SHARP MEAN-DISTANCE BOUNDS FOR SPECTRAL GAPS OF COMPACT METRIC GRAPHS AND TREES

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Abstract. Can the spectral gap of a compact metric graph be bounded above and below by universal multiples of the inverse squared mean distance? The question posed by Baptista, Kennedy and Mugnolo concerns the mean distance ρ between two independent points sampled uniformly in arclength, and the first positive eigenvalue μ_2 of the standard Kirchhoff Laplacian. Neither bound holds over all finite connected compact metric graphs. We give an explicit family of unit-edge trees of maximum degree three for which $\mu_2\rho^2$ tends to zero; unboundedness above follows from Mugnolo's expander examples. We prove two sharp replacements. On every metric tree, $\mu_2\rho^2 < \pi^2/4$, with equal-arm stars approaching the constant. On every compact metric graph of total length L, $\mu_2 L\rho > 2$, with the constant approached even by unit-edge trees of maximum degree three. The tree upper bound follows from a sharp rooted Dirichlet inequality proved by induction on branching. For unit-edge trees of maximum degree three with m edges, the minimum of $\mu_2\rho^2$ is of order $(\log m)/m$. An effective-resistance identity gives the exact deficit in the length-dependent bound.

1. Introduction

For a finite connected compact metric graph $\mathcal{G}$ of positive total length L, let

$$\rho = \rho(\mathcal{G}) = \frac{1}{L^2}\int_{\mathcal{G}}\int_{\mathcal{G}} d(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

where d is shortest-path distance. Write μ_2 for the first positive eigenvalue of the Laplacian with continuity and standard Kirchhoff conditions, including Neumann conditions at leaves. No vertex is Dirichlet. In particular, ρ is not the mean distance to a marked Dirichlet set considered in Schubr, *Mean distance and the first two Dirichlet eigenvalues of compact metric graphs* (Archive Edition 1, DOI 10.5281/zenodo.24519135). The Rayleigh characterization is

$$\mu_2 = \min_{\substack{0\neq u\in H^1(\mathcal{G})\\ \int_{\mathcal{G}} u=0}} \frac{\int_{\mathcal{G}} |u'|^2}{\int_{\mathcal{G}} |u|^2}, \tag{1}$$

where $H^1(\mathcal{G})$ consists of continuous functions that are H^1 on each edge.

Baptista, Kennedy and Mugnolo [1, Theorem 3.5] proved $\mu_2 L\rho \geq 1$. Their Open Problem 3.4 asks for graph-independent positive lower and finite upper bounds for $\mu_2\rho^2$. Mugnolo (*The role of expanders in the spectral geometry of metric graphs*, arXiv:2607.14312v2, Proposition 3.9) ruled out the upper bound with expander graphs. The positive lower bound already fails by the fixed-diameter counterexamples of [2, Example 5.1], since $\rho \leq \operatorname{diam}(\mathcal{G})$. We give a unit-edge, bounded-degree tree counterexample and show that including total length gives the optimal constant 2. The tree upper bound π^2 in [1, Corollary 3.3] is not sharp: we show that the star limit $\pi^2/4$ is the supremum. The lower-bound construction concentrates arclength near the two ends of a joining path, analogous to the endpoint-mass method for the distinct marked-Dirichlet problem in Schubr, *Sharp length–inradius bounds and heat-content stability on compact metric graphs* (Archive Edition 17, DOI 10.5281/zenodo.24440362).

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Theorem 1. *Every finite connected compact metric graph of positive length satisfies*

$$\mu_2(\mathcal{G})L\rho(\mathcal{G}) > 2.$$

The infimum is 2*, even over trees consisting entirely of unit edges and of maximum degree at most three. On the same sequence of trees,* $\mu_2\rho^2 \to 0$.

2. The lower bound

Let $u \in H^1(\mathcal{G})$ be nonzero with mean zero, and set $E = \int_{\mathcal{G}} |u'|^2 > 0$. A shortest path γ from x to y has no repeated segments of positive length. Cauchy–Schwarz along γ gives

$$|u(x) - u(y)|^2 \le d(x,y) \int_\gamma |u'|^2 \le d(x,y)E.$$

The second inequality is strict on a positive-measure set of pairs. Indeed, choose an interval I strictly inside an edge with $\int_I |u'|^2 < E$, shorter than its distance to either endpoint of the edge. For distinct $x, y \in I$ the direct segment is shortest, since a route leaving the edge interior must first reach an endpoint. The path energy is at most $\int_I |u'|^2$. By the variance identity,

$$2L \int_{\mathcal{G}} |u|^2 = \int_{\mathcal{G}} \int_{\mathcal{G}} |u(x) - u(y)|^2 \,\mathrm{d}x \,\mathrm{d}y < L^2 \rho E.$$

Taking a first positive eigenfunction in (1) gives the strict bound. In particular no finite graph attains the constant.

3. Sharpness on unit-edge trees

For each integer $h \ge 1$, join the roots of two full binary trees of depth h by a path of $b = h^2$ edges, all edges of length one. Each binary tree has length $M = 2^{h+1} - 2$ and $L = 2M + b$. All degrees are at most three. Take the function equal to -1 on one binary tree and $+1$ on the other, affine on the joining path. It has mean zero, energy $4/b$, and squared norm $2M + b/3$. Therefore

$$\mu_2 \le \frac{4}{b(2M + b/3)}. \tag{2}$$

For a uniform pair of points, opposite binary trees occur with probability $2M^2/L^2$, and on this event the distance lies in $[b, b + 2h]$. Inside a single binary tree the distance is at most $2h$. At least one point lies on the joining path with probability at most $2b/L$, and every distance is at most $b + 2h$. Thus

$$\frac{2M^2}{L^2} \le \frac{\rho}{b} \le \frac{2M^2}{L^2} + \frac{2h}{b} + \frac{2b}{L}.$$

Now $h/b \to 0$ and $b/M \to 0$, so $\rho/b \to 1/2$. Equation (2) gives $\limsup \mu_2 L\rho \le 2$; the universal lower bound then gives $\mu_2 L\rho \to 2$. Since $\rho/L \to 0$, we also have $\mu_2\rho^2 \to 0$.

Together with Mugnolo's upper-bound counterexamples (arXiv:2607.14312v2, Proposition 3.9), this gives infimum 0 and supremum $+\infty$ for $\mu_2\rho^2$ over finite connected compact metric graphs, resolving both sides of [1, Open Problem 3.4].

4. A sharp upper bound for metric trees

For a rooted tree (T, o) let $r_o = |T|^{-1} \int_T d(o,x) \,\mathrm{d}x$ and let $\lambda_D(T, o)$ be the first eigenvalue with Dirichlet condition at o and natural conditions elsewhere. We first prove the rooted inequality

$$\sqrt{\lambda_D(T,o)}\, r_o \le \frac{\pi}{4}. \tag{3}$$

Equality holds precisely when T is a union of equal-length intervals joined only at o, including a single interval rooted at its endpoint, up to degree-two subdivision.

Suppose first that the root has degree one. Rescale distances so that $\lambda_D = 1$, and let u be the positive first eigenfunction away from o. Positivity follows from the ground state principle on the

connected Dirichlet-cut tree (or from taking a nonnegative minimizer and using uniqueness of the edge ODE and Kirchhoff conditions at a putative zero). Orient the edges away from o. Integration of $-u'' = u$ on the subtree downstream of an interior point, with cancellation of Kirchhoff fluxes, gives $u' > 0$ there. The phase $\theta = \arctan(u'/u)$ drops at unit speed along an edge, since $\theta' = -1$. At a vertex the incoming phase θ and outgoing phases θ_i satisfy

$$\tan\theta = \sum_i \tan\theta_i.$$

At a leaf the phase is zero; at o the one-sided phase is $\pi/2$.

Put $a = \pi/4$. For a downstream rooted piece, let M be its length and J its first arclength moment about its own root. At an incoming cut use the phase just before the split; at the root of an outgoing piece use that piece's outgoing phase. We claim, by induction from the leaves, that

$$M \geq \theta, \qquad J \leq (\theta - a)M + a\theta - \theta^2/2. \tag{4}$$

The empty piece at a leaf has $M = J = \theta = 0$. On adjoining a stem of length ℓ, (θ, M, J) becomes $(\theta+\ell, M+\ell, J+\ell M+\ell^2/2)$. Both deficits in (4) are unchanged. At a split the lengths and moments add. The inequality $\arctan(\sum_i \tan\theta_i) \leq \sum_i \theta_i$ follows from superadditivity of the tangent when the sum of angles is below $\pi/2$, and is automatic otherwise. This proves the length bound.

For the moment bound at a split, use $M_i \geq \theta_i$ and $\theta_i \leq \theta$ to obtain

$$J - (\theta - a)M \leq \sum_i \left[(a-\theta)\theta_i + \theta_i^2/2\right].$$

For $0 < y \leq \theta < \pi/2$, the function $[(a-\theta)y + y^2/2]/\tan y$ is strictly increasing in y. Indeed its derivative multiplied by $\sin^2 y$ is

$$(a - \theta + y)\sin y\cos y - (a-\theta)y - y^2/2 \;\geq\; y^2/2 - a(y - \sin y\cos y) > 0.$$

For the last step, $(y - \sin y\cos y)/y^2$ increases strictly to $2/\pi$: its derivative has the sign of $2\cos y(\sin y - y\cos y) > 0$ on this interval. Thus the sum is at most $(a-\theta)\theta + \theta^2/2 = a\theta - \theta^2/2$ by the Kirchhoff tangent sum above. The inequality is strict at a genuine split, where every positive outgoing angle is smaller than the incoming one. This proves (4), with strictness propagated toward the root. At o, $\theta = \pi/2$ yields $J \leq aM$, with equality only on an interval. If o has several branches, the Dirichlet problem is a direct sum with ground energy equal to the smallest of the branch ground energies. Apply the single-branch bound to each and average by branch lengths. Equality then requires both interval branches and identical branch eigenvalues, hence equal lengths. This proves (3).

Theorem 2. *Every finite metric tree T of positive length satisfies*

$$\mu_2(T)\rho(T)^2 < \frac{\pi^2}{4}.$$

The supremum is $\pi^2/4$ and is not attained.

Proof. Choose a real mean-zero eigenfunction at μ_2, and choose a zero o along the unique path between a point of positive value and a point of negative value. There are at least two distinct components of $T \setminus \{o\}$ carrying nonzero sign parts of the eigenfunction. For each of those, the corresponding positive or negative part restricted to that component and extended by zero has Rayleigh quotient μ_2 by the weak eigenfunction equation. Each such test has nonzero integral. Thus $\lambda_D(T, o) \leq \mu_2$. If it were strictly smaller, choose a ground state on a branch attaining λ_D, and choose one of the two sign tests supported on a different branch. A mean-zero combination using both tests has quotient strictly below μ_2, since their supports are disjoint up to o. This contradicts (1). This argument also covers eigenfunctions that vanish on entire edges.

The triangle inequality through o gives $\rho(T) < 2r_o$; strictness holds for pairs in a short subinterval of any edge away from o. Together with (3) and $\lambda_D(T, o) = \mu_2$ this gives the upper bound. For the star with $m \geq 2$ equal arms of length ℓ, $\mu_2 = \pi^2/(4\ell^2)$: subtracting the value at the center from a

mean-zero function gives the Dirichlet lower bound, and taking opposite mixed-interval ground states on two arms gives the matching upper bound. Direct averaging over pairs on the same arm and on different arms yields $\rho = \ell(1 - 2/(3m))$. The product tends to $\pi^2/4$. □

5. The lower range for unit-edge trees of bounded degree

Let m be the number of unit edges. Among such trees with maximum degree at most three, the infimum of $\mu_2\rho^2$ is of order $(\log m)/m$ as $m \to \infty$. To see the lower bound, the arclength of every ball of radius $r \geq 1$ is at most $C2^r$ for an absolute constant C. Start from a vertex within one unit of its center; the number of vertices at combinatorial distance $j \geq 1$ is at most $3 \cdot 2^{j-1}$, and any edge intersecting the ball has an endpoint within distance $r + 2$ of that vertex. With $r = \frac{1}{2}\log_2 m$, the probability that a random pair is at distance at most r is at most $C/\sqrt{m}$. Thus $\rho \geq c \log m$, and Theorem 1 gives $\mu_2\rho^2 > 2\rho/m$.

For the reverse bound at every sufficiently large integer m, take an integer b within two of $\log_2 m$ with the same parity as m, and use two identical binary trees each with $(m - b)/2$ edges. Such a tree can be made by filling levels in breadth-first order and has depth $O(\log m)$. Join their roots by a unit-edge path of length b. The same affine test as in (2) gives $\mu_2 \leq C/(m \log m)$, and the diameter is $O(\log m)$. This proves the matching upper order for the infimum.

6. An exact resistance decomposition

Define $R(x, y) = \sup_{E(u)>0} |u(x) - u(y)|^2/E(u)$, with $E(u) = \int_{\mathcal{G}} |u'|^2$, and write $\bar{R}$ for its pairwise arclength average. This is effective resistance with edge resistance equal to length; see, for example, [3]. Let $(\mu_j, \phi_j)_{j\geq 2}$ be the positive eigenvalues with multiplicity and their L^2-orthonormal eigenfunctions. In the mean-zero energy space, $(\phi_j/\sqrt{\mu_j})_{j\geq 2}$ is an orthonormal basis and $u \mapsto u(x) - u(y)$ is a bounded functional by the shortest-path inequality. Parseval and Tonelli therefore give

$$R(x, y) = \sum_{j\geq 2} \frac{|\phi_j(x) - \phi_j(y)|^2}{\mu_j}, \qquad L\bar{R} = 2\sum_{j\geq 2} \frac{1}{\mu_j}.$$

In particular the original bound has the exact deficit

$$\mu_2 L\rho - 2 = 2\sum_{j\geq 3} \frac{\mu_2}{\mu_j} + \mu_2 L(\rho - \bar{R}).$$

Both terms are nonnegative since $R \leq d$. On a tree, $R = d$: take a function of unit slope along the unique path and extend it constantly on branches off that path to realize the supremum. Thus near equality forces an isolated first positive eigenvalue: in particular $\mu_3/\mu_2 \geq 2/(\mu_2 L\rho - 2)$. For graphs with cycles the formula also measures the separate loss from replacing resistance by shortest-path distance; it does not give a topology-independent upper bound.

Statements and Declarations

Data and code availability. No external datasets were generated or analyzed. The analytic proofs are included in the manuscript. Supplementary scripts for the illustrative calculations accompany the source package and are not required for the proofs.

Use of generative AI. OpenAI's ChatGPT assisted with developing and checking mathematical arguments, drafting the manuscript, and preparing the supplementary checks.

References

[1] L. N. Baptista, J. B. Kennedy and D. Mugnolo, *Mean distance on metric graphs*, J. Geom. Anal. **34** (2024), Article 137. https://doi.org/10.1007/s12220-024-01574-0.

[2] J. B. Kennedy, P. Kurasov, G. Malenová and D. Mugnolo, *On the spectral gap of a quantum graph*, Ann. Henri Poincaré **17** (2016), 2439–2473. https://doi.org/10.1007/s00023-016-0460-2.

[3] M. Baker and X. Faber, *Metrized graphs, Laplacian operators, and electrical networks*, in *Quantum Graphs and Their Applications*, Contemp. Math. **415**, AMS, 2006, pp. 15–33. https://doi.org/10.1090/conm/415/07857.