A sharp lower bound for adjacent-edge correlations in random subtrees

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Abstract

Choose uniformly a nonempty subtree of the labeled complete graph, including the single-vertex subtrees. We prove that the conditional probability of one edge, given an adjacent edge, exceeds three quarters of its unconditional probability. This establishes the lower-bound conjecture stated by Luo, Xu and Tian in their author manuscript. The proof uses the classical fixed-forest extension counts, Jensen's inequality, and a count of four-vertex trees. The constant is optimal by the asymptotic limit already obtained by Luo, Xu and Tian.

Background. Luo, Xu and Tian [2] prove the adjacent-edge upper bound and obtain the limiting ratio 3/4. We establish the strict finite lower bound proposed as Conjecture 5.1 on printed page 18 of their publicly available author manuscript [3].[1] The fixed-subtree counts are classical; our argument combines them with a reciprocal-moment bound to prove the finite inequality.

Write $\mathcal{T}_n$ for the set of all nonempty connected acyclic subgraphs of K_n. Thus a subtree is not required to be induced or spanning, and the n single-vertex trees belong to $\mathcal{T}_n$. Let T be uniform on $\mathcal{T}_n$. An edge contained in T means an edge of its subgraph.

Theorem 1. *For $n \geq 3$ and distinct adjacent edges e, f of K_n,*

$$\mathbb{P}(e \in T \mid f \in T) > \frac{3}{4}\mathbb{P}(e \in T).$$

The constant $3/4$ cannot be replaced by a larger constant valid for all n.

Proof. Put $K = |V(T)|$, $\mu = \mathbb{E}(K-1)$, and $N = |\mathcal{T}_n|$. Cayley's formula gives

$$b_n(k) := |\{T \in \mathcal{T}_n : |V(T)| = k\}| = \binom{n}{k} k^{k-2} \quad (2 \leq k \leq n), \qquad b_n(1) = n.$$

Conditional on $K = k$, the vertex set is uniform and the tree is a uniform spanning tree on that set.

The classical forest-extension formula says that a fixed forest on k labeled vertices, with component orders $s_1, \ldots, s_c$, is contained in $k^{c-2}\prod_i s_i$ spanning trees. In the present two cases, this gives $2k^{k-3}$ extensions of an edge and $3k^{k-4}$ extensions of an adjacent pair. These standard local

[1] Conjecture numbering refers to the 2023 author manuscript. The literature comparison uses the accessible predecessor statements.

counts are also part of the framework of Chin et al. [1]; see their Theorem 6.5 for fixed-subtree extension counts. Multiplying by the probabilities that the required vertices are selected yields

$$\mathbb{P}(e \in T \mid K = k) = \frac{2(k-1)}{n(n-1)}, \tag{1}$$

$$\mathbb{P}(e, f \in T \mid K = k) = \frac{3(k-1)(k-2)}{k\,n(n-1)(n-2)}. \tag{2}$$

The first expression is zero for $k = 1$; the second is zero for $k = 1, 2$, so the displayed formulas cover all possible orders.

By symmetry and averaging (1)–(2), the desired ratio is

$$R_n := \frac{\mathbb{P}(e, f \in T)}{\mathbb{P}(e \in T)\mathbb{P}(f \in T)} = \frac{3n(n-1)}{4(n-2)\mu^2}\mathbb{E}\left(K - 3 + \frac{2}{K}\right).$$

Since $x \mapsto 1/x$ is convex on $(0, \infty)$, Jensen's inequality gives

$$\mathbb{E}\left(K - 3 + \frac{2}{K}\right) \geq \mu - 2 + \frac{2}{\mu + 1} = \frac{\mu(\mu - 1)}{\mu + 1}.$$

Consequently,

$$R_n \geq \frac{3n(n-1)}{4(n-2)} \frac{\mu - 1}{\mu(\mu + 1)}. \tag{3}$$

For $n \geq 4$ we have $\mu > 2$. Indeed, retaining the negative contributions at orders one and two and the positive contribution at order four gives

$$\begin{aligned} N\mathbb{E}(K - 3) &= \sum_{k=1}^{n} (k - 3) b_n(k) \\ &\geq 16\binom{n}{4} - 2n - \binom{n}{2} \\ &\geq 4n(n - 3) - \frac{n(n + 3)}{2} \\ &= \frac{n(7n - 27)}{2} > 0. \end{aligned}$$

Here the second inequality follows from $(n - 1)(n - 2) \geq 6$. Also $\mu < n - 1$, since proper subtrees have positive probability. Hence

$$\frac{n}{n - 2} < \mu < n - 1,$$

where the left inequality uses $n/(n - 2) \leq 2$. The factorization

$$n(n - 1)(\mu - 1) - (n - 2)\mu(\mu + 1) = (n - 2)(n - 1 - \mu)\left(\mu - \frac{n}{n - 2}\right) > 0$$

now shows from (3) that $R_n > 3/4$.

Finally, for $n = 3$ there are nine subtrees, three containing a specified edge and one containing the specified adjacent pair. Thus $R_3 = 1$. Optimality follows from the already established limit $R_n \to 3/4$ in Luo, Xu and Tian [2, 3]. □

Scope. The result concerns uniform subtrees of labeled complete graphs with singleton subtrees included.

Use of AI tools. OpenAI's ChatGPT/Codex was used for literature searches, proof exploration, drafting, and computational consistency checks.

References

[1] A. J. Chin, G. Gordon, K. J. MacPhee and C. Vincent, Subtrees of graphs, *Journal of Graph Theory* **89** (2018), 413–438. doi:10.1002/jgt.22359.

[2] Z. Luo, K. Xu and J. Tian, Random subtrees and unimodal sequences in graphs, *Discrete Mathematics* **347** (2024), no. 1, 113654. doi:10.1016/j.disc.2023.113654.

[3] Z. Luo, K. Xu and J. Tian, *Random subtrees and unimodal sequences in graphs*, 20-page author manuscript, publicly posted August 2, 2023; Conjecture 5.1, p. 18. Public manuscript record. This is the version used to identify the conjecture, not a verified copy of the final typeset article.