Sharp Quadratic Smoothing for Truncated Simplex Sums

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Abstract

We prove joint log-concavity of the raw lattice-simplex sum $\sum_{i+j+k\le r}(A-i)_+^2(B-j)_+^2(C-k)_+^2$ for every integer radius $r \ge 0$ and arbitrary positive real endpoints. The proof combines exhaustive cell coverage with exact positive-coefficient certificates over unbounded parameter orthants. Independent real exponents can all be at least two, and this coordinatewise threshold is sharp. In every dimension we also prove the corresponding theorem with up to two unrestricted occupancies and arbitrarily many binary matroid companions, including activities in $[0,1]$. This uses degree-elevated homogenization, the classical complete-log-concavity criterion and compensated composition.

1 Statements and previous work

For $r \in \mathbb{N}_0$, $d \ge 1$, $p_i > 0$ and $A \in \mathbb{R}^d$ define

$$T_{r,p}(A) = \sum_{\substack{n\in\mathbb{N}_0^d \\ |n|\le r}} \prod_{i=1}^{d} (A_i - n_i)_+^{p_i}, \qquad x_+ = \max(x, 0). \tag{1}$$

The radius is integer, whereas endpoints and exponents are real. Occupancy vectors have unit weights: no factorial or multinomial normalization is implicit.

Theorem 1.1 (Three unrestricted coordinates)**.** *For $d = 3$ and every $r \in \mathbb{N}_0$, $T_{r,p}$ is jointly log-concave on $\mathbb{R}^3$ whenever $p_1, p_2, p_3 \ge 2$. Conversely, log-concavity for every radius for a fixed positive exponent vector requires each $p_i \ge 2$.*

The positive support is the positive orthant; zero extension is included. The quadratic endpoint is proved by exact computer-assisted polynomial identities, not numerical sampling or a parameter cutoff.

For $[m] = \{1, \dots, m\}$ define

$$U_{r,p}(x, A) = \sum_{\substack{j\in\mathbb{N}_0,\ S\subseteq[m] \\ j+|S|\le r}} (x-j)_+^{p_0} \prod_{i=1}^{m} (A_i - \mathbf{1}_{i\in S})_+^{p_i}. \tag{2}$$

Theorem 1.2 (One unrestricted occupancy)**.** *For every $m, r \in \mathbb{N}_0$ and independent real exponents $p_i \ge 2$ $(0 \le i \le m)$, $U_{r,p}$ is jointly log-concave, including zero extension. Hence the full simplex sum is log-concave on the convex cylinder $x > 0$, $0 < A_i \le 2$, with one unrestricted endpoint, in every dimension.*

On this cylinder all excluded companion occupancies vanish, so the fixed binary-companion support and full-simplex sums agree.

More generally let $\mathcal{M}$ be a matroid on $[m]$, let $\lambda_i \in [0,1]$, and for $s \in \{1,2\}$ put

$$W_{r,p}^{\mathcal{M}}(x, A; \lambda) = \sum_{\substack{n \in \mathbb{N}_0^s,\ S \in \mathcal{I}(\mathcal{M}) \\ |n|+|S| \le r}} \lambda^S \prod_{j=1}^{s} (x_j - n_j)_+^{p_j} \prod_{i=1}^{m} (A_i - \mathbf{1}_{i \in S})_+^{q_i}. \tag{3}$$

Here $\lambda^S = \prod_{i \in S} \lambda_i$, including the empty product one.

Theorem 1.3 (Two unrestricted occupancies and matroid companions)**.** *For every $r, m \in \mathbb{N}_0$, $s \in \{1,2\}$, matroid $\mathcal{M}$ and activities $\lambda_i \in [0,1]$, (3) is jointly log-concave when all independent real exponents p_j, q_i are at least two. Zero extension is included. Thus the full simplex kernel in every dimension is log-concave on any convex cylinder with at most two unrestricted endpoints and all other endpoints in $(0,2]$.*

The cylinder conclusion uses the free matroid and activities one. The concavity statement fixes the unrestricted coordinate.

The earlier two-dimensional theorem and real-power lift are in the unpublished companion [6]. Appendix A gives the required exact baseline; it supplies the inherited two-dimensional input. Another companion [7] gives dimension-dependent high-power smoothing for decreasing arrays and a binary matroid transform, which do not imply the endpoint above. The classical complete-log-concavity criterion and matroid instance are due to Anari, Liu, Oveis Gharan and Vinzant [1, Theorems 3.2 and 4.1]; see also the Lorentzian framework of Brändén and Huh [2]. Our extra work in Theorems 1.2 and 1.3 is joint endpoint–activity concavity for raw shifted-square rows, including the mixed Hessian blocks. Degree elevation and the classical criterion are credited tools. Prékopa's marginal theorem [3] supplies the real-power lift.

For the unrestricted simplex sum at integer endpoints and integer powers there is important earlier lattice coverage. The total-unimodularity framework of Leake and Mohammadi Yekta [4] applies to the auxiliary variables $n_i, u_{ij} \ge 0$, $\sum_i n_i \le r$, $n_i + u_{ij} \le A_i - 1$, $1 \le j \le p_i$. The base occupancy matrix consists of a global all-ones row and repeated unit rows and is totally unimodular; adding unit auxiliary and slack columns preserves that property and gives full row rank. All fibers are finite. Counting auxiliary variables gives exactly (1) at integer endpoints. This supplies prior integral-midpoint coverage in every dimension. Speyer's framework [5] provides related lattice ancestry and the two-dimensional difference-bound lift explained in [6]; we do not attribute an arbitrary-dimensional difference-bound model to that result.

2 Exact moments and the deep region

Let

$$F_m(t) = \sum_{u=0}^{m} (t+u)^2 = (m+1)t^2 + m(m+1)t + \frac{m(m+1)(2m+1)}{6}.$$

For $m \ge 1$, writing $F_m = at^2 + bt + c$ gives $b^2 - 2ac = m(m-1)(m+1)^2/3 \ge 0$, hence $F_m F_m'' - (F_m')^2 \le 0$ for $t \ge 0$. The case $F_0 = t^2$ is immediate. Products of row sums are jointly log-concave. For positive integer h define

$$\mathcal{T}(h; t, u, v) = \sum_{a+b+c \le h-1} (t+a)^2 (u+b)^2 (v+c)^2. \tag{4}$$

All summation variables are nonnegative integers. Its exact reconstruction uses

$$\sum_{a+b+c\le h-1} (a)_{\underline{i}}(b)_{\underline{j}}(c)_{\underline{k}} = i!j!k!\binom{h+2}{i+j+k+3}. \tag{5}$$

Indeed the generating series of $\binom{a}{i}$ is $z^i/(1-z)^{i+1}$; include a slack variable and extract the coefficient of z^{h-1}. Expand squares using $x = (x)_{\underline{1}}$, $x^2 = (x)_{\underline{2}} + (x)_{\underline{1}}$. Only positive-size tetrahedra are evaluated as polynomials below; a geometric correction of size at most zero is absent.

Lemma 2.1 (Untruncated region)**.** *For every $r \in \mathbb{N}_0$, $T_{r,(2,2,2)}$ is jointly log-concave on $A, B, C \ge r$ within the positive orthant.*

Proof. Radius zero is a product. Otherwise normalize $f = T/\binom{r+3}{3}$, set $r = R+1$, $(A,B,C) = (r+x, r+y, r+z)$, and drop positive parts. Uniform-simplex moments satisfy, for $0 \le a_j \le 2$,

$$\mathbb{E}\prod_{j=1}^{3} I_j^{a_j} = \sum_{0\le l_j\le a_j} \frac{6(r)_{\underline{l_1+l_2+l_3}}}{(l_1+l_2+l_3+3)!}\prod_{j=1}^{3}\left\{{a_j \atop l_j}\right\} l_j!. \tag{6}$$

This follows from (5), cardinality normalization and Stirling expansion. For derivatives in x, y, z put $N = \nabla f \nabla f^{\mathsf{T}} - f\nabla^2 f$. The checker `verify_deep.py` reconstructs every coefficient and expands

	f	N_{ii}	$\det N_{\{i,j\}}$	$\det N$
nonzero coefficients	108	446	4837	21263.

Each diagonal or pair has the stated count. All nonzero coefficients are positive; missing coefficients vanish. Hence all principal minors are nonnegative on the full nonnegative parameter orthant and $N \succeq 0$. The zero-index term gives $f > 0$, and $\nabla^2 \log f = -N/f^2$ proves the lemma. Derivatives hold r fixed. The unbounded-radius scope follows from complete identities, not sampled radii. □

3 Complete three-dimensional cell coverage

On an open unit cell put $A_i = m_i + t_i$, $m_i \in \mathbb{N}_0$, $0 < t_i < 1$, and $c_i = m_i + 1$, $s = m_1 + m_2 + m_3 - r$. Sort $c_1 \le c_2 \le c_3$. If $m_i = 0$, only occupancy zero contributes there, giving a squared factor times the two-dimensional theorem of Appendix A, or a lower product. Thus assume $c_i \ge 2$. Radius zero is a product; $r \le \min m_i$ is deep. If $s \le 0$, reversing indices leaves the product of row sums. For $s > 0$ the reversed cell polynomial is

$$f = \sum_{\substack{0\le U_i\le m_i \\ \sum U_i \ge s}} \prod_i (t_i + U_i)^2. \tag{7}$$

Start with $\prod_i F_{m_i}(t_i) - \mathcal{T}(s;t)$. Singleton i adds $\mathcal{T}(s-c_i; t+c_ie_i)$ if $s > c_i$; pair i,j subtracts $\mathcal{T}(s-c_i-c_j; t+c_ie_i+c_je_j)$ if $s > c_i+c_j$. Vector notation shifts the three endpoint arguments. There is no triple correction since $s \le c_1+c_2+c_3-3$. This is ordinary cap inclusion–exclusion, shifting the excluded coordinate by c_i. All free parameters below are nonnegative integers. Checker names identify exact formulas. A parameter orthant may exceed the sorted region, which only strengthens the sufficient sign check.

I: $0 < s \le c_1$**.** For $s = 1$, put $(m_1, m_2, m_3) = (M+1, N+1, L+1)$; subtract only $t_1^2t_2^2t_3^2$ (`s1`). For $s \ge 2$ put

$$s = R+2, \qquad (m_1, m_2, m_3) = (R+1+M, R+1+N, R+1+L)$$

(`no-correction`). The entire family $m_i = s-1$ is included.

II: $c_1 < s \le c_2$**.** Use $m_1 = M+1$, $s = M+R+3$, $m_2 = M+R+N+2$, $m_3 = M+R+L+2$. Here $R = s-c_1-1, N = c_2-s, L = c_3-s$. Add only singleton1 (`one-correction`); $r = 2M+R+N+L+2$.

III: $c_2 < s \le c_3$**.** If $s \le c_1 + c_2$, let $R = s - c_1 - 1, M = s - c_2 - 1, e = c_1 + c_2 - s, L = c_3 - s$. Then

$$c_1 = M + e + 1, \quad c_2 = R + e + 1, \quad s = R + M + e + 2, \quad c_3 = s + L. \tag{8}$$

For $e \ge 1$ set $e = N + 1$ (`two-single`). For $e = 0$ the nonzero caps force $R, M \ge 1$; shift both by one in (8) (`two-single-zero`). Add both singletons and no pair. Before substitution $r = R + M + 2e + L - 1$. If instead $c_1 + c_2 < s \le c_3$, put

$$m_1 = M + 1, \quad m_2 = N + 1, \quad s = M + N + R + 5, \quad m_3 = M + N + R + L + 4.$$

The pair12 size is $R + 1$, $c_3 - s = L$, and $r = M + N + L + 1$. Add singletons1,2 and subtract pair12 (`pair12-only`).

IVa: $c_3 < s \le c_1 + c_2$**.** Define $D = s - c_3 - 1, E = c_1 + c_2 - s, j = c_3 - c_2, k = c_2 - c_1$. Then

$$\begin{aligned} c_1 &= D + E + j + 1, & c_2 &= D + E + j + k + 1, \\ c_3 &= D + E + 2j + k + 1, & s &= 2D + E + 2j + k + 2, \\ r &= D + 2E + 2j + k - 2. \end{aligned} \tag{9}$$

The valid domain is covered by

$$\begin{aligned} (D, E, j, k) &= (R, M + 1, N, L) && (\texttt{three-single-E}), \\ &= (R, 0, N + 1, L) && (\texttt{three-single-j}), \\ &= (R + 2, 0, 0, L) && (\texttt{three-single-D}), \\ &= (1, 0, 0, L + 1) && (\texttt{three-single-k}). \end{aligned}$$

For $E = j = 0$, $c_1 \ge 2$ forces $D \ge 1$, and $r \ge 0$ forces $D + k \ge 2$. If $D = E = j = 0$, dimension reduction already applies. Add all singletons, no pair.

IVb: $c_1 + c_2 < s \le c_1 + c_3$**.** Let $D = s - c_3 - 1, E = s - c_1 - c_2 - 1, J = c_1 + c_3 - s, K = c_2 - c_1$. Then

$$\begin{aligned} c_1 &= D + J + 1, & c_2 &= D + J + K + 1, \\ c_3 &= D + E + 2J + K + 2, & s &= 2D + E + 2J + K + 3, \\ r &= D + K + 2J - 2. \end{aligned} \tag{10}$$

The complete domain is covered by

$$\begin{aligned} (D, E, J, K) &= (R, M, N + 1, L) && (\texttt{three-one-pair-J}), \\ &= (R + 2, M, 0, L) && (\texttt{three-one-pair-D}), \\ &= (1, M, 0, L + 1) && (\texttt{three-one-pair-k}). \end{aligned}$$

For $J = 0$, the same cap/radius argument proves exhaustion. Add all singletons and subtract pair12 of size $E + 1$. At equality $s = c_1 + c_3$ the new pair13 correction has size zero.

IVc: $c_1+c_3 < s \le c_2+c_3$**.** Put $a = c_1 \ge 2, e = s-c_1-c_3 \ge 1, h = c_2+c_3-s \ge 0, c = c_3-c_2 \ge 0$. Then $c_2 = a+e+h, c_3 = a+e+h+c, s = 2a+2e+h+c, r = a+h-3$. If $h \le 2$, $r \le a-1 = m_1$, so the deep lemma applies. Otherwise set $a = R+2, e = M+1, h = N+3, c = L$ (`three-two-pairs`). Thus $c_2 = R+M+N+6, c_3 = c_2+L, s = 2R+2M+N+L+9, r = R+N+2$. Add all singletons and subtract pairs12 and13 of sizes $M+L+1$ and $M+1$. Pair23 is absent.

IVd: $s > c_2+c_3$**.** Here $r = c_1+c_2+c_3-3-s < c_1-3 < m_1$, so the deep lemma applies. Equality $s = c_2+c_3$ is retained in IVc with $h = 0$. These intervals exhaust all geometries. Every single-cap or pair equality belongs to the preceding closed upper interval, with the newly empty correction explicitly absent. No fractional cell is discarded.

3.1 Uniform polynomial certificates

Write $g = \nabla f, H = \nabla^2 f, N = gg^{\mathsf{T}} - fH$. Besides the diagonals, put

$$\begin{aligned} D_{ij} &= f(H_{ii}H_{jj} - H_{ij}^2) - H_{ii}g_j^2 - H_{jj}g_i^2 + 2H_{ij}g_ig_j, \\ D_3 &= -f \det H + g^{\mathsf{T}} \operatorname{adj}(H) g. \end{aligned} \tag{11}$$

Determinant expansion gives $\det N_{\{i,j\}} = fD_{ij}$ and $\det N = f^2 D_3$. Since $f > 0$ on its cell, nonnegative diagonals, three D_{ij} and D_3 prove $N \succeq 0$. The checker `verify_cells.py` constructs every cell polynomial from (5) and the stated cap corrections and expands all seven polynomials. Table 1 records the full coefficient inventories. Every nonzero coefficient is positive; missing coefficients vanish. All free parameters, including fractional endpoint variables, have no upper cutoff in these identities. Arithmetic uses rational construction followed by integer coefficient polynomials. For the largest triple identities the first free parameter is represented as a univariate with six-variable polynomial coefficients, an exact polynomial-ring isomorphism. Original-index finite sums guard transcription. The determinant coefficient signs, together with the complete geometry cover, are the unbounded proof obligations.

Quadratic case of Theorem 1.1. The cover, certificates and reductions give a negative semidefinite log-Hessian on every open unit cell. The zero-index term gives positivity on the positive orthant. Products of squared positive parts and their finite sum are C^1, hence so is the logarithm there. On a generic compact line segment its derivative is nonincreasing within each of finitely many cells and continuous across integer walls. Wall-contained segments follow by limits. Thus the logarithm is globally concave. Boundary limits vanish, so zero extension from the convex positive orthant is log-concave. □

4 A joint endpoint–activity theorem in every dimension

We first prove Theorem 1.2 at power two. Let $e_k(z)$ be the elementary symmetric polynomial in m activity variables, with $e_0 = 1$ and $e_k = 0$ outside $0 \le k \le m$. For an integer $L \ge 2$ put

$$P_L(t,z) = \sum_{k=0}^{m} a_k(t) e_k(z), \qquad a_k(t) = \sum_{j=0}^{\min(L,r-k)} (t+L-j)^2, \tag{12}$$

where empty sums vanish. We prove joint log-concavity on all $t, z \ge 0$, which is stronger than the application on $0 \le t \le 1$. Write $a_k(t) = \sum_{l=0}^{2} a_{k,l} t^l$ and homogenize to degree $D = m+6$:

$$G_L(y,t,z) = \sum_{k,l} a_{k,l} t^l y^{D-k-l} e_k(z). \tag{13}$$

The elevated degree is part of the construction.

We use the classical criterion [1, Theorem 3.2]: a homogeneous polynomial with nonnegative coefficients is completely log-concave if all nonzero partial derivatives of remaining degree at least two are indecomposable and all degree-two partial derivatives are log-concave. Indecomposability is checked by connectedness of the active-variable Hessian graph. A nonnegative homogeneous quadratic is log-concave exactly when its Hessian has at most one positive eigenvalue. Zero derivatives are omitted.

Any nonzero derivative of (13) differentiates k distinct z variables, $h \le 2$ copies of t, and some copies of y. For remaining degree at least two, the pure-y term from $a_{k,h} > 0$ survives. Each other surviving variable has a positive mixed term with y, so its active graph has a positive y hub. Here $a_{k,h} > 0$ whenever $k \le \min(r, m)$ and $0 \le h \le 2$: the $j = 0$ summand and $L \ge 2$ give strictly positive coefficients. Other derivative types are zero. This proves the indecomposability condition.

4.1 All fifteen quadratic types

Differentiate k distinct z variables, h times in t, and $D - 2 - k - h$ times in y. Set $N = m - k$ and $n = N + 6 - h$. For $N \ge 1$, after division by $h!(n-2)!$, the Hessian on the $(y, t, \text{normalized all-}z)$ subspace is

$$\begin{pmatrix} n(n-1)a_{k,h} & (h+1)(n-1)a_{k,h+1} & \sqrt{N}(n-1)a_{k+1,h} \\ (h+1)(n-1)a_{k,h+1} & (h+1)(h+2)a_{k,h+2} & \sqrt{N}(h+1)a_{k+1,h+1} \\ \sqrt{N}(n-1)a_{k+1,h} & \sqrt{N}(h+1)a_{k+1,h+1} & (N-1)a_{k+2,h} \end{pmatrix}. \tag{14}$$

Coefficients with second index above two vanish. When $N = 0$, omit all z rows and columns. For $N \ge 1$ the other $N - 1$ eigenvalues are $-a_{k+2,h} \le 0$. For $N \ge 1$ scale the last coordinate by $\sqrt{N}$; call the resulting rational-polynomial matrix M. Its mixed z entries have N instead of $\sqrt{N}$, and its last diagonal is $N(N-1)a_{k+2,h}$. The yy pivot is positive. It has at most one positive eigenvalue if

$$-\det M_{\{y,t\}} \ge 0, \qquad -\det M_{\{y,z\}} \ge 0, \qquad \det M \ge 0. \tag{15}$$

These are precisely the diagonal and determinant signs of the negative-semidefinite Schur complement under the positive pivot. For $N = 0$ check only the (y, t) block.

For consecutive square sums define the coefficient vector

$$R(q, b) = \left((q+1)b^2 + bq(q+1) + \frac{q(q+1)(2q+1)}{6}, 2(q+1)b + q(q+1), q+1 \right).$$

Exactly five patterns occur for (a_k, a_{k+1}, a_{k+2}):

1. (c, c, c), $c = R(L, 0)$, $L \ge 2$;
2. $(c, c, c - (0, 0, 1))$, with the same c;
3. $(c, c - (b^2, 2b, 1), c - (b^2 + (b+1)^2, 4b + 2, 2))$, $c = R(q, b)$, $q \ge 2, b \ge 0$;
4. $((2b^2 + 2b + 1, 4b + 2, 2), ((b+1)^2, 2(b+1), 1), 0)$, $b \ge 1$;
5. $((L^2, 2L, 1), 0, 0)$, $L \ge 2$.

Indeed $q = r - k \ge L + 2$ gives three full rows; $q = L + 1$ removes only the third row's t^2 term. For $2 \le q \le L$, successive deletions have residuals $t + b, t + b + 1$ with $b = L - q$. Values $q = 1, 0$ give

the last patterns. Formal coefficients beyond available z degree are harmless because their matrix factors vanish if fewer than two z variables remain.

For every pattern and $h = 0, 1, 2$, `verify_lorentzian.py` expands all three expressions (15) after $N = K + 1$, $K \geq 0$. Use $L = Q + 2$ in patterns 1 and 2; $q = Q + 2, b = B$ in pattern 3; $b = B+1$ in pattern 4; and $L = B+2$ in pattern 5. Every coefficient is nonnegative in (K, Q, B); the largest determinant has 130 nonzero terms. The checker separately verifies the $N = 0$ minor and the strictly positive yy pivot. These fifteen whole polynomial identities cover unbounded caps, radii and numbers of companions, rather than fifteen numerical instances. Thus every quadratic derivative is log-concave. The classical criterion proves complete log-concavity of G_L; affine specialization $y = 1$ proves joint log-concavity of P_L.

4.2 The spatial mixed block

On $x = L + t$, $L \geq 2$, factor only the binary companions:

$$U_{r,2}(x, A) = \prod_i A_i^2 \, P_L(x - L, \phi(A_1), \ldots, \phi(A_m)), \qquad \phi(A) = (1 - 1/A)_+^2. \tag{16}$$

Since $a_{k+1}(t) \leq a_k(t)$, deletion of a binary coordinate gives $0 \leq \partial_i \log P_L \leq 1$. Namely $P_L = P_0 + z_i P_i$ with coefficientwise $0 \leq P_i \leq P_0$. The full log-Hessian is a congruence of $\nabla^2 \log P_L$, including all x–A cross terms, plus companion diagonal entries

$$-2/A_i^2 + (\partial_i \log P_L)\phi''(A_i).$$

These are nonpositive. For $A \leq 1$, ϕ is constant; for $A > 1$, $\phi''(A) = 2(3 - 2A)/A^4$. Where positive it is at most $2/A^2$, since $3 - 2A \leq A^2$; where negative the sign is immediate. Thus the spatial sum is jointly log-concave on every cell $L \geq 2$.

For $0 < x \leq 2$, terms $j \geq 2$ vanish. The support is the independent sets of the uniform matroid on $m + 1$ binary coordinates. Its independent-set polynomial is log-concave by [1, Theorem 4.1]; the same deletion bound and compensated composition apply. All squared positive parts are C^1, so the linewise gluing argument proves global log-concavity of $U_{r,2}$. Its zero extension is nondecreasing and log-concave.

5 Two unrestricted occupancies

We first prove Theorem 1.3 for the free matroid and activities one. At power two, write the unrestricted endpoints as $x = L + t, u = M + v$, with integer $L, M \geq 2$. Define

$$b_q(t, v) = \sum_{\substack{0 \leq i \leq L,\ 0 \leq j \leq M \\ i+j \leq q}} (t + L - i)^2 (v + M - j)^2, \qquad b_q = 0 \quad (q < 0).$$

Write $b_{q,h,g} = [t^h v^g] b_q$, and homogenize $Q(t, v, z) = \sum_k b_{r-k}(t, v) e_k(z)$ to degree $D = m + 12$:

$$G(y, t, v, z) = \sum_{k,h,g} b_{r-k,h,g} t^h v^g y^{D-k-h-g} e_k(z).$$

Every nonzero derivative of remaining degree at least two has the same positive y hub as in Section 4. The occupancy term $i = j = 0$ supplies every coefficient in the spatial degree box $0 \leq h, g \leq 2$ strictly positively, since $L, M \geq 2$; when a binary variable is active, its y edge also survives. Repeated binary derivatives or negative residual rank give zero. Thus the classical indecomposability hypothesis holds.

5.1 The complete quadratic reduction

After k binary derivatives and h, g spatial derivatives, let $N = m-k$, $q = r-k$, $n = N+12-h-g$ and differentiate $D-2-k-h-g$ times in y. Divide the Hessian by $h!g!(n-2)!$. For $N \geq 1$ use the unnormalized all-ones binary vector. The resulting congruent block H has entries

$$\begin{aligned}
H_{yy} &= n(n-1)b_{q,h,g}, & H_{yt} &= (h+1)(n-1)b_{q,h+1,g},\\
H_{yv} &= (g+1)(n-1)b_{q,h,g+1}, & H_{tt} &= (h+1)(h+2)b_{q,h+2,g},\\
H_{vv} &= (g+1)(g+2)b_{q,h,g+2}, & H_{tv} &= (h+1)(g+1)b_{q,h+1,g+1},\\
H_{yz} &= N(n-1)b_{q-1,h,g}, & H_{tz} &= N(h+1)b_{q-1,h+1,g},\\
H_{vz} &= N(g+1)b_{q-1,h,g+1}, & H_{zz} &= N(N-1)b_{q-2,h,g}.
\end{aligned} \tag{17}$$

An exponent above two means coefficient zero. The remaining $N-1$ binary eigenvalues are $-b_{q-2,h,g} \leq 0$. When $N = 0$, discard z. The pivot H_{yy} is strictly positive. If J is a nonempty subset of $\{t, v, z\}$, the principal determinant of the negative Schur-complement numerator is

$$\det(H_{Jy}H_{yJ} - H_{yy}H_{JJ}) = (-1)^{|J|}H_{yy}^{|J|-1}\det H_{\{y\}\cup J}.$$

It therefore suffices to check all seven signs $(-1)^{|J|}\det H_{\{y\}\cup J} \geq 0$, not merely leading minors. For $N = 0$ the three signs with $J \subseteq \{t, v\}$ suffice.

5.2 All eight rank/cap chambers

For an integer $w \geq 0$ define the unbounded-simplex polynomial

$$S(w; X, Y) = \sum_{a,b=0}^{2} c_a(X)c_b(Y)a!b!\binom{w+2}{a+b+2}, \qquad (c_0, c_1, c_2)(X) = (X^2, 1-2X, 1). \tag{18}$$

The falling-factorial identity proves that this is $\sum_{i+j\leq w}(X-i)^2(Y-j)^2$. Its polynomial extension satisfies $S(-1) = S(-2) = 0$, so two consecutive rank shifts may cross an empty-cut boundary without extra cases. Assume $L \leq M$ by symmetry. The following eight chambers cover all $q \geq 0, L, M \geq 2$; Q, B, C are nonnegative integers. Put $S_j = S(q-j; t+L, v+M)$ for $j = 0, 1, 2$.

1. $q = 0$, $L = B+2, M = C+2$: retain only $i = j = 0$ in b_q and zero rows at negative ranks.
2. $q = 1$, $L = B+2, M = C+2$: use the three occupancy vectors of rank at most one, with the appropriate shifted rows.
3. $2 \leq q \leq L$: set $q = Q+2, L = q+B, M = q+C$ and use $b_{q-j} = S_j$.
4. $L < q \leq M$: set $L = Q+2, q = L+B+1, M = q+C$ and use $b_{q-j} = S_j - S(q-j-L-1; t-1, v+M)$.
5. $M < q \leq L+M$, $q-M \geq 2$: set $a = Q+2, L = a+B, M = L+C, q = M+a$ and subtract both $S(q-j-L-1; t-1, v+M)$ and $S(q-j-M-1; t+L, v-1)$ from S_j.
6. $q = M+1$: set $L = B+2, M = L+C$ and use the same two-cut formula.
7. $q \geq L+M+2$: all three rows equal $F_L(t)F_M(v)$, with $L = B+2, M = C+2$.
8. $q = L+M+1$: the first two rows equal $F_L(t)F_M(v)$ and the third is this product minus t^2v^2, with $L = B+2, M = C+2$.

Chamber 3 need not impose the ordering and includes its ordered subset. In chamber 5, $a = q - M$ and $B = L - a \geq 0$; chamber 6 supplies the remaining $a = 1$ possibility. Equality $q = L + M$ is chamber 5 with $B = 0$. Every subtracted S argument is at least -2, so the polynomial empty-sum convention is valid. The excluded-cap intersection is empty for $q \leq L + M$; no pair correction was omitted.

For each chamber and each $h, g \in \{0, 1, 2\}$, `verify_two_rows.py` reconstructs all seven signed bordered principal determinants after $N = R+1$, and the three $N = 0$ determinants. All coefficients are nonnegative directly, with seven exceptions in chamber 5. For these multiply the polynomial by $(1 + Q + B + C + R)^j$, a strictly positive factor, with the following exponents:

h	g	J	j
0	0	$\{t, z\}$	1
0	0	$\{v, z\}$	1
0	0	$\{t, v, z\}$	2
0	1	$\{t, z\}$	4
0	1	$\{t, v, z\}$	3
1	0	$\{v, z\}$	4
1	0	$\{t, v, z\}$	4

Every resulting coefficient is nonnegative. All $N = 0$ cases require no multiplier. The largest certificate has 11517 nonzero terms, for the chamber 5 full bordered determinant at $h = g = 0$ after its square multiplier. These are full rational-polynomial identities on unbounded parameter orthants: 72 derivative types, with all their bordered minors, rather than 72 numerical tests. Pivot positivity follows directly from the zero-occupancy term, as above. The Schur-complement criterion and the classical local-to-global theorem therefore prove complete log-concavity of G, hence joint log-concavity of Q.

5.3 Returning to spatial endpoints

The row deletion inequality $b_{q-1} \leq b_q$ holds coefficientwise. Thus $0 \leq \partial_i \log Q \leq 1$ for every binary activity. Factoring $\prod A_i^2$ and substituting $z_i = \phi(A_i)$ gives exactly the spatial kernel with two unrestricted coordinates. Its full log-Hessian is the congruence of $\nabla^2 \log Q$ by $\mathrm{diag}(1, 1, \phi'(A_1), \ldots)$ plus the same nonpositive companion diagonal corrections already proved above. Both unrestricted cross blocks are included. When one unrestricted endpoint is at most two, it is binary, and Theorem 1.2 supplies the remaining region. Positivity and C^1 gluing prove global log-concavity for the uniform binary-companion support.

6 Matroid companion transfer

We give the degree bookkeeping explicitly; the classical product and parallel-class principles themselves are not new. Suppose $a_q(t)$ are polynomials in a fixed spatial degree box with every coefficient strictly positive for $q \geq 0$, and $a_q = 0$ for $q < 0$. Assume that for every ground size ℓ and rank q, the homogeneous polynomial

$$H_{\ell,q}(y, t, z) = \sum_{T \subseteq [\ell], h} a_{q-|T|,h} t^h y^{\ell+\delta-|T|-|h|} z^T$$

is completely log-concave, where δ is fixed and at least the sum of the spatial degree bounds. For a matroid $\mathcal{M}$ on m elements, replace the unrestricted subsets by its independent sets, obtaining $G_{\mathcal{M}}$ of degree $m + \delta$.

Every nonzero derivative again has a positive y hub. After binary derivatives indexed by an independent set, contract that set; the remaining ground size is N and residual rank is q. For a quadratic derivative, truncate this contraction to rank at most q. Ignore loops. Its nonloops partition into ℓ parallel classes, and its binary pair-independence matrix is

$$\mathbf{1}\mathbf{1}^{\mathsf{T}} - \sum_{\text{parallel classes } C} \mathbf{1}_C \mathbf{1}_C^{\mathsf{T}}.$$

This is the classical rank-two decomposition used in [1, Lemma 4.2]. Binary linear and spatial–binary coefficients are constant over nonloops. The quadratic form consequently depends on each class only through the sum of its coordinates; all within-class zero-sum directions and loops have zero contribution, including cross terms. It is a pullback of the uniform-ℓ quadratic with the same row coefficients, but total residual degree $N + \delta$, rather than $\ell + \delta$.

The degree discrepancy is exactly supplied by $y^{N-\ell} H_{\ell,q}$. A nonnegative monomial is Lorentzian, and products of homogeneous completely log-concave polynomials remain completely log-concave by the classical equivalence/product results [2, Theorem 2.30 and Corollary 2.32]. The corresponding spatial and y derivative of this product has precisely the reduced quadratic's coefficients and factorial factors. It therefore has at most one positive Hessian eigenvalue, as does its pullback. Rank zero gives $\ell = 0$; rank one gives one class of nonloops. Loops are not counted as independent classes. The classical criterion now proves complete log-concavity of $G_{\mathcal{M}}$.

Apply this argument to the one-row coefficients of Section 4 with $\delta = 6$, and the two-row coefficients of Section 5 with $\delta = 12$. The zero-occupancy term gives strict positivity of every coefficient in the spatial degree box, and increasing the remaining rank increases coefficients. Matroid deletion therefore gives $Q_i \le Q|_{z_i=0}$ after $y = 1$. Substitution $z_i = \lambda_i \phi(A_i)$ for $0 \le \lambda_i \le 1$ retains the compensated bound: the derivative with respect to $\phi(A_i)$ is at most $\lambda_i \le 1$. The full spatial Hessian congruence, including all unrestricted–binary cross terms, is negative semidefinite.

If an unrestricted endpoint drops below two, make its occupancy a free binary element by direct sum with the companion matroid; the same overall cardinality budget stays in the row rank. Thus the one-free case ends in the classical binary matroid transform, and the two-free case reduces to the already proved one-free matroid case. The binary transform follows from ALOV's independent-set theorem and the same compensated calculation. The C^1 gluing argument proves the global quadratic case of Theorem 1.3. The real-power lift below completes it.

Fixed finite occupancy caps on either free coordinate are also allowed: beyond the final capped cell, the proved cell polynomial with that cap is valid for all nonnegative residual endpoints. This is a final-cell argument. Caps one and zero reduce respectively to a free binary element and a positive coordinate factor.

7 Real powers and sharpness

Lemma 7.1 (Monotone fractional marginal)**.** *Let $f : \mathbb{R}^d \to [0, \infty)$ be finite, measurable, coordinatewise nondecreasing and log-concave, vanishing whenever some coordinate is nonpositive. For independent $\alpha_i > 0$, the function*

$$I_\alpha f(A) = \int_{[0,\infty)^d} f(A - s) \prod_i s_i^{\alpha_i - 1}\, ds$$

is finite and log-concave. An order zero means omitting its integration.

Proof. For $\alpha_i \geq 1$ retain s_i and its log-concave power factor. For $0 < \alpha_i < 1$ put $s_i = u_i^{1/\alpha_i}$; then $s_i^{\alpha_i - 1} ds_i = du_i/\alpha_i$ and $A_i - u_i^{1/\alpha_i}$ is concave on $u_i \geq 0$. A coordinatewise nondecreasing log-concave function composed with concave coordinate maps is log-concave: the image of an interpolated point dominates the interpolation of the images, then monotonicity and log-concavity apply. Multiply by the remaining log-concave power factors and orthant indicators, and apply Prékopa marginalization [3, Theorem 6]. The original fibers satisfy $0 \leq s_i \leq (A_i)_+$ and the integrand is bounded by $f(A) \prod_i s_i^{\alpha_i - 1}$, so the integrals are finite. The same proof omits zero-order coordinates. □

For $p > 2$, the beta integral gives

$$\int_0^{z_+} (z - s)_+^2 s^{p-3}\, ds = \frac{2\Gamma(p-2)}{\Gamma(p+1)} z_+^p. \tag{19}$$

Apply Lemma 7.1 independently with orders $p_i - 2$, omitting integration at power two. All the globally proved quadratic fixed-support kernels are nondecreasing. Tonelli and (19) give a positive constant times the desired fixed-support sum. This completes the sufficient parts of Theorems 1.1, 1.2 and 1.3. The lift is applied to a globally proved function, not an isolated cell. It uses classical monotone composition and marginalization.

To prove sharpness, set all but one endpoint in $(0, 1]$ and take $r = 1$. Up to a positive constant the remaining function is $f_p(x) = x_+^p + (x - 1)_+^p$. At $1 - \varepsilon, 1, 1 + \varepsilon$ its endpoint product minus squared midpoint is

$$(1 - \varepsilon^2)^p + \varepsilon^p (1 - \varepsilon)^p - 1.$$

For $0 < p < 2$, division by ε^p gives a limit of one as $\varepsilon \downarrow 0$. Thus a genuine midpoint violation occurs. This proves coordinatewise necessity in Theorem 1.1 and the same universal threshold across these fixed-support families. Radius zero by itself is log-concave for every positive exponent.

Remark 7.2 (No positive-power strengthening). The scalar quadratic sum $f(x) = x^2 + (x - 1)_+^2$ admits no positive power concave on the entire positive half-line. For $x > 1$,

$$(f^\eta)'' = \eta f^{\eta - 2} [4f + (\eta - 1)(4x - 2)^2],$$

whose right limit at one is $4\eta^2 > 0$ for every $\eta > 0$. Thus logarithmic concavity at this endpoint cannot be upgraded to a universal positive-power assertion, already in dimension one.

8 Reproducibility and remaining scope

The checkers are part of the computer-assisted proof.

- `verify_cells.py`: all fourteen cell families and seven principal-minor certificates per family;
- `verify_deep.py`: the deep lemma;
- `verify_lorentzian.py`: all fifteen uniform one-row quadratic types;
- `verify_two_rows.py`: all 72 two-row types and their bordered signs;
- `verify_baseline.py`: the two-dimensional dependency below.

The first uses exact python-flint polynomial arithmetic. Runtime requirements and commands are in the README. All correctness gates are explicit and remain active under optimized Python. Finite sample sums check transcription only; the unbounded proofs use full symbolic reconstruction, coefficient signs, complete geometry coverage and the stated classical analytic steps.

Theorem 1.1 covers every mixed cell, not only the untruncated region or finitely many radii. Theorems 1.2 and 1.3 give all-dimensional fixed-support families with up to two unrestricted occupancies and matroid binary companions, controlling the endpoint–activity cross blocks. A factorial-weighted generating-polynomial conclusion is not silently identified with the raw smoothing Hessian.

A The two-dimensional dependency

We reproduce the baseline [6] required for dimension reduction. For $r \in \mathbb{N}_0$, the function

$$B_r(A,B) = \sum_{i+j\le r} (A-i)_+^2 (B-j)_+^2$$

is jointly log-concave, including zero extension. Here are the full reconstruction and sign checks. Write $A = m+t, B = n+u$, $m \le n$, $0 < t, u < 1$, and $s = m+n-r$. Put $S_j(l) = \sum_{i=0}^{l} i^j$. For $s \le 0$ the sum is $F_m(t)F_n(u)$. For $0 < s \le m$,

$$f = F_m(t)F_n(u) - \sum_{i=0}^{s-1} (t+i)^2 F_{s-1-i}(u). \tag{20}$$

Use $s = R+1, m = s+M, n = s+N$, all parameters nonnegative. For $m < s \le n$, set $L = s-m \ge 1, N = n-s \ge 0$. Then

$$\begin{aligned} f = \sum_{U=0}^{m} (t+U)^2 \big\{ &(N+U+1)u^2 + 2u[S_1(n) - S_1(s-U-1)] \\ &+ S_2(n) - S_2(s-U-1) \big\}. \end{aligned} \tag{21}$$

For $m \ge 1$ use $m = M+1, L = l+1$; for $m = 0$ use the expression separately. The summand has degree at most five in U. Expanding into falling factorials and using $\sum_{U=0}^{m} (U)_{\underline{j}} = j!\binom{m+1}{j+1}$ reconstructs it exactly. The removed triangle in (20) similarly uses

$$\sum_{i+j\le s-1} (i)_{\underline{a}} (j)_{\underline{b}} = a!b! \binom{s+1}{a+b+2}.$$

If $n < s \le m+n$, then $r < m \le n$ and every term is untruncated. Radius zero is a product; otherwise divide by $(r+1)(r+2)/2$ and obtain

$$\begin{aligned} f = A^2B^2 - \tfrac{2r}{3}(A^2B + AB^2) + \tfrac{r(r+1)}{6}(A^2+B^2) + \tfrac{r(r-1)}{3}AB \\ - \tfrac{r(r-1)(2r+1)}{30}(A+B) + \tfrac{r(r-1)(2r^2+2r+3)}{180}. \end{aligned} \tag{22}$$

Set $r = R+1, A = r+x, B = r+y$. This formula also follows directly from the uniform-simplex moments. Let $X = f_t^2 - f f_{tt}, Y = f_u^2 - f f_{uu}$, and let D_{tu} be the reduced pair polynomial in (11). The determinant identity is $XY - (f_t f_u - f f_{tu})^2 = f D_{tu}$. The full coefficient inventories are

case	X	Y	last polynomial	kind
shallow	1419	1419	5662	D_{tu}
middle, $m \ge 1$	1121	705	3306	D_{tu}
middle, $m = 0$	65	22	80	D_{tu}
deep	58	58	337	fD_{tu}.

Every listed coefficient is positive. The standard-library checker `verify_baseline.py` reconstructs and verifies them. The cases exhaust the integer-cell geometries; in particular $m = s - 1$ belongs to the middle case. Positivity, the Hessian test, C^1 gluing and zero extension work as in the three-dimensional proof. This appendix makes the dependency explicit rather than assigning it new contribution value.

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Case	diagonal counts	reduced pair counts $(12, 13, 23)$	D_3
`s1`	6956,6956,6956	41759,41759,41759	168898
`no-correction`	51975,51975,51975	422652,422652,422652	2147315
`one-correction`	66684,52850,52850	490923,490923,410200	2342532
`two-single`	85135,85135,71850	718731,634886,634886	3378431
`two-single-zero`	15670,15670,12625	99591,84301,84301	362442
`pair12-only`	36860,36860,22025	306058,208992,208992	1163668
`three-single-E`	117055,114690,114690	952700,952700,936670	4814985
`three-single-j`	28700,27700,27700	174965,174965,169925	707217
`three-single-D`	5550,5200,5200	24675,24675,23450	78547
`three-single-k`	650,600,600	2065,2065,1925	5096
`three-one-pair-J`	73165,70165,51075	584360,467005,451885	2459114
`three-one-pair-D`	14775,13725,9775	86450,67900,64225	283318
`three-one-pair-k`	1800,1650,1150	7560,5845,5425	19012
`three-two-pairs`	49790,38515,29425	347900,286825,231560	1371510

Table 1: Nonzero coefficients in complete exact expansions. Diagonals are in coordinate order. The checker also verifies exact minimum coefficients.