# Sharp linearized Darcy inference at a two-block saddle

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**Abstract**

We study linear conductivity functionals when the Darcy reference has an interior saddle and perturbations are supported in a fixed neighborhood of it. At a saddle with one equally expanding block and one equally contracting block, the larger block dimension determines whether an $L^2$ representer exists. For equal rates and a larger expanding block of dimension $m$, the linearized minimax risk is $N^{-(2s+m)/(4s+8)}$ for $m < 4$, $\sqrt{\log N/N}$ at $m = 4$, and parametric for $m > 4$. The matching lower alternatives follow complete finite saddle transits inside the permitted support. Their collar sources have zero marginal in the thin coordinates, while variation in transit time leaves a nonzero target gap. The lower rates also hold in nonlinear Darcy at sufficient smoothness. A smooth positive-source example has several critical points. This answers the saddle-type part of Liu, Chang and Hu, Section 6, second bullet under 'Further open problems,' in the supported linearized experiment at an exact two-block saddle. It gives the representer criterion for arbitrary positive block rates and the sharp minimax rates when the rates agree and the expanding block is larger.

## 1 Setting and results

At a Darcy saddle, the local adjoint can use either the expanding coordinates or the contracting coordinates to accumulate the time spent near the critical point. The larger block supplies the exact dimension criterion for square-integrable representers, even when the two rates differ. For equal rates and a larger expanding block, we also obtain matching minimax orders. The lower bound uses full transits contained in the allowed perturbation support.

The saddle problem is one of the extensions posed by Liu, Chang and Hu [1, §6]. We study an exact local two-block field and distinguish the sharp linearized results from the lower bounds carried into the nonlinear experiment.

Let $\Omega \subset \mathbb{R}^{m+n}$ be smooth, bounded and connected, with $m, n \geq 1$. Work at constant conductivity $f_0 = 1$. Let $u_0$ be a smooth reference potential with zero Dirichlet boundary condition and known source $g = \Delta u_0$. Suppose that in a product neighborhood of zero, in orthonormal coordinates $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$,

$$b(x, y) = \nabla u_0(x, y) = (\lambda x, -\mu y), \qquad \lambda, \mu > 0.$$

The potential may have other critical points outside this neighborhood. Let $K$ be a fixed compact subset of the product neighborhood with $0 \in \operatorname{int} K$. Fix $\psi \in C_c^\infty(\Omega)$ with $\psi(0) \neq 0$. The parameter perturbations satisfy $\operatorname{supp} h \subset K$ and $\|h\|_{H^s(\Omega)} \leq \eta$. The linearized Gaussian white-noise mean is $Ah = -\Delta_D^{-1} \operatorname{div}(hb)$, the noise level is $N^{-1/2}$, and the target is $\int h\psi$. In the nonlinear experiment the white-noise mean is $u_f$ with $\operatorname{div}(f \nabla u_f) = g$ and zero trace, where $f = 1 + h > 0$, and the target is $\int f\psi$. Risks are minimax mean absolute errors.

**Theorem 1.** *For arbitrary positive $\lambda, \mu$, the supported linearized problem has an $L^2$ representer if and only if $\max(m,n) > 4$. If $\max(m,n) \le 4$, the minimax risk times $\sqrt{N}$ diverges in the linearized experiment for every $s \ge 0, \eta > 0$, and in the nonlinear experiment for $s > (m+n)/2+1$ and small fixed $\eta > 0$.*

**Theorem 2.** *Suppose $\lambda = \mu$ and $m > n$. For all $s \ge 0$ and fixed $\eta > 0$, the linearized risk has the sharp order*

$$\begin{cases} N^{-(2s+m)/(4s+8)}, & m < 4, \\ \sqrt{\log N / N}, & m = 4, \\ N^{-1/2}, & m > 4. \end{cases}$$

*These orders are also lower bounds for the nonlinear experiment over the supported Sobolev class when $s > (m+n)/2+1$ and $\eta > 0$ is sufficiently small.*

Liu, Chang and Hu [1, §6] leave several critical points and saddle-type critical points as open extensions of their nondegenerate minimum theory. We solve the linearized problem in the equal-rate configuration above and give an exact representer criterion for the two-rate configuration. Theorem 2 also gives lower bounds in the supported nonlinear experiment.

## 2 Local $H^2$ representers at a saddle

By the Dirichlet inverse and integration by parts, a representer has the form $\Delta w$ with $w \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $b \cdot \nabla w = \psi$ almost everywhere on $K$. Necessity already concerns the weak equation near zero.

For the upper part, suppose first that $m > 4$. Write $r = |x|$ and choose an exit radius $R$ larger than the maximal $r$ on $K$. Extend $\psi$ smoothly by zero and use the auxiliary linear saddle on the whole coordinate space. Its forward exit potential for $r > 0$ is

$$w(x,y) = -\frac{1}{\lambda} \int_r^R \psi\big(tx/r, (r/t)^{\mu/\lambda} y\big) \, \frac{dt}{t}.$$

Following the flow changes only the lower endpoint in its equivalent time parametrization, so $b \cdot \nabla w = \psi$. With $c = \psi(0)$ this potential differs from $(c/\lambda) \log r$ by a bounded function, since $|\psi(tx/r, (r/t)^{\mu/\lambda} y) - c| \le C(t + (r/t)^{\mu/\lambda} |y|)$. Derivatives of orders one and two in $x$ cost at most $Cr^{-1}$ and $Cr^{-2}$, respectively. Indeed each derivative of $x/r$ costs one power of $r^{-1}$, while derivatives of $(r/t)^{\mu/\lambda}$ carry the same power and the factor $(r/t)^{\mu/\lambda}$. Angular derivatives of the integrand carry $t$ and derivatives in $y$ carry $(r/t)^{\mu/\lambda}$, both integrable with $dt/t$. The lower endpoint derivatives obey the same bounds. First and second derivatives in $y$ are bounded, and the mixed derivative is $O(r^{-1})$. For $m > 4$ these estimates give $H^2$ across $r = 0$ by cutting off near that subspace and taking the limit. More explicitly, fill the inner ball in $x$ with $(c/\lambda) \log \epsilon$ and join it smoothly at $r \asymp \epsilon$. The derivative errors are supported on a set of volume $O(\epsilon^m)$ and have squared integral $O(\epsilon^{m-4})$ at order two. The smoothened potentials converge in $H^2$. Multiplying by a smooth cutoff equal to one near $K$ gives a Dirichlet representer. If instead $n > 4$, reverse the direction of the flow and use the exit in $y$, changing the sign of $\psi$.

For necessity assume that $m, n \le 4$ and an $H^2$ solution exists near the saddle. Average it over the rotations $O(m) \times O(n)$ on a smaller product ball. This preserves $H^2$ and gives the same equation with the averaged score, whose value at zero is still $c \neq 0$. Write the averaged solution and score as $W(r,t)$ and $\Psi(r,t)$ for the two radii. On the positive quadrant $W$ is locally $H^2$ and continuous, with

$$\lambda r W_r - \mu t W_t = \Psi.$$

For almost every fixed $t_0 > 0$ in a compact radial collar, $\int_0^{r_0} |W_r(r,t_0)|^2 r^{m-3}\,dr < \infty$. For $m \geq 2$ this follows from the tangential Hessian component $W_r/r$. For $m = 1$ rotation averaging makes the corresponding one-dimensional slice even, so its first derivative has zero trace at zero and the one-dimensional Hardy inequality gives the same integral. For almost every fixed $r_0 > 0$ there is an analogous bound for the $t$ derivative. By Cauchy–Schwarz these slices grow at most $O(1+\sqrt{|\log r|})$ as $r \downarrow 0$, and similarly in $t$, because $\int_r^{r_0} v^{3-m}\,dv$ is bounded for $m < 4$ and logarithmic for $m = 4$.

Now join the points with $t = t_0$ and $r = r_0$ along the characteristic of invariant label $z = r^\mu t^\lambda$. As $z \downarrow 0$, the starting $r$ and ending $t$ tend to zero, and the transit time is $\tau = (\lambda\mu)^{-1}\log(1/z)+O(1)$. The weak transport equation integrates in the interior along almost every such characteristic, by changing to the smooth flow coordinates and using the one-dimensional weak chain rule. The variation of $W$ between the two endpoints is $c\tau + O(1)$ because $|\Psi - c| \leq C(r+t)$ and the integral of $r+t$ along the transit is uniformly bounded. This contradicts the slice bounds, which give an endpoint difference $o(\log(1/z))$.

The nonexistence of a local solution implies that $|\int h\psi|/\|Ah\|_2$ is unbounded over fixed smooth directions supported in the interior of $K$. Otherwise the target on the range of $A$ would have a bounded linear functional extension and thus an $L^2$ representer, whose weak equation would hold near zero. Directions in the kernel with nonzero target already give the same lower-bound claim: their linearized means are equal, and their nonlinear mean gaps are $o(t)$ at perturbation amplitude $t$. For nonzero denominator, scale a fixed direction by $c_1/(\sqrt{N}\|Ah\|_2)$, with $c_1 > 0$ small. Its Gaussian Kullback–Leibler divergence is bounded and its root-sample target gap can be arbitrarily large. For a zero-denominator direction one instead uses an arbitrarily large fixed multiple of $N^{-1/2}$, with vanishing divergence. The same arguments hold in the nonlinear model because along each fixed smooth direction its mean is differentiable at zero with derivative $Ah$. A fixed supported smooth direction of nonzero target also gives the ordinary parametric lower bound in either experiment. This proves Theorem 1.

## 3 An upper bound by a normal moment correction

Suppose $\lambda = \mu$ and $m > n$. Use the exit potential $w$ from the expanding $x$ block. Let $\chi_0(x/\epsilon)$ be a smooth radial cutoff equal to zero for $r \leq \epsilon$ and one for $r \geq 2\epsilon$. Put

$$\widetilde{w}_\epsilon = \frac{c}{\lambda}\log\epsilon + \chi_0(x/\epsilon)\left(w - \frac{c}{\lambda}\log\epsilon\right).$$

The defect $\rho_\epsilon = \psi - b\cdot\nabla\widetilde{w}_\epsilon$ is supported in the tube $r \leq 2\epsilon$. It and its first two derivatives in $y$ are bounded uniformly on a fixed smaller product chart containing $K$. To see this, use

$$\rho_\epsilon = (1-\chi_0(x/\epsilon))\psi - (b\cdot\nabla\chi_0(x/\epsilon))\left(w - \frac{c}{\lambda}\log\epsilon\right).$$

The second term is on $r \asymp \epsilon$, where the quantity in parentheses and its first two $y$ derivatives are bounded. Let $\eta_K$ be a smooth cutoff equal to one near $K$, supported in this product chart. By the derivative estimates for $w$, the Laplacian of $\eta_K\widetilde{w}_\epsilon$ has $L^2$ norm bounded by

$$C\begin{cases}\epsilon^{m/2-2}, & m < 4,\\ (1+\log(1/\epsilon))^{1/2}, & m = 4,\\ 1, & m > 4.\end{cases} \tag{1}$$

The leading integral is $\int_\epsilon^R r^{m-5}\,dr$ for its squared second-derivative norm. Cutoff terms where $r$ is very small and a derivative falls on $\eta_K$ are controlled by the same estimate and local integrability of $\log r$.

Fix $L \geq 0$ with $L + 1 > s - m/2$. For each multi-index $\alpha$ in the $x$ block of order at most $L$, define the scaled moment

$$F_\alpha(y) = \int_{\mathbb{R}^m} X^\alpha \rho_\epsilon(\epsilon X, y)\, dX.$$

We use a correction $U(x/\epsilon, y)$ to the potential. Its moments $M_\alpha(y) = \int X^\alpha U(X, y)\, dX$ need to satisfy

$$-\lambda(m + |\alpha| + y \cdot \nabla_y) M_\alpha(y) = F_\alpha(y).$$

This identity for the moments of the transport derivative follows by integration by parts in $X$. One solution is

$$M_\alpha(y) = -\lambda^{-1} \int_0^1 v^{m+|\alpha|-1} F_\alpha(vy)\, dv.$$

The moments and their first two $y$ derivatives are uniformly bounded. Choose fixed smooth functions supported in the $X$ ball of radius two, dual to the finite set of moments $\int X^\alpha(\cdot)\, dX$. They can be obtained by inverting a polynomial Gram matrix with a positive smooth interior weight. Use their linear combination with coefficients $M_\alpha(y)$ for $U$. Taking $w_\epsilon = \eta_K(\widetilde{w}_\epsilon + U(x/\epsilon, y))$ gives the Dirichlet potential. The added term has Laplacian norm at most $C\epsilon^{m/2-2}$, preserving (1).

On $K$ the corrected defect equals the scaled, uniformly bounded, compactly supported $X$ function with all its moments through $L$ zero, for every $y$. Its homogeneous $H^{-s}(\mathbb{R}^m)$ norm in the physical $x$ variable is $O(\epsilon^{s+m/2})$ uniformly in $y$. This follows by Taylor expansion of its Fourier transform at zero and Plancherel at high frequencies, then scaling. Extend $h$ by zero using its interior support. Its $L^2_y H^s_x$ norm is at most $C\|h\|_{H^s(\Omega)}$ by an ambient Sobolev extension and Fourier comparison. Cauchy–Schwarz in $y$ on the bounded chart then bounds the paired defect by $C\epsilon^{s+m/2}$.

The estimator pairs the linearized observation with $\Delta w_\epsilon$. Its bias is the paired defect and its Gaussian standard deviation is $N^{-1/2}\|\Delta w_\epsilon\|_2$. Choose $\epsilon = N^{-1/(2s+4)}$ and apply (1). This proves all upper bounds in Theorem 2.

## 4 Lower bounds from interior saddle transits

Write $r = |x|$ and $t = |y|$. Choose a single smooth radial cutoff $\chi$ equal to one near zero and supported at small fixed radii, so the product of its supports lies inside $\operatorname{int} K$. Choose $\vartheta \in C_c^\infty(1, 2)$ with

$$\int \vartheta(z)\, \frac{dz}{z} = 0, \qquad \int \vartheta(z) \log z\, \frac{dz}{z} \neq 0. \tag{2}$$

If $m = 4$, also impose $\int \vartheta(z) z^{-2}\, dz/z = 0$. These conditions are compatible by linear independence of the indicated functions on the interval.

For small $\epsilon > 0$ use the smooth supported perturbation

$$H_\epsilon(x, y) = r^{-m} t^{-n} \vartheta(rt/\epsilon) \chi(r) \chi(t).$$

On its support both radii are bounded below by positive multiples of $\epsilon$. The product $rt$ is invariant under the saddle flow. The prefactor $r^{-m} t^{-n}$ absorbs the divergence of $b$. Consequently the divergence source $S_\epsilon = \operatorname{div}(H_\epsilon b)$ is supported only on the two outer transition collars of $\chi$. There it is a sum of two sources, one thin in $x$ and one thin in $y$. More precisely, for all sufficiently small $\epsilon$,

$$S_\epsilon = S^\epsilon_x + S^\epsilon_y, \qquad S^\epsilon_x(x, y) = \epsilon^{-m} S_x(x/\epsilon, y), \qquad S^\epsilon_y(x, y) = \epsilon^{-n} S_y(x, y/\epsilon),$$

for fixed smooth compactly supported profiles. Indeed in the $x$-thin term $t$ stays on the transition collar of $\chi$, so $r = O(\epsilon)$ and $\chi(r) = 1$. The opposite term is symmetric. Each source is radial in its

thin coordinates and has zero integral in those coordinates for every fixed other coordinate. The cancellation follows by the change $z = rt/\epsilon$ and the first condition in (2).

We record the Sobolev norm bound

$$\|H_\epsilon\|_{H^s(\Omega)} \le C_s \epsilon^{-s-m/2}. \tag{3}$$

For derivatives of orders $a$ in $x$ and $b'$ in $y$, a pointwise bound is $Cr^{-m-a}t^{-n-b'}$ on the same invariant band. Use polar measure and $t = \epsilon z/r$. Its squared integral is bounded by

$$C\epsilon^{-n-2b'} \int_{c_0\epsilon}^{C_0} r^{n+2b'-m-2a}\, \frac{dr}{r}.$$

For $a+b' \le k$ this is $O(\epsilon^{-m-2k})$: when exponents coincide the endpoint bound acquires a logarithm, but the common exponent is at most $n+2k < m+2k$. All integer $k \ge 0$ are allowed. Interpolating consecutive integer bounds gives (3) for real $s \ge 0$.

To bound the observation norm, solve in the thin variables $\Delta_x V_x^\epsilon = S_x^\epsilon$ and $\Delta_y V_y^\epsilon = S_y^\epsilon$ using the radial inverse. Zero mass in the thin coordinates makes each inverse compactly supported in the same outer thin ball: its radial derivative is the interior mass divided by a power of radius, and vanishes outside the source. Choose its constant value there to be zero. For the two profiles this gives

$$V_x^\epsilon(x,y) = \epsilon^{2-m} V_x(x/\epsilon, y), \qquad V_y^\epsilon(x,y) = \epsilon^{2-n} V_y(x, y/\epsilon)$$

with fixed smooth compact profiles. By Dirichlet $H^2$ regularity, duality gives

$$\|AH_\epsilon\|_2 \le C(\|V_x^\epsilon\|_2 + \|V_y^\epsilon\|_2) \le C(\epsilon^{2-m/2} + \epsilon^{2-n/2}). \tag{4}$$

When $m = 4$, the additional moment in (2) also makes $V_x^\epsilon$ zero inside the inner thin radius, uniformly for $y$ on its collar. Specifically, for a zero-mass radial four-dimensional source $F(r)$, the inverse which vanishes outside the source has inner constant $-\frac{1}{2}\int F(r) r\, dr$. Substituting this source into the formula gives exactly a multiple of the additional moment. Thus the $x$-thin potentials of sufficiently separated geometric scales have disjoint supports.

It remains to show nonvanishing signal despite the zero marginal. Let $\Psi(r,t)$ be the angular average of $\psi$. After changing variables to $z = rt/\epsilon$, the signal is a fixed positive angular constant times

$$\int_1^2 \vartheta(z) \frac{dz}{z}\, I(\epsilon z), \qquad I(P) = \int_0^\infty \chi(r)\chi(P/r)\Psi(r, P/r)\, \frac{dr}{r}.$$

As $P \downarrow 0$, $I(P) = c\log(1/P) + C_1 + o(1)$. For the constant part of $\Psi$ this follows by splitting the integral at $\sqrt{P}$: on either outer half the small-coordinate cutoff equals one and the remaining integral is logarithmic plus a fixed constant. For the varying part write

$$\Psi(r,t) - c = (\Psi(r,0) - c) + (\Psi(0,t) - c) + O(rt).$$

The two single-coordinate terms have limits by dominated convergence, after changing $t = P/r$ for the second. The remaining term is bounded by $O(P\log(1/P))$. The expansion therefore holds uniformly when $P = \epsilon z$ with $1 \le z \le 2$. By (2), $\int H_\epsilon \psi$ tends to a nonzero constant. Fix the orientation of $\vartheta$ to make this limit positive.

For $m < 4$, choose $\epsilon = N^{-1/(2s+4)}$ and compare zero to $h = \gamma \epsilon^{s+m/2} H_\epsilon$ with fixed small $\gamma > 0$. Formula (3) gives the Sobolev constraint and (4) gives bounded small Gaussian divergence. The target gap is of order $\epsilon^{s+m/2}$, as required.

If $m = 4$, choose $\epsilon_j = \epsilon_0 B^{-j}$ with a fixed $B > 1$ large enough for the critical $V_x^{\epsilon_j}$ to have disjoint supports. The other potentials have summable norm since $n < 4$. Hence

$$\left\| A \sum_{j=1}^{J} H_{\epsilon_j} \right\|_2 \le C(\sqrt{J} + 1)$$

by the same Dirichlet duality argument. Let $J = \lfloor \log N / [4(s+2) \log B] \rfloor$ and use

$$h = \frac{\gamma}{\sqrt{NJ}} \sum_{j=1}^{J} H_{\epsilon_j}.$$

Its Sobolev norm is $O(\gamma N^{-1/4}/\sqrt{J})$ by (3), and its Gaussian divergence is bounded by $C\gamma^2$. Each summand has a uniformly positive target gap for small fixed $\epsilon_0$, so the total gap is at least $c_2 \gamma \sqrt{J/N}$. The two-point Gaussian bound gives the critical lower rate. For $m > 4$ the ordinary parametric lower bound was proved in Theorem 1.

The nonlinear lower bounds use the same alternatives when $s > (m+n)/2 + 1$ and the Sobolev radius is sufficiently small. For $L_f = \operatorname{div}(f \nabla)$ one has exactly

$$L_f(u_f - u_0) = -\operatorname{div}(hb) = \Delta A h.$$

The Dirichlet operator $\Delta L_f^{-1}$ is uniformly bounded on $L^2$ because $f = 1 + h$ is close to 1 in $W^{1,\infty}$. For instance factor $L_f \Delta_D^{-1}$ as the identity plus an $L^2$ operator of small norm using the Dirichlet $H^2$ estimate, and invert by a Neumann series. By self-adjointness and Dirichlet integration by parts, duality gives $\|u_f - u_0\|_2 \le C \|Ah\|_2$. Thus the same Gaussian divergence bounds apply up to constants, and the functional gap is unchanged. This proves Theorem 2.

## 5 A positive-source example with several critical points

For $m > n$ choose a smooth even function $V$ on the line with $V'' \ge -1$, quadratic behavior $V(y) = V(0) - y^2/2$ near zero, two equal minima at $\pm a$ of value zero, and no other critical points besides zero. It can be constructed by taking $V'' = -1 + \alpha$ with $\alpha$ smooth, even, zero near zero, increasing for positive $y$ where it transitions, and equal to a constant greater than one for all sufficiently large $|y|$. Integrate with $V'(0) = 0$ and shift the minimum to zero. This gives a confining double well with nondegenerate minima. Put

$$U(x, y) = \tfrac{1}{2}|x|^2 + \sum_{j=1}^{n} V(y_j), \qquad \Omega = \{U < L\}, \qquad u_0 = U - L,$$

where $L > nV(0)$. The domain is bounded and $\partial\Omega$ is smooth since every critical value of $U$ is at most $nV(0)$. It is connected: one can first reduce $x$ to zero and descend each $y_j$ to one of the minima. Any two such minima can be joined by changing one coordinate at a time through its barrier while all others remain at their minima. These paths stay below $L$. The source is

$$g = \Delta U = m + \sum_{j=1}^{n} V''(y_j) \ge m - n > 0.$$

There are $3^n$ critical points, including the saddle at zero with $b = (x, -y)$ on a product neighborhood. Thus the nonparametric and critical linearized rates occur even for strictly positive source and several nondegenerate critical points on a smooth connected domain.

## References

[1] Y. Liu, X. Chang and Q. Hu, *Minimax estimation of linear functionals in the Darcy inverse problem*, preprint, arXiv:2611.01290v1 (2026). Article.