# SHARP MOMENT ORDER FOR DETECTING A NEGATIVE PARTIAL TRANSPOSE

SCOTT SCHUBR

Abstract. We determine how many exact moments distinguish a positive partial transpose from a negative eigenvalue of magnitude at least $\delta$. For total bipartite dimension $D$, let $N = \min\{D, \delta^{-1/3}\log^{2/3}(1/\delta)\}$. For small $\delta$, the optimal order is $\Theta(N)$, uniformly in the two local dimensions. A polynomial that vanishes at the large eigenvalues of a candidate positive spectrum gives the upper bound. A contraction argument moves repeated Chebyshev nodes to construct separable and entangled states with matching moments, proving that the logarithmic factor is necessary. The required absolute moment precision has scale $\exp[-\Theta(N\log(1/\delta))]$. We also give explicit physical pairs agreeing through moment $D-1$.

## 1. Moment data and the gap promise

Let $d_A, d_B \geq 2$ and $D = d_A d_B$. For a density operator $\rho$ on $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}$, write $\rho^\Gamma$ for its partial transpose on the second factor. Put

$$p_j(\rho) = \mathrm{Tr}[(\rho^\Gamma)^j], \qquad j \geq 1.$$

In particular $p_1 = 1$. The PPT class consists of states for which $\rho^\Gamma$ is positive semidefinite. The alternative class consists of states with $\lambda_{\min}(\rho^\Gamma) \leq -\delta$. A PPT spectrum can always be reproduced by a separable state diagonal in a product basis. Moment data shared by a PPT state and an alternative state therefore cannot certify entanglement.

Let $M_{d_A,d_B}(\delta)$ be the least $m$ for which the first $m$ moment vectors of the two promised classes are disjoint. This is an information-theoretic order in the exact moment model. Let

$$A(\delta) = \delta^{-1/3}\log^{2/3}(1/\delta), \qquad N = \min\{D, A(\delta)\}.$$

All logarithms are natural.

**Theorem 1.** *There are absolute constants $c, C, \delta_0 > 0$ such that, for every $d_A, d_B \geq 2$ and $0 < \delta \leq \delta_0$,*

$$cN \leq M_{d_A,d_B}(\delta) \leq CN.$$

*Date*: November 24, 2026.

2020 *Mathematics Subject Classification.* 81P15, 44A60, 41A10.

*Key words and phrases.* Partial transpose, entanglement witnesses, moment matching, Chebyshev polynomials.

**Theorem 2.** *With the same range of parameters and absolute constants, the following hold.*

(1) *There is a separable state and an alternative state with negative eigenvalue $-\delta$ such that, simultaneously for every $j \geq 1$,*

$$|p_j(\rho) - p_j(\sigma)| \leq \exp[-cN \log(1/\delta)].$$

(2) *There is $m \leq CN$ such that every pair from opposite promised classes satisfies*

$$\max_{1 \leq j \leq m} |p_j(\rho) - p_j(\sigma)| \geq \exp[-CN \log(1/\delta)].$$

Given finite moments, the PPT tests of Yu, Imai, and Gühne [8], and the hierarchy of Neven et al. [4], establish the general testing framework. Theorems 1 and 2 quantify the order and precision required uniformly under an absolute spectral-gap promise. Dou, Wu, and Chen characterize the two-qubit inverse spectrum and its low-order moment bounds [2]. Lancien and McShane study fixed-order thresholds for induced random states [3]. Spectral moment bounds for distance to the positive cone appear in [1]. Deflating large eigenvalues in matrix-vector testing is studied in [5], under a different observation model. The Chebyshev level-set construction in [7, Theorem 36] matches two positive spectra. We use a shift of those levels to put one spectrum across zero, and use displaced nodes and integer multiplicities for the gap-dependent lower bound. The latter construction is motivated by polynomial sign masking in [6].

## 2. A polynomial upper bound

Let $y = (y_i)_{i=1}^{D}$ be a nonnegative candidate spectrum of trace one. Fix $B \in (0, 1)$, and form a factor for each $y_i > B$, counting multiplicity. Their number $\ell$ is at most $1/B$. With $T_r$ the Chebyshev polynomial of the first kind, define

$$P(t) = \prod_{y_i > B} \frac{y_i - t}{y_i + \delta} \frac{T_r(1 - 2t/B)}{T_r(1 + 2\delta/B)}, \qquad f(t) = t^2 P(t)^2.$$

For every $t \leq -\delta$, all factors and the Chebyshev ratio are at least one. Thus for an alternative spectrum $x$, $\sum_i f(x_i) \geq \delta^2$. At a candidate eigenvalue $y_i > B$ the polynomial is zero. At a candidate eigenvalue $0 \leq y_i \leq B$ each product factor has absolute value at most one and $|T_r(1 - 2y_i/B)| \leq 1$. Hence

$$\sum_i f(y_i) \leq \frac{\sum_{y_i \leq B} y_i^2}{T_r(1 + 2\delta/B)^2} \leq \frac{B}{T_r(1 + 2\delta/B)^2}.$$

For $B$ of order $\delta^{1/3}/\log^{2/3}(1/\delta)$, take $r$ of order $\sqrt{B/\delta}\log(1/\delta)$, with a sufficiently large absolute prefactor. Since $T_r(1 + u) = \cosh(r \operatorname{arcosh}(1 + u))$ for $u > 0$, the candidate sum is at most $\delta^2/2$. The degree $2(\ell + r) + 2$ is $O(A(\delta))$. Matching that many moments is therefore impossible. The first $D$

moments also suffice, by Newton's identities. This proves the upper bound in Theorem 1.

## 3. A physical realization of matched spectra

The following elementary realization will be used twice. Suppose a real trace-one spectrum has exactly one negative entry $-\tau$ and has three positive entries $t, a, e$ with $a, e \geq t \geq \tau$. Embed a two-by-two product block in the bipartite space and put

$$\rho_{\text{block}} = \begin{pmatrix} a & 0 & 0 & c \\ 0 & (t-\tau)/2 & 0 & 0 \\ 0 & 0 & (t-\tau)/2 & 0 \\ c & 0 & 0 & e \end{pmatrix}, \qquad c = (t+\tau)/2.$$

The block is positive semidefinite, since $c^2 \leq t^2 \leq ae$. Its partial transpose moves the coherence to the central block and has eigenvalues $a, e, t, -\tau$. Put the remaining nonnegative weights on the unused product basis vectors. The comparator nonnegative trace-one spectrum gives a diagonal separable state in the same space.

**Proposition 3.** *For every nontrivial bipartite dimension $D$, a separable state and a state with exactly one negative partial-transpose eigenvalue agree through moment $D-1$. The negative eigenvalue has magnitude*

$$\tau_D = \frac{\sec(\pi/D) - 1}{D} \sim \frac{\pi^2}{2D^3}.$$

*Proof.* Let $b = \cos(\pi/D)$. Take the root multisets of $T_D(z) = (-1)^D$ and $T_D(z) = -(-1)^D$, respectively, and transform each root by $x = (z+b)/(Db)$. The root multisets are real including multiplicities. The transformed monic polynomials differ only in their constant terms, so the spectra sum to one and their first $D-1$ moments agree. The second spectrum is nonnegative. In the first spectrum the unique negative entry comes from $z = -1$, while the least positive entry comes from $z = -\cos(2\pi/D)$. This latter entry is at least $\tau_D$ because $2b \geq 1 + \cos(2\pi/D) = 2b^2$. The physical realization above applies. Newton's identity at degree $D$ gives the first differing moment explicitly, in absolute value, $D\,2^{2-D}/(Db)^D$. □

## 4. Displacing repeated nodes

**Lemma 4.** *For every sufficiently large integer $n$ there are two real trace-one spectra of size $D_n = n + O(n^{1/4}\log^3 n)$, one nonnegative and the other with one strictly negative entry*

$$-\tau_n = -\frac{(\log n)^2}{64n^3}.$$

*Their first $n$ power sums agree. Apart from a common residual entry, all eigenvalues have absolute value at most $1/n$, and at least three positive entries on the negative side have values of order $1/n$.*

*Proof.* Set $B = 1/n$, $\theta_j = (j - \frac{1}{2})\pi/n$, $z_j = \cos\theta_j$, $y_j = B(1 - z_j)/2$, and $h_j = B\sin\theta_j/(2n)$, for $1 \le j \le n$. Let $L_i(t)$ be the cardinal interpolation polynomials for the distinct nodes $y_j$, so that $L_i(y_j) = \mathbf{1}_{i=j}$. Define

$$H = T_n(1 + 2\tau_n/B), \qquad E_i = \frac{\tau_n H \sin\theta_i}{n(1 - z_i)},$$
$$\beta_i = \int_{-\tau_n}^{0} L_i(t)\,dt, \qquad \eta = \frac{c_0}{\log(n+1)}, \qquad q_i = \left\lceil \frac{2E_i}{\eta h_i} \right\rceil,$$

where $c_0 > 0$ is an absolute constant chosen below. The explicit cardinal formula in the variable $z = 1 - 2t/B$ is

$$L_i(t) = \frac{T_n(z)}{(z - z_i)T_n'(z_i)}.$$

As $|T_n'(z_i)| = n/\sin\theta_i$, this formula on $[-\tau_n, 0]$ gives $|\beta_i| \le E_i$.

For $|t - y_j| \le \eta h_j$, write $z = \cos(\theta_j + u)$. For a sufficiently small absolute $\eta$, one has $|u| \le 2\eta/n$. This follows by integrating $\sin\theta$ near $\theta_j$ and using $n\sin\theta_j \ge 1$. The cardinal formula gives

$$|L_j(t) - 1| \le C_0\eta, \qquad |L_i(t)| \le \frac{C_0\eta\sin\theta_i}{n|z_i - z_j|} \quad (i \ne j). \tag{1}$$

For the diagonal bound, both the numerator and denominator in

$$\frac{\sin(nu)/(nu)}{(\cos\theta_j - \cos(\theta_j + u))/(u\sin\theta_j)}$$

are within $O(\eta)$ of one. For the off-diagonal bound, $|T_n(\cos(\theta_j + u))| = |\sin(nu)| \le 2\eta$, and $|z - z_j| \le \eta\sin\theta_j/n$ is at most $\eta|z_i - z_j|$. Indeed the cosine difference identity gives

$$\frac{\sin\theta_j}{n|z_i - z_j|} \le \frac{1}{|i - j|}.$$

Also $\sin\theta_j/[n(1 - z_j)] \le C_1/j$. By $(1 - z_i) \le |z_i - z_j| + (1 - z_j)$, the estimates in (1) consequently give

$$\sup_i \sum_j \frac{E_j}{E_i} \sup_{|t - y_j| \le \eta h_j} |L_i(t) - \mathbf{1}_{i=j}| \le C_2\eta\log(n+1). \tag{2}$$

Use the weighted norm $\|s\| = \max_i |s_i|/E_i$. On $\|s\| \le 2$ define the map

$$F_i(s) = \beta_i - \sum_{j=1}^{n} q_j \int_{y_j}^{y_j + s_j/q_j} [L_i(t) - \mathbf{1}_{i=j}]\,dt.$$

The displacement is at most $\eta h_j$. By (2), for small enough $c_0$ the map has Lipschitz constant at most $1/2$. Since $\|F(0)\| \le 1$, it maps the closed ball to itself and has a fixed point $s$. Set $x_j = y_j + s_j/q_j$. The small displacement

implies $x_j \in (0, B)$: both $y_j/h_j$ and $(B - y_j)/h_j$ are bounded below by an absolute positive constant. At the fixed point,

$$\sum_j q_j \int_{y_j}^{x_j} L_i(t)\,dt = \int_{-\tau_n}^{0} L_i(t)\,dt.$$

The $L_i$ span the polynomials of degree at most $n-1$. Applying the identity to $kt^{k-1}$, $1 \le k \le n$, gives $\sum_j q_j(x_j^k - y_j^k) = -(-\tau_n)^k$. Thus the multisets of $q_j$ copies of each $x_j$ plus $-\tau_n$, and $q_j$ copies of each $y_j$ plus zero, match the first $n$ moments.

The trace before adding a common residual is at most

$$\sum_j q_j y_j \le \frac{1}{2} + \frac{2n\tau_n H}{\eta} = \frac{1}{2} + o(1).$$

Here $\operatorname{arcosh}(1 + 2\tau_n/B) \le 2\sqrt{\tau_n/B}$ gives $H \le n^{1/4}$. Also

$$\sum_j (q_j - 1) \le \frac{4\tau_n H}{\eta B} \sum_j \frac{1}{1 - z_j} = O(n^{1/4} \log^3 n).$$

The last sum is $O(n^2)$ by $1 - \cos\theta_j \gtrsim j^2/n^2$. Add the same positive entry to each list to make their traces one. At least three $x_j$ with $\theta_j$ bounded away from zero and $\pi$ are of order $B$. This proves the lemma. □

*Completion of the proof of Theorem 1.* For $N$ large, choose $n = \lfloor c_1 N \rfloor$ with $c_1 > 0$ sufficiently small. The lemma gives $D_n \le D$. The function $\log^2(n)/n^3$ is eventually decreasing, and its value at $c_1 A(\delta)$ is asymptotic to $\delta/(9c_1^3)$. After accounting for the factor $1/64$, this gives $\tau_n \ge \delta$. Pad both spectra by zeros. The three positive entries of order $1/n$ in the alternative permit the physical realization, over any shape $d_A, d_B \ge 2$ of total size $D$. The pair matches the first $n$ moments and proves the lower bound. For bounded $N$, $M \ge 1$ and adjustment of the absolute constant suffices. □

## 5. Precision for raw moments

*Proof of Theorem 2.* For the first claim use the spectra from Lemma 4 and the same $n$ as above. Scale only their varying entries by $s = \delta/\tau_n \le 1$ and reset their identical residuals to make the traces one. The first $n$ moments still match. For $k \ge n + 1$ the difference is at most

$$2D_n(sB)^k.$$

Moreover $\log(1/(sB)) = \log n + \log(\tau_n/\delta)$ is at least $c_2 \log(1/\delta)$. For example, if $n \ge \delta^{-1/6}$, the first term suffices. Otherwise the formula for $\tau_n$ gives the claim from the second term. The physical construction survives the scaling. For bounded $N$ and small $\delta$, a two-qubit block $(1-u)I/4 + u|\Phi\rangle\langle\Phi|$, with $|\Phi\rangle$ maximally entangled, gives the analogous pair at $u = 1/3$ and $u = 1/3 + 4\delta/3$, padded with zeros. The difference in moments is uniformly $O(\delta)$ by the mean value theorem, since near $u = 1/3$ all partial-transpose eigenvalues have absolute value bounded by a constant strictly less than

one. At $u = 1/3$ a diagonal separable state with the same partial-transpose spectrum can be substituted.

For the second claim, use the polynomial in Section 2 when $A(\delta) \le D$. Its coefficient $\ell^1$-norm is bounded by $(C_3/B)^{2(\ell+r)+2}$. Each deflation factor has coefficient norm at most $2/B$, and the Chebyshev bound follows directly from its three-term recurrence and $\|1-2t/B\|_1 = 1+2/B$. A moment vector distance $\epsilon$ makes the difference in its two sums at most $\epsilon\|f\|_1$, whereas that difference is at least $\delta^2/2$. This yields the asserted exponential bound.

When $D < A(\delta)$, use the first $D$ moments. The elementary symmetric polynomials are the coefficients through degree $D$ of

$$\exp\left(\sum_{k=1}^{D}(-1)^{k-1}p_k\frac{z^k}{k}\right).$$

On $|z| = 1/2$ this generating function is uniformly bounded because $|p_k| \le 1$. Indeed partial transpose preserves the Hilbert–Schmidt norm, and $p_1 = 1$. A maximum moment difference $\epsilon$ changes the generating function there by at most $C_4\epsilon$. Cauchy's estimate bounds the coefficient changes at degree $j$ by $C_4\epsilon 2^j$. Evaluate the characteristic polynomials at a negative eigenvalue $t \le -\delta$ of the alternative. It belongs to $[-1, 0)$. The alternative polynomial vanishes there, whereas the candidate polynomial has magnitude at least $\delta^D$. Hence $\delta^D \le C_5\epsilon 2^D$, proving the claim. □

## 6. Scope of the result

The bounds concern partial-transpose moment data and a promise on the least eigenvalue. They apply to exact real observations and to absolute adversarial perturbations of those observations as stated. The separate question of the number of quantum state copies needed under a specified measurement protocol depends on the variances and correlations of the moment estimators. The PPT side of the test includes PPT entangled states.

## Author contributions and AI assistance

Scott Schubr provided the research direction and the source Math Lab program on polynomial sign recovery. AI-assisted tools were used for literature searches, mathematical derivation and proof checks, and manuscript preparation.

## References

[1] G. de las Cuevas, T. Fritz, and T. Netzer, *Optimal bounds on the positivity of a matrix from a few moments*, arXiv:1808.09462.

[2] R. Dou, S. Wu, and Z.-B. Chen, *The inverse eigenvalue problem for partial transposes of two-qubit states and sharp entanglement bounds from their low-order moments*, arXiv:2608.29068.

[3] C. Lancien and K. McShane, *Moment-based PPT criteria for random bipartite states*, arXiv:2607.11369.

[4] A. Neven, J. Carrasco, V. Vitale, C. Kokail, A. Elben, M. Dalmonte, P. Calabrese, P. Zoller, B. Vermersch, R. Kueng, and B. Kraus, *Symmetry-resolved entanglement detection using partial transpose moments*, npj Quantum Inf. **7** (2021), 152. doi:10.1038/s41534-021-00487-y.
[5] D. Needell, W. Swartworth, and D. P. Woodruff, *Testing positive semidefiniteness using linear measurements*, arXiv:2204.03782.
[6] S. Schubr, *Sharp degree exponents for polynomial detection of isolated negative atoms*, Luminary Research Archive, edition 2 (2026). doi:10.5281/zenodo.24473032.
[7] Q. Ye, Z. Liu, and D.-L. Deng, *Exponential advantage from one more replica in estimating nonlinear properties of quantum states*, arXiv:2509.24000.
[8] X.-D. Yu, S. Imai, and O. Gühne, *Optimal entanglement certification from moments of the partial transpose*, Phys. Rev. Lett. **127** (2021), 060504. doi:10.1103/PhysRevLett.127.060504.

Luminary Oddities Math Lab