Sharp noise transitions for an adaptive single-copy stabilizer-learning process

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Abstract

We determine the noise-dependent behavior of the adaptive single-copy stabilizer-learning update of Bittel, Eisert, Gong, Mele and Schatzki on globally depolarized stabilizer states. For every fixed prime local dimension p, the number of diagonal stabilizers still missing follows an exact birth–death chain. Writing c for the retained pure-state weight, starting with all n diagonal stabilizers missing, the expected time to record the optimizing stabilizer is linear in n when $c^2 > p/(p+1)$, quadratic at equality, and exponential when $c^2 < p/(p+1)$. We obtain the exact supercritical speed, subcritical exponential prefactors, and a critical-window mean profile uniform over all initial missing ranks, with an $O(n)$ remainder and the exact leading recording penalty. An exact regeneration identity distinguishes first exposure of the computational-basis target from its actual recording. A finite-time bound shows that below threshold the unchanged algorithm has exponentially small success probability in any polynomial round budget, even on this simple noise family. The obstruction concerns this update and candidate generation rule. Bittel and coauthors ask whether the requirement that the stabilizer fidelity exceed $\sqrt{2/3}$, under which their learner is tolerant to noise, can be removed. For qubits the transition lies exactly at retained weight $c = \sqrt{2/3}$, which is the stabilizer fidelity up to $(1-c)2^{-n}$, so for their unchanged update this requirement is sharp: no fixed fidelity threshold below $\sqrt{2/3}$ suffices.

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1 Introduction

Stabilizer states admit compact classical descriptions and are central to quantum error correction, simulation and learning. The distinction between single-copy measurements and measurements coupling several unknown copies is important both algorithmically and experimentally. Bittel et al. [1] recently gave an adaptive single-copy learner for exact stabilizer states, extended it to prime-dimensional qudits, and proved a tolerant mixed-qubit guarantee when stabilizer fidelity exceeds $\sqrt{2/3}$. Their algorithm changes the measurement frame using the difference of two separately measured computational-basis samples. On exact stabilizers, this monotonically exposes diagonal stabilizers.

Noise can destroy that monotonicity. Bittel et al. close with the question whether the restriction $F_{\mathrm{stab}}(\rho) \geq f_0 > \sqrt{2/3}$ of their tolerant learner [1, Theorem III.1] can be removed, the goal being a stabilizer state whose fidelity with an arbitrary input is within a prescribed accuracy of the best

stabilizer fidelity [1, Section VII, third item, "Fully agnostic stabilizer learning"]. For their own update this raises the question whether the high-fidelity restriction is only a feature of a sufficient proof or reflects actual behavior of the update. We answer this question for global depolarization. The algorithm itself is unchanged. Its latent progress variable becomes an exactly computable one-dimensional Markov chain, whose transition from linear to exponential cost occurs at precisely the asymptotic qubit threshold already suggested by the positive theorem.

The result has three components. First, we identify the exact transition law, rather than bound progress and regression separately. Second, we solve its mean asymptotics, including the critical window and exponential prefactors. Third, we relate latent exposure to actual candidate recording, accounting for the dependence between a failed recording and the following frame update. The last step matters at leading order below threshold.

1.1 Relationship to earlier work

The update and its noiseless prime-dimensional analysis belong to Bittel et al. [1, Algorithm 1 and Appendix A]. Their Theorem III.1 supplies the positive high-fidelity qubit learner. Their proof already introduces an auxiliary birth–death chain and its expected-hitting-time potential to control candidate recording. We compute the exact chain and recording law for the depolarizing family. Our contribution is an exact noise-dependent analysis and an algorithm-specific failure theorem for its unchanged candidate generator. The birth–death recurrences, gambler's-ruin calculation and convergence arguments used below are classical probability methods; see, for example, Norris [2] and Levin and Peres [3].

Stabilizer bootstrapping gives proper agnostic learning with multi-copy measurements for qubits [4]; Qi et al. [5] give a pure-input odd-prime extension. The author's mixed-input odd-prime learner [6] uses coherent four- or six-copy sampling. Tang and Steckmann [7], in their second version, use a Bell-sampling finder–verifier architecture for noise-robust stabilizer structure. At stabilizer nullity zero, their commuting-superset criterion recovers the same underlying unsigned stabilizer group. Their coherent Bell measurements and guarded finder provide a different discovery mechanism from the prescribed single-copy update analyzed here. Cho and Kim [8] give worst-case single-copy lower bounds for Pauli symmetry recovery under a stabilizer-dimension promise; our result quantifies one fixed update on the explicit depolarizing family.

All regime statements concern the explicit candidate-generation update in Section 2. Candidate-generation round budgets and fresh-copy fidelity-validation costs are given separately.

1.2 The model and main conclusions

Fix a prime p and an n-qudit pure stabilizer $|s\rangle$, and consider

$$\rho_n = c|s\rangle\langle s| + (1-c)\frac{I}{p^n}, \qquad 0 < c \le 1. \tag{1}$$

The parameter c is the retained pure-state weight. It is not the finite-dimensional stabilizer fidelity, which is

$$F_n = c + (1-c)p^{-n}. \tag{2}$$

Let r be the codimension of the target's diagonal stabilizer space in the current measurement frame. Initially we take $|s\rangle = |+_p\rangle^{\otimes n}$, so $r = n$. The same formulas apply to any initial target with a specified initial r.

Let τ_0 be the first time at which $r = 0$, and let σ be the first pre-update round index at which the recorded candidate is exactly $|s\rangle$. Indices start at zero; the actual number of measured rounds

up to and including recording is $\sigma+1$. Neither time is observable by the learner, because the target is unknown.

Put

$$s = c^2, \qquad R = \frac{p(1-s)}{s}, \qquad \mu = \frac{s}{p} - (1-s). \tag{3}$$

For fixed p, c, the mean candidate time has the following transition:

Noise regime	Condition	$\mathbb{E}_n\sigma$
Supercritical retained signal	$s > p/(p+1)$	$n/\mu + o(n)$
Critical retained signal	$s = p/(p+1)$	$(p+1)n^2/2 + o(n^2)$
Subcritical retained signal	$s < p/(p+1)$	$K_{p,c}R^n(1+o(1))$

Here $K_{p,c}$ is a strictly positive explicit convergent-series constant given in Theorem 6.2. At $c = 1$, the already known pure-state mean is $pn + O_p(1)$.

The exact correction $\mathbb{E}_n\sigma - \mathbb{E}_n\tau_0 = H_n$ in Theorem 4.1 is bounded above threshold and lower order in the critical window, while contributing to the leading exponential constant below threshold. Theorem 5.2 also quantifies the order-n critical recording penalty and gives a uniform error for every initial rank. Theorems 5.2, 6.1 and 6.2 give the quantitative limits. Corollary 4.2 supplies deterministic confidence budgets; Theorem 7.1 gives the corresponding finite-round subcritical obstruction. The proof proceeds from the exact rank law to recording, then to these asymptotic and operational conclusions.

For qubits the critical weight is $\sqrt{2/3}$. The equality with the external sufficient threshold is asymptotic: $F_n \neq c$ at finite n. At critical c, F_n is slightly larger than c, but the external high-fidelity drift parameter then tends to zero exponentially. Thus its nonuniform upper bound does not contradict the quadratic behavior proved here.

For the unchanged update this locates the restriction asked about in [1, Section VII] exactly: for qubits it cannot be relaxed to any fixed threshold $f_1 < \sqrt{2/3}$. Indeed, choose c with $f_1 < c < \sqrt{2/3}$. The inputs (1) with $|s\rangle = |+_2\rangle^{\otimes n}$, so that initially $r = n$, then have stabilizer fidelity $F_n > c > f_1$, the target being their unique closest stabilizer state (Lemma 7.2), while $R > 1$. Theorem 7.1 makes the probability that the update ever records the target within any polynomial round budget exponentially small in n, and Corollary 7.3 turns this into failure of the unchanged candidate-and-select learner at every additive accuracy below $c/2$. Removing the restriction, as Bittel et al. ask, therefore requires a different candidate generator; at $c = \sqrt{2/3}$ itself the mean recording time is quadratic (Theorem 5.1). For prime p the same argument, with the gap $c(1-1/p)$ of Lemma 7.2, shows that no fixed threshold below $\sqrt{p/(p+1)}$ suffices for the unchanged update.

This fifth edition relates the threshold theorems to the question of Bittel et al. and states their consequence for it; the theorems and proofs of the fourth edition are unchanged.

2 Stabilizer geometry and the fixed update

2.1 Unsigned labels and computational support

Let $V = \mathbb{F}_p^{2n}$, with labels (a, b) representing the Pauli operator $X(a)Z(b)$ up to scalar phase, where

$$X(a)|x\rangle = |x+a\rangle, \qquad Z(b)|x\rangle = e^{2\pi i b\cdot x/p}|x\rangle.$$

The symplectic form is $[(a,b),(a',b')] = a\cdot b' - b\cdot a'$. A pure stabilizer state determines an n-dimensional isotropic subspace $L \leq V$, together with compatible eigenvalue phases. Define

$$D = \{b : (0,b) \in L\}, \qquad r = n - \dim D, \qquad A = D^{\perp} \leq \mathbb{F}_p^n,$$

where $\perp$ in this definition is the ordinary dot-product orthogonal complement.

Lemma 2.1. *The X-projection of L equals A. The computational-basis distribution of its pure stabilizer state is uniform on an affine coset $z + A$ of size p^r.*

Proof. Every $(a, b) \in L$ commutes with every $(0, d) \in L$, so $a \cdot d = 0$. The projection onto X-coordinates therefore lies in $D^\perp$. Its kernel has dimension $\dim D$, and hence its image has dimension r, equal to $\dim D^\perp$; the two spaces coincide. The diagonal stabilizer eigenvalue equations restrict nonzero amplitudes to one affine coset $z + A$. Every $a \in A$ occurs as the X-part of a stabilizer, whose action translates amplitudes by a while preserving their absolute values. Transitivity on the coset makes all these absolute values equal. Normalization proves uniformity. □

2.2 One adaptive round

The current Clifford frame is C, and the transformed target is $C|s\rangle$. In each round the algorithm does the following, exactly as in [1]:

1. Measure two fresh copies of $C\rho_n C^\dagger$, separately, in the computational basis, obtaining $x, y \in \mathbb{F}_p^n$.
2. Record the candidate $C^\dagger|x\rangle$, and set $u = x - y$.
3. If $u = 0$, leave C unchanged.
4. If $u \neq 0$, use the pivoting linear Clifford F_u to send its X-label to e_i. Then apply one uniformly chosen local Clifford H_a, $a \in \mathbb{F}_p$, such that $H_a X Z^a H_a^\dagger$ is proportional to Z. Replace C by $H_a F_u C$.

For $p = 2$, the two local choices are H and HS, up to the same slope convention. In general, F_u is implemented by the controlled-SUM and multiplication gates specified in [1, Appendix A]. Its action on X-coordinates is an invertible matrix M_u with $M_u u = e_i$, and its action on Z-coordinates is M_u^{-T}. All statements below are independent of the permitted choice of pivot or representative H_a.

Recording does not alter the state update, and no target-membership test is used. One round uses two copies measured singly, rather than a coherent two-copy measurement.

Lemma 2.2 (Exact rank change)**.** *For a current target with support direction A:*

(i) If $u = 0$, the rank variable r stays fixed.

(ii) If $u \in A \setminus \{0\}$, exactly one of the p local corrections decreases r by one, and the others leave it fixed.

(iii) If $u \notin A$, every local correction increases r by one.

Proof. For $d \in D$, the pivot component of the transformed diagonal label is

$$(M_u^{-T} d)_i = d \cdot M_u^{-1} e_i = d \cdot u.$$

If $u \in A$, every old diagonal stabilizer is therefore trivial on the pivot. Also, some stabilizer has X-part u, and after F_u it has X-part e_i. Modulo old diagonal stabilizers its pivot Pauli has one well-defined nonvertical slope: two choices with X-part e_i differ by an old diagonal label, trivial on the pivot. Exactly one H_a turns this slope diagonal. The resulting generator is independent of the old diagonal space because it acts nontrivially on the pivot. At most one generator can be

added: the preimages of new diagonal labels have X-part in the one-dimensional span of e_i. The remaining corrections preserve the old space and add no generator.

If $u \notin A$, the functional $d \mapsto d \cdot u$ on D is nonzero. Its kernel has codimension one, and consists precisely of the old diagonal labels that are trivial on the pivot after F_u. Every H_a preserves these labels. It takes a nontrivial pivot Z-label to a nondiagonal label: an invertible symplectic map cannot send both the distinct lines Z and XZ^a to the Z line.

No new diagonal generator can replace the lost one. Before the local correction, such a generator must have X-part supported only at the pivot. If that part is nonzero, pulling it back through F_u puts a nonzero multiple of u in the target X-projection A, impossible. If it is zero, the generator was already diagonal and must lie in the kernel just identified. Thus exactly one diagonal generator is lost. The zero-difference case makes no update by definition. □

3 The exact chain and a finite mean formula

Theorem 3.1 (Exact transition law)**.** *For input* (1)*, conditional on any complete history, the current rank variable has a time-homogeneous birth–death law depending only on r. For $1 \le r \le n$, its downward and upward probabilities are*

$$a_r = \frac{1 - p^{-r}}{p}\big[s + (1 - s)p^{r-n}\big], \tag{4}$$

$$b_r = (1 - s)(1 - p^{r-n}). \tag{5}$$

The remaining probability is holding, and $b_n = 0$.

Proof. Clifford conjugation preserves the form (1). By Lemma 2.1, the measured distribution is $cU_{z+A} + (1-c)U_{\mathbb{F}_p^n}$. Expanding two independent draws shows that their difference has distribution

$$sU_A + (1 - s)U_{\mathbb{F}_p^n}. \tag{6}$$

The target–target difference is uniform on A; each term involving a uniform full-space draw is uniform on the full space. The probability of $u \in A \setminus \{0\}$ is $(1 - p^{-r})[s + (1 - s)p^{r-n}]$, and the probability of $u \notin A$ is $(1 - s)(1 - p^{r-n})$. Lemma 2.2 proves the formulas and their history independence. □

For hitting-time calculations we stop the chain at zero. For $0 < c < 1$, all a_r are positive, all b_r with $r < n$ are positive, and the finite chain reaches zero in finite mean. Write

$$g_n(r) = \mathbb{E}_r \tau_0, \qquad g_n(0) = 0, \qquad d_{n,r} = g_n(r) - g_n(r-1).$$

Proposition 3.2 (Exact mean identity)**.** *For $0 < c < 1$,*

$$d_{n,r} = \sum_{k=r}^{n} \frac{1}{a_k} \prod_{j=r}^{k-1} \frac{b_j}{a_j}, \tag{7}$$

$$g_n(n) = \sum_{1 \le r \le k \le n} \frac{1}{a_k} \prod_{j=r}^{k-1} \frac{b_j}{a_j}. \tag{8}$$

Empty products equal one; in particular $d_{n,r} > 0$.

Proof. First-step conditioning gives $a_r d_{n,r} - b_r d_{n,r+1} = 1$ for $r < n$, while $a_n d_{n,n} = 1$. Backward substitution gives (7), and summing the increments gives (8). □

Put $q = 1/p$, $t = (1-s)/s$, and

$$C_p = \prod_{j=1}^{\infty}(1-p^{-j}). \tag{9}$$

This product is positive and $C_p \geq 1/4$. Indeed its factors are at least those for $p = 2$, whose first factor is $1/2$ and whose remaining product is at least $1 - \sum_{j\geq 2} 2^{-j} = 1/2$.

Lemma 3.3 (Interval-product bounds)**.** *For $1 \leq r \leq k \leq n$ and $0 < c < 1$,*

$$C_p e^{-t/(p-1)} R^{k-r} \leq \prod_{j=r}^{k-1} \frac{b_j}{a_j} \leq C_p^{-1} R^{k-r}. \tag{10}$$

Moreover, with $S_n(R) = \sum_{\ell=0}^{n-1}(n-\ell)R^{\ell}$,

$$pC_p e^{-t/(p-1)} S_n(R) \leq g_n(n) \leq \frac{p}{s(1-1/p)C_p} S_n(R). \tag{11}$$

Proof. For $1 \leq j < n$, direct division gives

$$\frac{b_j}{a_j} = R\frac{1-p^{j-n}}{(1-p^{-j})(1+tp^{j-n})}. \tag{12}$$

Any interval product of the factors $1-p^{j-n}$ or $1-p^{-j}$ lies in $[C_p, 1]$. Also

$$1 \leq \prod_{j=r}^{k-1}(1+tp^{j-n}) \leq \exp\left(t\sum_{h=1}^{\infty} p^{-h}\right) = e^{t/(p-1)}.$$

This proves (10), including empty products. Since $s(1-1/p)/p \leq a_k \leq 1/p$, substitution into (8) gives (11). □

4 Actual recording and finite-confidence budgets

First exposure of a computational-basis target does not guarantee that the next measurement records it. This distinction can change the subcritical leading constant.

Theorem 4.1 (Exact recording correction)**.** *Let $h_n = (1-c)p^{-n}$ and $F_n = c + h_n$. For every initial rank r,*

$$\mathbb{E}_r\sigma = g_n(r) + H_n, \qquad H_n = \frac{1-F_n}{F_n}\big[1 + (1-h_n)g_n(1)\big]. \tag{13}$$

At $c = 1$ this reads $H_n = 0$.

Proof. An exact target candidate is impossible before rank zero. At a rank-zero frame the target is a computational vector z. Its first measurement is $x = z$ with probability F_n, in which case the recording index is zero. Conditional on failure $x \neq z$, the independent second measurement equals this particular x with probability h_n. Then $u = 0$, and the next rank is zero. Otherwise $u \neq 0$ is outside the zero-dimensional target X-projection, so the next rank is one by Lemma 2.2.

Let H_n be the expected recording index from rank zero and let $H_{n,1}$ be that from rank one. Strong Markov conditioning gives $H_{n,1} = g_n(1) + H_n$. First-step conditioning, with the failed recording and update kept jointly, yields

$$H_n = (1-F_n)\{1 + h_n H_n + (1-h_n)H_{n,1}\}.$$

Solving gives (13). These expectations are finite without assuming the identity: return times to zero have finite mean in the finite chain, and each visit has conditional recording probability $F_n > 0$. Successive excursions therefore give a geometric bound on their expected number. The same argument at the first rank-zero visit proves the formula for arbitrary r. □

Corollary 4.2 (Deterministic round budgets)**.** *Let*

$$M_n = g_n(n) + H_n, \qquad B_n = \lceil 2(M_n + 1) \rceil .$$

For every initial frame and integer $k \geq 1$,

$$\mathbb{P}(\sigma + 1 > kB_n) \leq 2^{-k}. \tag{14}$$

Thus running kB_n rounds for $k = \lceil \log_2(1/\delta) \rceil$ includes the target candidate with probability at least $1 - \delta$.

Proof. By Proposition 3.2, $g_n(r) \leq g_n(n)$. Conditional on any complete history of unsuccessful rounds, the current input is again a depolarized stabilizer in some frame and the expected additional inclusive round count is at most $M_n + 1$. Markov's inequality bounds the probability of needing more than B_n further rounds by $1/2$. Iterating this conditional bound over k blocks proves (14). □

This is an implementable fixed-budget statement; it does not require an oracle announcing τ_0 or σ. It costs twice the stated number of rounds in unknown-state copies for candidate generation. For this parameterized budget, p, n, c are supplied; alternatively any certified upper bound on M_n may be used. The finite sums above determine the budget from these parameters without knowledge of the target. Final selection from the list requires additional fresh copies and computation. For example, direct independent measurements of each listed stabilizer projector give additive accuracy ε for all N candidates using $O(N\varepsilon^{-2} \log(N/\delta))$ copies by Hoeffding's inequality. This elementary validation cost is separate from our exposure analysis; the sharper shadow-based validation in [1] may instead be used when its hypotheses and implementation are supplied.

5 Three regimes and the critical window

Theorem 5.1 (Mean phase transition)**.** *Fix p and $0 < c < 1$. If $R > 1$, then $g_n(n)$ and $\mathbb{E}_n\sigma$ are $\Theta_{p,c}(R^n)$. If $R = 1$, both are $\Theta_p(n^2)$. If $R < 1$, both are $\Theta_{p,c}(n)$. At the pure endpoint $c = 1$, $\mathbb{E}_n\sigma = pn + O_p(1)$.*

Proof. For $R < 1$, $n \leq S_n(R) \leq n/(1 - R)$. For $R = 1$, $S_n(R) = n(n + 1)/2$. For $R > 1$, setting $j = n - 1 - \ell$ shows

$$R^{n-1} \leq S_n(R) = R^{n-1} \sum_{j=0}^{n-1} (j + 1)R^{-j} \leq \frac{R^{n-1}}{(1 - R^{-1})^2}.$$

Lemma 3.3 gives the asserted orders for $g_n(n)$. Its same estimates applied to (7) at $r = 1$ give $g_n(1) = O_{p,c}(R^n)$ when $R > 1$, $O_p(n)$ when $R = 1$, and $O_{p,c}(1)$ when $R < 1$. Since $F_n \geq c > 0$, the correction (13) has these same upper orders. The lower orders transfer from $\sigma \geq \tau_0$. If $c = 1$, the chain is pure death with $a_r = (1 - p^{-r})/p$, so

$$\mathbb{E}_n\sigma = \sum_{r=1}^{n} \frac{p}{1 - p^{-r}} = pn + p \sum_{r=1}^{n} \frac{1}{p^r - 1} = pn + O_p(1).$$

This endpoint is already the noiseless analysis of [1]. □

Theorem 5.2 (Uniform critical-window profile)**.** *Fix p and $\Lambda < \infty$. For $c_n \in (0,1)$, put*

$$R_n = \frac{p(1-c_n^2)}{c_n^2}, \qquad \lambda_n = n \log R_n, \qquad \Phi_p(\lambda, \alpha) = (p+1) \int_0^\alpha \int_x^1 e^{\lambda(y-x)}\, dy\, dx.$$

If $|\lambda_n| \le \Lambda$, then for all sufficiently large n,

$$\max_{0 \le r \le n} \left\{ |g_n(r) - n^2 \Phi_p(\lambda_n, r/n)|, |\mathbb{E}_r \sigma - n^2 \Phi_p(\lambda_n, r/n)| \right\} \le C_{p,\Lambda} n. \tag{15}$$

Uniformly on the same window, the exact recording correction satisfies

$$H_n = n \frac{1-c_*}{c_*} \frac{p+1}{C_p} \int_0^1 e^{\lambda_n y}\, dy + O_{p,\Lambda}(1), \qquad c_* = \sqrt{\frac{p}{p+1}}. \tag{16}$$

Consequently, if $\lambda_n \to \lambda$ and $r_n/n \to \alpha \in [0,1]$, both $g_n(r_n)/n^2$ and $\mathbb{E}_{r_n}\sigma/n^2$ tend to $\Phi_p(\lambda, \alpha)$. At full missing rank this is

$$\Phi_p(\lambda, 1) = \begin{cases} (p+1)(e^\lambda - 1 - \lambda)/\lambda^2, & \lambda \ne 0, \\ (p+1)/2, & \lambda = 0. \end{cases} \tag{17}$$

The integral in (16) *is $(e^\lambda - 1)/\lambda$ when $\lambda \ne 0$, and one at zero.*

Proof. Write $s_n = c_n^2$, $t_n = (1-s_n)/s_n$ and $s_* = p/(p+1)$. The window bound gives $s_n = p/(p+R_n)$, hence $|s_n - s_*| \le C_{p,\Lambda}/n$, while t_n stays in a fixed positive compact interval. Let $T_{n,i,k}$ be the summand in (7) beginning at i and ending at k. Direct combination of (4) and (12) gives

$$T_{n,i,k} = \frac{p}{s_n} R_n^{k-i} \frac{\prod_{j=i}^{k-1}(1-p^{j-n})}{\prod_{j=i}^{k}(1-p^{-j}) \prod_{j=i}^{k}(1+t_n p^{j-n})}. \tag{18}$$

All inverse products here are bounded uniformly by the positive product C_p and by the exponential bound of Lemma 3.3. The sums of the omitted lower- and upper-boundary factors satisfy

$$\sum_{j=i}^{k} p^{-j} \le \frac{p}{p-1} p^{-i}, \qquad \sum_{j=i}^{k} p^{j-n} \le \frac{p}{p-1} p^{k-n},$$

Using $|1 - \prod(1-u_j)| \le \sum u_j$ for $0 \le u_j \le 1$, and $\prod(1+u_j) \le e^{\sum u_j}$, therefore yields

$$|T_{n,i,k} - (p+1) R_n^{k-i}| \le C_{p,\Lambda} R_n^{k-i} \big(n^{-1} + p^{-i} + p^{k-n}\big). \tag{19}$$

Because $R_n^{k-i} \le e^\Lambda$, summing over $1 \le i \le r$ and $i \le k \le n$ gives an $O_{p,\Lambda}(n)$ error, uniformly in r. Indeed each geometric boundary sum contributes at most $O_p(n)$, and the n^{-1} term occurs at most n^2 times. The function $e^{\lambda_n(y-x)}$ is uniformly bounded and Lipschitz on the unit square. Its mesh-$1/n$ Riemann sum on $0 \le x \le r/n$, $x \le y \le 1$ thus has error $O_\Lambda(1/n)$, uniformly in the integer r. Since $g_n(r) = \sum_{i=1}^{r} \sum_{k=i}^{n} T_{n,i,k}$, this proves the first estimate in (15). The product bounds also give $g_n(1) = O(n)$; Theorem 4.1 then gives $H_n = O(n)$ and the second estimate.

For the precise recording penalty, keep $i = 1$ in (18). The reciprocal lower-boundary product differs from $1/C_p$ by $O_p(p^{-k})$: its missing infinite tail has geometric sum $O_p(p^{-k})$. The upper-boundary numerator and reciprocal denominator differ from one by $O_p(p^{k-n})$. Therefore

$$\left| T_{n,1,k} - \frac{p+1}{C_p} R_n^{k-1} \right| \le C_{p,\Lambda} R_n^{k-1} (n^{-1} + p^{-k} + p^{k-n}).$$

The right side sums to $O_{p,\Lambda}(1)$. A one-dimensional Riemann sum, uniform for $|\lambda_n| \le \Lambda$, gives

$$g_n(1) = n\frac{p+1}{C_p}\int_0^1 e^{\lambda_n y}\,dy + O_{p,\Lambda}(1).$$

Here $c_n - c_* = O_{p,\Lambda}(n^{-1})$, $h_n = O(p^{-n})$ and $F_n - c_* = O_{p,\Lambda}(n^{-1})$. Substitution in (13) changes the displayed leading coefficient by at most $O_{p,\Lambda}(1)$ and proves (16). Continuity of Φ_p gives the initial-rank limits, and integrating at $\alpha = 1$ gives (17). □

The regime statements are for fixed p. The constants and exact formulas remain available at every specified finite prime.

6 Exact fixed-regime constants

Theorem 6.1 (Supercritical speed)**.** *For fixed p, c with $c^2 > p/(p+1)$,*

$$\frac{g_n(n)}{n},\ \frac{\mathbb{E}_n\sigma}{n} \longrightarrow \frac{1}{c^2/p - (1-c^2)}. \tag{20}$$

Proof. Here $R < 1$. The summands in (7), indexed by $h = k - r$, are bounded uniformly by a constant times R^h. For each fixed h, when $r \to \infty$ and $n - r \to \infty$, their factors satisfy $1/a_{r+h} \to p/s$ and $b_j/a_j \to R$. Dominated convergence against the geometric majorant gives, uniformly away from the two boundaries,

$$d_{n,r} \to \frac{p}{s}\sum_{h=0}^{\infty} R^h = \frac{p}{s(1-R)} = \frac{1}{\mu}.$$

Explicitly, first truncate h so its geometric tail is uniformly small, then remove a fixed number of indices at each boundary to control all remaining coefficients. Every $d_{n,r}$ is uniformly bounded, so those boundary indices have vanishing contribution to $n^{-1}\sum_r d_{n,r}$. This proves the first limit. Here $g_n(1) = O_{p,c}(1)$, and the bounded recording correction proves the second. □

For the subcritical constants let

$$P_m = \prod_{j=1}^{m}(1-q^j), \quad P_0 = 1, \qquad Q_m = \prod_{j=0}^{m}(1+tq^j), \quad Q_{-1} = 1, \tag{21}$$

and $Q_\infty = \prod_{j=0}^{\infty}(1+tq^j)$.

Theorem 6.2 (Subcritical prefactors)**.** *Fix p, c with $0 < c^2 < p/(p+1)$, and put $z = 1/R \in (0,1)$. Define the positive finite constants*

$$A = \sum_{r=1}^{\infty} z^r P_{r-1}, \qquad B = \sum_{\ell=0}^{\infty} z^\ell \frac{Q_{\ell-1}}{P_\ell}. \tag{22}$$

Then

$$\frac{g_n(n)}{R^n} \longrightarrow \frac{p}{s}\frac{AB}{Q_\infty}, \tag{23}$$

$$\frac{g_n(1)}{R^n} \longrightarrow \frac{p}{s}\frac{zB}{Q_\infty}, \tag{24}$$

$$\frac{\mathbb{E}_n\sigma}{R^n} \longrightarrow K_{p,c} := \frac{p}{s}\frac{B}{Q_\infty}\left[A + \frac{1-c}{c}z\right]. \tag{25}$$

Proof. Let $T_{n,r,k}$ denote the summand in (8). Combining $1/a_k$ with the product in (12) gives exactly

$$T_{n,r,k} = \frac{p}{s} R^{k-r} \frac{\prod_{j=r}^{k-1}(1-q^{n-j})}{\prod_{j=r}^{k}(1-q^j)\prod_{j=r}^{k}(1+tq^{n-j})}. \tag{26}$$

Set $k = n - \ell$. For each fixed $r \geq 1$ and $\ell \geq 0$, the numerator product tends to C_p/P_ℓ, and the two denominator products tend to C_p/P_{r-1} and $Q_\infty/Q_{\ell-1}$, respectively. Thus

$$\frac{T_{n,r,n-\ell}}{R^n} \longrightarrow \frac{p}{s} z^{r+\ell} \frac{P_{r-1}Q_{\ell-1}}{P_\ell Q_\infty}.$$

Extend this expression by zero when $r + \ell > n$. Lemma 3.3 bounds it by a constant times $z^{r+\ell}$, summable on $r \geq 1, \ell \geq 0$. Dominated convergence for the double series proves (23). Restricting the same calculation to $r = 1$ proves (24). Finally $F_n \to c$, $h_n \to 0$, and $R > 1$, so the constant term in (13) divided by R^n vanishes. Substituting (24) into that identity proves (25). □

The prefactors are explicit approximable constants, not only qualitative existence statements. The elementary bounds

$$1 \leq Q_\infty \leq e^{t/(1-q)}, \qquad C_p \geq 1/4$$

give, for $M \geq 0$,

$$\begin{aligned}
0 &\leq \sum_{r=M+1}^{\infty} z^r P_{r-1} \leq \frac{z^{M+1}}{1-z}, \\
0 &\leq \sum_{\ell=M+1}^{\infty} z^\ell \frac{Q_{\ell-1}}{P_\ell} \leq \frac{Q_\infty}{C_p} \frac{z^{M+1}}{1-z}, \\
1 &\leq \frac{Q_\infty}{Q_M} \leq \exp\left(\frac{tq^{M+1}}{1-q}\right).
\end{aligned}$$

They allow certified truncation using outward arithmetic if numerical prefactors are desired.

7 A finite-time failure theorem

Theorem 7.1 (Full-height exponential barrier)**.** *Fix p and $0 < c < 1$ with $R > 1$. For $n \geq 2$ and every integer $T \geq 0$,*

$$\mathbb{P}_n(\sigma \leq T) \leq \mathbb{P}_n(\tau_0 \leq T) \leq T C_p^{-1} e^{t/(p-1)} R^{-(n-1)}. \tag{27}$$

In particular, the unchanged candidate generator has exponentially small target-inclusion probability in any polynomial round budget.

Proof. For the chain stopped upon hitting 0 or n, let v_r be the probability of hitting zero first from r. Holding probabilities cancel from its first-step equations:

$$b_r(v_{r+1} - v_r) = a_r(v_r - v_{r-1}), \qquad v_0 = 1,\ v_n = 0.$$

With $w_0 = 1$ and $w_j = \prod_{i=1}^{j}(a_i/b_i)$ for $j < n$, solving the increment recurrence gives

$$v_{n-1} = \frac{w_{n-1}}{\sum_{j=0}^{n-1} w_j} \leq w_{n-1} \leq C_p^{-1} e^{t/(p-1)} R^{-(n-1)},$$

where the last inequality is the lower product bound (10) inverted on $1, \dots, n-1$. Every path from n to zero must succeed during an excursion beginning with a downward step n to $n-1$. There are at most T such excursions in T transitions. Applying the strong Markov property at their entrance times and a union bound proves the second inequality in (27). Recording requires prior exposure, proving the first. The statement remains valid when its right side exceeds one. □

The mean lower bound has the same exponential rate. Indeed the factors $1 - p^{j-n}$ and $1 - p^{-j}$ cancel over the full interval, giving

$$\prod_{j=1}^{n-1} \frac{b_j}{a_j} = \frac{R^{n-1}}{\prod_{j=1}^{n-1}(1 + tp^{-j})}, \qquad g_n(n) \geq pe^{-t/(p-1)} R^{n-1}.$$

The finite-time bound is stronger than a mean lower bound alone: it rules out success hidden in a large mass of short trajectories with a small number of extremely long failures.

Lemma 7.2 (Stabilizer overlap gap)**.** *Distinct pure n-qudit stabilizers in prime dimension satisfy $|\langle s'|s\rangle|^2 \leq 1/p$. For (1), the target is the unique maximizing stabilizer, and every other stabilizer's overlap is smaller by at least $c(1 - 1/p)$.*

Proof. The stabilizer projector is the average of its p^n phase-corrected Pauli stabilizers. Orthogonality of Pauli operators shows that the trace of two such projectors is either zero, if their characters disagree on the intersection, or p^{k-n}, where k is the intersection dimension of their unsigned spaces. If the states are distinct and the trace is nonzero, $k \leq n-1$; when $k = n$ and the characters agree, the states are the same. Thus the trace is at most $1/p$. The identity component in (1) contributes the same $(1-c)p^{-n}$ to every pure-state overlap, so the gap is at least $c(1 - 1/p)$. □

Corollary 7.3 (Failure of the unchanged agnostic extension)**.** *For a fixed subcritical c and any $0 < \varepsilon < c(1 - 1/p)$, success of the unchanged candidate-and-select procedure at additive accuracy ε requires target inclusion. Consequently any polynomial candidate round budget has success probability tending to zero exponentially, even if final selection from the list is perfect.*

Proof. Every non-target listed stabilizer fails the required overlap inequality by Lemma 7.2. Apply Theorem 7.1. □

8 Interpretation and reproducibility

The limiting bulk downward probability is c^2/p, while the upward probability is $1 - c^2$. Their difference is μ in (3). This explains the location of the transition. The finite-dimensional boundary factors do more than perturb this drift: they determine the critical mean profile and the subcritical prefactors. The exact recording correction further separates exposure of a suitable frame from obtaining the correct observed eigenvalue.

The critical result gives an actual finite-confidence inclusion guarantee of $O_{p,K}(n^2 \log(1/\delta))$ rounds for this promised input family, uniformly in critical windows with $|n \log R_n| \leq K$ for fixed K. The generic block argument gives $O_{p,c}(n \log(1/\delta))$ rounds above threshold. Below threshold, the finite-round failure bound applies to the prescribed update and output-list construction from initial missing rank n. The distinct discovery mechanisms discussed in the introduction place this process-specific transition in the wider learning landscape.

Known classical obstacles also require precise scope. Over $\mathbb{F}_2$, recovering an unknown hyperplane from a mixture of its uniform distribution and uniform full-space noise is exactly a version of

learning parity with noise; Chen, De and Vijayaraghavan [9, Section 6] give the explicit reduction. This classical reduction identifies the computational content of a proposed noisy-support recovery subroutine.

The analytic proofs here establish all-n statements. Finite symplectic checks of gate actions and exact transition probabilities, and direct evaluations of the finite recurrence, provide reproducible supporting tests. The supplied finite controls use exact arithmetic; the asymptotic claims rest on the analytic proofs.

AI assistance

OpenAI Codex assisted with derivation, programming, literature comparison, proof checking and manuscript preparation. Anthropic Claude identified the answered question, edited this edition and checked the argument. The author takes responsibility for the mathematics and accompanying files.

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Data and materials availability. The results are analytic. The accompanying source package contains the manuscript and the verification code and outputs supporting its reproducible calculations.

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