OPTIMAL MOMENT ORDER FOR DETECTING A NEGATIVE PARTIAL TRANSPOSE

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Abstract. We determine, up to absolute constants, how many exact moments distinguish a positive partial transpose from a negative eigenvalue of magnitude at least δ. For total bipartite dimension D, let $N = \min\{D, \delta^{-1/3}\log^{2/3}(1/\delta)\}$. For small δ, the optimal order is $\Theta(N)$, uniformly in the two local dimensions. When D is the minimum, the upper bound is the classical fact that D moments fix the spectrum. A polynomial that vanishes at the large eigenvalues of a candidate positive spectrum gives the upper bound. A contraction argument moves repeated Chebyshev nodes to construct separable and entangled states with matching moments, proving that the logarithmic factor is necessary. The required absolute moment precision has scale $\exp[-\Theta(N\log(1/\delta))]$. We also give explicit physical pairs agreeing through moment $D-1$.

1. Moment data and the gap promise

Let $d_A, d_B \geq 2$ and $D = d_A d_B$. For a density operator ρ on $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}$, write ρ^Γ for its partial transpose on the second factor. Put

$$p_j(\rho) = \mathrm{Tr}[(\rho^\Gamma)^j], \qquad j \geq 1.$$

In particular $p_1 = 1$. The PPT class consists of states for which ρ^Γ is positive semidefinite. The alternative class consists of states with $\lambda_{\min}(\rho^\Gamma) \leq -\delta$. A PPT spectrum can always be reproduced by a separable state diagonal in a product basis, so certification requires moment vectors that no PPT state shares.

Let $M_{d_A,d_B}(\delta)$ be the least m for which the first m moment vectors of the two promised classes are disjoint. This is an information-theoretic order in the exact moment model. Let

$$A(\delta) = \delta^{-1/3}\log^{2/3}(1/\delta), \qquad N = \min\{D, A(\delta)\}.$$

All logarithms are natural.

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Theorem 1. *There are absolute constants $c, C, \delta_0 > 0$ such that, for every $d_A, d_B \geq 2$ and $0 < \delta \leq \delta_0$,*

$$cN \leq M_{d_A,d_B}(\delta) \leq CN.$$

Thus $\Theta(N)$ is the optimal order up to absolute constants.

Theorem 2. *With the same range of parameters and absolute constants, the following hold.*

(1) *There is a separable state and an alternative state with negative eigenvalue $-\delta$ such that, simultaneously for every $j \geq 1$,*

$$|p_j(\rho) - p_j(\sigma)| \leq \exp[-cN \log(1/\delta)].$$

(2) *There is $m \leq CN$ such that every pair from opposite promised classes satisfies*

$$\max_{1 \leq j \leq m} |p_j(\rho) - p_j(\sigma)| \geq \exp[-CN \log(1/\delta)].$$

Elben et al. [3] introduced partial-transpose moments measured by local randomized measurements and the p_3-PPT condition, which originated this programme. The PPT tests of Yu, Imai, and Gühne [11], and the hierarchy of Neven et al. [5], develop the general testing framework from finitely many moments. Theorems 1 and 2 quantify the order and precision required uniformly under an absolute spectral-gap promise. Dou, Wu, and Chen characterize the two-qubit inverse spectrum and its low-order moment bounds [2]. Lancien and McShane study fixed-order thresholds for induced random states [4]. Spectral moment bounds for distance to the positive cone appear in [1]. Deflating large eigenvalues in matrix-vector testing is studied in [6], under a different observation model. The Chebyshev level-set construction in [10, Lemma 36] matches two probability vectors through the roots of $T \mp \delta$. We use a shift of those levels to put one spectrum across zero, and use displaced nodes and integer multiplicities for the gap-dependent lower bound. The latter construction is motivated by the polynomial detection of isolated negative atoms in [7], where the least detecting degree of a signed measure with a negative point mass separated by a gap is determined. The continuous-measure analogues, with tails in place of a trace constraint, are treated in [8] and [9]; the latter also handles several negative directions.

This second edition adds complete computations for the node displacement, the row-sum estimate and the absorption of constants in the precision theorem, credits the randomized-measurement origin of the programme, and states the results as optimal in order up to absolute constants.

2. A polynomial upper bound

Let $y = (y_i)_{i=1}^{D}$ be a nonnegative candidate spectrum of trace one. Fix $B \in (0, 1)$, and form a factor for each $y_i > B$, counting multiplicity. Their

number ℓ is at most $1/B$. With T_r the Chebyshev polynomial of the first kind, define

$$P(t) = \prod_{y_i > B} \frac{y_i - t}{y_i + \delta} \frac{T_r(1 - 2t/B)}{T_r(1 + 2\delta/B)}, \qquad f(t) = t^2 P(t)^2.$$

For every $t \le -\delta$, all factors and the Chebyshev ratio are at least one. Thus for an alternative spectrum x, $\sum_i f(x_i) \ge \delta^2$. At a candidate eigenvalue $y_i > B$ the polynomial is zero. At a candidate eigenvalue $0 \le y_i \le B$ each product factor has absolute value at most one and $|T_r(1 - 2y_i/B)| \le 1$. Hence

$$\sum_i f(y_i) \le \frac{\sum_{y_i \le B} y_i^2}{T_r(1 + 2\delta/B)^2} \le \frac{B}{T_r(1 + 2\delta/B)^2}.$$

For B of order $\delta^{1/3}/\log^{2/3}(1/\delta)$, take r of order $\sqrt{B/\delta}\log(1/\delta)$, with a sufficiently large absolute prefactor. Since $T_r(1+u) = \cosh(r \operatorname{arcosh}(1+u))$ for $u > 0$, the candidate sum is at most $\delta^2/2$. The degree $2(\ell + r) + 2$ is $O(A(\delta))$. Matching that many moments is therefore impossible. The first D moments also suffice, by Newton's identities. This proves the upper bound in Theorem 1.

3. A physical realization of matched spectra

The following elementary realization will be used twice. Suppose a real trace-one spectrum has exactly one negative entry $-\tau$ and has three positive entries t, a, e with $a, e \ge t \ge \tau$. Embed a two-by-two product block in the bipartite space and put

$$\rho_{\text{block}} = \begin{pmatrix} a & 0 & 0 & c \\ 0 & (t-\tau)/2 & 0 & 0 \\ 0 & 0 & (t-\tau)/2 & 0 \\ c & 0 & 0 & e \end{pmatrix}, \qquad c = (t + \tau)/2.$$

The block is positive semidefinite, since $c^2 \le t^2 \le ae$. Its partial transpose moves the coherence to the central block and has eigenvalues $a, e, t, -\tau$. Put the remaining nonnegative weights on the unused product basis vectors. The comparator nonnegative trace-one spectrum gives a diagonal separable state in the same space.

Proposition 3. *For every nontrivial bipartite dimension D, a separable state and a state with exactly one negative partial-transpose eigenvalue agree through moment $D - 1$. The negative eigenvalue has magnitude*

$$\tau_D = \frac{\sec(\pi/D) - 1}{D} \sim \frac{\pi^2}{2D^3}.$$

Proof. Let $b = \cos(\pi/D)$. Take the root multisets of $T_D(z) = (-1)^D$ and $T_D(z) = -(-1)^D$, respectively, and transform each root by $x = (z+b)/(Db)$. The root multisets are real including multiplicities. The transformed monic polynomials differ only in their constant terms, so the spectra sum to one

and their first $D-1$ moments agree. The second spectrum is nonnegative. In the first spectrum the unique negative entry comes from $z=-1$, while the least positive entry comes from $z=-\cos(2\pi/D)$. This latter entry is at least τ_D because $2b \geq 1+\cos(2\pi/D) = 2b^2$. The root $z=-\cos(2\pi/D)$ is a double root of $T_D(z)-(-1)^D$, so it gives two equal entries t. Since $D \geq 4$, the largest root ($\cos(\pi/D)$ for odd D, 1 for even D) is a further root, and every root other than -1 and $-\cos(2\pi/D)$ exceeds $-\cos(2\pi/D)$. So a third positive entry $e \geq t \geq \tau_D$ exists, and with $a=t$ the physical realization above applies. Newton's identity at degree D gives the first differing moment explicitly, in absolute value, $D\,2^{2-D}/(Db)^D$. □

4. Displacing repeated nodes

Lemma 4. *For every sufficiently large integer n there are two real trace-one spectra of size $D_n = n + O(n^{1/4}\log^3 n)$, one nonnegative and the other with one strictly negative entry*

$$-\tau_n = -\frac{(\log n)^2}{64n^3}.$$

Their first n power sums agree. Apart from a common residual entry, all eigenvalues have absolute value at most $1/n$, and at least three positive entries on the negative side have values of order $1/n$.

Proof. Set $B=1/n$, $\theta_j=(j-\frac12)\pi/n$, $z_j=\cos\theta_j$, $y_j=B(1-z_j)/2$, and $h_j = B\sin\theta_j/(2n)$, for $1\leq j\leq n$. Let $L_i(t)$ be the cardinal interpolation polynomials for the distinct nodes y_j, so that $L_i(y_j)=\mathbf{1}_{i=j}$. Define

$$H = T_n(1+2\tau_n/B), \qquad E_i = \frac{\tau_n H \sin\theta_i}{n(1-z_i)},$$
$$\beta_i = \int_{-\tau_n}^{0} L_i(t)\,dt, \qquad \eta = \frac{c_0}{\log(n+1)}, \qquad q_i = \left\lceil \frac{2E_i}{\eta h_i} \right\rceil,$$

where $c_0>0$ is an absolute constant chosen below. The explicit cardinal formula in the variable $z=1-2t/B$ is

$$L_i(t) = \frac{T_n(z)}{(z-z_i)T_n'(z_i)}.$$

As $|T_n'(z_i)| = n/\sin\theta_i$, this formula on $[-\tau_n,0]$ gives $|\beta_i|\leq E_i$.

We take $c_0\leq 1/8$, so that $\eta\leq 1/4$. For $|t-y_j|\leq \eta h_j$ the variable $z=1-2t/B$ satisfies $|z-z_j|\leq \eta\sin\theta_j/n$. Since $\theta_j\geq\pi/(2n)$ and $\pi-\theta_j\geq\pi/(2n)$, every v with $|v|\leq\pi/(4n)$ has $|v|\leq\min(\theta_j,\pi-\theta_j)/2$, and concavity of sin on $[0,\pi]$ gives $\sin(\theta_j+v)\geq\frac12\sin\theta_j$. By the mean value theorem, $u\mapsto\cos(\theta_j+u)$ on $|u|\leq\pi/(4n)$ covers every point within $\pi\sin\theta_j/(8n) > \eta\sin\theta_j/n$ of z_j, and $|\cos(\theta_j+u)-z_j|\geq\frac12|u|\sin\theta_j$ there. Hence $z=\cos(\theta_j+u)$ with

$$\tfrac12|u|\sin\theta_j \leq \frac{\eta\sin\theta_j}{n}, \qquad \text{that is,} \qquad |u|\leq\frac{2\eta}{n}.$$

The cardinal formula gives

$$|L_j(t)-1| \le C_0\eta, \qquad |L_i(t)| \le \frac{C_0\eta \sin\theta_i}{n|z_i-z_j|} \quad (i\neq j). \tag{1}$$

For the diagonal bound, both the numerator and denominator in

$$\frac{\sin(nu)/(nu)}{(\cos\theta_j-\cos(\theta_j+u))/(u\sin\theta_j)}$$

are within $O(\eta)$ of one. For the off-diagonal bound, $|T_n(\cos(\theta_j+u))| = |\sin(nu)| \le 2\eta$, and $|z-z_j| \le \eta\sin\theta_j/n$ is at most $\eta|z_i-z_j|$ by (2) below.

We now prove (3). The cosine difference identity gives

$$|z_i-z_j| = 2\sin\frac{\theta_i+\theta_j}{2}\,\sin\frac{|i-j|\pi}{2n}.$$

Concavity of sin on $[0,\pi]$ gives $2\sin\frac{\theta_i+\theta_j}{2} \ge \sin\theta_i+\sin\theta_j$, and $\sin x \ge 2x/\pi$ on $[0,\pi/2]$ gives $\sin\frac{|i-j|\pi}{2n} \ge |i-j|/n$. Therefore, for both $k=i$ and $k=j$,

$$\frac{\sin\theta_k}{n|z_i-z_j|} \le \frac{1}{|i-j|}. \tag{2}$$

Also $\sin\theta_j/[n(1-z_j)] = \cot(\theta_j/2)/n \le 2/(n\theta_j) \le C_1/j$ with $C_1 = 4/\pi$. Since $E_j/E_i = \sin\theta_j(1-z_i)/[\sin\theta_i(1-z_j)]$, the off-diagonal bound in (1) gives, for $i\neq j$,

$$\frac{E_j}{E_i}\sup_{|t-y_j|\le\eta h_j}|L_i(t)| \le \frac{C_0\eta\sin\theta_j(1-z_i)}{n(1-z_j)|z_i-z_j|} \le C_0\eta\left[\frac{\sin\theta_j}{n(1-z_j)}+\frac{\sin\theta_j}{n|z_i-z_j|}\right],$$

where the second step uses $1-z_i \le |z_i-z_j|+(1-z_j)$. By (2) with $k=j$, the bracket is at most $C_1/j+1/|i-j|$. Summing over $j\neq i$, and adding the diagonal term $C_0\eta$, gives $C_0\eta[1+(C_1+2)(1+\log n)]$. Hence

$$\sup_i\sum_j\frac{E_j}{E_i}\sup_{|t-y_j|\le\eta h_j}|L_i(t)-\mathbf{1}_{i=j}| \le C_2\eta\log(n+1). \tag{3}$$

Use the weighted norm $\|s\| = \max_i|s_i|/E_i$. On $\|s\|\le 2$ define the map

$$F_i(s) = \beta_i - \sum_{j=1}^n q_j\int_{y_j}^{y_j+s_j/q_j}[L_i(t)-\mathbf{1}_{i=j}]\,dt.$$

The displacement is $|s_j|/q_j \le 2E_j/q_j \le \eta h_j$. For s,s' in the ball, $|F_i(s)-F_i(s')| \le \sum_j|s_j-s'_j|\sup|L_i-\mathbf{1}_{i=j}| \le E_i\|s-s'\|\,C_2\eta\log(n+1)$ by (3), so with $c_0 \le 1/(2C_2)$ the map has Lipschitz constant at most $1/2$. Since $\|F(0)\|\le 1$, it maps the closed ball to itself and has a fixed point s. Set $x_j = y_j+s_j/q_j$. The small displacement implies $x_j\in(0,B)$. Indeed $y_j = B\sin^2(\theta_j/2)$ and $h_j = B\sin(\theta_j/2)\cos(\theta_j/2)/n$, so

$$\frac{y_j}{h_j} = n\tan\frac{\theta_j}{2} \ge n\tan\frac{\pi}{4n} \ge \frac{\pi}{4}, \qquad \frac{B-y_j}{h_j} = n\cot\frac{\theta_j}{2} \ge \frac{\pi}{4},$$

and $\eta \le 1/4 < \pi/4$. At the fixed point,

$$\sum_j q_j \int_{y_j}^{x_j} L_i(t)\,dt = \int_{-\tau_n}^{0} L_i(t)\,dt.$$

The L_i span the polynomials of degree at most $n-1$. Applying the identity to kt^{k-1}, $1 \le k \le n$, gives $\sum_j q_j(x_j^k - y_j^k) = -(-\tau_n)^k$. Thus the multisets of q_j copies of each x_j plus $-\tau_n$, and q_j copies of each y_j plus zero, match the first n moments.

The trace before adding a common residual is at most

$$\sum_j q_j y_j \le \frac{1}{2} + \frac{2n\tau_n H}{\eta} = \frac{1}{2} + o(1).$$

Here $\operatorname{arcosh}(1 + 2\tau_n/B) \le 2\sqrt{\tau_n/B}$ gives $H \le n^{1/4}$. Also

$$\sum_j (q_j - 1) \le \frac{4\tau_n H}{\eta B} \sum_j \frac{1}{1 - z_j} = O(n^{1/4} \log^3 n).$$

The last sum is $O(n^2)$ by $1 - \cos\theta_j \gtrsim j^2/n^2$. Add the same positive entry to each list to make their traces one. At least three x_j with θ_j bounded away from zero and π are of order B. This proves the lemma. □

Completion of the proof of Theorem 1. For N large, choose $n = \lfloor c_1 N \rfloor$ with $c_1 > 0$ sufficiently small. The lemma gives $D_n \le D$. The function $\log^2(n)/n^3$ is eventually decreasing, and its value at $c_1 A(\delta)$ is asymptotic to $\delta/(9c_1^3)$. After accounting for the factor $1/64$, this gives $\tau_n \ge \delta$. Pad both spectra by zeros. The three positive entries of order $1/n$ in the alternative permit the physical realization, over any shape $d_A, d_B \ge 2$ of total size D. The pair matches the first n moments and proves the lower bound. For bounded N, $M \ge 1$ and adjustment of the absolute constant suffices. □

5. Precision for raw moments

Proof of Theorem 2. For the first claim use the spectra from Lemma 4 and the same n as above. Scale only their varying entries by $s = \delta/\tau_n \le 1$ and reset their identical residuals to make the traces one. The first n moments still match. For $k \ge n+1$ the difference is at most

$$2D_n(sB)^k \le 2D_n(sB)^{n+1},$$

since every varying entry has absolute value at most $sB \le 1$ and the common residuals cancel. Moreover $\log(1/(sB)) = \log n + \log(\tau_n/\delta)$ is at least $c_2 \log(1/\delta)$ with $c_2 = 1/6$ for $\delta \le \delta_0$. Indeed, if $n \ge \delta^{-1/6}$ the first term is at least $\frac{1}{6}\log(1/\delta)$ and the second is nonnegative because $\tau_n \ge \delta$. If $n < \delta^{-1/6}$, then $\tau_n/\delta \ge \delta^{-1/2}/64$, and $\frac{1}{2}\log(1/\delta) - \log 64 \ge \frac{1}{6}\log(1/\delta)$ for small δ. Next we absorb the prefactor. As $D_n \le 2n \le 2A(\delta)$, we have $\log(2D_n) \le \log(4A(\delta)) \le \frac{1}{2}\log(1/\delta)$ for $\delta \le \delta_0$. Let N_0 be the threshold above which

the completion proof applies, enlarged so that $c_1 N \geq 2\max(1, 1/c_2)$ and $n \geq c_1 N/2$ when $N \geq N_0$. Then $c_2(n+1) \geq 1$, and

$$\begin{aligned}\log\bigl(2D_n(sB)^{n+1}\bigr) &\leq \log(1/\delta)\Bigl[\tfrac{1}{2} - c_2(n+1)\Bigr] \\ &\leq -\tfrac{c_2}{2}(n+1)\log(1/\delta) \leq -\tfrac{c_1 c_2}{4} N \log(1/\delta).\end{aligned}$$

The physical construction survives the scaling. For $N < N_0$, so that $D < N_0$, and small δ, a two-qubit block $(1-u)I/4 + u|\Phi\rangle\langle\Phi|$, with $|\Phi\rangle$ maximally entangled, gives the analogous pair at $u = 1/3$ and $u = 1/3 + 4\delta/3$, padded with zeros. The partial-transpose eigenvalues are $(1+u)/4$, three times, and $(1-3u)/4$, which equals 0 and $-\delta$ at the two parameters. The difference in moments is at most $C_W\delta$ uniformly in j, by the mean value theorem, since near $u = 1/3$ all partial-transpose eigenvalues have absolute value at most $\kappa < 1$ and $\sup_j j\kappa^{j-1} < \infty$. At $u = 1/3$ a diagonal separable state with the same partial-transpose spectrum can be substituted. With $c \leq 1/(2N_0)$ one has $cN \leq \frac{1}{2}$, so $\exp[-cN\log(1/\delta)] \geq \delta^{1/2} \geq C_W\delta$ once $\delta \leq C_W^{-2}$. Both branches hold with

$$c = \min\Bigl(\frac{c_1 c_2}{4}, \frac{1}{2N_0}\Bigr).$$

For the second claim, use the polynomial in Section 2 when $A(\delta) \leq D$. Its coefficient ℓ^1-norm is bounded by $(C_3/B)^{2(\ell+r)+2}$. Each deflation factor has coefficient norm at most $2/B$, and the Chebyshev bound follows directly from its three-term recurrence and $\|1-2t/B\|_1 = 1+2/B$. A moment vector distance ϵ makes the difference in its two sums at most $\epsilon\|f\|_1$, whereas that difference is at least $\delta^2/2$. This yields the asserted exponential bound.

When $D < A(\delta)$, use the first D moments. The elementary symmetric polynomials are the coefficients through degree D of

$$\exp\left(\sum_{k=1}^{D}(-1)^{k-1}p_k\frac{z^k}{k}\right).$$

On $|z| = 1/2$ this generating function is uniformly bounded because $|p_k| \leq 1$. Indeed partial transpose preserves the Hilbert–Schmidt norm, and $p_1 = 1$. A maximum moment difference ϵ changes the generating function there by at most $C_4\epsilon$. Cauchy's estimate bounds the coefficient changes at degree j by $C_4\epsilon 2^j$. Evaluate the characteristic polynomials at a negative eigenvalue $t \leq -\delta$ of the alternative. It belongs to $[-1, 0)$. The alternative polynomial vanishes there, whereas the candidate polynomial has magnitude at least δ^D. Hence $\delta^D \leq C_5\epsilon 2^D$, proving the claim. □

6. Range of the theorems

The theorems are stated for exact moments and for adversarial absolute perturbations of them. They apply to every PPT state, including PPT entangled states, and give the moment order that any estimation protocol must reach.

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Luminary Oddities