. Thus a child is positive definite when all its predecessors are.

Here are all steps, including the types whose sign is not needed.

degree	direct positive inputs	positive inherited types
7	$7, 61, 52$	type 43 need not be positive
8	53	$8 \leftarrow 7$, $71 \leftarrow 7, 61$, $62 \leftarrow 61, 52$
9	54	$9 \leftarrow 8$, $81 \leftarrow 8, 71$,
		$72 \leftarrow 71, 62$, $63 \leftarrow 62, 53$

At degree eight the untreated type is 44; the direct 54 certificate at degree nine pays the only child that could inherit that obstruction. The degree-nine list $9, 81, 72, 63, 54$ is exhaustive. Their highestweight dimensions are $715, 1760, 2475, 2400, 1470$ and their column dimensions are $10, 8, 6, 4, 2$,

summing to 48620. This proves positive-definiteness of (3). Its monomial evaluation vector is nonzero at every nonzero Y. Since $r > 0$, division by r^5 proves Theorem 1.1 and the positive assertion of Theorem 1.2.

6 An explicit trace-four obstruction

We now prove the lower trace-degree assertion. In the center chart use indices $1, \dots, 5$ and let

$$\mathcal{A}q = \sum_{\sigma \in S_5} \operatorname{sgn}(\sigma) q(u_{\sigma(1)}, \dots, v_{\sigma(5)}).$$

The permutation acts on the row indices of both colors. Define three polynomials q_j by applying $\mathcal{A}$ to (15) with rows

$$\begin{array}{c|c} (1,1,1,2) & (2,2,3,4) \\ (1,1,1,2) & (2,3,3,4) \\ (1,1,2,2) & (3,4,4,5). \end{array} \tag{16}$$

These belong to the type 44 sign range. The explicit integer receiving polynomial is

$$q = -q_1 + 6q_2 + 2q_3. \tag{17}$$

The expanded 1080 coefficients and their equality to (17) are included in `witness.json` and checked by `trace_four.py`.

After the physical transformation and separate positive dual-column reductions, the complete three-dimensional form of $r^4 h$ is a positive multiple of $A + \sqrt{2}B$, where

$$A = \begin{pmatrix} 938640 & 2408820 & -735090 \\ 2408820 & 14844180 & -6217680 \\ -735090 & -6217680 & 2713755 \end{pmatrix}, \qquad B = \begin{pmatrix} -6308 & -17644 & 5668 \\ -17644 & -64748 & 23552 \\ 5668 & 23552 & -9086 \end{pmatrix}.$$

For $w = (-1, 1, 2)^T$, exact multiplication gives

$$w^T A w = -110160, \qquad w^T B w = -576. \tag{18}$$

The direct check of this small form uses arbitrary-integer FLINT contractions, independently of the modular reconstruction above, with 663 nonzero γ items. The dual-column gcds are $1296000000, 216000000, 1296000000$. Writing $L = 1296000000$, undoing them gives precisely (17); its physical value is the negative expression in (18) multiplied by

$$\frac{3566440179302400000\, L^2}{48\, 60^{16} (8!)^2} > 0.$$

The sign requires no approximate eigenvector or radical estimate.

A Hermitian bihomogeneous polynomial is HSOS exactly when its unique raw Hermitian coefficient matrix is PSD. Hence $r^4 h \notin \mathrm{HSOS}$. If $r^j h \in \mathrm{HSOS}$ for an integer $j \le 4$, multiplication by $r^{4-j} \in \mathrm{HSOS}$ would imply $r^4 h \in \mathrm{HSOS}$, a contradiction. Conversely (3) and $1000 r^9 \in \mathrm{HSOS}$ imply $r^5 h \in \mathrm{HSOS}$. This proves the sharp assertion in Theorem 1.2. The negative receiving vector is a coefficient-operator direction, not an actual negative value at a PSD matrix. The lower obstruction is therefore compatible with Theorem 1.1.

7 Propagation to full phases and the remaining boundary

We state the exact predecessor theorem used here [1]. For nonzero centered $K = R + iS \succeq 0$, $S \neq 0$, there are $0 < \kappa \leq 1$ and a saturated phase endpoint $K_* = R + iS_* \succeq 0$ with $S = \kappa S_*$ and

$$\operatorname{rank} K_* \leq \operatorname{rank} R - 1 \leq 4.$$

Writing $y = \kappa^2$ and $r = \operatorname{tr} R$, its paid chord is

$$F_4(K) \geq yF_4(K_*) + (1-y)M(R), \qquad M(R) > 2016r^4, \tag{19}$$

where, with $\alpha_j(R) = [x^j] \operatorname{per}(J + xR)$ and $\delta_j(R) = \partial_v \alpha_j(R + vW)|_{v=0}$,

$$M(R) = (20160 - 464\sqrt{2})\delta_5(R) + 14800r\delta_4(R) + (24744 - 10368\sqrt{2})r^2 \operatorname{tr} R^2.$$

This is the actual-function chord, rather than a bound on an isolated quadratic phase coefficient. Its proof retains the real-section reserve and the quartic phase allowance; it includes singular R on its support.

Proof of Corollary 1.3. If $S = 0$, the established real-section bound is $F_4(K) > 5000r^4$ [2]. Otherwise $\operatorname{rank} R \leq 3$ gives $\operatorname{rank} K_* \leq 2$. Theorem 1.1 applies to that endpoint, with unchanged trace. For $0 < y < 1$, both bounds on the right of (19) are strictly larger than $1000r^4$; at $y = 1$ strictness is supplied by the endpoint bound itself. No inverse of a singular R outside its support is used. □

The new domains include arbitrary phases and ill-conditioned Gram factors; no eigenvalue separation is a hypothesis. The generic real-support-four and -five cases have saturated endpoints of complex rank three and four, respectively. The minimal trace exponent concerns this particular unrestricted holomorphic SOS hierarchy, not arbitrary denominators or every possible positivity certificate. The lower constant 1000 is sufficient. The full Chollet conjecture, and the other compensation obligations in that program, remain separate.

8 Reproduction and exact-proof scope

The accompanying files are a complete generative supplement. Run `python verify.py` after installing the pinned numerical-array and FLINT dependencies in `requirements.txt`. The array calculations in acceptance use exact native integers with checked nonoverflow bounds. The CAS calculations use rational or arbitrary integer arithmetic. No floating eigenvalue is an acceptance criterion. The program reconstructs the original identity, generates the base operator, checks the full pure type 7, all 28 primitive bases/forms, both endpoint algorithms and the expanded negative witness. It writes `verification.json`; normal and optimized Python runs must give identical mathematical output. The README provides build instructions and the exact member manifest.

This is a computer-assisted proof with fully specified finite data and algorithms, rather than formal proof-assistant verification. The classical representation-theoretic decomposition and the published phase-boundary theorem are stated explicitly above; numerical experiments are not their premises.

Assistance. AI systems assisted with algebraic development, source comparison, exact-code checking and manuscript preparation. The author is responsible for the statements and attribution.

References

[1] S. Schubr, *Phase-boundary reduction and low-rank compensation for permanent derivatives*, Luminary Research Archive, 2026. Published predecessor.

[2] S. Schubr, *Real-section permanent-derivative compensation and quadratic phase control*, Luminary Research Archive, 2026. Published predecessor.

[3] A. Drouot and M. Zworski, *A quantitative version of Catlin–D'Angelo–Quillen theorem*, arXiv:1205.3248v2, 2012.

[4] J. Halfpap and J. Lebl, *Signature pairs of positive polynomials*, arXiv:1211.0997v3, 2013.

[5] B. Lovitz and N. Johnston, *A hierarchy of eigencomputations for polynomial optimization on the sphere*, Mathematical Programming **219** (2026), 113–156. doi:10.1007/s10107-025-02251-y.

[6] A. T. Blomenhofer and M. Laurent, *Moment-sos and spectral hierarchies for polynomial optimization on the sphere and quantum de Finetti theorems*, arXiv:2412.13191v2.

[7] W. Fulton and J. Harris, *Representation Theory: A First Course*, Graduate Texts in Mathematics 129, Springer, 1991.