# Sharp mean-distance bounds for spectral gaps of compact metric graphs and trees

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November 25, 2026 (edition 2)

**Abstract**

Baptista, Kennedy and Mugnolo (J. Geom. Anal., 2024) proved that the spectral gap $\mu_2$ of the standard Kirchhoff Laplacian on a compact metric graph of total length $L$ and the mean distance $\rho$ between two uniform random points satisfy $\mu_2 L\rho \geq 1$, showed that $\mu_2\rho^2 \leq \pi^2$ on trees, and asked (their Open Problem 3.4) whether $\mu_2\rho^2$ is bounded above and below by absolute constants. Both answers are negative: the fixed-diameter dumbbells of Kennedy, Kurasov, Malenová and Mugnolo settle the lower side, and Mugnolo's unilateral expander graphs settle the upper side. We determine the sharp constants in the two published bounds. Every compact metric graph satisfies $\mu_2 L\rho > 2$, and the constant 2 is approached by unit-edge trees of maximum degree three, on which $\mu_2\rho^2$ tends to zero. Every metric tree satisfies $\mu_2\rho^2 < \pi^2/4$, the limit value of equilateral stars observed by Baptista, Kennedy and Mugnolo. The first bound follows from the variance identity and a Cauchy–Schwarz inequality along shortest paths. The second follows from a sharp rooted Dirichlet inequality, proved by a phase induction on branching and transferred to the spectral gap through a nodal zero. For unit-edge trees of maximum degree three with $m$ edges, the minimum of $\mu_2\rho^2$ has order $(\log m)/m$, and an effective-resistance trace identity gives the exact deficit $\mu_2 L\rho - 2$.

**Keywords.** Metric graph, quantum graph, spectral gap, mean distance, metric tree, effective resistance.
**MSC 2020.** Primary 34B45. Secondary 35P15, 05C12.

## 1 Introduction

For a finite connected compact metric graph $\mathcal{G}$ of positive total length $L$, let

$$\rho_{\mathrm{pr}} = \rho_{\mathrm{pr}}(\mathcal{G}) = \frac{1}{L^2}\int_{\mathcal{G}}\int_{\mathcal{G}} d(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

where $d$ is shortest-path distance. This is the mean distance $\rho(\mathcal{G})$ of Baptista, Kennedy and Mugnolo [2]; the subscript separates it from the mean distance $\rho_D$ to a Dirichlet set, which appears below. Write $\mu_2$ for the first positive eigenvalue of the Laplacian with continuity and standard Kirchhoff conditions, including Neumann conditions at leaves; no vertex is Dirichlet. The Rayleigh characterization is

$$\mu_2 = \min_{\substack{0\neq u\in H^1(\mathcal{G})\\ \int_{\mathcal{G}} u=0}} \frac{\int_{\mathcal{G}} |u'|^2}{\int_{\mathcal{G}} |u|^2}, \tag{1}$$

where $H^1(\mathcal{G})$ consists of continuous functions that are $H^1$ on each edge.

*This edition* renames the pairwise mean distance $\rho_{\mathrm{pr}}$, credits the known answers to [2, Open Problem 3.4], writes out the flux identity of the phase induction and the counting step for bounded-degree trees, and adds the Dirichlet analogues; edition 1 is [7].

**The question and its known answers.** Baptista, Kennedy and Mugnolo proved

$$\frac{1}{L} \le \mu_2(\mathcal{G})\rho_{\mathrm{pr}}(\mathcal{G})$$

for every compact metric graph [2, Theorem 3.5], and, for a graph with $N_E$ edges and first Betti number $\beta$, the two-sided bound $\pi^2(2N_E-1)^2/(16N_E^4) \le \mu_2\rho_{\mathrm{pr}}^2 \le \min\{\pi^2 N_E^2/9,\ \pi^2(1+\beta)^2\}$ [2, Corollary 3.3], which gives $\mu_2\rho_{\mathrm{pr}}^2 \le \pi^2$ on trees. They observe that $\mu_2\rho_{\mathrm{pr}}^2$ increases to $\pi^2/4$ along equilateral stars and along equilateral flowers as the number of edges grows [2, Section 3, discussion preceding Open Problem 3.4], and they ask [2, Open Problem 3.4]:

> *Do there exist absolute constants $C, c > 0$ such that $c \le \mu_2(\mathcal{G})\rho_{\mathrm{pr}}(\mathcal{G})^2 \le C$ for all compact graphs $\mathcal{G}$?*

Both answers are negative in the literature. For the lower side, the flower dumbbells of fixed diameter of Kennedy, Kurasov, Malenová and Mugnolo [3, Example 5.1] have spectral gap tending to zero, and $\rho_{\mathrm{pr}} \le \operatorname{diam}(\mathcal{G})$. For the upper side, Mugnolo shows with unilateral expander graphs that no universal $C$ satisfies $\mu_2 \le C/\rho_{\mathrm{pr}}^2$ [4, Proposition 3.9].

**Results.** This article determines the sharp constants in the two published bounds of [2].

- Theorem 1 improves [2, Theorem 3.5] from 1 to the sharp, unattained constant 2: every compact metric graph satisfies $\mu_2 L\rho_{\mathrm{pr}} > 2$, and the constant is approached by unit-edge trees of maximum degree three, on which also $\mu_2\rho_{\mathrm{pr}}^2 \to 0$.
- Theorem 2 improves the tree case of [2, Corollary 3.3] from $\pi^2$ to the sharp, unattained supremum $\pi^2/4$, so the star limit observed in [2] is the supremum of $\mu_2\rho_{\mathrm{pr}}^2$ over all metric trees, approached but not attained.
- Section 5 shows that the minimum of $\mu_2\rho_{\mathrm{pr}}^2$ over unit-edge trees of maximum degree three with $m$ edges has order $(\log m)/m$, and Section 6 gives an exact effective-resistance formula for the deficit $\mu_2 L\rho_{\mathrm{pr}} - 2$.

As a consequence of the deficit formula, every graph that is close to extremal in Theorem 1 has an isolated first positive eigenvalue: $\mu_3/\mu_2 \ge 2/(\mu_2 L\rho_{\mathrm{pr}} - 2)$.

For the Dirichlet problem with a nonempty marked set $D$ and $\rho_D = L^{-1}\int_{\mathcal{G}} d(x,D)\,\mathrm{d}x$, the analogue of Theorem 1 is Plümer's lower bound $1/(L\rho_D) \le \lambda_1(\mathcal{G},D)$ [5, Theorem 4.4.3], as cited in [4, Section 4.2]. The matching upper bound in the Dirichlet setting is the sharp constant $\lambda_1\rho_D^2 < C_* \approx 0.66167$ of [8, Theorem 1.1], which proves Mugnolo's Conjecture 4.7 [4]. The lower-bound construction below concentrates arclength near the two ends of a joining path, as does the endpoint-mass method for the marked-Dirichlet problem in [6].

**Route.** Theorem 1 follows from the variance identity and the Cauchy–Schwarz inequality along shortest paths (Section 2), with sharpness from two binary trees joined by a long path (Section 3). Theorem 2 reduces, through a nodal zero, to the rooted Dirichlet inequality (3), which we prove by an induction on branching that tracks the phase of the ground state (Lemma 1 and (5)).

**Theorem 1.** *Every finite connected compact metric graph of positive length satisfies*

$$\mu_2(\mathcal{G}) L \rho_{\mathrm{pr}}(\mathcal{G}) > 2.$$

*The infimum is* 2*, even over trees consisting entirely of unit edges and of maximum degree at most three. On the same sequence of trees,* $\mu_2 \rho_{\mathrm{pr}}^2 \to 0$.

## 2 The lower bound

Let $u \in H^1(\mathcal{G})$ be nonzero with mean zero, and set $E = \int_{\mathcal{G}} |u'|^2 > 0$. A shortest path $\gamma$ from $x$ to $y$ has no repeated segments of positive length, so Cauchy–Schwarz along $\gamma$ gives

$$|u(x) - u(y)|^2 \le d(x,y) \int_\gamma |u'|^2 \le d(x,y) E.$$

The second inequality is strict on a positive-measure set of pairs. Indeed, choose an interval $I$ strictly inside an edge with $\int_I |u'|^2 < E$, shorter than its distance to either endpoint of the edge. For distinct $x, y \in I$ the direct segment is shortest, since a route leaving the edge interior must first reach an endpoint, and the path energy is at most $\int_I |u'|^2$. By the variance identity,

$$2L \int_{\mathcal{G}} |u|^2 = \int_{\mathcal{G}} \int_{\mathcal{G}} |u(x) - u(y)|^2 \, \mathrm{d}x \, \mathrm{d}y < L^2 \rho_{\mathrm{pr}} E.$$

Taking a first positive eigenfunction in (1) gives the strict bound. In particular no finite graph attains the constant.

## 3 Sharpness on unit-edge trees

For each integer $h \ge 1$, join the roots of two full binary trees of depth $h$ by a path of $b = h^2$ edges, all edges of length one. Each binary tree has length $S = 2^{h+1} - 2$, and $L = 2S + b$. All degrees are at most three. Take the function equal to $-1$ on one binary tree and $+1$ on the other, affine on the joining path. It has mean zero, energy $4/b$, and squared norm $2S + b/3$. Therefore

$$\mu_2 \le \frac{4}{b(2S + b/3)}. \tag{2}$$

For a uniform pair of points, opposite binary trees occur with probability $2S^2/L^2$, and on this event the distance lies in $[b, b + 2h]$. Inside a single binary tree the distance is at most $2h$. At least one point lies on the joining path with probability at most $2b/L$, and every distance is at most $b + 2h$. Thus

$$\frac{2S^2}{L^2} \le \frac{\rho_{\mathrm{pr}}}{b} \le \frac{2S^2}{L^2} + \frac{2h}{b} + \frac{2b}{L}.$$

Now $h/b \to 0$ and $b/S \to 0$, so $\rho_{\mathrm{pr}}/b \to 1/2$. Equation (2) gives $\limsup \mu_2 L \rho_{\mathrm{pr}} \le 2$, and the universal lower bound then gives $\mu_2 L \rho_{\mathrm{pr}} \to 2$. Since $\rho_{\mathrm{pr}}/L \to 0$, we also have $\mu_2 \rho_{\mathrm{pr}}^2 \to 0$.

Together with [3, Example 5.1] and [4, Proposition 3.9], this answers both sides of [2, Open Problem 3.4] in the negative; the family above adds a bounded-degree unit-edge example on which the sharp length-weighted constant of Theorem 1 is attained in the limit.

## 4 A sharp upper bound for metric trees

For a rooted tree $(T, o)$ let $r_o = |T|^{-1} \int_T d(o, x)\, \mathrm{d}x$ be the mean distance to the root, and let $\lambda_D(T, o)$ be the first eigenvalue with Dirichlet condition at $o$ and natural conditions elsewhere. We first prove the rooted inequality

$$\sqrt{\lambda_D(T, o)}\, r_o \le \frac{\pi}{4}. \tag{3}$$

Equality holds precisely when $T$ is a union of equal-length intervals joined only at $o$, including a single interval rooted at its endpoint, up to degree-two subdivision.

Suppose first that the root has degree one. Rescale distances so that $\lambda_D = 1$, and let $u$ be the first eigenfunction, chosen positive on $T \setminus \{o\}$. Positivity follows from the ground state principle on the connected Dirichlet-cut tree, or from taking a nonnegative minimizer and using uniqueness for the edge equation and the Kirchhoff conditions at a putative zero. Orient every edge away from $o$, and write $u'$ for the derivative in that direction.

**Lemma 1** (Flux identity)**.** *For every point $x \neq o$ that is not a vertex, let $T_x$ be the closed subtree of points whose path to $o$ passes through $x$. Then*

$$u'(x) = \int_{T_x} u(y)\, \mathrm{d}y > 0.$$

*Consequently the phase $\theta = \arctan(u'/u)$ lies in $(0, \pi/2)$ at every such point, it satisfies $\theta' = -1$ along each edge, it tends to $\pi/2$ at $o$ and to $0$ at every leaf, and at a vertex with incoming phase $\theta$ and outgoing phases $\theta_i$,*

$$\tan\theta = \sum_i \tan\theta_i, \qquad 0 \le \theta_i \le \theta < \pi/2. \tag{4}$$

*Proof.* Integrate $-u'' = u$ over every edge piece of $T_x$. On an edge piece $[\sigma, \tau]$ oriented away from $o$ the integral of $-u''$ is $u'(\sigma) - u'(\tau)$. Summing, the contribution of every internal vertex $v$ of $T_x$ is $\sum_i u_i'(v) - u_{\mathrm{in}}'(v)$, the outgoing derivatives minus the incoming one, which vanishes by the Kirchhoff condition; every leaf contributes $-u' = 0$ by the Neumann condition. The only remaining term is $u'(x)$ at the cut point, so $u'(x) = \int_{T_x} u$, which is positive since $u > 0$ on $T_x$. Hence $u'/u \in (0, \infty)$ and $\theta \in (0, \pi/2)$. On an edge, $\theta' = (u''u - u'^2)/(u^2 + u'^2) = -1$. At $o$, $u \to 0$ and $u' > 0$, so $\theta \to \pi/2$; at a leaf, $u' \to 0$ and $u > 0$, so $\theta \to 0$. At a vertex, $u$ is continuous and $u_{\mathrm{in}}' = \sum_i u_i'$, which is the tangent sum in (4). All terms are nonnegative, so $\tan\theta \ge \tan\theta_i$ and $\theta_i \le \theta$. □

Put $a = \pi/4$. For a downstream rooted piece, let $M$ be its length and $J$ its first arclength moment about its own root. At an incoming cut use the phase just before the split; at the root of an outgoing piece use that piece's outgoing phase. We claim, by induction from the leaves, that

$$M \ge \theta, \qquad J \le (\theta - a)M + a\theta - \theta^2/2. \tag{5}$$

The empty piece at a leaf has $M = J = \theta = 0$. On adjoining a stem of length $s$, by $\theta' = -1$ the triple $(\theta, M, J)$ becomes $(\theta + s, M + s, J + sM + s^2/2)$, and both deficits in (5) are unchanged. At a split the lengths and moments add. By (4),

$$\theta = \arctan\Big(\sum_i \tan\theta_i\Big) \le \sum_i \theta_i \le \sum_i M_i = M,$$

where the first inequality follows from superadditivity of the tangent when the sum of the angles is below $\pi/2$ and is automatic otherwise. This proves the length bound.

For the moment bound at a split, use $M_i \geq \theta_i$ and $\theta_i \leq \theta$ from (4) to obtain

$$J - (\theta - a)M \leq \sum_i [(a-\theta)\theta_i + \theta_i^2/2].$$

For $0 < t \leq \theta < \pi/2$, the function $f(t) = [(a-\theta)t + t^2/2]/\tan t$ is strictly increasing in $t$. Indeed its derivative multiplied by $\sin^2 t$ is

$$(a - \theta + t)\sin t \cos t - (a-\theta)t - t^2/2 \;\geq\; t^2/2 - a(t - \sin t \cos t) > 0.$$

For the last step, $(t - \sin t \cos t)/t^2$ increases strictly to $2/\pi$: its derivative has the sign of $2\cos t(\sin t - t\cos t) > 0$ on this interval. Thus each summand is at most $f(\theta)\tan\theta_i$, and by the tangent sum (4) the sum is at most $f(\theta)\tan\theta = (a-\theta)\theta + \theta^2/2 = a\theta - \theta^2/2$. The inequality is strict at a genuine split, where every positive outgoing angle is smaller than the incoming one. This proves (5), with strictness propagated toward the root.

At $o$, $\theta = \pi/2$ yields $J \leq aM$, with equality only on an interval. If $o$ has several branches, the Dirichlet problem is a direct sum with ground energy equal to the smallest of the branch ground energies. Apply the single-branch bound to each and average by branch lengths. Equality then requires interval branches with identical branch eigenvalues, hence equal lengths. This proves (3).

**Theorem 2.** *Every finite metric tree $T$ of positive length satisfies*

$$\mu_2(T)\rho_{\mathrm{pr}}(T)^2 < \frac{\pi^2}{4}.$$

*The supremum is $\pi^2/4$ and is not attained.*

*Proof.* Choose a real mean-zero eigenfunction at $\mu_2$, and choose a zero $o$ along the unique path between a point of positive value and a point of negative value. At least two distinct components of $T \setminus \{o\}$ carry nonzero sign parts of the eigenfunction. For each of those, the corresponding positive or negative part restricted to that component and extended by zero has Rayleigh quotient $\mu_2$ by the weak eigenfunction equation, and nonzero integral. Thus $\lambda_D(T, o) \leq \mu_2$. If it were strictly smaller, choose a ground state on a branch attaining $\lambda_D$, and one of the two sign tests supported on a different branch. A mean-zero combination of the two has quotient strictly below $\mu_2$, since their supports are disjoint up to $o$, which contradicts (1). The argument also covers eigenfunctions that vanish on entire edges.

The triangle inequality through $o$ gives $\rho_{\mathrm{pr}}(T) < 2r_o$; strictness holds for pairs in a short subinterval of any edge away from $o$. Together with (3) and $\lambda_D(T, o) = \mu_2$ this gives the upper bound. For the star with $n \geq 2$ equal arms of length $\ell$, $\mu_2 = \pi^2/(4\ell^2)$: subtracting the value at the center from a mean-zero function gives the Dirichlet lower bound, and opposite mixed-interval ground states on two arms give the matching upper bound. Averaging over pairs on the same arm and on different arms gives $\rho_{\mathrm{pr}} = \ell(1 - 2/(3n))$, in agreement with [2, Example 4.1]. The product tends to $\pi^2/4$. □

## 5 The lower range for unit-edge trees of bounded degree

Let $m$ be the number of unit edges, so $L = m$. Among such trees with maximum degree at most three, the infimum of $\mu_2\rho_{\mathrm{pr}}^2$ is of order $(\log m)/m$ as $m \to \infty$.

For the lower bound, fix a point $x$ and $r \geq 1$, and let $v$ be a vertex within one unit of $x$. The number of vertices at combinatorial distance $j \geq 1$ from $v$ is at most $3 \cdot 2^{j-1}$, and every edge meeting the ball $B(x, r)$ has an endpoint within distance $r+2$ of $v$. Each vertex has at most three edges, so

$$|B(x,r)| \leq 3\Big(1 + \sum_{j=1}^{\lceil r \rceil + 2} 3 \cdot 2^{j-1}\Big) \leq C_0 2^r$$

for an absolute constant $C_0$. For a uniform pair $(x, y)$, the probability that $d(x, y) \leq r$ is therefore at most $C_0 2^r/m$, and Markov's inequality in the form $\rho_{\mathrm{pr}} \geq r \Pr[d(x,y) > r]$ gives

$$\rho_{\mathrm{pr}} \geq r\Big(1 - \frac{C_0 2^r}{m}\Big).$$

With $r = \frac{1}{2}\log_2 m$ this reads $\rho_{\mathrm{pr}} \geq \frac{1}{2}(\log_2 m)(1 - C_0/\sqrt{m})$, so $\rho_{\mathrm{pr}} \geq c_0 \log m$ for all large $m$ and an absolute $c_0 > 0$. Theorem 1 then gives $\mu_2 \rho_{\mathrm{pr}}^2 > 2\rho_{\mathrm{pr}}/m \geq 2c_0(\log m)/m$.

For the reverse bound at every sufficiently large integer $m$, take an integer $b$ within two of $\log_2 m$ with the same parity as $m$, and use two identical binary trees, each with $(m-b)/2$ edges. Such a tree can be made by filling levels in breadth-first order and has depth $O(\log m)$. Join their roots by a unit-edge path of length $b$. The affine test of (2) gives $\mu_2 \leq C_1/(m \log m)$, and the diameter is $O(\log m)$, so $\mu_2 \rho_{\mathrm{pr}}^2 \leq C_2(\log m)/m$ for absolute constants $C_1, C_2$. This proves the matching upper order for the infimum.

## 6 An exact resistance decomposition

Define $R(x,y) = \sup_{E(u)>0} |u(x) - u(y)|^2/E(u)$, with $E(u) = \int_{\mathcal{G}} |u'|^2$, and write $\bar{R}$ for its pairwise arclength average. This is effective resistance with edge resistance equal to length; see, for example, [1]. Let $(\mu_j, \phi_j)_{j \geq 2}$ be the positive eigenvalues with multiplicity and their $L^2$-orthonormal eigenfunctions. In the mean-zero energy space, $(\phi_j/\sqrt{\mu_j})_{j \geq 2}$ is an orthonormal basis and $u \mapsto u(x) - u(y)$ is a bounded functional by the shortest-path inequality. Parseval and Tonelli therefore give

$$R(x,y) = \sum_{j \geq 2} \frac{|\phi_j(x) - \phi_j(y)|^2}{\mu_j}, \qquad L\bar{R} = 2\sum_{j \geq 2} \frac{1}{\mu_j}.$$

In particular the bound of Theorem 1 has the exact deficit

$$\mu_2 L \rho_{\mathrm{pr}} - 2 = 2\sum_{j \geq 3} \frac{\mu_2}{\mu_j} + \mu_2 L(\rho_{\mathrm{pr}} - \bar{R}).$$

Both terms are nonnegative since $R \leq d$. On a tree, $R = d$: a function of unit slope along the unique path, extended constantly on the branches off that path, realizes the supremum. Thus near equality forces an isolated first positive eigenvalue, $\mu_3/\mu_2 \geq 2/(\mu_2 L \rho_{\mathrm{pr}} - 2)$. For graphs with cycles the second term measures exactly the loss from replacing resistance by shortest-path distance.

## Statements and Declarations

**Data and code availability.** No external datasets were generated or analyzed. The analytic proofs are included in the article. The scripts `check_bounds.py` (exact rational check of the binary-tree construction) and `check_tree_phase.py` (randomized check of the phase induction) accompany the article.

**AI assistance.** OpenAI ChatGPT assisted with derivations, proof development and the supplementary checks, and OpenAI Codex with source checks. Anthropic Claude edited and cross-referenced this edition, expanded the flux and counting steps, and checked computations. Scott Schubr directed the research and is responsible for the content.

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