A sharp lower bound for adjacent-edge correlations in random subtrees

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Abstract

For a uniformly chosen nonempty subtree of the labeled complete graph K_n, with singleton subtrees included, the conditional probability of an edge given an adjacent edge is strictly greater than three quarters of its unconditional probability for every $n \geq 3$. This proves the lower-bound conjecture stated by Luo, Xu and Tian in their 2023 author manuscript. Classical fixed-forest extension counts reduce the inequality to a reciprocal moment of the subtree order; Jensen's inequality and a count of four-vertex trees then establish the strict bound. A self-contained derivation of the previously known limiting ratio 3/4 establishes optimality of the constant.

Background. Luo, Xu and Tian [2] prove the adjacent-edge upper bound and obtain the limiting ratio 3/4. We establish the strict finite lower bound proposed as Conjecture 5.1 on printed page 18 of their publicly available author manuscript [3]. The fixed-subtree counts are classical; our argument combines them with a reciprocal-moment bound to prove the finite inequality.

Write $\mathcal{T}_n$ for the set of all nonempty connected acyclic subgraphs of K_n. Thus a subtree is not required to be induced or spanning, and the n single-vertex trees belong to $\mathcal{T}_n$. Let T be uniform on $\mathcal{T}_n$. An edge contained in T means an edge of its subgraph.

The proof reduces the adjacent-edge probability ratio to the single mean parameter $\mu = \mathbb{E}(K-1)$, where K is the random subtree order. Classical fixed-forest counts supply the conditional probabilities; Jensen's reciprocal-moment estimate and a strict mean bound supply the finite inequality. For completeness, a direct derivation of the predecessor's asymptotic limit is included to establish optimality of the constant.

Theorem 1. *For $n \geq 3$ and distinct adjacent edges e, f of K_n,*

$$\mathbb{P}(e \in T \mid f \in T) > \frac{3}{4}\mathbb{P}(e \in T).$$

The constant 3/4 cannot be replaced by a larger constant valid for all n.

Proof. Put $K = |V(T)|$, $\mu = \mathbb{E}(K-1)$, and $N = |\mathcal{T}_n|$. Cayley's formula gives

$$b_n(k) := |\{T \in \mathcal{T}_n : |V(T)| = k\}| = \binom{n}{k} k^{k-2} \quad (2 \leq k \leq n), \qquad b_n(1) = n.$$

Conditional on $K = k$, the vertex set is uniform and the tree is a uniform spanning tree on that set.

The classical forest-extension formula says that a fixed forest on k labeled vertices, with component orders $s_1, \ldots, s_c$, is contained in $k^{c-2} \prod_i s_i$ spanning trees. In the present two cases, this

gives $2k^{k-3}$ extensions of an edge and $3k^{k-4}$ extensions of an adjacent pair. These standard local counts are also part of the framework of Chin et al. [1]; see their Theorem 6.5 for fixed-subtree extension counts. Multiplying by the probabilities that the required vertices are selected yields

$$\mathbb{P}(e \in T \mid K = k) = \frac{2(k-1)}{n(n-1)}, \tag{1}$$

$$\mathbb{P}(e, f \in T \mid K = k) = \frac{3(k-1)(k-2)}{k\,n(n-1)(n-2)}. \tag{2}$$

The first expression is zero for $k = 1$; the second is zero for $k = 1, 2$, so the displayed formulas cover all possible orders.

By symmetry and averaging (1)–(2), the desired ratio is

$$R_n := \frac{\mathbb{P}(e, f \in T)}{\mathbb{P}(e \in T)\mathbb{P}(f \in T)} = \frac{3n(n-1)}{4(n-2)\mu^2}\mathbb{E}\left(K - 3 + \frac{2}{K}\right).$$

Since $x \mapsto 1/x$ is convex on $(0, \infty)$, Jensen's inequality gives

$$\mathbb{E}\left(K - 3 + \frac{2}{K}\right) \geq \mu - 2 + \frac{2}{\mu + 1} = \frac{\mu(\mu - 1)}{\mu + 1}.$$

Consequently,

$$R_n \geq \frac{3n(n-1)}{4(n-2)} \frac{\mu - 1}{\mu(\mu + 1)}. \tag{3}$$

For $n \geq 4$ we have $\mu > 2$. Indeed, retaining the negative contributions at orders one and two and the positive contribution at order four gives

$$\begin{aligned} N\mathbb{E}(K - 3) &= \sum_{k=1}^{n} (k - 3) b_n(k) \\ &\geq 16\binom{n}{4} - 2n - \binom{n}{2} \\ &\geq 4n(n - 3) - \frac{n(n + 3)}{2} \\ &= \frac{n(7n - 27)}{2} > 0. \end{aligned}$$

Here the second inequality follows from $(n-1)(n-2) \geq 6$. Also $\mu < n - 1$, since proper subtrees have positive probability. Hence

$$\frac{n}{n-2} < \mu < n - 1,$$

where the left inequality uses $n/(n-2) \leq 2$. The factorization

$$n(n-1)(\mu - 1) - (n-2)\mu(\mu + 1) = (n-2)(n - 1 - \mu)\left(\mu - \frac{n}{n-2}\right) > 0$$

now shows from (3) that $R_n > 3/4$.

Finally, for $n = 3$ there are nine subtrees, three containing a specified edge and one containing the specified adjacent pair. Thus $R_3 = 1$. This proves the finite inequality.

For completeness, the limiting ratio already obtained by Luo, Xu and Tian [2, 3] can be recovered directly from the subtree counts. For $3 \le k \le n$,

$$\frac{b_n(k-1)}{b_n(k)} = \frac{(1-1/k)^{k-3}}{n-k+1} \le \frac{1}{n-k+1}, \qquad \frac{b_n(1)}{b_n(2)} = \frac{2}{n-1}.$$

Consequently

$$\frac{b_n(n-j)}{b_n(n)} \le \frac{1}{j!} \quad (0 \le j \le n-2), \qquad \frac{b_n(1)}{b_n(n)} \le \frac{2}{(n-1)!}.$$

Since $N \ge b_n(n)$, these estimates give

$$\mathbb{E}(n-K) \le \frac{b_n(n)}{N}\left(\sum_{j=1}^{n-2}\frac{j}{j!} + \frac{2}{(n-2)!}\right) \le e+2.$$

It follows that $\mu = n-1-\mathbb{E}(n-K) = n+O(1)$ and $\mathbb{E}(K-3+2/K) = n+O(1)$, because $0 < 2/K \le 2$. The exact formula for R_n therefore gives $R_n \to 3/4$. Consequently no larger constant can hold uniformly in n.

□

Scope. The result concerns uniform subtrees of labeled complete graphs with singleton subtrees included.

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References

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[3] Z. Luo, K. Xu and J. Tian, *Random subtrees and unimodal sequences in graphs*, 20-page author manuscript, publicly posted August 2, 2023; Conjecture 5.1, p. 18. Public manuscript record.