# Sharp length–inradius bounds and heat-content stability on compact metric graphs

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Luminary Oddities

**Abstract**

We determine how total length and Dirichlet inradius constrain heat retention on compact metric graphs, refining the all-times length-only comparison with the mixed Dirichlet–Neumann interval proved in a companion paper. For every stationary $p$-torsion problem, $1 < p < \infty$, we obtain the exact joint upper envelope, and we determine the sharp envelope of every positive linear resolvent. For finite metric trees with one Dirichlet root, the same endpoint-mass interval gives the exact heat-content supremum at every positive time; one sequence of finite brooms approaches the entire envelope uniformly on positive compact time windows. The resulting marked measured geometric stability estimate has the optimal exponent 1/3, uniformly in tree complexity and minimum edge length. A positive averaged mass-relocation principle extends the heat comparison to trees with matching edge energy and speed weights at least one, and to cyclic graphs with exact root-preserving tree quotients, where the resistance height of the quotient sharpens the envelope. For every finite connected compact metric graph the same endpoint-mass interval gives the exact heat-content supremum at fixed length and inradius, approached by finite brooms, and the cubic one-time deficit holds with a constant uniform on compact time windows, so the marked measured stability exponent 1/3 is attained and sharp on the whole class. We also determine the infimum of the first $p$-Laplacian eigenvalue at fixed length and inradius for every $1 < p < \infty$, extending Kennedy's linear result, with the first correction near maximal inradius and the sharp geometric stability power, and a sharp inverse-edge rate for the gap above this infimum on graphs with finitely many edges. For the nonlinear $p$-Laplacian heat flow from datum one, the same endpoint-mass interval gives the sharp one-step mass for every $1 < p < \infty$, and for $p > 2$ the sharp early-time heat content and the sharp coefficient of $t^{-1/(p-2)}$ at late times. Finally, the deficit from the sharp torsion and resolvent envelopes controls the missing path length and the first moment of the remaining arclength, giving an inverse-complexity bound, sharp for every $p > 1$, with complexity twice the cycle rank plus the number of leaves; equal-arm brooms are the unique maximizers under an arm cap. The exact maximum of $p$-torsional rigidity is determined among rooted trees with a leaf cap (an equal-arm broom) and, for all sufficiently large edge and topological caps, among general graphs with arbitrary Dirichlet sets, where the edge-capped extremizer is a rooted stem ending in equal loops. The sharp first-order coefficients of the finite-complexity gaps for the first $p$-eigenvalue and the positive linear resolvent are computed, with the edge-cap coefficient exactly half the topological one. At every positive time the sharp first-order heat-content deficit under an edge or topological cap is computed as an explicit convolution of absorption-time densities, and the heat and torsion deficits are shown to be quantitatively equivalent. The exact capped optimizers persist for the datum-one resolvent at every nonnegative mass: the equal-arm broom for rooted trees under a leaf cap, and both capped torsion optima for arbitrary topologies throughout the finite-cap range, with all equality cases and the complete two-edge phase diagram. For the nonlinear $p$-Laplacian resolvent with homogeneous absorption, an equal-arm broom solves the rooted problem under a leaf cap for every absorption strength, one explicit complexity threshold gives the exact optimum under an edge cap or a combined cycle and leaf cap uniformly in the strength, and in the compact boundary-layer regime the unrestricted value is computed with its equality cases. The leading cap deficit is derived below and at the threshold, with its full crossover on the

scale $1/n$, and the optimization extends to continuous nonincreasing absorption laws. Under small complexity caps the first eigenvalue is determined exactly: the equal three-arm broom uniquely minimizes the linear eigenvalue among rooted trees with at most three leaves, exact lower envelopes hold for general graphs under the topological cap $2b_1+\ell \le 3$ for all $p > 1$ and $2b_1 + \ell \le 4$ for $p = 2$, and the two-edge phase diagram is complete at every feasible inradius. Near maximal volume the exact rooted optimizer is the equal broom for $1 < p \le 2$ and a binary broom for $p > 2$, with every intermediate arm count selected by an exact adjacent phase transition in the scaling window around $p = 2$; near path volume the equal broom is exactly optimal for all $p > 1$ and all finite leaf caps. The proofs combine shortest-path variational bounds, positive absorption-time response identities and a nonnegative heat-deficit source.

# 1 Introduction

How much heat can a graph retain when both its total length and its largest distance from the absorbing boundary are prescribed? The total length alone singles out the interval with one reflecting and one absorbing endpoint. Keeping the Dirichlet inradius adds information about how the graph departs from that interval. We determine the sharp stationary and resolvent envelopes for these two geometric parameters, and use their deficits to quantify the geometry of near-maximal heat retention at a single positive time.

### The model and the sharp envelopes

Let $G$ be a finite compact connected metric graph of total length $L > 0$, with a nonempty set $D$ of Dirichlet vertices. At all other vertices we impose continuity and the unweighted Kirchhoff condition; an unmarked terminal vertex is therefore reflecting. Loops and multiple edges are allowed, and $G \setminus D$ need not be connected. Let $H_G$ be the nonnegative Dirichlet–Kirchhoff Laplacian and write

$$R = \max_{x\in G} \operatorname{dist}(x, D), \qquad \mathcal{Q}_G(t) = \langle \mathbf{1}, e^{-tH_G}\mathbf{1}\rangle.$$

Integration and the inner product use arclength measure. The comparison interval $I_L$ has length $L$, with one Dirichlet and one Neumann endpoint. The weak operator formulations and the geometric distance used below are defined in the corresponding proof sections.

Bifulco and Täufer [1] asked whether $\mathcal{Q}_G(t) \le \mathcal{Q}_{I_L}(t)$ at every time. For $G \setminus D$ connected, they proved it for large times and, when the edge lengths are rationally dependent, for small times [1, Theorem 2.6], and they recorded the all-times question in their abstract and in [1, Remark 2.9]. The companion paper [A, Theorem 1.2] proves the inequality for every graph in the present model at every time, with no condition on the edge lengths; the decomposition at Dirichlet vertices recorded in [2, Remark 2.7] also accommodates disconnected $G \setminus D$. Theorem 5.3 below rederives it for initial datum one through a nonnegative heat-deficit source, whose quantitative form the stability theory requires. This all-times inequality uses only the total length. Prescribing the Dirichlet inradius as well, we determine the exact supremum at every positive time on rooted trees,

$$\mathcal{Q}_T(t) \le Q_{R,L-R}(t) < \mathcal{Q}_{I_L}(t) \qquad (t > 0,\ R < L),$$

where $Q_{R,L-R}$ is the endpoint-mass interval model described below. The supremum is approached by finite brooms uniformly on positive compact time windows. The pitchforks singled out in [1, Remark 2.8] as the delicate small-time candidates obey the comparison at every time through an explicit positive convolution formula (Proposition 9.1, also [A, Proposition 6.1]). At fixed positive times their deficit is exactly cubic in the arm length (Theorem 10.1), which fixes the optimal geometric stability exponent 1/3. The double-Dirichlet counterpart of the

domain comparison discussed in [1, Remark 2.7] is the subject of [A, Theorem 5.1], under the condition that every bridge has Dirichlet vertices on both sides.

For $1 < p < \infty$, let $u_p$ solve the datum-one stationary problem $-(|u_p'|^{p-2}u_p')' = 1$, with zero Dirichlet values and natural Kirchhoff $p$-flux conditions, and put

$$S_p(G) = \int_G u_p \, \mathrm{d}x, \qquad q = \frac{p}{p-1}.$$

Our first sharp envelope is

$$S_p(G) \le \frac{L^{q+1} - (L-R)^{q+1}}{q+1}. \tag{1}$$

The normalization is important: $S_p$ is the torsion integral, not its $(p-1)$st power. In particular $S_2(I_L) = L^3/3$. At each prescribed $0 < R < L$, the right-hand side is an exact supremum. It is approached by finite brooms and is not attained by a finite graph. We prove the full weak formulation, the bound, nonattainment and the fixed-parameter sharpness in Section 2.

The second envelope retains a positive resolvent parameter. Set $J_G(z) = \langle \mathbf{1}, (H_G + z)^{-1}\mathbf{1} \rangle$ for $z > 0$ and $s = \sqrt{z}$. Then

$$J_G(z) \le \frac{L}{z} - \frac{\tanh(sR) + s(L-R)}{z^{3/2}\big(1 + s(L-R)\tanh(sR)\big)}. \tag{2}$$

This is again the exact supremum at fixed $L, R$, approached by the same broom limit. Section 3 proves the bound and an exact integral formula for its deficit from $J_{I_L}(z)$. The stationary $p = 2$ bound is recovered as $z \downarrow 0$. The resolvent family supplies additional information needed for the heat-stability argument. On rooted trees and exact tree quotients, the pointwise heat comparison with the limiting broom model is obtained separately, through positive inverse kernels (Section 7).

## From deficit to marked measured geometry

Normalize length by $L$ and arclength measure by $L^{-1}$, and put $\beta = 1 - R/L$. We compare $G/L$ with the unit mixed interval by a common-embedding distance $d_m$. This distance records the metric support, the Dirichlet marked set and the arclength probability measure through Hausdorff and Prokhorov distances. Thus it detects more than an unmarked metric approximation. The shortest-path argument gives

$$d_m(G/L, I_1) \le 2\beta.$$

Combined with (1), it yields

$$d_m(G/L, I_1) \le 2\left(\frac{L^{q+1} - (q+1)S_p(G)}{L^{q+1}}\right)^{1/(q+1)}. \tag{3}$$

The exponent $1/(q+1)$ is optimal, even on a fixed graph topology; the coefficient 2 comes from an explicit common embedding. The distance and the proof of geometric stability are given in Section 4.

For heat content, the normalized one-time deficit is

$$\delta_G(\tau) = \frac{\mathcal{Q}_{I_L}(L^2\tau) - \mathcal{Q}_G(L^2\tau)}{L}.$$

For every $\varepsilon > 0$ and every window $0 < \tau_0 \le \tau_1 < \infty$, we prove

$$\delta_G(\tau) \ge C_{\varepsilon,\tau_0,\tau_1}\beta^{3+\varepsilon} \qquad (\tau \in [\tau_0, \tau_1]), \tag{4}$$

where the constant is positive and independent of the number of edges, vertices and minimum edge length. Consequently, for each $0 < \alpha < 1/3$ there is a uniform estimate

$$d_m(G/L, I_1) \le C_{\alpha,\tau_0,\tau_1} \delta_G(\tau)^\alpha.$$

For general graphs the positive-time proof uses a nonnegative heat-deficit source to control an earlier integrated deficit, followed by the sharp resolvent reserve and a weighted-tail estimate. The necessary comparison and source identities for constant initial datum are proved inside this article; they overlap with [A, Sections 3 and 4], as detailed at the start of Section 5.

A family of small-arm pitchforks shows that the heat deficit is cubic in its geometric departure from the interval. The full asymptotic is proved in Section 10, uniformly on positive compact time windows. It rules out every exponent $\alpha > 1/3$, even at a fixed positive observation time. Together with the preceding upper stability bounds, this identifies $1/3$ as the exact supremum of the admissible exponents. For singly rooted trees, the endpoint itself is valid by the stronger heat-envelope comparison below.

## The sharp tree heat envelope and its positive extensions

For a finite tree with one Dirichlet root, the positive-time comparison retains both geometric parameters exactly. Let $Q_{r,\beta}$ be the heat content on $[0, r]$ with Dirichlet condition at zero, energy $\int |f'|^2$ and speed measure $\mathrm{d}x + \beta\delta_r$; its initial heat includes the endpoint atom. We prove

$$\mathcal{Q}_T(t) \le Q_{R,L-R}(t) \qquad (t > 0).$$

At every prescribed $0 < R < L$ this is the exact supremum, approached by the same finite brooms uniformly on positive compact time windows. Each finite tree has strict inequality there; the DN interval attains the case $R = L$. The endpoint model has a positive radius-variation formula, which yields a cubic deficit and hence the optimal admissible geometric exponent $1/3$ for this tree class.

The proof uses absorption-time distributions to merge parallel rooted branches, and positive junction-response factors to propagate their heat-content order along a common stem. An averaged mass-relocation lemma then moves excess speed mass from ancestors to descendant leaves. In resistance coordinates this gives the same heat envelope for weighted trees whose edge energy and speed have identical multiplicities $c_e \ge 1$. Exact invariant tree quotients transfer the result to a specified cyclic class; equilateral complete graphs with one Dirichlet vertex provide examples of unbounded cycle rank. These results are proved in Section 7.

## Relation to existing bounds and methods

Mugnolo and Plümer [4] establish the sharp length-only stationary bound $S_2(G) \le L^3/3$ and discuss bounds retaining the Dirichlet inradius, including $S_2(G) < RL^2$ [4, Remark 5.2(1)]. Our joint envelope (1) for $p = 2$ is strictly smaller than $RL^2$ when $0 < R < L$, and determines the exact supremum rather than an inradius scaling alone. Their stower constructions also supply important ancestry for concentrating graph length in short arms; their explicit stower formula already yields torsion of order $R$ at fixed length, incompatible with an upper bound proportional to $LR^2$ (see the discussion after Corollary 2.4). The new ingredient here is the exact joint envelope. Their work also develops surgery principles and Kohler–Jobin-type bounds for the ground-state energy.

Özcan [5] develops sharp metric-graph $p$-torsion inequalities using level sets and variational methods. Comparisons with that work must account for the powered convention for torsional rigidity: our quantity is $S_p$, and the corresponding powered quantity is $S_p^{p-1}$. The extension here determines the exact joint length–inradius envelope and its fixed-parameter sharpness

for every $1 < p < \infty$. The linear resolvent theorem and the positive-time stability conclusions concern the linear case $p = 2$.

For heat content, Bifulco and Täufer [1] prove the interval comparison for large times and, under rational dependence of the edge lengths, for small times, and give an all-times comparison formula for suitable pairs of equilateral graphs [1, Corollary 4.9]. Bifulco and Mugnolo [2] give a closed formula for the heat content at every positive time as a sum over paths between Dirichlet vertices, a small-time expansion, and surgery principles, including operations that preserve heat content at all times. The companion paper [A] proves the length-only all-times comparison, together with a concentration order for general nonnegative $L^2$ data. The present question is quantitative: how much does prescribed inradius reduce the extremal stationary and resolvent values, and how does a small positive-time deficit constrain marked measured geometry? The sharp envelopes, effective heat reserve and cubic obstruction answer successive parts of that question.

The analytic inputs have classical antecedents. Friedlander's graph coarea argument [3] and parabolic symmetrization [6, 7, 8] underlie the constant-datum heat comparison, here and in [A]. Our proofs retain the graph-specific positive deficit source and the shortest-path variational information needed for stability. The paper first proves the stationary and resolvent envelopes and their geometric consequence. It then supplies the necessary heat-source argument, the exact rooted-tree heat envelope and its mass-relocation extensions, the general-graph effective positive-time estimate, and the sharpness obstruction.

## 2 Sharp stationary bounds in length and inradius

Let $G$ be a finite compact connected metric graph with strictly positive edge lengths, total length $L > 0$, and a nonempty set $D$ of Dirichlet vertices. Integrals use ordinary edge-length measure. All functions are continuous at vertices; at vertices outside $D$ we impose the natural Kirchhoff condition. In particular, a degree-one vertex outside $D$ has a Neumann condition. The Dirichlet inradius is

$$R = \max_{x \in G} \operatorname{dist}_G(x, D), \qquad 0 < R \le L.$$

There is no bound on the number of edges or the vertex degrees.

Fix $1 < p < \infty$ and put $q = p/(p-1)$. Write $V_p(G, D)$ for the continuous, edgewise $W^{1,p}$ functions vanishing on $D$. We use the unpowered normalization

$$S_p(G, D) = \int_G u_p \, \mathrm{d}x,$$

where $u_p$ is the weak solution of

$$\int_G |u_p'|^{p-2} u_p' f' \, \mathrm{d}x = \int_G f \, \mathrm{d}x \qquad (f \in V_p(G, D)). \tag{5}$$

Thus the edge equation is $-(|u_p'|^{p-2}u_p')' = 1$, with conservation of outward $p$-flux at every non-Dirichlet vertex.

The length-only Saint–Venant comparison is classical for $p = 2$ [4] and has a nonlinear counterpart in [5]. The latter uses the powered quantity $\mathcal{T}_p = S_p^{p-1}$. We retain $S_p$ throughout the stationary argument and give the conversion explicitly below. The new comparison fixes both $L$ and $R$.

**Lemma 2.1** (Weak problem and normalization)**.** *The problem* (5) *has a unique nonnegative, nonzero solution. Moreover*

$$S_p(G,D) = \int_G |u_p'|^p \,\mathrm{d}x, \tag{6}$$

$$S_p(G,D) = \frac{1}{p-1} \sup_{f \in V_p(G,D)} \left\{ p \int_G f \,\mathrm{d}x - \int_G |f'|^p \,\mathrm{d}x \right\}, \tag{7}$$

$$\mathcal{T}_p(G,D) := \sup_{0 \neq f \in V_p(G,D)} \frac{|\int_G f \,\mathrm{d}x|^p}{\int_G |f'|^p \,\mathrm{d}x} = S_p(G,D)^{p-1}. \tag{8}$$

*Proof.* For any $x \in G$, integrate $f'$ along a simple path from $D$ to $x$. Hölder's inequality gives

$$|f(x)| \le L^{1/q} \left( \int_G |f'|^p \,\mathrm{d}x \right)^{1/p}.$$

Consequently the derivative norm controls the full Sobolev norm on $V_p(G,D)$. This space is a closed subspace of the finite product of the edge Sobolev spaces. The functional

$$f \longmapsto \frac{1}{p} \int_G |f'|^p \,\mathrm{d}x - \int_G f \,\mathrm{d}x$$

is coercive and weakly lower semicontinuous. Reflexivity gives a minimizer, and strict convexity in derivatives, together with the Dirichlet anchor, gives uniqueness. Its Euler equation is (5). Replacing a real minimizer by its absolute value preserves energy and does not increase the functional, so uniqueness gives nonnegativity. A small positive multiple of a nonzero nonnegative edge-interior bump has negative functional value; hence the minimizer is nonzero.

On an edge its $p$-flux has distributional derivative $-1$, so that flux is affine. Inverting the map $t \mapsto |t|^{p-2}t$ gives a continuous edge derivative. Testing with functions having prescribed vertex values yields the Kirchhoff flux law. No twice-differentiability assertion at a zero of $u_p'$ is needed.

Testing by $u_p$ proves (6). The minimized functional at $u_p$ equals $-(p-1)S_p/p$, proving (7). Finally (5) and Hölder give

$$\left| \int_G f \,\mathrm{d}x \right| \le S_p^{(p-1)/p} \left( \int_G |f'|^p \,\mathrm{d}x \right)^{1/p}.$$

Equality in the resulting quotient bound is attained by $f = u_p$, which proves (8). □

**Theorem 2.2** (Sharp joint stationary envelope)**.** *For every graph in the stated class and every* $1 < p < \infty$*,*

$$S_p(G,D) \le A_p(L,R) := \frac{L^{q+1} - (L-R)^{q+1}}{q+1}. \tag{9}$$

*For each prescribed* $0 < R < L$*, the right side is the exact supremum over finite graphs with those values of* $L, R$*, but is not attained by a finite graph. At* $R = L$ *the Dirichlet–Neumann interval attains the bound. In the powered convention the inequality is* $\mathcal{T}_p(G,D) \le A_p(L,R)^{p-1}$*.*

*Proof.* The supremum in (7) may be restricted to nonnegative real functions. Let $f$ be such a function, not identically zero, and choose a point $x$ where it reaches its maximum. A shortest path $\gamma$ from a closest Dirichlet vertex to $x$ has length $r = \mathrm{dist}_G(x,D) \le R$. It is simple and has edge measure $r$; if $x$ lies inside an edge we subdivide that edge. Parameterize it by

arclength from its Dirichlet endpoint and put $g = f|_\gamma$. Then $g(0) = 0$ and $g(r) = \max_G f$. The complement has length $L - r$, whence

$$p\int_G f\,\mathrm{d}x - \int_G |f'|^p\,\mathrm{d}x \le p\int_0^r g(s)\,\mathrm{d}s + p(L-r)g(r) - \int_0^r |g'(s)|^p\,\mathrm{d}s$$
$$= \int_0^r \left[p(L-s)g'(s) - |g'(s)|^p\right]\,\mathrm{d}s.$$

The equality is Fubini's formula for an absolutely continuous $g$ with $g(0) = 0$; it requires no monotonicity of $g$. For $a \ge 0$ and real $z$,

$$paz - |z|^p \le (p-1)a^q.$$

Applying this sharp Young inequality pointwise gives an upper bound $(p-1)\int_0^r (L-s)^q\,\mathrm{d}s$, and then $(p-1)\int_0^R (L-s)^q\,\mathrm{d}s$. Equation (7) proves (9).

If $R < L$, apply the argument to $u_p$. The complement of its selected path has positive length and contains an open edge subinterval. Equality would force the discarded energy there to be zero, so $u_p$ would be constant on that subinterval. This contradicts the edge equation that its $p$-flux has derivative $-1$. Thus the bound is strict on every finite graph when $R < L$. Exact sharpness follows from the brooms constructed next. For $R = L$, the DN interval gives $S_p = L^{q+1}/(q+1)$ by direct integration. □

**Proposition 2.3** (Exact brooms and sharp deficit scale)**.** *Join a Dirichlet stem of length $a > 0$ to $n \ge 2$ equal arms of length $b > 0$, with Neumann conditions at their terminal endpoints. Then $L = a + nb$, $R = a + b$, and*

$$S_p = \frac{L^{q+1} - (n^{q+1} - n)b^{q+1}}{q+1}. \tag{10}$$

*For every fixed $0 < R < L$, these brooms approach $A_p(L, R)$ while retaining exactly $L$ and $R$. The stationary deficit exponent $q + 1$ in*

$$\frac{L^{q+1}}{q+1} - S_p \ge \frac{(L-R)^{q+1}}{q+1}$$

*cannot be replaced by any smaller exponent with a positive uniform coefficient at fixed $L$.*

*Proof.* With $s$ measured along the stem from its Dirichlet endpoint and $t$ along an arm from the junction, define

$$u_{\mathrm{stem}}(s) = \frac{L^q - (L-s)^q}{q}, \qquad 0 \le s \le a,$$
$$u_{\mathrm{arm}}(t) = u_{\mathrm{stem}}(a) + \frac{b^q - (b-t)^q}{q}, \qquad 0 \le t \le b.$$

Their derivatives have $p$-fluxes $L - s$ and $b - t$, respectively, since $(q-1)(p-1) = 1$. The forcing equation and endpoint conditions hold. At the junction the sum of outgoing fluxes is $-(L-a) + nb = 0$. Thus these functions solve (5), and uniqueness identifies them with $u_p$. Equation (6) now gives

$$S_p = \int_0^a (L-s)^q\,\mathrm{d}s + n\int_0^b (b-t)^q\,\mathrm{d}t,$$

which is (10).

For fixed $L, R$, put $b = (L-R)/(n-1)$ and $a = R - b$, positive for sufficiently large $n$. Then

$$\frac{L^{q+1}/(q+1) - S_p}{(L-R)^{q+1}} = \frac{n^{q+1} - n}{(q+1)(n-1)^{q+1}} \longrightarrow \frac{1}{q+1}.$$

This proves the asserted exact supremum. For the exponent assertion keep $L$ fixed, take $n = 2$, and let $b \downarrow 0$ with $a = L - 2b$. Here $L - R = b$ and the deficit is exactly $(2^{q+1} - 2)b^{q+1}/(q+1)$. Dividing by any smaller power of $b$ proves the claim. □

**Corollary 2.4** (Failure of an inradius-power upper bound)**.** *For each fixed $1 < p < \infty$, there is no finite constant $C_p$ such that*

$$S_p(G, D) \le C_p L R^q$$

*for every graph in this class. Equivalently, there is no universal bound $\mathcal{T}_p(G, D) \le C_p L^{p-1} R^p$ in the powered convention. The leading small-inradius scale $L^q R$ has optimal coefficient one.*

*Proof.* For integers $k \ge 2$, take

$$n = k^2, \qquad a = \frac{1}{k+1}, \qquad b = \frac{1}{k(k+1)}.$$

These are finite brooms with $L = 1$ and $R = 1/k$. Their exact value is

$$S_p = \frac{1}{q+1}\left[1 - \left(\frac{k}{k+1}\right)^{q+1} + \frac{k^{1-q}}{(k+1)^{q+1}}\right].$$

For fixed $q > 1$, differentiability of $(1-t)^{q+1}$ at zero gives $kS_p \to 1$; the final term is $O(k^{-2q}) = o(k^{-1})$. Consequently $S_p/R^q \sim k^{q-1} \to \infty$, whereas $S_p/R \to 1$. Raising to $p-1$ proves the powered statement, because $(q-1)(p-1) = 1$. Finally (9) implies $S_p \le L^q R$, and the same sequence proves the optimal leading coefficient. □

For $p = 2$ the failed form is $T \le CLR^2$. A particular constant-four analogy with Makai's domain inequality was raised in [4, Remark 4.7(1)]. The negative example has classical network ancestry: the explicit stower formula in [4, Example 3.10], with many small loop petals on one Dirichlet stem, also yields $T \asymp R$ at fixed length as $R \downarrow 0$. The broom calculation above gives the elementary Neumann-arm version. The sharp fixed-$(L, R)$ envelope, rather than the computation of torsion on a star or stower, is the joint geometric comparison established here. All statements concern a fixed exponent $1 < p < \infty$.

## 3 The sharp positive-resolvent envelope

Return to the linear operator $H = -\,\mathrm{d}^2/\,\mathrm{d}x^2$ with Dirichlet conditions on $D$ and standard continuity/Kirchhoff conditions elsewhere. For $z > 0$, write

$$J_G(z) = \langle \mathbf{1}, (H+z)^{-1}\mathbf{1}\rangle, \qquad s = \sqrt{z},$$

and define

$$F_{L,R}(z) = \frac{L}{z} - \frac{\tanh(sR) + s(L-R)}{z^{3/2}\big[1 + s(L-R)\tanh(sR)\big]}. \tag{11}$$

The following is a resolvent comparison; the pointwise heat comparison with the endpoint-mass model is proved for rooted trees in Section 7.

**Theorem 3.1.** *For every $z > 0$,*

$$J_G(z) \le F_{L,R}(z).$$

*For each prescribed $0 < R < L$ and $z > 0$, this is the exact supremum over finite graphs with those $L, R$. The DN interval attains the endpoint $F_{L,L}(z) = L/z - \tanh(sL)/z^{3/2}$.*

*Proof.* The anchored form is coercive, and its resolvent solution $u = (H + z)^{-1}\mathbf{1}$ satisfies

$$\int_G u'f' \, \mathrm{d}x + z\int_G uf \, \mathrm{d}x = \int_G f \, \mathrm{d}x \qquad (f \in V_2(G, D)).$$

Testing with the negative part gives $u \ge 0$. Testing with $(u - 1/z)_+$, which vanishes on $D$, gives

$$\int_G |((u - 1/z)_+)'|^2 \, \mathrm{d}x + z\int_G (u - 1/z)_+^2 \, \mathrm{d}x = 0.$$

Hence $0 \le u \le 1/z$. Testing by $u$ and completing the form square also gives

$$J_G(z) = \sup_{f \in V_2(G,D)} \left\{ 2\int_G f \, \mathrm{d}x - \int_G |f'|^2 \, \mathrm{d}x - z\int_G f^2 \, \mathrm{d}x \right\}. \tag{12}$$

The supremum is attained by $u$.

Choose a maximum point of $u$, and a shortest anchored path to it of length $r \le R$. Set $g = u$ along that path, parameterized from its Dirichlet endpoint. On $[0, 1/z]$, the function $w(v) = 2v - zv^2$ is nondecreasing. Thus the complement contribution is at most $(L-r)w(g(r))$; discarding only its nonnegative gradient energy yields

$$J_G(z) \le \int_0^r (2g - zg^2 - |g'|^2) \, \mathrm{d}y + (L - r)\big[2g(r) - zg(r)^2\big]. \tag{13}$$

We may bound this by the supremum over all real $g \in H^1(0, r)$ with $g(0) = 0$. This strictly concave functional has the unique maximizer $v$ satisfying, with $\beta_r = L - r$,

$$-v'' + zv = 1, \qquad v(0) = 0, \qquad v'(r) = \beta_r\big(1 - zv(r)\big).$$

The boundary condition follows by integration by parts at the free endpoint; when $\beta_r = 0$ it is Neumann. Put $t = \tanh(sr)$ and $b = s\beta_r$. Direct solution gives

$$v(y) = \frac{1 - \cosh(sy) + A\sinh(sy)}{z}, \qquad A = \frac{t + b}{1 + bt}. \tag{14}$$

All denominators are positive. Integration of its equation and the endpoint condition gives

$$z\left[\int_0^r v \, \mathrm{d}y + \beta_r v(r)\right] + v'(0) = L.$$

Testing the endpoint-mass weak equation by $v$ shows that the maximum of (13) is $\int_0^r v \, \mathrm{d}y + \beta_r v(r)$. Since $v'(0) = A/s$, this maximum is $F_{L,r}(z)$.

It remains to vary $r$ while keeping the total length $L$ fixed. Put $x = sr$, $\ell = sL$, $b = \ell - x$ and $t = \tanh x$. Differentiation gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{t + b}{1 + bt} = -\frac{b^2(1 - t^2)}{(1 + bt)^2}.$$

Therefore $F_{L,r}(z)$ is nondecreasing in $r$, and $J_G(z) \le F_{L,r}(z) \le F_{L,R}(z)$. This does not assume that the resolvent maximum occurs at a point realizing the inradius.

For sharpness, use a broom with stem $a$ and $n$ equal arms $b$. An arm with junction value $c$ has the solution

$$u_{\mathrm{arm}}(t) = \frac{1}{z} + \left(c - \frac{1}{z}\right) \frac{\cosh(s(b-t))}{\cosh(sb)}.$$

Its outgoing derivative at the junction is $(1-zc)\tanh(sb)/s$. Kirchhoff gives the stem condition $u'_{\mathrm{stem}}(a) = n\tanh(sb)(1-zu(a))/s$. Solving the stem as in (14) yields

$$u'_{\mathrm{stem}}(0) = \frac{A_n}{s}, \qquad A_n = \frac{\tanh(sa) + n\tanh(sb)}{1 + n\tanh(sb)\tanh(sa)}.$$

Integration of the graph equation cancels all non-Dirichlet fluxes and gives $u'_{\mathrm{stem}}(0) + zJ_G(z) = L$. Hence its exact resolvent is

$$J_G(z) = \frac{L}{z} - \frac{A_n}{z^{3/2}}. \tag{15}$$

At fixed $0 < R < L$, take $b = (L-R)/(n-1)$ and $a = R - b > 0$. Then $a \to R$ and $n\tanh(sb) \to s(L-R)$, proving convergence to (11). Every finite graph in the sequence has exactly the prescribed $L, R$. At $R = L$, direct solution on the DN interval gives the stated endpoint value. □

**Corollary 3.2** (An explicit weighted heat deficit)**.** *Let $I_L$ be the length-L DN interval. Then, for $z > 0$,*

$$\begin{aligned} J_{I_L}(z) - J_G(z) &\geq J_{I_L}(z) - F_{L,R}(z) \\ &= \int_R^L \frac{(L-r)^2\operatorname{sech}^2(sr)}{[1 + s(L-r)\tanh(sr)]^2}\,\mathrm{d}r \qquad (16) \\ &\geq \frac{\operatorname{sech}^2(sL)}{3(1+sL)^2}(L-R)^3. \qquad (17) \end{aligned}$$

*The middle expression is the sharp envelope deficit. It is positive for $R < L$ and tends to $(L-R)^3/3$ as $z \downarrow 0$.*

*Proof.* The derivative computed above gives exactly

$$\frac{\partial F_{L,r}(z)}{\partial r} = \frac{(L-r)^2\operatorname{sech}^2(sr)}{[1 + s(L-r)\tanh(sr)]^2}.$$

Integrating from $R$ to $L$ proves (16). On this interval the numerator's hyperbolic factor is at least $\operatorname{sech}^2(sL)$, while the denominator is at most $(1+sL)^2$. Integration of $(L-r)^2$ proves (17). As $z \downarrow 0$, the integrand tends to $(L-r)^2$ and is bounded by it. Dominated convergence proves the limit. Also $J_{I_L}(z) \to L^3/3$, so the stationary $p = 2$ envelope is recovered without requiring cancellation of divergent terms as a proof step. □

If $\mathcal{Q}_G(t)$ denotes the heat content for generator $-H$, the spectral semigroup–resolvent identity gives

$$J_G(z) = \int_0^\infty e^{-zt}\mathcal{Q}_G(t)\,\mathrm{d}t.$$

Thus (16) is an exponentially weighted all-time heat-content deficit bound. Pointwise heat ordering against the endpoint-mass model is obtained on rooted trees and exact tree quotients by the positive-kernel arguments of Section 7, and fixed-time stability rates on every graph by the parabolic argument of Section 8. The stationary and resolvent proofs use classical variational duality, the maximum principle and elementary interval equations.

# 4 Marked measured geometry

Throughout this section normalize total length to one. Distances are rescaled by $L^{-1}$ and arclength measure by $L^{-1}$ when starting from a graph of length $L$. Write

$$R = \max_{x \in G} \operatorname{dist}(x, D), \qquad \beta = 1 - R.$$

The interval used for geometric comparison is $[0, 1]$ marked at its Dirichlet endpoint 0. Reflection identifies its heat content with that of the physical interval in Section 5.

**Definition 4.1.** Let $d_m(G, I)$ be the infimum over common isometric embeddings of the maximum of the Hausdorff distance between the supports, the Hausdorff distance between their closed Dirichlet marked sets, and the Prokhorov distance between their arclength probability measures. The Prokhorov distance is the infimum of $\varepsilon > 0$ such that $\mu(A) \le \nu(A^\varepsilon) + \varepsilon$ and $\nu(A) \le \mu(A^\varepsilon) + \varepsilon$ for every Borel set $A$; $A^\varepsilon$ denotes its open $\varepsilon$-neighborhood.

**Theorem 4.2** (Geometric consequence of the inradius reserve)**.** *For every graph in the stated model,*

$$d_m(G, I) \le 2\beta.$$

*Consequently, for every fixed $1 < p < \infty$, $q = p/(p-1)$, and every original length $L$,*

$$d_m(G/L, I) \le 2 \left[ \frac{L^{q+1} - (q+1) S_p(G)}{L^{q+1}} \right]^{1/(q+1)}. \tag{18}$$

*The exponent $1/(q+1)$ cannot be increased in an estimate uniform over these graphs, even with a fixed graph topology.*

*Proof.* Choose $x_*$ at distance $R$ from $D$ and a shortest path $\gamma$ from a closest $d_0 \in D$ to $x_*$. This path is an isometric interval: a shortcut between two of its points would shorten the whole path. Its complement has total length $\beta$. Every off-path point connects to its first attachment on $\gamma$ by a simple path of length at most $\beta$. The same statement applies to another Dirichlet vertex $d$. If its first attachment is $\gamma(s)$ and that connecting path has length $\ell \le \beta$, then

$$R \le \operatorname{dist}(x_*, d) \le R - s + \ell.$$

Thus $s \le \ell$ and $\operatorname{dist}(d, d_0) \le 2\beta$. A Dirichlet vertex already on $\gamma$ must be $d_0$, by the same inequality with $\ell = 0$.

Attach a new dead-end segment of length $\beta$ at $x_*$. In the resulting space both $G$ and the interval formed by $\gamma$ followed by this segment embed isometrically. Their support Hausdorff distance is at most $\beta$ and their marked-set Hausdorff distance is at most $2\beta$. Their measures agree on $\gamma$, of mass $1-\beta$, and the remaining mass is $\beta$ for each. Their total-variation distance, and hence their Prokhorov distance, is at most $\beta$. This proves the embedding bound. The stationary reserve gives $\beta^{q+1} \le [L^{q+1} - (q+1)S_p(G)]/L^{q+1}$, proving (18).

For the exponent, take a length-one broom with three equal arms of length $b$ and a stem of length $1 - 3b$, where $0 < b < 1/4$. Its diameter is $1 - 2b$, so its ordinary Gromov–Hausdorff distance from the unit interval is at least $b$: any common embedding has Hausdorff distance at least half the difference of diameters. Its exact torsion deficit is

$$\frac{1}{q+1} - S_p = \frac{3^{q+1} - 3}{q+1} b^{q+1}.$$

Since $d_m$ dominates the ordinary Gromov–Hausdorff distance, no larger deficit exponent can hold uniformly as $b \downarrow 0$. □

## 5 The constant-data heat deficit and its positive source

**Relation to the companion paper.** This section overlaps with [A]. The all-times inequality $\mathcal{Q}_G \le \mathcal{Q}_{I_L}$ is [A, Theorem 1.2], where it follows from a concentration comparison for arbitrary nonnegative $L^2$ data [A, Theorem 1.1]. Lemma 5.1, Lemma 5.2, Theorem 5.3 and Corollary 5.4 correspond to [A, Lemma 4.1, Lemma 3.2, Theorem 4.2 and Corollary 4.5]. Only initial datum one is needed here, and the argument below obtains $W \ge 0$ directly from the source representation. The section is therefore self-contained, and its quantitative source identities are the ones used in Sections 6 and 8.

Let $G$ be a finite compact connected metric graph obtained by identifying endpoints of finitely many intervals of positive lengths $\ell_e$. Its total length is $L = \sum_e \ell_e > 0$, and integration uses arclength; vertices have measure zero. Loops and multiple edges are allowed, and a loop contributes two germs at its vertex. Let $D$ be a nonempty set of vertices. The form

$$a_G(f,g) = \sum_e \int_e f'\overline{g'}\,\mathrm{d}x, \qquad V = \{f \in H^1(G) : f|_D = 0\},$$

where $H^1(G)$ means continuous and edgewise $H^1$, defines the self-adjoint operator $H_G$. It acts as $-d^2/dx^2$, with continuity and unweighted Kirchhoff conditions outside $D$. There is no additional flux condition at a Dirichlet vertex. A non-Dirichlet leaf is Neumann. The heat generator is $+d^2/dx^2$, without a factor $1/2$.

The Dirichlet anchor gives a useful elementary coercivity estimate. A simple path from $D$ to any $x \in G$ has length at most $L$, so

$$|f(x)| \le \sqrt{L}\,\|f'\|_{L^2(G)}, \qquad \|f\|^2_{L^2(G)} \le L^2 a_G(f,f) \quad (f \in V).$$

Thus $H_G$ has a positive spectral gap. The finite-edge Sobolev embedding also makes its resolvent compact. Inserting or suppressing a non-Dirichlet vertex of degree two changes neither this form nor its heat semigroup.

Cutting every Dirichlet incidence separates the form into finitely many positive-length components, each anchored at a Dirichlet endpoint. A path from any point to $D$ proves that each component is anchored. In particular $G \setminus D$ need not be connected. The semigroup $T_G(t) = e^{-tH_G}$ is positive and sub-Markovian, by the usual form contraction property. On each cut component it is positivity improving. Here one can use the classical strong parabolic maximum principle on the open edges: an interior zero at a positive time forces vanishing backwards on that edge. At a zero non-Dirichlet vertex all outgoing derivatives are nonnegative. If an adjacent edge solution is positive, the boundary Hopf lemma makes its outgoing derivative strictly positive, contradicting the Kirchhoff sum. Vanishing therefore propagates through the connected component. A nonzero initial datum cannot have a zero evolved component, since the spectral multipliers $e^{-t\lambda}$ are nonzero. This proves strict positivity off the Dirichlet endpoints for any nonzero nonnegative component datum. This is the standard positivity mechanism for these graph forms; see also [9, Theorem A.2].

Only the initial datum $\mathbf{1}$ is used in this section. Write

$$u(t) = T_G(t)\mathbf{1}, \qquad \mathcal{Q}_G(t) = \langle \mathbf{1}, T_G(t)\mathbf{1}\rangle.$$

Thus $\mathcal{Q}_G$ is heat content, not heat trace. Let $I_L = [0,L]$ have Neumann condition at 0 and Dirichlet condition at $L$, and write $v(t) = T_{I_L}(t)\mathbf{1}$. Its explicit heat content is

$$\mathcal{Q}_{I_L}(t) = \frac{8L}{\pi^2} \sum_{n\ge 0} \frac{e^{-(2n+1)^2\pi^2 t/(4L^2)}}{(2n+1)^2}. \tag{19}$$

Its torsion solution is $(L^2 - x^2)/2$, and consequently

$$\int_0^\infty \mathcal{Q}_{I_L}(t)\,\mathrm{d}t = \frac{L^3}{3}. \tag{20}$$

Reflection gives the same heat content when the Dirichlet mark is placed at 0 for later geometric comparisons.

### 5.1 Level coordinates for the survival solution

For a nonnegative function $f$ on the graph, define its top-mass profile by

$$\mathcal{M}f(s) = \sup_{|E|=s} \int_E f\,\mathrm{d}x = \int_0^s f^*(y)\,\mathrm{d}y, \qquad 0 \le s \le L,$$

where $f^*$ is its decreasing rearrangement. The supremum allows measurable parts of a level set. Directly from this formula,

$$\|\mathcal{M}f - \mathcal{M}g\|_\infty \le \|f - g\|_1 \le \sqrt{L}\,\|f - g\|_2 \qquad (f, g \ge 0). \tag{21}$$

Set

$$U(t,s) = \mathcal{M}u(t)(s), \qquad F(t,s) = \int_0^s v(t,x)\,\mathrm{d}x, \qquad W = F - U.$$

No sign of $W$ is assumed at this point.

**Lemma 5.1** (Positive-time survival profiles)**.** *For every $t > 0$, $u(t) > 0$ off $D$ and $u_{xx}(t,x) = u_t(t,x) < 0$ on the open edges. Each edge profile is strictly concave and has at most one interior critical point. All edge derivatives needed below are bounded locally uniformly for positive time.*

*Proof.* Positivity and sub-Markovianity give

$$u(t+h) = T_G(t)T_G(h)\mathbf{1} \le T_G(t)\mathbf{1} = u(t).$$

Spectral calculus puts $u(t)$ in every power domain of $H_G$ for $t > 0$; the edge equation and one-dimensional Sobolev regularity give smooth edgewise representatives and locally uniform derivative bounds. Therefore $w(t) = -u_t(t) = H_G u(t) \ge 0$. On each anchored cut component, $u(r)$ is nonzero by injectivity and the initial component datum $\mathbf{1}$. If $w(r) = 0$ on that component, its trivial operator kernel would give $u(r) = 0$, a contradiction. For $0 < r < t$, $w(t) = T_G(t-r)w(r)$, so positivity improvement gives $w(t) > 0$ on every open edge and at non-Dirichlet vertices. The edge heat equation yields $u_{xx} = u_t = -w < 0$. Positivity of $u$ follows from the same component positivity argument applied directly to $u(t)$. Strict decrease of its edge derivative proves the critical-point assertion. □

At each positive time there are therefore only finitely many vertex and critical levels, and every level set has length zero. The coarea argument below is the graph rearrangement mechanism of [3, Lemma 3]; the integrated parabolic formulation has classical symmetrization ancestry [7, 8]. We print the needed constant-data argument.

**Lemma 5.2** (Inverse-distribution regularity)**.** *For each $t > 0$, the decreasing rearrangement $q(t,\cdot) = u(t)^*$ extends by $q(t,0) = \max u(t)$ and $q(t,L) = 0$ to a Lipschitz function. Consequently $U(t,\cdot) \in W^{2,\infty}(0,L)$, with $U(t,0) = 0$ and $U_s(t,L) = 0$, locally uniformly for positive times.*

*Proof.* Fix a time, let $M = \max_e \|u_e'\|_\infty > 0$, and set $\mu(z) = |\{u > z\}|$. A simple arclength path from a point of $D$ to a maximizer crosses every level between zero and $\max u$. For $0 \le \alpha < \beta \le \max u$, choose the first $\beta$ crossing on that path and the last preceding $\alpha$ crossing. The intervening open segment has $\alpha < u < \beta$, is counted only once in edge measure, and has length at least $(\beta - \alpha)/M$. Hence

$$\mu(\alpha) - \mu(\beta) \ge (\beta - \alpha)/M.$$

The distribution is continuous because all level sets have zero length; it strictly decreases from $L$ to zero. Its inverse is therefore $M$-Lipschitz, including the endpoint extensions. Integrating that inverse gives the asserted spatial regularity. The derivative bounds from Lemma 5.1 are uniform on compact positive-time intervals. No connectedness of the Dirichlet-cut interior was used. □

For a regular level $z$ of a positive-time profile, let $s = \mu(z)$ and sum over all its crossings:

$$A = \sum_{u=z} |u_x|, \qquad B = \sum_{u=z} |u_x|^{-1}, \qquad N = \#\{u = z\}.$$

The path in Lemma 5.2 ensures $N \ge 1$. Coarea on the finitely many monotone edge pieces and inverse differentiation give

$$-\mu'(z) = B, \qquad U_{ss}(t,s) = -1/B, \qquad AB \ge N^2 \ge 1. \tag{22}$$

The top-$s$ set is the hot set $E(t,s) = \{u(t) > q(t,s)\}$, up to null sets. Its envelope derivative is

$$U_t(t,s) = \int_{E(t,s)} u_t(t,x)\,\mathrm{d}x = -A. \tag{23}$$

To justify the first equality, bracket the time difference with the old and new maximizing sets. Uniform convergence of $u$ and continuity of its inverse distribution place their symmetric difference in a shrinking band around a null level set. The two difference quotients have the same limit. For the second equality integrate $u_t = u_{xx}$ over the hot set. Every hot non-Dirichlet vertex includes all its germs and its flux cancels by Kirchhoff. Dirichlet vertices do not occur at a positive hot level. Each boundary crossing contributes $-|u_x|$. Disconnected hot sets give the same cancellation. Also $|U(t,s) - U(r,s)| \le \|u(t) - u(r)\|_1$ gives local Lipschitz regularity in time, uniformly in $s$.

### 5.2 The positive heat-deficit source

On the mass interval $[0, L]$ let

$$\mathsf{A} = -d^2/ds^2, \qquad f(0) = 0, \quad f'(L) = 0, \qquad \mathcal{S}(t) = e^{-t\mathsf{A}},$$

and let $K(t; s, y)$ be its kernel. This semigroup is positive and an $L^\infty$ contraction by the mixed maximum principle. Its positive-time kernel is smooth and

$$K(t; s, y) = K_{DD,2L}(t; s, y) + K_{DD,2L}(t; s, 2L - y) > 0 \quad (t > 0,\ s, y \in (0, L]),$$

by reflection at $L$, where $K_{DD,2L}$ is the usual Dirichlet kernel on $(0, 2L)$. Its Green kernel is

$$\int_0^\infty K(h; s, y)\,\mathrm{d}h = \min(s, y), \tag{24}$$

as follows by solving the mixed one-dimensional inverse problem. At the Dirichlet mass endpoint, the inward boundary derivative is strictly positive by the parabolic Hopf lemma. Thus $K(h; L, y)/y$ has a positive continuous extension at $y = 0$, uniformly for $h$ in any compact positive window. These facts concern this fixed interval, not heat kernels on a varying graph.

**Theorem 5.3** (Exact constant-data source)**.** *There is a jointly measurable nonnegative function $d(t,s)$, bounded on compact positive-time cylinders, such that at almost every mass coordinate of every positive-time profile,*

$$d = A - \frac{1}{B} = \frac{N^2-1}{B} + \frac{1}{B}\sum_{i<j}\frac{(\alpha_i-\alpha_j)^2}{\alpha_i\alpha_j}, \qquad \alpha_i = |u_x| \text{ at the crossings.} \tag{25}$$

*The deficit satisfies*

$$W_t - W_{ss} = d, \qquad W(t,0) = 0, \quad W_s(t,L) = 0,$$

*and, for every $t > 0$ and $0 \le s \le L$,*

$$W(t,s) = \int_0^t \int_0^L K(t-r;s,y)d(r,y)\,\mathrm{d}y\,\mathrm{d}r. \tag{26}$$

*The integral is the finite increasing limit from source times $r \ge \varepsilon$ as $\varepsilon \downarrow 0$. In particular $W \ge 0$, so $\mathcal{Q}_G(t) \le \mathcal{Q}_{I_L}(t)$ for all $t \ge 0$, which is [A, Theorem 1.2].*

*Proof.* Equations (22)–(23) give $U_t - U_{ss} = -(A - 1/B)$. Direct expansion gives

$$AB - N^2 = \sum_{i<j}\left(\frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} - 2\right),$$

proving (25) and nonnegativity. Every regular level has at most two crossings per edge. If $M(t) = \max_e \|u_x(t)\|_\infty$, then $0 \le d \le A \le 2|\mathsf{E}(G)|M(t)$. The weak time and spatial derivatives define a jointly measurable version. At each fixed time there are finitely many exceptional mass coordinates. The established Sobolev and local Lipschitz bounds exclude any singular residual in the space-time equation.

For the physical interval, Lemma 5.1 gives $v_{xx} < 0$ at positive times, while its reflecting condition is $v_x(t,0) = 0$. Thus $v$ is decreasing from that endpoint. Integrating its heat equation gives $F_t = F_{ss}$, with $F(t,0) = 0$ and $F_s(t,L) = v(t,L) = 0$.

Fix $0 < \varepsilon < t$. Variation of constants for the weak mixed heat problem on $[\varepsilon, t]$ yields

$$\begin{aligned} W(t,s) = {} & (\mathcal{S}(t-\varepsilon)W(\varepsilon,\cdot))(s) \\ & + \int_\varepsilon^t \int_0^L K(t-r;s,y)d(r,y)\,\mathrm{d}y\,\mathrm{d}r. \end{aligned} \tag{27}$$

The identity first holds in $L^2$; the continuous initial profiles and positive-time kernel smoothing make the initial term continuous. For the source integral, split off a short final-time strip: its sup norm is bounded by the strip length times the local $L^\infty$ source bound, while on the remaining strip the kernel is smooth. This gives a continuous representative and equality for every $s$, including $L$.

Strong $L^2$ continuity of both heat semigroups and (21) imply $U(\varepsilon,\cdot) \to s$ and $F(\varepsilon,\cdot) \to s$ uniformly. Hence $\|W(\varepsilon,\cdot)\|_\infty \to 0$. The contraction property gives

$$\|\mathcal{S}(t-\varepsilon)W(\varepsilon,\cdot)\|_\infty \le \|W(\varepsilon,\cdot)\|_\infty \longrightarrow 0.$$

No sign of this initial term is required. The source integral in (27) is nonnegative and increases as $\varepsilon$ decreases. That same identity bounds it by $|W(t,s)| + \|W(\varepsilon,\cdot)\|_\infty$, so its limit is finite and equals $W(t,s)$. This proves (26), and derives $W \ge 0$ rather than assuming a previous concentration theorem. At $t = 0$ both heat contents equal $L$. □

**Corollary 5.4** (Integrated constant-data source)**.** *For $0 \le s \le L$,*

$$\int_0^\infty W(t,s)\,\mathrm{d}t = \int_0^\infty \int_0^L \min(s,y) d(r,y)\,\mathrm{d}y\,\mathrm{d}r.$$

*In particular*

$$\frac{L^3}{3} - \langle \mathbf{1}, H_G^{-1}\mathbf{1}\rangle = \int_0^\infty \int_0^L y\, d(r,y)\,\mathrm{d}y\,\mathrm{d}r. \tag{28}$$

*For $L = 1$, $0 \le \Delta_G(t) = \mathcal{Q}_{I_1}(t) - \mathcal{Q}_G(t) \le \mathcal{Q}_{I_1}(t)$ and $\int_0^\infty \mathcal{Q}_{I_1}(t)\,\mathrm{d}t = 1/3$.*

*Proof.* Tonelli in (26) and (24) gives the identity. Its left side is finite because $0 \le W(t,s) \le F(t,s) \le \mathcal{Q}_{I_L}(t)$ and (20) holds. At $s = L$, integrate both heat contents; spectral calculus and coercivity give $\int_0^\infty \mathcal{Q}_G(t)\,\mathrm{d}t = \langle \mathbf{1}, H_G^{-1}\mathbf{1}\rangle$. □

# 6 A backward integral estimate

In this section normalize total length to one and write $I = I_1$. Put $\Delta_G(t) = \mathcal{Q}_I(t) - \mathcal{Q}_G(t)$. For datum one, Theorem 5.3 gives

$$\Delta_G(t) = \int_0^t \int_0^1 K(t-r;1,y) d(r,y)\,\mathrm{d}y\,\mathrm{d}r, \qquad d \ge 0. \tag{29}$$

This is the fixed mixed kernel in the mass coordinate, not the graph kernel. Its Green integral is

$$\int_0^\infty K(h;1,y)\,\mathrm{d}h = y.$$

For $0 < a \le b < \infty$ define

$$c(a,b) = \min_{\substack{a\le h\le b\\ 0\le y\le 1}} \frac{K(h;1,y)}{y} > 0, \tag{30}$$

where the quotient at $y = 0$ is the inward derivative. To justify positivity, positive-time smoothness and positivity of this fixed interval kernel give positivity for $y > 0$, including its Neumann endpoint. Reflection across the Neumann endpoint reduces the Dirichlet endpoint to the ordinary interval kernel; the parabolic boundary Hopf lemma gives a strictly positive inward derivative. The quotient therefore extends continuously and positively on the displayed compact set. These facts involve only a fixed interval.

**Lemma 6.1** (Early integrated deficit from one observation)**.** *Fix $0 < \tau_0 \le \tau_1 < \infty$, set $T = \tau_0/2$ and $c = c(\tau_0/2, \tau_1)$. For every $t_0 \in [\tau_0, \tau_1]$,*

$$\int_0^T \Delta_G(t)\,\mathrm{d}t \le \frac{\Delta_G(t_0)}{c}. \tag{31}$$

*Proof.* Tonelli and the Green integral give

$$\int_0^T \Delta_G(t)\,\mathrm{d}t \le \int_0^T \int_0^1 y\, d(r,y)\,\mathrm{d}y\,\mathrm{d}r.$$

In (29) at $t_0$, retain $r \le T$. Then $t_0 - r \in [\tau_0/2, \tau_1]$, and the retained integral is at least $c$ times the right side. All integrands are nonnegative; the monotone interpretation at initial time from Theorem 5.3 makes both Tonelli steps valid without a separate initial integrability assumption. □

# 7 The sharp heat envelope on rooted trees

We now retain the endpoint mass as a positive heat model. For $r > 0$ and $\beta \geq 0$, let $E(r, \beta)$ have speed measure $\mu_{r,\beta} = \mathrm{d}x + \beta\delta_r$ on $[0, r]$ and quadratic form

$$\mathcal{E}(f, g) = \int_0^r f'\overline{g'}\ \mathrm{d}x, \qquad f, g \in H^1(0, r), \quad f(0) = g(0) = 0.$$

Its total mass is $r + \beta$. At $\beta = 0$ it is the ordinary DN interval. For $\beta > 0$ the heat equation is $u_t = u_{xx}$ in the interior, with $\beta u_t(t, r) = -u_x(t, r)$ at the endpoint. Write $Q_{r,\beta}(t) = \langle \mathbf{1}, e^{-tH_{r,\beta}}\mathbf{1}\rangle_{\mu_{r,\beta}}$. The initial datum and this integral both include the endpoint atom. The resolvent calculation in Section 3 gives

$$\widehat{Q}_{r,\beta}(z) = F_{r+\beta,r}(z),$$

where $F$ is defined by (11).

## 7.1 A positive model variation

**Lemma 7.1** (Endpoint absorption densities)**.** *For every $r > 0$ and $\beta \geq 0$ there is a continuous strictly positive probability density $k_{r,\beta}$ on $(0, \infty)$ with transform*

$$h_{r,\beta}(z) = \frac{1}{\cosh(r\sqrt{z}) + \beta\sqrt{z}\sinh(r\sqrt{z})}. \tag{32}$$

*The densities and their self-convolutions are jointly continuous for $r$ in a compact positive interval, $\beta$ in a compact nonnegative interval, and time in a compact subinterval of $(0, \infty)$.*

*Proof.* The form is dense, closed and Markovian. For $\beta > 0$, thin boundary layers approximate the endpoint coordinate independently in $L^2(\mu)$; thus the trace constraint in the form domain does not prevent density. The bounds $|f(r)|^2 \leq r\mathcal{E}(f, f)$ and $\int |f|^2 \leq r^2\mathcal{E}(f, f)$ give coercivity. Compactness of the $H^1$ embedding, including its endpoint trace, gives compact resolvent and strictly positive spectral bottom.

Put $v(t) = e^{-tH}\mathbf{1}$. The Markov property gives $0 \leq v \leq 1$ and $v(t + s) \leq v(t)$. For positive time, $g(t) = -v_t(t) = He^{-tH}\mathbf{1}$ is nonnegative and nonzero. Interior positivity follows from the strong maximum principle applied to $g(t) = e^{-tH/2}g(t/2)$. It holds at $r$ as well. Indeed, if a nonnegative positive-time solution vanished at $r$, its time derivative there would vanish, whereas the boundary Hopf lemma would give a strictly negative outward derivative. This contradicts $\beta g_t(r) = -g_x(r)$, or the Neumann condition when $\beta = 0$. Positive-time smoothing gives the continuous traces used here. Thus $k_{r,\beta}(t) = g(t, r) > 0$. For $\beta > 0$, strong initial $L^2(\mu)$ continuity gives $v(0+, r) = 1$; at $\beta = 0$ the ordinary DN kernel gives the same limit. Coercivity and smoothing give $v(\infty, r) = 0$, so $\int k_{r,\beta} = 1$.

For $u = (H + z)^{-1}\mathbf{1}$, integration by parts in time gives $\widehat{k}(z) = 1 - zu(r)$. The function $a = 1 - zu$ solves

$$a'' = za, \qquad a(0) = 1, \qquad a'(r) = -\beta z a(r).$$

Its endpoint value is (32). To check parameter continuity, take $z = \sigma + i\omega$, $\sigma > 0$, and the square root $w$ with positive real part. For $p(x) = \cosh(wx)/\cosh(wr)$, integration of $p'' = zp$ gives

$$w\tanh(rw) = \int_0^r |p'|^2\ \mathrm{d}x + z\int_0^r |p|^2\ \mathrm{d}x.$$

Its real part is nonnegative. Hence $|1 + \beta w\tanh(rw)| \geq 1$ and $|h_{r,\beta}(z)| \leq |\operatorname{sech}(rw)|$. For $r \geq a > 0$,

$$|\operatorname{sech}(rw)| \leq \frac{2e^{-r\operatorname{Re} w}}{1 - e^{-2a\sqrt{\sigma}}}.$$

Since $\operatorname{Re}\sqrt{\sigma+i\omega}$ grows like $\sqrt{|\omega|/2}$, this is an integrable vertical-line majorant for both $h$ and $h^2$, uniformly over the stated parameters. Absolute Bromwich inversion gives the asserted joint continuity, including $\beta = 0$. Tonelli and transform uniqueness identify the inverse of $h^2$ with $k * k$. □

**Proposition 7.2** (Positive radius variation)**.** *For fixed $L > 0$, $0 < R < L$ and $t > 0$,*

$$\mathcal{Q}_{I_L}(t) - Q_{R,L-R}(t) = \int_R^L (L-r)^2 (k_{r,L-r} * k_{r,L-r})(t)\ \mathrm{d}r. \tag{33}$$

*For each window $0 < \tau_0 \le \tau_1 < \infty$ there is $c > 0$ such that, at unit total mass,*

$$\mathcal{Q}_{I_1}(\tau) - Q_{R,1-R}(\tau) \ge \frac{c}{24}(1-R)^3 \qquad (0 < R < 1,\ \tau \in [\tau_0, \tau_1]). \tag{34}$$

*Proof.* The derivative in the proof of Corollary 3.2 is

$$\partial_r F_{L,r}(z) = (L-r)^2 h_{r,L-r}(z)^2.$$

Integrate from $R$ to $L$. Lemma 7.1 supplies the positive inverse convolution for every integrand. Tonelli and transform uniqueness prove (33); continuity upgrades the identity to every positive time. This uses an identified positive inverse, rather than an implication from scalar transform order. At $L = 1$ put

$$c = \min_{\substack{1/2 \le r \le 1 \\ \tau_0 \le \tau \le \tau_1}} (k_{r,1-r} * k_{r,1-r})(\tau) > 0.$$

Positivity, continuity and compactness give the strict inequality. Restrict (33) to $r \in [(1+R)/2, 1]$. The remaining weight integral is $(1-R)^3/24$. □

## 7.2 Positive rooted responses and parallel merging

For a finite singly rooted graph, or a rooted interval model with endpoint mass, let $\tau_x$ be absorption time and $v_z(x) = \mathbb{E}_x e^{-z\tau_x}$. For real $z > 0$, it solves $v_z'' = z v_z$, has value one at the root and natural conditions elsewhere. Let $p(z)$ be the negative sum of its outward root derivatives, and let $M$ be its total speed mass. Integrating the equation cancels the other vertex fluxes, including the dynamic endpoint contribution, and gives

$$p(z) = z \int v_z\ \mathrm{d}\mu, \qquad J(z) = \frac{M}{z} - \frac{p(z)}{z^2}. \tag{35}$$

If $U(t)$ is the absorption CDF with initial distribution $\mu/M$, then

$$\frac{p(z)}{z^2} = M \int_0^\infty e^{-zt} U(t)\ \mathrm{d}t.$$

For $E(r, \beta)$, direct solution gives, with $k = \sqrt{z}$,

$$p_{r,\beta}(z) = k \frac{\tanh(kr) + k\beta}{1 + k\beta \tanh(kr)}. \tag{36}$$

Its endpoint-start absorption law has transform $h_{r,\beta}$.

We will use a pathwise monotonicity of those endpoint laws. Let $B$ be standard Brownian motion and set

$$Y_u = \sqrt{2} B_u + \Lambda_u, \qquad \Lambda_u = \max\{0, -\inf_{0 \le v \le u} \sqrt{2} B_v\}, \qquad \sigma_r = \inf\{u : Y_u = r\}.$$

The boundary regulator $\Lambda$ is normalized for generator $d^2/dx^2$. The time

$$T_{r,\beta} = \sigma_r + \beta\Lambda_{\sigma_r} \tag{37}$$

has transform $h_{r,\beta}$. To verify the coefficient of the clock, apply Itô's formula to $e^{-z(u+\beta\Lambda_u)}f(Y_u)$, stopped at $\sigma_r$. The interior drift is $(f'' - zf)\,\mathrm{d}u$ and its boundary drift is $[f'(0) - z\beta f(0)]\,\mathrm{d}\Lambda_u$. Take

$$f(r) = 1, \qquad f'' = zf, \qquad f'(0) = z\beta f(0).$$

The resulting expectation is (32), which also fixes the endpoint local-time convention. On the same path, both $\sigma_r$ and $\Lambda_{\sigma_r}$ increase with $r$. Therefore

$$r \le R, \quad \gamma \le \eta \quad \Longrightarrow \quad T_{r,\gamma} \le T_{R,\eta} \quad \text{on this coupling.} \tag{38}$$

Absorption from any starting point of $E(r,\gamma)$ is stochastically no larger than absorption from its far endpoint: the endpoint-start path must first hit that point, after which the strong Markov property supplies its remaining absorption time. Averaging includes the atom and gives

$$\tau_{\mathrm{unif},r,\gamma} \le_{\mathrm{st}} T_{r,\gamma}, \qquad \text{initial law } \frac{\mathrm{d}x + \gamma\delta_r}{r+\gamma}.$$

**Lemma 7.3** (A positive parallel merge)**.** *Take two endpoint models of heights $R \ge r > 0$, with endpoint masses $\beta, \gamma \ge 0$. Their sum of heat contents is at most that of the single model of height $R$ and total mass $R + \beta + r + \gamma$. The inequality is strict at every positive time.*

*Proof.* Put $\ell = r + \gamma$. Fractional subtraction in (36) gives

$$p_{R,\beta+\ell} - p_{R,\beta} = \ell z h_{R,\beta} h_{R,\beta+\ell}.$$

By (35), the heat-transform gap for the proposed merge is

$$\frac{p_{R,\beta} + p_{r,\gamma} - p_{R,\beta+\ell}}{z^2} = \frac{p_{r,\gamma}}{z^2} - \frac{\ell h_{R,\beta} h_{R,\beta+\ell}}{z}.$$

Its inverse is the concrete CDF difference

$$\ell\left[\mathbb{P}(\tau_{\mathrm{unif},r,\gamma} \le t) - \mathbb{P}(T_{R,\beta} + T'_{R,\beta+\ell} \le t)\right], \tag{39}$$

where the two terms in the sum are independent. The stochastic chain

$$\tau_{\mathrm{unif},r,\gamma} \le_{\mathrm{st}} T_{r,\gamma} \le_{\mathrm{st}} T_{R,\beta+\ell} \le_{\mathrm{st}} T_{R,\beta} + T'_{R,\beta+\ell}$$

uses $\gamma \le \beta + \ell$ and (38). It proves the sign of (39). Its sign is strict: the last endpoint law has a strictly positive density by Lemma 7.1, whereas the independent added time $T_{R,\beta}$ is strictly positive almost surely. Adding it strictly reduces that CDF at every $t > 0$. Transform uniqueness and continuity identify (39) with the actual heat gap. □

The tree induction consumes this merge in the response form used by the next lemma. For a finite rooted load $X$, that is, a finite union of singly rooted graphs and endpoint models $E(r,\beta)$ glued only at a common root, write $p_X$ for its root response and $M_X$ for its total speed mass. Write $X \vee Y$ for two loads glued only at their common root.

**Lemma 7.4** (Merge lemma for root responses)**.**

*(i)* $p_{X\vee Y} = p_X + p_Y$ *and* $M_{X\vee Y} = M_X + M_Y$.

*(ii) If loads $X_i, X_i'$, $1 \le i \le m$, satisfy $M_{X_i} = M_{X_i'}$ and $(p_{X_i} - p_{X_i'})/z^2$ is the Laplace transform of a nonnegative function $K_i$, then $X_1 \vee \cdots \vee X_m$ and $X_1' \vee \cdots \vee X_m'$ have equal masses and their response difference divided by $z^2$ is the transform of $\sum_i K_i \ge 0$.*

*(iii) Let $R \ge r > 0$, $\beta, \gamma \ge 0$ and $\ell = r + \gamma$. Then $E(R, \beta) \vee E(r, \gamma)$ and $E(R, \beta + \ell)$ have equal mass $R + \beta + \ell$, and*

$$\frac{p_{R,\beta}(z) + p_{r,\gamma}(z) - p_{R,\beta+\ell}(z)}{z^2} = \int_0^\infty e^{-zt} K_{\mathrm{m}}(t)\,\mathrm{d}t \qquad (z > 0),$$

*where $K_{\mathrm{m}}$ is the function (39). It is nonnegative, and strictly positive at every $t > 0$.*

*For two loads of equal mass, $(p_X - p_{X'})/z^2$ is the transform of $\mathcal{Q}_{X'} - \mathcal{Q}_X$, by (35). Thus (iii) is the hypothesis of Lemma 7.5 for the pair $E(R, \beta) \vee E(r, \gamma)$, $E(R, \beta + \ell)$, including its strict form.*

*Proof.* (i) The absorption solution $v_z$ equals one at the common root, and the two loads meet only there. Hence it solves the separate problems on $X$ and on $Y$, and the outward root derivatives add. Masses add because the root carries no speed mass. (ii) Apply (i) repeatedly and use linearity of the Laplace transform. (iii) The mass identity is immediate. By (i) the response of the glued pair is $p_{R,\beta} + p_{r,\gamma}$. The displayed transform identity is the computation in the proof of Lemma 7.3: there $p_{r,\gamma}/z^2$ is the transform of $\ell$ times the uniform-start absorption CDF, by (35), and $\ell h_{R,\beta} h_{R,\beta+\ell}/z$ is the transform of $\ell$ times the CDF of $T_{R,\beta} + T'_{R,\beta+\ell}$. The sign and strictness are proved there. The final assertion follows from (35), since $J_X - J_{X'} = (p_{X'} - p_X)/z^2$ when $M_X = M_{X'}$. □

**Lemma 7.5** (Preservation through a common stem)**.** *Suppose two rooted loads have equal mass and their response difference $(p_1 - p_2)/z^2$ has a nonnegative inverse kernel $K$. Attaching the same stem preserves this property. If $K(t) > 0$ for every $t > 0$, its propagated kernel has that property as well.*

*Proof.* For a stem of length $a > 0$, direct solution gives the new response

$$\mathcal{M}_a(p) = \frac{k \tanh(ka) + p}{1 + (p/k)\tanh(ka)}.$$

Its difference factors exactly as

$$\mathcal{M}_a(p_1) - \mathcal{M}_a(p_2) = (p_1 - p_2) h_1 h_2, \qquad h_i = \frac{1}{\cosh(ka) + (p_i/k)\sinh(ka)}.$$

Each $h_i$ is the absorption-time transform from the stem/load junction in the corresponding positive rooted object. The load's boundary value multiplies its unit-root absorption solution, so its flux is that value times $p_i$; the stem equation yields the displayed formula. The new inverse kernel is therefore $K$ convolved with two probability measures. Equal masses make it exactly the new heat-content gap.

For strictness those measures have mass in $(0, t)$ for every $t > 0$. Indeed a load of mass $M$ has $0 \le p_i(z) \le zM$, by (35) and $0 \le v_z \le 1$. Thus

$$\frac{e^{-a\sqrt{z}}}{1 + M\sqrt{z}} \le h_i(z) \le \operatorname{sech}(a\sqrt{z}) \qquad (z > 0).$$

The upper bound excludes an atom at zero. The lower bound gives $-\log h_i(z)/z \to 0$ and excludes a positive support gap, since support in $[\varepsilon, \infty)$ would force $h_i(z) \le e^{-\varepsilon z}$. Their convolution also has mass in every $(0, t)$, proving the strict assertion. A zero-length stem leaves the kernel unchanged. □

**Theorem 7.6** (Sharp rooted-tree heat envelope)**.** *Let $T$ be a finite compact connected metric tree of length $L > 0$, with one Dirichlet vertex and Kirchhoff conditions at every other vertex. Let $R$ be its maximum distance from the Dirichlet vertex. Then*

$$\mathcal{Q}_T(t) \le Q_{R,L-R}(t) \qquad (t > 0). \tag{40}$$

*For each fixed $0 < R < L$, the right side is the exact supremum over these trees, approached uniformly on positive compact time windows by the same finite broom sequence. Every finite tree satisfies strict inequality when $R < L$. At $R = L$ the DN interval attains the envelope.*

*Proof.* A rooted edge is the endpoint model with zero endpoint mass. Induct from the leaves toward the root. At each branch, replace the child branches by their endpoint models with the same masses and heights. Their already nonnegative response-gap kernels add under parallel gluing, by Lemma 7.4(ii). Lemma 7.4(iii), repeated from a branch of maximum height, replaces these models by one endpoint model with the sum of their masses and the maximum of their heights; each repetition adds a further nonnegative kernel. Lemma 7.5 propagates the comparison along the next stem. Appending a stem of length $a$ to $E(r, \beta)$ gives exactly $E(r + a, \beta)$, so the actual subtree height and mass are preserved. The finite induction also covers a root of degree greater than one, where the last operation is a parallel merge at the root. Each step supplies a nonnegative inverse response kernel, proving (40). If $R < L$ there is a genuine parallel merge; otherwise the rooted tree is a DN path up to degree-two subdivision. The strict merge and common-stem lemmas propagate strictness to the root.

For sharpness fix $\beta = L - R > 0$, and take the brooms already used in (15), with $b_n = \beta/(n-1)$, $a_n = R - b_n > 0$ and $n$ arms. Their length and height are exactly $L, R$. Their responses converge for every $z > 0$ to $p_{R,\beta}(z)$. Their absorption laws for uniform initial arclength have transforms $p_n(z)/(zL)$, which converge to the actual model law with initial distribution $(\,\mathrm{d}x + \beta\delta_R)/L$. The limiting probability transform is continuous at zero, with value one, because $p_{R,\beta}(z) \sim zL$. The Laplace continuity theorem for probability measures gives weak convergence of these absorption laws. The model CDF is continuous: its survival is $Q_{R,\beta}(t)/L$, and this heat content is the Laplace integral of a finite positive spectral measure of total mass $L$. Thus $\mathcal{Q}_{T_n}(t) \to Q_{R,\beta}(t)$ for every $t > 0$.

All the corresponding spectral measures have mass $L$, so

$$|\mathcal{Q}'_{T_n}(t)| \le \frac{L}{et}, \qquad |Q'_{R,\beta}(t)| \le \frac{L}{et}.$$

These follow from $\sup_{\lambda \ge 0} \lambda e^{-t\lambda} = 1/(et)$. A finite-grid argument upgrades pointwise convergence to uniform convergence on every compact positive time window. This proves the stated supremum with a single sequence. At $R = L$ use the DN interval. □

**Corollary 7.7** (The optimal cubic tree endpoint)**.** *For every $0 < \tau_0 \le \tau_1 < \infty$ there is $c > 0$ such that every rooted tree in Theorem 7.6 satisfies, with $\beta = 1 - R/L$ and $\delta_T(\tau) = [\mathcal{Q}_{I_L}(L^2\tau) - \mathcal{Q}_T(L^2\tau)]/L$,*

$$\delta_T(\tau) \ge \frac{c}{24}\beta^3, \qquad d_m(T/L, I_1) \le 2\left(\frac{24\delta_T(\tau)}{c}\right)^{1/3} \qquad (\tau \in [\tau_0, \tau_1]).$$

*The exponent $1/3$ is optimal, already on a fixed tree topology.*

*Proof.* Normalize length to one. Theorem 7.6 and (34) give the first inequality. Theorem 4.2 gives $d_m \le 2\beta$, proving the second. The pitchforks in Theorem 10.1 belong to this class, have deficit asymptotic to $2b^3 f(\tau) > 0$ and $d_m \ge b/2$ by (64). For any exponent greater than $1/3$, the corresponding ratio diverges as $b \downarrow 0$. Thus $1/3$ itself is admissible and optimal. □

The same conclusions hold when finitely many selfloops are attached at vertices of a singly rooted tree. Cutting each loop at its midpoint into two equal reflecting stubs preserves constant-data heat content, as in [2, Proposition 6.4] (numbering of arXiv:2502.09461v1). Directly, reflection along the loop fixes the rest of the graph and the initial datum; uniqueness makes the heat solution symmetric, with zero derivative at the midpoint. Its two attachment fluxes are unchanged after the cut. Length is preserved, and the maximum root distance on a loop of length $\ell$ is the distance of its attachment plus $\ell/2$, also preserved by the two stubs. Repeating this operation gives a rooted tree with the same $L, R$ and heat content.

### 7.3 Moving excess speed mass toward the leaves

The response argument also applies after a positive redistribution of speed mass. We first keep the energy and rooted metric tree fixed. Let $\mathcal{E}(f,f) = \sum_e \int_e |f'|^2 \, \mathrm{d}y$, with zero trace at the single root, and let the speed measure $\mu \geq \mathrm{d}y$ have a positive piecewise bounded density and finitely many nonroot atoms. Heat content uses initial datum one and the full measure $\mu$. These are classical speed-measure realizations of one-dimensional diffusions; endpoint reservoirs and response reduction have established antecedents [10, 11].

**Lemma 7.8** (Ancestor-to-descendant mass relocation)**.** *Let $x$ be an ancestor of $v$ in the rooted tree. Moving a nonnegative amount of excess speed mass from $x$ to $v$, while retaining $\mu \geq \mathrm{d}y$, cannot decrease heat content at any positive time. More generally the same conclusion holds when a finite nonnegative excess measure supported on an edge is moved to a leaf descended from that edge.*

*Proof.* For $z > 0$ put $u_\mu = (H_\mu + z)^{-1}\mathbf{1}$ and $h_\mu(w) = 1 - z u_\mu(w) = \mathbb{E}_w e^{-z\tau_0}$, where $\tau_0$ is absorption at the root. The common anchored $H^1$ variational identity is

$$J_\mu(z) = \sup_f \left\{ 2\int f \, \mathrm{d}\mu - \mathcal{E}(f,f) - z\int f^2 \, \mathrm{d}\mu \right\}.$$

Its unique maximizer gives the measure derivative

$$D_\mu J[\delta_w] = 2u_\mu(w) - z u_\mu(w)^2 = \frac{1 - h_\mu(w)^2}{z}.$$

A path from $v$ to the root must first hit $x$. The strong Markov property therefore gives $h_\mu(v) = \eta_\mu(v \to x; z) h_\mu(x)$, where $\eta$ is the first-hitting-time probability transform. Hence

$$D_\mu J[\delta_v - \delta_x] = h_\mu(x)^2 \frac{1 - \eta_\mu(v \to x; z)^2}{z}. \tag{41}$$

The first factor is the transform of the sum of two independent root absorption times started at $x$. The second is the transform of the survival function of the sum of two independent $v$-to-$x$ hitting times. Thus the derivative has a specified nonnegative inverse convolution. All these laws refer to the same current measure $\mu$.

For the distributed assertion, let $\lambda$ be the mass to be moved and interpolate

$$\mu_\theta = \mu_{\mathrm{base}} + (1-\theta)\lambda + \theta\lambda(T)\delta_v, \qquad 0 \leq \theta \leq 1, \qquad \mu_{\mathrm{base}} \geq \mathrm{d}y.$$

The forms $\mathcal{E} + z\mu_\theta$ act on one anchored $H^1$ space and are uniformly coercive. Point traces and integration against these finite bounded measures are bounded there. Differentiating their Lax–Milgram equations, or taking the variational difference quotient, justifies

$$\frac{\mathrm{d}J_{\mu_\theta}(z)}{\mathrm{d}\theta} = \int \frac{h_{\mu_\theta}(x)^2 - h_{\mu_\theta}(v)^2}{z} \, \lambda(\mathrm{d}x).$$

For real $z > 0$, the common form equations give continuous $u_{\mu_\theta}(x)$ and positive continuous $h_{\mu_\theta}(x; z)$. The quotient $\eta = h(v)/h(x)$ is continuous as well. Probability Laplace continuity makes the corresponding absorption and hitting laws weakly continuous in $(\theta, x)$, so their survival-convolution kernels are Borel measurable. Each inverse kernel in (41) lies between zero and one. Tonelli over the finite measure $\lambda$ and $0 \le \theta \le 1$ therefore identifies a nonnegative inverse for $J_{\mu_1} - J_{\mu_0}$. The actual heat contents are continuous at positive times, and transform uniqueness proves their pointwise order there. This is an averaged heat-content comparison; no ordering of every individual clock trajectory is required. □

**Theorem 7.9** (Weighted trees and resistance height)**.** *Let a finite metric tree have one Dirichlet root and, on each physical edge of length $\ell_e > 0$, identical energy and speed density $c_e \ge 1$:*

$$\mathcal{E}(f, f) = \sum_e c_e \int_e |f_x|^2 \, \mathrm{d}x, \qquad \mu = \sum_e c_e \, \mathrm{d}x.$$

*Set*

$$L = \sum_e c_e \ell_e, \qquad S = \max_v \sum_{e \in [0,v]} \frac{\ell_e}{c_e}, \qquad R = \max_v \sum_{e \in [0,v]} \ell_e.$$

*Then, for every $t > 0$,*

$$\mathcal{Q}_T(t) \le Q_{S,L-S}(t) \le Q_{R,L-R}(t).$$

*Proof.* On each edge put $y = x/c_e$. The energy becomes ordinary arclength energy on a tree of edge lengths $\ell_e/c_e$, while the speed measure becomes $c_e^2 \, \mathrm{d}y \ge \mathrm{d}y$. Its total mass is still $L$ and its rooted height is $S \le R \le L$. The actual interior generator in this coordinate is $c_e^{-2} d^2/dy^2$; the reference Brownian normalization in the time change is $d^2$, with no factor $1/2$.

Apply Lemma 7.8 to move the excess measure $(c_e^2 - 1) \, \mathrm{d}y$ on each edge to any leaf descended from that edge. There are finitely many edges. Energy, rooted height and total mass remain fixed, and heat content cannot decrease. The result is an ordinary energy tree with nonnegative leaf masses. The preceding tree response induction applies verbatim: a leaf edge of length $a$ with mass $m$ is the base endpoint model $E(a, m)$; parallel merging and common-stem preservation retain the same mass and height. Thus its heat content is at most $Q_{S,L-S}$. Positive radius variation gives the second inequality. □

**Corollary 7.10** (Exact tree quotients)**.** *Suppose a finite standard metric graph with a single Dirichlet root admits a finite root-preserving isometry group whose metric quotient, after subdividing edge-reflection midpoints, is a tree. Then*

$$\mathcal{Q}_G(t) \le Q_{R,L-R}(t) \qquad (t > 0),$$

*where $L$ is its original total length and $R$ its original root inradius. The uniform cubic heat-deficit and exponent-$1/3$ geometric estimates hold for this class as well.*

*Proof.* The constant initial datum and rooted operator are group invariant, so uniqueness makes the heat solution invariant. On each quotient edge, energy and speed are both multiplied by its orbit multiplicity $c_e \ge 1$. Summing the original Kirchhoff equations gives precisely the weighted quotient flux condition; an edge-reflection midpoint gives a Neumann leaf. Thus the invariant form and its $L^2$ space are exactly the weighted tree of Theorem 7.9, and their heat contents agree. Total mass equals the original length. Since the group fixes the root, quotient distance from its root orbit is the original distance from the root. Hence its physical height is $R$. Apply the theorem and the positive model cubic reserve, followed by the already proved bound $d_m \le 2(1 - R/L)$ after normalization. □

For example, take an equilateral complete graph $K_n$, $n \geq 3$, with one Dirichlet vertex and edge length $\ell$. Permuting the other $n-1$ vertices gives a quotient stem of length $\ell$ and multiplicity $n-1$. The nonroot edges fold at their midpoints into one pendant edge of length $\ell/2$ and multiplicity $(n-1)(n-2)$. These give

$$L = \frac{n(n-1)}{2}\ell, \qquad R = \frac{3}{2}\ell.$$

At the quotient junction, the weighted flux equation is the sum of the original equations. This supplies cyclic examples of arbitrarily large cycle rank under the exact quotient hypothesis.

The quotient also carries the resistance height of Theorem 7.9, which sharpens the envelope.

**Corollary 7.11** (Resistance-height bound on tree quotients)**.** *In the setting of Corollary 7.10, let the quotient tree have edge lengths $\ell_e$ and orbit multiplicities $c_e \geq 1$, and let*

$$S = \max_v \sum_{e \in [0,v]} \frac{\ell_e}{c_e}$$

*be its resistance height, the maximum over quotient vertices $v$ of the sum along the path from the root. Then $0 < S \leq R$ and*

$$\mathcal{Q}_G(t) \leq Q_{S,L-S}(t) \leq Q_{R,L-R}(t) \qquad (t > 0). \tag{42}$$

*For every $0 < \tau_0 \leq \tau_1 < \infty$, with the constant $c$ of Proposition 7.2,*

$$\frac{\mathcal{Q}_{I_L}(L^2\tau) - \mathcal{Q}_G(L^2\tau)}{L} \geq \frac{c}{24}\Big(1 - \frac{S}{L}\Big)^3 \qquad (\tau \in [\tau_0, \tau_1]).$$

*Proof.* The proof of Corollary 7.10 identifies the heat content of $G$ with that of the weighted tree of Theorem 7.9 whose edge lengths are $\ell_e$ and whose common energy and speed multiplicities are $c_e$. Its total mass is $L$ and its physical height is $R$. The quantity $S$ is exactly the resistance height in that theorem, which gives (42); $S \leq R$ because every $c_e \geq 1$.

For the cubic estimate, scaling lengths and endpoint masses by $L^{-1}$ and time by $L^{-2}$ multiplies the heat content of the endpoint model by $L^{-1}$, so $Q_{S,L-S}(L^2\tau) = LQ_{S/L,1-S/L}(\tau)$; likewise $\mathcal{Q}_{I_L}(L^2\tau) = L\mathcal{Q}_{I_1}(\tau)$. If $S < L$, (42) and (34) with $R$ replaced by $S/L \in (0,1)$ give the estimate; the constant in (34) depends only on the window. If $S = L$, the right side is zero and the estimate follows from Theorem 5.3. □

For the complete graph $K_n$ above,

$$S = \frac{\ell}{n-1} + \frac{\ell/2}{(n-1)(n-2)} = \frac{(2n-3)\,\ell}{2(n-1)(n-2)}, \qquad \frac{S}{L} = \frac{2n-3}{n(n-1)^2(n-2)},$$

whereas $R/L = 3/(n(n-1))$. Thus $S/R = (2n-3)/(3(n-1)(n-2)) < 1$ for every $n \geq 3$. For $n = 3$, $1 - S/L = 3/4$ against $1 - R/L = 1/2$, so the cubic reserve of Corollary 7.11 is $27/8$ times that of Corollary 7.10.

## 8 Effective almost-cubic heat stability

We retain the graph model and the marked measured distance of Theorem 4.2. First normalize total length to one and set

$$\Delta_G(t) = \mathcal{Q}_I(t) - \mathcal{Q}_G(t), \qquad R = \max_{x \in G} \operatorname{dist}(x, D), \qquad \beta = 1 - R.$$

The initial datum is $\mathbf{1}$, and the generator is $d^2/dx^2$. Fix a positive compact observation window $0 < \tau_0 \le \tau_1 < \infty$, and put

$$T = \frac{\tau_0}{2}, \qquad c = \min_{\substack{T \le h \le \tau_1 \\ 0 \le y \le 1}} \frac{K(h;1,y)}{y} > 0. \tag{43}$$

Here $K$ is the fixed mixed mass-coordinate kernel from Theorem 5.3; at $y = 0$ the quotient is its positive inward derivative. Lemma 6.1 justifies this constant and gives

$$\int_0^T \Delta_G(t)\,\mathrm{d}t \le \frac{\Delta_G(t_0)}{c}, \qquad t_0 \in [\tau_0, \tau_1]. \tag{44}$$

No graph-dependent heat-kernel minimum occurs in (43).

**Theorem 8.1** (An effective almost-cubic deficit)**.** *For $0 < \beta < 1$, define*

$$\ell = \log(1/\beta), \qquad s_\beta = \frac{2 + \sqrt{4 + T(3\ell + \log 2)}}{T}. \tag{45}$$

*Every graph in the stated model satisfies, uniformly for $t_0 \in [\tau_0, \tau_1]$,*

$$\Delta_G(t_0) \ge \frac{c}{6}\,\beta^3 \frac{e^{-2s_\beta}}{(1+s_\beta)^2}. \tag{46}$$

*At $\beta = 0$ the right side is defined to be zero. For every $\varepsilon > 0$, the positive constant*

$$C_\varepsilon = \frac{c}{6} \exp\left( -\frac{16}{T} - 4\sqrt{\frac{\log 2}{T}} - \frac{12}{\varepsilon T} \right) \tag{47}$$

*gives the simpler estimate*

$$\Delta_G(t_0) \ge C_\varepsilon \beta^{3+\varepsilon}. \tag{48}$$

*The constants depend only on the prescribed time window and $\varepsilon$, not on the number of edges, vertex degrees, or smallest edge length.*

*Proof.* The positive comparison in Theorem 5.3 gives $0 \le \Delta_G(t) \le \mathcal{Q}_I(t)$. The unit DN interval has torsion $\int_0^\infty \mathcal{Q}_I(t)\,\mathrm{d}t = 1/3$. Corollary 3.2, with $z = s^2 > 0$, supplies

$$\int_0^\infty e^{-s^2 t} \Delta_G(t)\,\mathrm{d}t \ge \frac{\beta^3 \operatorname{sech}^2 s}{3(1+s)^2} \ge \frac{\beta^3 e^{-2s}}{3(1+s)^2}. \tag{49}$$

The last inequality uses $\cosh s \le e^s$. Independently, splitting this weighted integral at $T$ and using (44) yields

$$\begin{aligned} \int_0^\infty e^{-s^2 t} \Delta_G(t)\,\mathrm{d}t &\le \int_0^T \Delta_G(t)\,\mathrm{d}t + e^{-s^2 T} \int_T^\infty \mathcal{Q}_I(t)\,\mathrm{d}t \\ &\le \frac{\Delta_G(t_0)}{c} + \frac{e^{-s^2 T}}{3}. \end{aligned} \tag{50}$$

Combining (49) and (50) proves, for every $s > 0$,

$$\Delta_G(t_0) \ge \frac{c}{3} \left[ \frac{\beta^3 e^{-2s}}{(1+s)^2} - e^{-Ts^2} \right]. \tag{51}$$

This is a time split combined with a separate backward estimate; the pointwise lower bound comes from the backward estimate at the observation time.

For $\beta > 0$, the number in (45) solves

$$Ts_\beta^2 - 4s_\beta = 3\ell + \log 2.$$

Thus

$$e^{-Ts_\beta^2} = \frac{\beta^3}{2} e^{-4s_\beta} \le \frac{\beta^3}{2} \frac{e^{-2s_\beta}}{(1+s_\beta)^2},$$

because $\log(1+s) \le s$ for $s \ge 0$. Substitution into (51) proves (46) for all $0 < \beta < 1$, without a smallness threshold. The case $\beta = 0$ follows from nonnegativity.

For the algebraic form, the elementary square-root inequality gives

$$s_\beta \le \frac{4}{T} + \sqrt{\frac{3\ell}{T}} + \sqrt{\frac{\log 2}{T}}.$$

Also $2s + 2\log(1+s) \le 4s$, and Young's inequality gives

$$4\sqrt{\frac{3\ell}{T}} \le \varepsilon\ell + \frac{12}{\varepsilon T}.$$

Consequently

$$2s_\beta + 2\log(1+s_\beta) \le \frac{16}{T} + 4\sqrt{\frac{\log 2}{T}} + \varepsilon\ell + \frac{12}{\varepsilon T}.$$

Exponentiating this inequality in (46), and using $e^{-\varepsilon\ell} = \beta^\varepsilon$, proves (48) with precisely (47). □

**Corollary 8.2** (Every geometric Hölder exponent below one third)**.** *For every $0 < \alpha < 1/3$, let $\varepsilon = 1/\alpha - 3 > 0$. Then*

$$\beta \le C_\varepsilon^{-\alpha} \Delta_G(t_0)^\alpha, \qquad d_m(G, I) \le 2C_\varepsilon^{-\alpha} \Delta_G(t_0)^\alpha \quad (t_0 \in [\tau_0, \tau_1]). \tag{52}$$

*For a graph of original length $L$, replace $\beta$ by $1 - R/L$, use the dimensionless observation window for $t/L^2$, and replace $\Delta_G(t)$ by*

$$\frac{\mathcal{Q}_{I_L}(t) - \mathcal{Q}_G(t)}{L}.$$

*Distances and arclength measure are normalized by $L$ before using $d_m$. The same constants then apply.*

*Proof.* Raise (48) to the power $\alpha = 1/(3+\varepsilon)$, then apply Theorem 4.2, which gives $d_m \le 2\beta$. Metric rescaling multiplies lengths by $L^{-1}$, time by $L^{-2}$, and heat content by $L^{-1}$, proving the final assertion. □

*Remark* 8.3 (Effective selection of the interval constant)*.* For computable positive window parameters, a positive rational number not exceeding the minimum in (43) can be certified from this fixed interval alone. With $k_n = (n + 1/2)\pi$,

$$\frac{K(h; 1, y)}{y} = 2\sum_{n \ge 0} (-1)^n \frac{\sin(k_n y)}{y} e^{-k_n^2 h}, \qquad h \ge T,$$

where the removable quotient at $y = 0$ is $k_n$. The uniform tail is bounded by $2\sum_{n>N} k_n e^{-Tk_n^2}$. Bounds for the derivatives with respect to $h$ and $y$ are respectively

$$2\sum_{n \ge 0} k_n^3 e^{-Tk_n^2}, \qquad \sum_{n \ge 0} k_n^2 e^{-Tk_n^2}.$$

For the latter, use $\sin(ky)/y = \int_0^k \cos(vy)\,\mathrm{d}v$, including its value at zero. Gaussian integral tails bound these series effectively. Finite rational grids, these Lipschitz and tail bounds, and interval enclosures of the finite trigonometric and exponential terms yield certified uniform lower bounds on the compact cylinder. Its strictly positive true minimum ensures that a finite stage certifies a positive rational lower bound. Any smaller positive rational may be used as $c$ throughout the theorem. This specifies an effective procedure for selecting the constant. For arbitrary real window parameters the inequalities hold for every positive $c$ below the minimum.

*Remark* 8.4 (Size of the subpower factor). Since $s_\beta = \sqrt{3\log(1/\beta)/T} + O_T(1)$, (46) differs from $\beta^3$ by the subpower factor $\exp(-O_T(\sqrt{\log(1/\beta)}))$. Its right side is increasing in $\beta$, so it also admits monotone one-dimensional inversion as an effective modulus. The factor comes from the absolute tail in (50), which is balanced against the reserve by the choice $s = s_\beta$. On rooted trees and exact tree quotients the cubic bound $\Delta_G(t_0) \geq C\beta^3$ itself holds, by Corollaries 7.7 and 7.10.

# 9 An independent formula for the asymmetric pitchfork

Consider a stem of length $a > 0$ joining a Dirichlet leaf to a Kirchhoff junction, with two further arms of lengths $b, c > 0$ ending at Neumann leaves. Write $P = P(a; b, c)$ and $L = a + b + c$. This example admits a separate positive convolution proof. The formulas and the identity of Proposition 9.1 are also [A, Section 6 and Proposition 6.1]; we include the proof because Section 10 uses its factorization. General pendant-edge surgery principles are treated in [2, Section 6] (numbering of arXiv:2502.09461v1).

For $s > 0$ put $k = \sqrt{s}$, $A = \tanh(ka)$, $B = \tanh(kb)$ and $C = \tanh(kc)$. Let $Z_P(s)$ be the Laplace transform of the absorption time for diffusion on $P$ started at the junction, and let $Z_I(s)$ be the corresponding transform on the mixed interval started at distance $a$ from its absorbing endpoint. This point has coordinate $L - a$ in our Neumann-at-zero convention. Solving $Z'' = sZ$, with $Z = 1$ at the absorbing endpoint and the usual conditions elsewhere, gives

$$Z_P(s) = \frac{1}{\cosh(ka) + (B + C)\sinh(ka)}, \qquad Z_I(s) = \frac{\cosh(k(b + c))}{\cosh(kL)}. \tag{53}$$

The same calculation gives the heat-content transforms

$$\widehat{\mathcal{Q}}_P(s) = \frac{L}{s} - \frac{A + B + C}{sk(1 + A(B + C))}, \qquad \widehat{\mathcal{Q}}_I(s) = \frac{L}{s} - \frac{\tanh(kL)}{sk}. \tag{54}$$

For completeness, $\widehat{u} = (1 - Z)/s$ solves the heat resolvent with initial datum one. Integrating $Z'' = sZ$ over the graph cancels all Kirchhoff and Neumann fluxes. On the stem, with distance $x$ from the absorbing endpoint, $Z(x) = \cosh(kx) + d\sinh(kx)$; the junction condition gives $d = -(A+B+C)/(1+A(B+C))$. Hence $\int_P Z = -Z'(0)/s = (A+B+C)/(k(1+A(B+C)))$, proving (54).

**Proposition 9.1** (Pitchfork convolution identity)**.** *For $\ell > 0$ define the positive locally integrable function*

$$h_\ell(t) = \frac{2}{\ell}\sum_{n=0}^{\infty} \exp\left(-\frac{(n + 1/2)^2\pi^2 t}{\ell^2}\right), \qquad t > 0.$$

*Let $\mu_P, \mu_I$ be the absorption-time probability measures whose Laplace transforms are* (53)*. Then*

$$\mathcal{Q}_I(t) - \mathcal{Q}_P(t) = \big(h_b * h_c * h_{b+c} * \mu_P * \mu_I\big)(t), \qquad t > 0. \tag{55}$$

*In particular the deficit is nonnegative, and*

$$\int_0^\infty (\mathcal{Q}_I(t) - \mathcal{Q}_P(t))\,\mathrm{d}t = bc(b+c).$$

*Proof.* The interval mixed eigenfunction expansion, or the partial fraction expansion of tanh, gives

$$\int_0^\infty e^{-st} h_\ell(t)\,\mathrm{d}t = \frac{\tanh(\ell\sqrt{s})}{\sqrt{s}}, \qquad \int_0^\infty h_\ell(t)\,\mathrm{d}t = \ell.$$

The transforms $Z_P, Z_I$ are those of probability measures: absorption is almost sure on these finite anchored graphs. This also follows by letting $s \downarrow 0$ in (53). Subtract (54), use $\tanh(k(b+c)) = (B+C)/(1+BC)$, and simplify to obtain

$$\widehat{\mathcal{Q}}_I(s) - \widehat{\mathcal{Q}}_P(s) = \frac{\tanh(kb)}{k}\frac{\tanh(kc)}{k}\frac{\tanh(k(b+c))}{k}\,Z_P(s)Z_I(s).$$

Tonelli and uniqueness of Laplace transforms prove the convolution identity almost everywhere. It holds everywhere by continuity: $h_\ell(t) = O(t^{-1/2})$ near zero and decays exponentially at infinity, so the triple convolution extends to a continuous bounded function on the real line when set to zero for $t \le 0$. Convolving with the finite measures preserves continuity. Finally integrate by Tonelli and multiply the five total masses. □

## 10 The cubic pitchfork profile and geometric sharpness

The positive convolution formula also determines the small-arm profile at fixed positive times. This supplies a heat-specific obstruction to improving the geometric exponents, rather than inferring such an obstruction from a stationary integral alone.

**Theorem 10.1.** *Let $P_b = P(1-2b; b, b)$, $0 < b \le 1/4$, and let $I$ be the unit mixed Dirichlet–Neumann interval. Put*

$$\Delta_b(t) = \mathcal{Q}_I(t) - \mathcal{Q}_{P_b}(t).$$

*Let $\mu$ be the absorption-time probability law from the reflecting endpoint of $I$, for the generator $+d^2/dx^2$. There is a continuous strictly positive probability density $f$ on $(0,\infty)$, namely the density of $\mu * \mu$, such that*

$$\frac{\Delta_b(t)}{2b^3} \longrightarrow f(t) \qquad (b \downarrow 0), \tag{56}$$

*uniformly on every compact subinterval of $(0,\infty)$. Its Laplace transform is*

$$\widehat{f}(z) = \operatorname{sech}^2\sqrt{z}, \qquad \Re z > 0, \tag{57}$$

*where the square root has positive real part.*

*Proof.* The graph has length one and inradius $1-b$. We use the functions $h_\ell$ and the absorption transforms from Proposition 9.1 and (53). Since $h_\ell$ has mass $\ell$, $h_\ell/\ell$ is a probability density. Its transform

$$\frac{\widehat{h}_\ell(z)}{\ell} = \frac{\tanh(\ell\sqrt{z})}{\ell\sqrt{z}}$$

has modulus at most one on $\Re z > 0$ and tends to one as $\ell \downarrow 0$ for each fixed such $z$. Write $a = 1-2b$, $k = \sqrt{z}$, and

$$Z_{P,b}(z) = \frac{1}{\cosh(ka) + 2\tanh(kb)\sinh(ka)}, \qquad Z_{I,b}(z) = \frac{\cosh(2kb)}{\cosh k}.$$

The positive convolution identity gives the probability density $f_b(t) = \Delta_b(t)/(2b^3)$, continuous for $t > 0$, and its transform

$$F_b(z) = \left(\frac{\widehat{h}_b(z)}{b}\right)^2 \frac{\widehat{h}_{2b}(z)}{2b}\, Z_{P,b}(z) Z_{I,b}(z). \tag{58}$$

The formulas originally obtained for positive real parameters extend to $\Re z > 0$ by analyticity of these probability transforms. Each absorption factor tends to $1/\cosh k$, so

$$F_b(z) \longrightarrow F(z) = \operatorname{sech}^2 k. \tag{59}$$

We next justify inversion uniformly in $b$. Fix $\sigma > 0$ and take $z = \sigma + i\omega$, $k = \rho + i\eta$. Then

$$\rho^2 = \frac{|z| + \sigma}{2}, \qquad \rho \geq \sqrt{\sigma}, \qquad \rho \geq \sqrt{|\omega|/2}.$$

The identity

$$\Re \tanh(bk) = \frac{\sinh(2b\rho)}{\cosh(2b\rho) + \cos(2b\eta)} > 0$$

shows that $w = 2\tanh(bk)$ has positive real part. Hence $|1+w| \geq 1$ and $|(1-w)/(1+w)| \leq 1$. Since

$$Z_{P,b}(z) = \frac{2e^{-ka}}{(1+w) + (1-w)e^{-2ka}},$$

and $a \geq 1/2$, we obtain

$$|Z_{P,b}(z)| \leq \frac{2e^{-a\rho}}{1 - e^{-2a\rho}} \leq \frac{2e^{-a\rho}}{1 - e^{-\sqrt{\sigma}}}. \tag{60}$$

Also $|\cosh(2bk)| \leq \cosh(2b\rho) \leq e^{2b\rho}$ and $|\cosh k| \geq \sinh \rho$. Therefore

$$|Z_{I,b}(z)| \leq \frac{2e^{-a\rho}}{1 - e^{-2\rho}} \leq \frac{2e^{-a\rho}}{1 - e^{-\sqrt{\sigma}}}. \tag{61}$$

The three normalized $h$ factors in (58) have modulus at most one. Thus, uniformly for $0 < b \leq 1/4$,

$$|F_b(\sigma + i\omega)| \leq C_\sigma e^{-2a\rho} \leq C_\sigma e^{-\rho} \leq C_\sigma e^{-\sqrt{|\omega|/2}}, \qquad C_\sigma = \frac{4}{(1 - e^{-\sqrt{\sigma}})^2}. \tag{62}$$

This is integrable over $\omega \in \mathbb{R}$. The pointwise limit $F$ has the same bound. Dominated convergence now gives

$$\int_{\mathbb{R}} |F_b(\sigma + i\omega) - F(\sigma + i\omega)|\, d\omega \longrightarrow 0. \tag{63}$$

For clarity, inversion here is ordinary Fourier inversion of an exponentially tilted density. Extend $f_b$ by zero to negative times. Then $e^{-\sigma t} f_b(t)$ is integrable and has Fourier transform $F_b(\sigma + i\omega)$, which is integrable by (62). Fourier inversion identifies its continuous inverse representative. Since the original $f_b$ is continuous for $t > 0$, the almost-everywhere identification holds at every positive time. In particular

$$f_b(t) = \frac{e^{\sigma t}}{2\pi} \int_{\mathbb{R}} e^{i\omega t} F_b(\sigma + i\omega)\, d\omega.$$

The inverse expression with $F$ defines a continuous function $f$ at positive times. For $0 < \tau_0 \leq \tau_1 < \infty$, (63) implies

$$\sup_{t \in [\tau_0, \tau_1]} |f_b(t) - f(t)| \leq \frac{e^{\sigma \tau_1}}{2\pi} \int_{\mathbb{R}} |F_b - F|\, d\omega \longrightarrow 0.$$

The argument uses only the fixed vertical line $\Re z = \sigma$.

It remains to identify the limit and its positivity. For this purpose use coordinates with Dirichlet endpoint 0 and reflecting endpoint 1. Solving $Z'' = zZ$, $Z(0) = 1$, $Z'(1) = 0$ gives $\widehat{\mu}(z) = \operatorname{sech}\sqrt{z}$. The absorption law has a continuous positive density $j(t)$ for $t > 0$. Indeed, if $K_t(1, y)$ is the mixed-interval heat kernel, survival from the reflecting endpoint is $S(t) = \int_0^1 K_t(1, y)\, dy$. The heat equation and its endpoint conditions give

$$j(t) = -S'(t) = \partial_y K_t(1, 0) > 0.$$

The derivative is strictly positive by the parabolic boundary Hopf lemma; positive-time smoothness gives continuity. Also $S(0+) = 1$ and $S(t) \to 0$ at infinity, so $j$ has mass one and there is no atom at zero.

One can verify the continuity needed for convolution directly on the transform side as well. On each line $\Re z = \sigma$, $|\operatorname{sech}\sqrt{z}| \le C'_\sigma e^{-\rho}$ is integrable in $\omega$. Fourier inversion for the exponentially tilted finite measure $\mu$ therefore gives a continuous bounded density on the whole real line, agreeing with $e^{-\sigma t} j(t)$ at positive times. Its self-convolution is continuous, since it is both bounded continuous and integrable. Undoing the tilt shows that $j * j$ is continuous on compact positive time intervals. It is strictly positive at every $t > 0$, because both integrand factors in

$$(j * j)(t) = \int_0^t j(s) j(t - s)\, ds$$

are positive on $(0, t)$. It is a probability density and its transform is $F(z)$. Uniqueness of Laplace transforms identifies it with the inverse $f$ above, proving (57) and (56). □

The mark set also gives the reverse geometric comparison

$$\beta/2 \le d_m(G, I) \le 2\beta. \tag{64}$$

Indeed, in any common embedding whose support and mark Hausdorff distances are at most $h$, match a support point to the other support within $h$. Its distance to the respective mark set changes by at most another $h$, because distance to a set is 1-Lipschitz and the mark sets are within $h$. Taking suprema in both directions gives $|R_G - R_I| \le 2h$. Infimizing over embeddings proves the lower bound; the upper bound is the explicit common embedding in Theorem 4.2.

**Corollary 10.2.** *At any fixed positive time, no graph-uniform bound*

$$d_m(G, I) \le C(t)[\mathcal{Q}_I(t) - \mathcal{Q}_G(t)]^\alpha$$

*can hold for an exponent $\alpha > 1/3$ in the normalized graph class. Together with Corollary 8.2, the supremum of the admissible geometric Hölder exponents on each compact positive time window is exactly $1/3$. On the rooted-tree and exact tree-quotient classes, the endpoint exponent itself is valid by Corollaries 7.7 and 7.10.*

*Proof.* For $P_b$, the longest root-to-leaf path has length $1 - b$, whereas the two-arm path has length $2b \le 1 - b$. Thus $\operatorname{diam} P_b = 1 - b$. Since $\operatorname{diam} I = 1$, the GH diameter inequality gives

$$d_m(P_b, I) \ge d_{\mathrm{GH}}(P_b, I) \ge b/2.$$

For every fixed $t > 0$, Theorem 10.1 gives $\Delta_b(t) \sim 2f(t)b^3$, with $f(t) > 0$. Hence for $\alpha > 1/3$,

$$\frac{d_m(P_b, I)}{\Delta_b(t)^\alpha} \ge c_t b^{1-3\alpha} \longrightarrow \infty$$

for some $c_t > 0$. This excludes every such exponent, even on this one fixed graph topology. The estimates in Corollary 8.2 provide every $\alpha < 1/3$ uniformly on each prescribed positive time window; together these two directions identify the general-graph supremum. Equivalently, the pitchfork family excludes a uniform lower heat-deficit bound proportional to $\beta^\gamma$ for any $\gamma < 3$, while the proved lower bounds cover every $\gamma > 3$. □

On any prescribed positive time window, (64) shows that an endpoint geometric estimate $d_m \le C\Delta^{1/3}$ and a cubic inradius lower bound $\Delta \ge c\beta^3$ are equivalent up to window-dependent constants. Corollaries 7.7 and 7.10 establish these endpoint estimates on the stated rooted-tree and quotient classes. Constants refer to a prescribed positive compact time window.

## 11 Sharp first $p$-eigenvalue bounds at fixed length and inradius

Let $G$ be a finite connected compact metric graph with total length $L > 0$ and nonempty Dirichlet vertex set $D$. Use continuous edgewise $W^{1,p}$ functions vanishing on $D$, with the natural nonlinear Kirchhoff condition at the other vertices. Its first eigenvalue is

$$\lambda_{p,1}(G) = \inf_{f \neq 0,\, f|_D = 0} \frac{\int_G |f'|^p}{\int_G |f|^p}.$$

Set $R = \max_{x \in G} \operatorname{dist}(x, D)$ and $q = p/(p-1)$. For $0 < r \le L$ define the endpoint-mass minimum

$$\lambda_p(L, r) = \inf_{g \neq 0,\, g(0) = 0} \frac{\int_0^r |g'|^p}{\int_0^r |g|^p + (L - r)|g(r)|^p}.$$

Here $g \in W^{1,p}(0, r)$. Del Pezzo and Rossi [13] give the sharp length-only bound for the first $p$-eigenvalue. For $p = 2$, Kennedy's broom comparison and limiting characteristic relation [12, Lemma 4.1, Proposition 1.4(i) and Equation (1.9)] already give the full joint envelope. The shortest-path estimate below parallels the nonlinear torsion envelope of Section 2. The sharp $p$-Kohler–Jobin inequality of [5] gives a related lower estimate through torsion. The endpoint model gives the exact joint infimum and its leading stability coefficient.

**Theorem 11.1.** *For every $1 < p < \infty$ and every $G$ as above,*

$$\lambda_{p,1}(G) \ge \lambda_p(L, R).$$

*When $0 < R < L$ the inequality is strict for each finite graph, and the right side is the exact infimum at fixed $L, R$. When $R = L$ the mixed interval attains equality.*

*Proof.* The graph and interval quotients attain their infima by coercivity from the Dirichlet anchor and compactness on the finitely many edges. The graph minimizer may be taken nonnegative. Take a shortest path from a nearest vertex of $D$ to a point attaining the maximum of this minimizer. Its length is $r \le R$. The restriction $g$ satisfies $g(0) = 0$, and $g(r)$ is the global maximum. Dropping energy outside the path and bounding its discarded mass by $(L - r)|g(r)|^p$ give the lower bound $\lambda_{p,1}(G) \ge \lambda_p(L, r)$.

The model minimum is nonincreasing in its radius at fixed $L$: for $r' > r$ extend any trial constantly from $r$ to $r'$. This preserves the energy and the total norm when the endpoint mass decreases by $r' - r$. In particular the comparison gives $\lambda_p(L, r) \ge \lambda_p(L, R)$. If equality holds for a finite graph with $R < L$, the graph minimizer is constant at its positive maximum on the part outside the path. That part contains an open edge interval of positive length. The weak $p$-eigenvalue equation with a positive eigenvalue prohibits a positive constant there.

For fixed $0 < R < L$ take $n$ arms of length $b_n = (L - R)/(n - 1)$ extending from a Dirichlet stem of length $R - b_n$, with Neumann conditions at the arm ends. The total length

and inradius are exactly $L, R$ for all sufficiently large $n$. Restrict the model minimizer at radius $R$ to the stem and extend it constantly on the arms. Its graph quotient tends to $\lambda_p(L,R)$ since $nb_n \to L-R$ and the model profile is continuous. This proves sharpness. At $R=L$ use the mixed interval. □

## 12 The model equation and the stability coefficient

Let $g$ be a nonnegative model minimizer at radius $R$ and let $\lambda = \lambda_p(L,R)$. The flux $w = |g'|^{p-2}g'$ satisfies

$$-w' = \lambda g^{p-1}, \qquad w(R) = \lambda(L-R)g(R)^{p-1}.$$

Thus $w \geq 0$ and $g$ is nondecreasing and nontrivial. Normalize $g(R)=1$, after which $g>0$ and $w>0$ in the open interval. The expression $(p-1)|g'|^p + \lambda g^p$ is constant, by the flux equation and differentiation of $|g'|^p = |w|^q$. Evaluate it at $R$ and separate variables to obtain the exact equation

$$R = \left(\frac{p-1}{\lambda}\right)^{1/p} \int_0^1 [1 + (p-1)\lambda^{q-1}(L-R)^q - y^p]^{-1/p}\, dy. \tag{65}$$

Its right side decreases strictly from infinity to zero as $\lambda$ runs from zero to infinity. For $p=2$ it is equivalent to $kR + \arctan(k(L-R)) = \pi/2$ with $k=\sqrt{\lambda}$, as in [12].

Writing $\beta = 1 - R/L$, $I_p = \int_0^1 (1-y^p)^{-1/p}\, dy$, and $\lambda_{0,p} = (p-1)I_p^p$, the normalized model has expansion

$$L^p\lambda_p(L,R) = \lambda_{0,p} + \frac{p(p-1)}{q+1}\lambda_{0,p}^q\, \beta^{q+1} + o(\beta^{q+1}). \tag{66}$$

To prove it scale to $L=1$. Continue the increasing model solution from the atom at $R$ to its first turning point $S = R + D_*$. The first integral or the separated equation gives $\lambda = (p-1)(I_p/S)^p$. The flux identity on the continuation gives

$$\beta = \int_R^S g(x)^{p-1}\, dx, \qquad w(x) = \lambda \int_x^S g(y)^{p-1}\, dy.$$

In particular $D_* \leq \beta$, $S \to 1$, and $\lambda \to \lambda_{0,p}$. The first integral bounds $g - 1 = O(\beta^q)$ on the continuation. Therefore

$$g'(x) = \lambda^{q-1}(S-x)^{q-1}(1 + O(\beta^q)), \qquad D_* \sim \beta.$$

Integrating first from $R$ to $x$ and then to $S$ gives

$$\beta - D_* = (p-1)\frac{\lambda_{0,p}^{q-1}}{q+1}\beta^{q+1} + o(\beta^{q+1}).$$

Expanding $S^{-p} = [1 - (\beta - D_*)]^{-p}$ now proves (66). At $p=2$ the reserve coefficient is $\pi^4/24$, also obtained by Taylor expansion of Kennedy's linear characteristic equation.

The positive coefficient in (66) and monotonicity of the model give $L^p\lambda_{p,1}(G) - \lambda_{0,p} \geq c_p\beta^{q+1}$ over the full radius range, for some $c_p > 0$. The marked measured distance $d_m$ of Section 4 satisfies $d_m(G/L, I_1) \leq 2\beta$. It follows that this distance is controlled by the dimensionless spectral deficit to power $1/(q+1)$. This power is sharp. Choose $\beta \downarrow 0$ and fine brooms whose eigenvalues differ from the exact model by $o(\beta^{q+1})$. In any common embedding, the difference of the inradii is bounded by the sum of the Hausdorff distances of the supports and of the Dirichlet marks. Hence these graphs also satisfy $d_m(G/L, I_1) \geq c\beta$, proving sharpness.

## 13 The sharp heat-content envelope on general graphs

Let $G$ be a finite connected compact metric graph of total length $L > 0$. Impose Dirichlet conditions on a nonempty set $D$ of vertices and the natural Kirchhoff conditions at all other vertices. Use the generator $d^2/dx^2$ on the edges. Let $H_G$ be the associated nonnegative Laplacian, and put $Q_G(t) = \langle 1, e^{-tH_G}1\rangle$ with arclength measure. Write $R = \max_{x\in G} \operatorname{dist}(x, D)$ and $\beta = 1 - R/L$. Let $I_L$ denote the length-$L$ Dirichlet–Neumann interval. For $r > 0$, $b \geq 0$ write $Q_{r,b}(t)$ for the datum-one heat content on $[0, r]$ with Dirichlet at zero, energy $\int_0^r |f'|^2$, and speed measure $dx + b\delta_r$. The initial heat and the heat-content integral both include the endpoint atom.

The effective estimate of Section 8 gives the normalized heat-content deficit lower bound with power $3 + \varepsilon$ for every $\varepsilon > 0$ on general graphs. It proves $Q_G \leq Q_{R,L-R}$ on singly rooted trees and exact tree quotients. We prove this sharper heat envelope for the general graph class. Earlier work includes the interval comparison and walk expansion of [1], the graph heat content formulas and surgery of [2], and an inradius bound in [14, Corollary 4.1.9].

**Theorem 13.1.** *For every graph in the stated class,*

$$Q_G(t) \leq Q_{R,L-R}(t), \qquad t > 0.$$

*When $R < L$ the inequality is strict for each finite graph. At fixed $L, R$ the right side is the exact supremum, approached by brooms uniformly on compact positive time windows. The mixed interval attains the case $R = L$.*

**Theorem 13.2.** *For $0 < \tau_0 \leq \tau_1 < \infty$ there exists $c > 0$, depending only on this time window, such that every graph in the stated class satisfies*

$$\frac{Q_{I_L}(L^2\tau) - Q_G(L^2\tau)}{L} \geq c(1 - R/L)^3, \qquad \tau \in [\tau_0, \tau_1].$$

*Consequently the marked measured distance of $G/L$ from the unit mixed interval is bounded by a constant times the one-time normalized heat deficit to power $1/3$.*

## 14 The fraction supplied by a shortest path

Normalize $L = 1$. Let $u(t, x) = (e^{-tH_G}1)(x)$ and $U(t, s)$ be the integral of the top $s$ arclength mass of $u(t, \cdot)$, for $0 \leq s \leq 1$. Use $U_I$ for the corresponding integral on $I_1$. For a regular heat level $z$, write $s = |\{u(t) > z\}|$ and define

$$A(t, s) = \sum_{u(t,x)=z} |u_x(t, x)|, \qquad B(t, s) = \sum_{u(t,x)=z} |u_x(t, x)|^{-1}.$$

At positive time the level and coarea results of Lemmas 5.1–5.2 and Theorem 5.3 apply. The inverse distribution derivative is $-1/B$ at almost every mass coordinate, while

$$-U_t(t, s) = A(t, s) = \int_{\{u(t)>z\}} -u_t(t, x)\, dx.$$

Since the datum-one heat profile is nonincreasing in time, $s \mapsto A(t, s)$ is nonnegative and nondecreasing by the nested sets in this integral. The positive source in the exact mass-coordinate heat equation for $U_I - U$ is $d(t, s) = A(t, s) - 1/B(t, s)$.

At each $t > 0$ choose a shortest path $\gamma$ from a nearest Dirichlet vertex to a maximizer of $u(t, \cdot)$. It has length at most $R$ and covers every intermediate level. At a regular level define

$B_\gamma$ as the sum of inverse slopes on the crossings belonging to $\gamma$. Then $0 < B_\gamma \le B$ and $A \ge 1/B_\gamma$, since the sum of slopes includes any one crossing on the path. Hence

$$d(t,s) \ge A(t,s)\, m(t,s), \qquad m(t,s) = 1 - \frac{B_\gamma(t,s)}{B(t,s)} \in [0,1]. \tag{67}$$

In changing between levels and mass coordinates, the absolute differential of $s$ is $B\,dz$. Coarea on the path therefore gives

$$\int_0^1 (1 - m(t,s))\, ds = \int B_\gamma(t,z)\, dz = |\gamma| \le R.$$

Critical and vertex levels have zero mass measure, and no monotonicity along the path is needed. In particular $\int m(t,s)\, ds \ge \beta$.

Since $sA(t,s)$ is nonnegative and nondecreasing, its integral against $m(t,s)$ is at least its integral over $(0,\beta)$. Indeed the removed mass inside $(0,\beta)$ is no greater than the added mass outside it, where the integrand is larger. Thus (67) gives

$$\int_0^1 s\, d(t,s)\, ds \ge \int_0^\beta s\, A(t,s)\, ds. \tag{68}$$

The path is chosen only to prove this inequality at each time. The functions remaining in (68) have the measurable versions established in the source theorem.

## 15 The exact fixed-inradius envelope

We prove Theorem 13.1. Work at unit length and take $0 < \beta < 1$. Put $b = 1 - m = B_\gamma/B$. The coarea bound and (67) are equivalently

$$0 \le b \le 1, \qquad \int_0^1 b(t,s)\, ds \le R, \qquad bU_t - U_{ss} \le 0.$$

We compare this profile with a fixed coefficient $b_*(s) = \mathbf{1}_{(\beta,1)}(s)$.

Let $v_*(t,x)$ be the heat profile for the physical interval $[0,R]$ with endpoint atom $\beta$, so $\beta v_{*,t}(t,R) = -v_{*,x}(t,R)$. In the mass coordinate set

$$U_*(t,s) = \begin{cases} s\, v_*(t,R), & 0 \le s \le \beta, \\ \beta v_*(t,R) + \displaystyle\int_{1-s}^{R} v_*(t,x)\, dx, & \beta \le s \le 1. \end{cases}$$

On the second interval the physical heat equation and endpoint condition give $U_{*,t} = -v_{*,x}(t,1-s) = U_{*,ss}$. On the first $U_{*,ss} = 0$. The value and first spatial derivative match at $\beta$, so $b_* U_{*,t} - U_{*,ss} = 0$. Both endpoints have the mass-coordinate conditions $U_*(t,0) = 0$, $U_{*,s}(t,1) = 0$, and $U_*(t,1) = Q_{R,\beta}(t)$. As $t \downarrow 0$, $U_*(t,s) \to s$ uniformly by continuity in the physical $L^2(dx + \beta\delta_R)$ norm.

For the graph put $F = b_* U_t - U_{ss}$. On compact positive time strips it is bounded by the graph profile regularity. At regular mass coordinates it obeys

$$F \le (b - b_*)A \tag{69}$$

because $U_t = -A$. On the interval $(\beta,1)$ let $K_\beta(h;x,s)$ and $S_\beta(h)$ be the heat kernel and semigroup with Robin condition $f'(\beta) = f(\beta)/\beta$ and Neumann condition $f'(1) = 0$. The section $K_\beta(h;1,s)$ is positive and nondecreasing in $s$. To see monotonicity, evolve smooth nonnegative nondecreasing profiles with integral one, supported near 1 and constant in a

smaller neighborhood of 1. Their spatial derivatives solve the heat equation with nonnegative value $f(h,\beta)/\beta$ at $\beta$ and zero at 1. The maximum principle preserves nonnegativity of the derivative. Let the initial profiles converge to the point mass at 1 and use kernel continuity and symmetry. Positivity at the Robin endpoint follows from the strong principle and boundary Hopf lemma, since a zero there would force zero inward derivative. Extend the section to the full mass interval by

$$\widetilde{K}_\beta(h;1,s) = \begin{cases} (s/\beta)K_\beta(h;1,\beta), & 0 \le s \le \beta, \\ K_\beta(h;1,s), & \beta \le s \le 1. \end{cases}$$

This extended section is nondecreasing and positive when $s > 0$ and $h > 0$.

On the left segment $-U_{ss} = F$ with $U(t,0) = 0$. Integrating once against $s$ gives

$$U_s(t,\beta) - U(t,\beta)/\beta = -\frac{1}{\beta}\int_0^\beta sF(t,s)\,ds.$$

The derivative at the interface agrees on both sides by the mass-coordinate Sobolev regularity. In the weak heat equation on $(\beta,1)$ the negative of this Robin inhomogeneity acts as the boundary source. In fact integrating by parts against a test function $\varphi$ brings $-U_s(t,\beta)\varphi(\beta)$ to the right side. Thus the volume and boundary sources together give, for $0 < a < t_0$,

$$\begin{aligned} &U(t_0,1) - U_*(t_0,1) \\ &\quad = \left[S_\beta(t_0-a)(U(a,\cdot) - U_*(a,\cdot))\right](1) + \int_a^{t_0}\int_0^1 \widetilde{K}_\beta(t_0-t;1,s)\,F(t,s)\,ds\,dt. \end{aligned}$$

Here the initial data are restricted to $[\beta,1]$. The boundary point functional produces $K_\beta(h;1,\beta)$ by the weak semigroup formula. The integrals are finite on fixed positive time strips. The volume term is bounded by sub-Markovianity, and the boundary term is integrable near the final time by the $O(h^{-1/2})$ fixed-interval kernel bound, for instance by comparison with the reflected Neumann kernel.

For every source time the product $A(t,s)\widetilde{K}_\beta(t_0-t;1,s)$ is nonnegative and nondecreasing in $s$. Among fractions between zero and one of integral at most $1-\beta$, its weighted integral is maximized by $b_*$, by moving any available mass towards larger $s$. Therefore (69) makes each time slice of the source pairing nonpositive. Let $a \downarrow 0$. The initial term tends to zero by uniform convergence of both profiles to $s$ and the Robin contraction property. This proves the envelope without imposing a time-zero bound on the signed volume or boundary source.

For strictness, the selected path has complement of positive length when $\beta > 0$. On any open edge subinterval in that complement the heat profile is strictly concave at each positive time. A positive measure of regular levels therefore has a crossing of nonzero slope off the path. There $A > 1/B_\gamma$, so $bU_t - U_{ss} < 0$. The kernel section is positive, and the time-slice pairing is strictly negative. Retaining any compact subinterval of $(0,t_0)$ of positive length gives a strictly negative Duhamel integral after letting $a \downarrow 0$. The brooms of Section 7 converge to the endpoint-mass content at fixed $L,R$, proving the sharpness claim and the stated time uniformity. At $R = L$ use the ordinary interval comparison. Scaling proves Theorem 13.1.

The same endpoint model maximizes every positive integral of the heat content at fixed $L,R$, whenever the model's integral is finite. Positivity gives the upper bound, and the same brooms give equality in the supremum by dominated convergence. In particular for any $a > 0$ and $z \ge 0$ the spectral quadratic $\langle 1, (H_G + z)^{-a}1\rangle$ has exactly this extremal model, by integration against $t^{a-1}e^{-zt}/\Gamma(a)$. The comparison is strict for finite graphs with $R < L$. The model has a positive spectral gap by the Dirichlet anchor, so these integrals are finite even at $z = 0$.

## 16 The cubic one-time bound on general graphs

Write $\Delta_G(t) = Q_{I_1}(t) - Q_G(t)$ and let $K(h; x, s)$ be the unit mixed-interval heat kernel in mass coordinates, Dirichlet at 0 and Neumann at 1. By Theorem 5.3,

$$\Delta_G(t_0) = \int_0^{t_0} \int_0^1 K(t_0 - t; 1, s) d(t, s)\, ds\, dt.$$

For any $0 < T < \tau_0$, define

$$c_0 = \min_{\tau_0 - T \le h \le \tau_1,\, 0 \le s \le 1} \frac{K(h; 1, s)}{s} > 0,$$

interpreting the quotient by its inward derivative at 0, as in (30). Restrict the nonnegative source to $0 < t < T$ and use (68). For $a > 0$, local absolute continuity of $U(t, s)$ gives $\int_a^T A(t, s)\, dt = U(a, s) - U(T, s)$ for almost every $s$. As $a \downarrow 0$, $U(a, s) \to s$ uniformly, by $L^2$ continuity of the heat semigroup at datum one and the concentration integral's $L^1$ Lipschitz bound. Tonelli then yields

$$\Delta_G(t_0) \ge c_0 \int_0^\beta s\,[s - U(T, s)]\, ds \ge c_0 \int_0^\beta s\,[s - U_I(T, s)]\, ds. \tag{70}$$

The last step is the source theorem's concentration comparison. The interval's datum-one heat profile has maximum $\theta_T < 1$ at any $T > 0$, by the strong maximum principle after reflection at the Neumann endpoint. Consequently $U_I(T, s) \le \theta_T s$ and

$$\Delta_G(t_0) \ge \frac{c_0(1 - \theta_T)}{3} \beta^3.$$

Taking $T = \tau_0/2$ and rescaling completes the heat bound. The marked measured comparison follows from $d_m(G/L, I_1) \le 2\beta$ in Section 4. The exact cubic pitchfork example in Theorem 10.1 shows sharpness of the exponent. Inequality (70) gives the cubic endpoint directly from the ordinary interval kernel. The sharp envelope also implies the endpoint using the positive endpoint-model variation in Section 7.1.

## 17 First $p$-eigenvalues at finite edge count

A broom $B$ consists of a stem of length $a > 0$, with a Dirichlet condition at its root, and Neumann arms of lengths $b_i > 0$ at the other stem endpoint. We use the continuous edgewise $W^{1,p}$ energy with the natural Kirchhoff flux at the junction. Write $L = a + \sum_i b_i$, $b = \max_i b_i$, $R = a + b$, and $\delta = L - R > 0$. Its first eigenvalue $\lambda(B)$ is the minimum of $\int |f'|^p / \int |f|^p$ over nonzero rooted trials. Throughout $1 < p < \infty$ and $q = p/(p - 1)$.

For the linear case Kennedy [12, Proposition 1.4(i)] constructed the equal-arm sequence and proved its monotonicity and endpoint-mass limit. The linear secular equation also appears in the symmetric dumbbell setting of [15, Lemma 7.6]. Kennedy's Lemma 3.6 compares allocations between the two equilateral clusters of a dumbbell. We keep the Dirichlet inradius fixed and allow all individual arm lengths, and hence the stem length, to vary.

**Theorem 17.1.** *Suppose $n \ge 2$ and $\delta/(n - 1) < R$. Among all brooms with total length $L$, Dirichlet inradius $R$, and at most $n$ arms, the unique minimum up to isometry is attained at $B_n$, with $n$ equal arms of length $\delta/(n - 1)$. In particular the minimum decreases strictly as the feasible arm cap increases.*

*Proof.* For a positive increasing solution on an edge, the coordinate $m = |u'|^{p-2}u'/(\lambda u^{p-1})$ satisfies $m' = -(1 + c_\lambda m^q)$, where $c_\lambda = (p-1)\lambda^{q-1}$. Define

$$F_\lambda(s) = 1 + c_\lambda s^q, \qquad \Phi_\lambda(s) = \int_0^s F_\lambda(t)^{-1}\,dt, \qquad S_\lambda = \Phi_\lambda(\infty), \qquad A_\lambda = \Phi_\lambda^{-1}.$$

Here $S_\lambda$ is the length of the mixed interval with first eigenvalue $\lambda$. If $\lambda_0$ is the first mixed eigenvalue of the unit interval, $S_\lambda = (\lambda_0/\lambda)^{1/p}$.

The nonnegative first eigenfunction is concave on each edge since its flux is nonincreasing. A zero at the junction would force all outgoing derivatives to be nonnegative and hence zero by Kirchhoff. Concavity would then make the eigenfunction identically zero. The arm fluxes are nonnegative, with zero flux at the tips. The stem flux is positive by Kirchhoff. The stem thus has coordinate $A_\lambda(S_\lambda - a)$ at the junction and the arms have $A_\lambda(b_i)$. The flux relation is

$$S_\lambda = a + \Phi_\lambda\left(\sum_i A_\lambda(b_i)\right). \tag{71}$$

The signed arm-minus-stem flux mismatch increases strictly with $\lambda$ on the domain $a, b_i < S_\lambda$. The arms increase with $c_\lambda$ since $\Phi_\lambda$ decreases with it. The stem decreases by the identity $a = \int_{m_{\text{stem}}}^\infty F_\lambda(t)^{-1}\,dt$.

Fix $\lambda$ at the first eigenvalue of a competitor. To bound the arm sum at its given longest length $b$, keep one longest arm and distribute the remaining length $\delta$ with cap $b$. The convexity of $A_\lambda$ gives

$$\sum_i A_\lambda(b_i) \le M(b) = (j+1)A_\lambda(b) + A_\lambda(\delta - jb), \qquad j = \lfloor \delta/b \rfloor.$$

Indeed transferring mass from a smaller unsaturated length to a larger unsaturated length increases the sum until all but one lengths are zero or $b$. Across the intervals where $j$ is constant,

$$M'(b) \le 1 + (j+1)c_\lambda A_\lambda(b)^q < F_\lambda(M(b)).$$

For $j \ge 1$ the strict inequality follows from $q > 1$. For $j = 0$ use $M(b) > A_\lambda(b)$. The formulas agree at the transition points, so $-b + \Phi_\lambda(M(b))$ decreases strictly with $b$ below $S_\lambda$. The arm cap forces $b \ge b_* = \delta/(n-1)$, with equality only for the stated equal arms. At $b_*$ the right side of (71) is at least the right side for the competitor, strictly unless $b = b_*$. If $R - b_* \ge S_\lambda$, the turning length at the eigenvalue of $B_n$ is larger than $S_\lambda$, so its eigenvalue is smaller. If $R - b_* < S_\lambda$, the signed mismatch at $\lambda$ is nonnegative and the strictly increasing mismatch gives the asserted comparison. □

# 18 Convergence to the endpoint-mass model

Let $\lambda_* = \lambda_p(L, R)$ be the minimum of the same quotient on the rooted interval $[0, R]$ with speed measure $dx + \delta\delta_R$, where $\delta_R$ denotes the unit mass at $R$. Normalize its eigenfunction $g$ by $g(R) = 1$. It is the exact infimum at fixed $(L, R)$ over finite graphs Section 11. Equivalently $S_{\lambda_*} = R + \Phi_{\lambda_*}(\delta)$. For brooms with $b \to 0$ at fixed $L, R$, convergence to $\lambda_*$ follows from the envelope and the trial given by restricting $g$ to the stem and extending constantly over the arms.

**Theorem 18.1.** *As the longest arm $b \to 0$ at fixed $L, R$,*

$$\lambda(B) - \lambda_* = \frac{(p-1)\lambda_*^q \delta^q}{\delta + \int_0^R g^p}\, b + o(b).$$

*For the joint limit normalize $L = 1$, put $\beta = 1 - R$, and assume $b + \beta \to 0$. Uniformly in all arm distributions,*

$$\lambda(B) - \lambda_* = \frac{p(p-1)\lambda_0^q}{q+1}\left[(\beta+b)^{q+1} - \beta^{q+1} - \sum_i b_i^{q+1}\right] \times \bigl(1 + O_p((b+\beta)^q)\bigr).$$

*In particular, for equal arms, if $\beta \downarrow 0$ and $n \to \infty$ at any relative rates, the gap is asymptotic to $p(p-1)\lambda_0^q \beta^{q+1}/(n-1)$.*

*Proof.* At fixed $L, R$, for $\lambda$ near $\lambda_*$ denote by $m_\lambda(R)$ the stem coordinate at $R$, which is positive in this neighborhood. Its value at $\lambda_*$ is $\delta$. The arm sum is $\delta + b + O(b^q)$ since $A_\lambda(t) = t + O(t^{q+1})$ and the sum of arm lengths is $\delta + b$. The stem coordinate at length $R - b$ is $m_\lambda(R) + bF_\lambda(m_\lambda(R)) + O(b^2)$. For the stem solution $u_\lambda(R) = 1$, its endpoint flux $\lambda m_\lambda(R) = \int |u_\lambda'|^p - \lambda \int u_\lambda^p$ has derivative $-\int u_\lambda^p$, by the Euler equation under zero trace variations at both endpoints. The positive flux makes the separated solution smooth in $\lambda$ here. Thus $\partial_\lambda m_\lambda(R)$ equals $-(\delta + \int g^p)/\lambda_*$ at $\lambda_*$. Equating the fluxes yields the first formula.

For the joint result let $X = b + \beta$, $m = \min(b, \beta)$ and $k = q + 1$. For $\lambda$ in a compact neighborhood of $\lambda_0$ put $u = \sum_i A_\lambda(b_i)$ and $H = \Phi_\lambda(\beta) + b - \Phi_\lambda(u)$. We prove

$$H = \frac{c_\lambda}{k}\left[X^k - \beta^k - \sum_i b_i^k\right] + O_p(mX^{2q}). \tag{72}$$

Use $A_\lambda(t) = t + c_\lambda t^k/k + O_p(t^{2q+1})$ and $\Phi_\lambda'(s) = 1 - c_\lambda s^q + O_p(s^{2q})$. If $b \le \beta$, the arm expansion has total remainder $O_p(Xb^{2q}) = O_p(bX^{2q})$ and $u - X = O_p(Xb^q)$. Integrate $\Phi_\lambda'$ from $\beta$ to $u$. The interval has length $O(b)$, and replacing $u^k$ by $X^k$ costs $O_p(X^{q+1}b^q) = O_p(bX^{2q})$. This gives (72).

If $b \ge \beta$, single out an arm of length $b$. Then $d = A_\lambda(b) = b + O_p(b^k)$ and $u - d = \beta + c_\lambda(\sum_i b_i^k - b^k)/k + O_p(\beta^{2q+1})$. Integrate from $d$ to $u$ using $\Phi_\lambda(d) = b$. The integral of $s^q$ on this segment differs from $\int_b^X s^q\,ds$ by $O_p(\beta X^{2q})$. Indeed the left endpoint shift is $O_p(b^k)$, the length error is $O_p(\beta^k)$, and the derivative of $s^q$ on the combined range is $O_p(X^{q-1})$. Combining with the expansion of $\Phi_\lambda(\beta)$ gives (72).

The bracket in (72) is bounded below by $(q-1)m\max(b,\beta)^q$. For $b \le \beta$ use $\sum_i b_i^k \le Xb^q \le 2\beta^q b$, and for $b \ge \beta$ use $\sum_i b_i^k \le b^k + \beta^k$, in each case expanding $X^k$ from the larger length by convexity. The remainder in (72) is therefore relative $O_p(X^q)$.

The model root satisfies $D(\lambda_*) := S_{\lambda_*} - R - \Phi_{\lambda_*}(\beta) = 0$, and the finite root satisfies $D(\lambda(B)) = -H$ with $H$ evaluated there. Both eigenvalues tend to $\lambda_0$: the length-only bound [13] gives $S \le 1$, and the stem is at least $1 - X$. The flux expansions give $S = 1 + O_p(X^{q+1})$ for both roots. Between them $D' = -1/(p\lambda_0) + O_p(X^{q+1})$, by differentiating the integral for $\Phi_\lambda(\beta)$. The mean-value theorem now proves the relative estimate. For equal arms the bracket is $(\beta+b)^{q+1} - \beta^{q+1} - (\beta+b)b^q$, asymptotic to $(q+1)\beta^q b$ as $b/\beta \to 0$. □

## 19 An inverse-edge bound on arbitrary graphs

Let $G$ now be any finite connected compact metric graph of length $L$, with a nonempty Dirichlet vertex set $D$ and inradius $R = \max_y \operatorname{dist}(y, D)$ in $(0, L)$. Write $\lambda_G$ for its first $p$-eigenvalue, $k_G = 2\,\mathrm{b}_1(G) + \ell(G)$ for twice its cycle rank plus its number of leaves, and $E$ for its number of edges. Kennedy's linear comparison [12, Lemma 4.1] uses a graph-dependent arm threshold involving the minimum edge length. Here the constants depend only on $L, R, p$.

**Theorem 19.1.** *There is $c > 0$ depending only on $L, R, p$ such that*

$$\lambda_G - \lambda_* \geq \frac{c}{k_G} \geq \frac{c}{2E}.$$

*Both inverse powers are sharp. More precisely, for any nonnegative first eigenfunction $f$ with maximum one at $x$, and any shortest path $\gamma$ from $D$ to $x$, setting $e = \lambda_G - \lambda_*$,*

$$R - |\gamma| + \int_{G\setminus\gamma} \operatorname{dist}(y, x)\, dy \leq Ce$$

*with $C$ depending only on $L, R, p$.*

*Proof.* Put $r = |\gamma|$, $\lambda_r = \lambda_p(L, r)$, and let $g_r$ be the radius-$r$ model eigenfunction normalized by $g_r(r) = 1$. The bound $e > 0$ and the monotonicity of $\lambda_r$ in $r$ are in Section 11. The flux equation $m_{\lambda_r}(r) = L - r$ and the derivatives from Sections 1–2 show that $\lambda_r$ has strictly negative derivative near $R$. Indeed its defining equation has $r$-derivative $1 - F_\lambda(m_\lambda(r)) < 0$ and $\lambda$-derivative strictly negative. As the Rayleigh quotient on $\gamma$ gives $\lambda_G \geq \lambda_r$, this bounds $R - r$ by $Ce$ near $R$. Monotonicity and $R - r \leq R$ extend the estimate to the full range of $e$.

The same path quotient gives the nonnegative defect

$$D_\gamma = \int_\gamma |f'|^p - \lambda_r \left( \int_\gamma f^p + L - r \right) \leq Le, \qquad \int_{G\setminus\gamma} (1 - f^p) \leq Le/\lambda_*.$$

To see both upper bounds, use the total energy $\lambda_G \int_G f^p$ and $\int_\gamma f^p + L - r = \int_G f^p + \int_{G\setminus\gamma}(1 - f^p)$. For sufficiently small $e$, $r \geq R/2$ and the total energy is at most $L(\lambda_* + 1)$. Hölder's inequality from $f(x) = 1$ gives $f \geq 1/2$ on a terminal segment of $\gamma$ of a fixed positive length $\ell_0$. There $v = f g_r'/g_r \geq d > 0$ uniformly because $g_r \leq 1$ and $g_r' \geq g_r'(r) = [\lambda_r(L - r)]^{1/(p-1)}$.

Test the model with $f^p/g_r^{p-1}$, whose trace is zero at the root and one at the terminal point. This is admissible: both functions have bounded derivatives near the root, $g_r'(0) > 0$, and the quotient $f/g_r$ and the test derivative are bounded there. The Picone identity [16] gives

$$D_\gamma = \int_0^r \left( |f'|^p + (p - 1)|v|^p - p|v|^{p-2} v f' \right) ds.$$

Convexity makes the integrand nonnegative everywhere and at least a positive constant where $v \geq d$ and $|f'| < d/2$. In particular the low-speed subset of the terminal segment has length at most $C_0 e$.

For $y \notin D$ take another shortest Dirichlet path $\sigma$ to $y$. Its model Rayleigh bound, with its own actual endpoint value included in the atom, gives

$$\int_\sigma |f'|^p \geq \lambda_* \left( \int_\sigma f^p + (L - |\sigma|) f(y)^p \right).$$

This also holds for a zero restriction. Hence the energy outside $\sigma$ is at most $Le + \lambda_* L(1 - f(y)^p)$. Set $s_y = e + 1 - f(y)^p$ and choose $H$ with $(d/2)^p(H - C_0) > L(1 + \lambda_*)$. If $H s_y \leq \ell_0$, then $\sigma$ meets the terminal segment of $\gamma$ of length $H s_y$ in positive arclength: otherwise the segment spends energy at least $(d/2)^p(H s_y - C_0 e)$ outside $\sigma$. For a common point $z$, its distance to $D$ is its coordinate on either path, by shortestness. It is within $H s_y$ of $x$ and within $R - r + H s_y$ of $y$. It follows that $\operatorname{dist}(y, x) \leq C s_y$. When $H s_y > \ell_0$ or $e$ is large, increase $C$ using the diameter bound $L$. Integration off $\gamma$, with the mass deficit above, proves the moment bound.

For completeness the ball about $x$ has length at most $k_G t$ at radius $t$, a local form of the radius–length bound in [17, Theorem 1]. At a regular level $t$ of distance from $x$, cut at the intersection points. The ball is connected by shortest paths. If the outside components have $N_i$ boundary points, Euler characteristic gives

$$\mathrm{b}_1(G) = \mathrm{b}_1(\overline{B}) + \sum_i \big(\mathrm{b}_1(G_i) + N_i - 1\big).$$

For $N_i \geq 2$, $N_i \leq 2(N_i - 1)$. If $N_i = 1$ a tree outside component has an original leaf, since a nontrivial tree has at least two leaves. Otherwise it has a cycle. Thus the total number of level crossings is at most $k_G$; distance has derivative of absolute value one almost everywhere on each edge, so integration of the level count proves the ball estimate. The off-path length is at least $\delta = L - R$. Layer cake now bounds its moment below by

$$\int_0^{\delta/k_G} (\delta - k_G t)\, dt = \delta^2/(2k_G).$$

Also $k_G \leq 2E$ by Euler's formula and $\ell(G) \leq |V(G)|$ for at least two vertices. With one vertex every edge is a loop, so $k_G = 2E$. The moment inequality proves both lower bounds. □

The equal brooms in Theorem 18.1 have $k_G = E = n + 1$ and show sharpness of the power. Kennedy also proposes linear stowers approaching the endpoint model [12, Remark 1.5(a)]. Here $m$ loops of length $2b$ on the junction of a rooted stem of length $R - b$, with $b = \delta/(2m - 1)$, give the same first eigenvalue as the $2m$-arm broom. Cutting each loop at its midpoint relaxes the equality of that pair of endpoint traces. Conversely, a ground state of the equal-arm broom has equal arm profiles by the scalar flux relation with prescribed junction value, so it reglues with the same quotient. These loops have $m + 1$ edges and therefore give half the leading edge-count gap of the equal-arm sequence at the same asymptotic edge count.

## 20 Sharp nonlinear heat content at fixed length and inradius

Let $G$ be a finite connected compact metric graph of total length $L > 0$, with a nonempty set $D$ of Dirichlet vertices. Its Dirichlet inradius is $R = \max_{x \in G} \operatorname{dist}(x, D)$. Write $\phi(a) = |a|^{p-2}a$ and $\nu = 1/(p - 1)$. The energy $\mathcal{E}_p(u) = p^{-1} \int_G |u'|^p$ is defined on continuous edgewise $W^{1,p}$ functions vanishing on $D$ and extended by $+\infty$ in $L^2(G)$. Its subgradient evolution is $u_t = \Delta_p u$, with Kirchhoff flux at vertices outside $D$. Write $Q_G(t) = \int_G S_G(t)1$.

The comparison space $I_*$ is $[0, R]$ with speed measure $\mu_* = dx + \beta\delta_R$, where $\beta = L - R$, and energy $p^{-1} \int_0^R |u'|^p$. The form domain imposes a Dirichlet trace at zero and identifies the atomic coordinate with the trace at $R$ when $\beta > 0$. Denote its flow by $S_*(t)$ and its heat content by $Q_*(t) = \int S_*(t)1 \, \mathrm{d}\mu_*$. For $\beta = 0$ this is the mixed Dirichlet–Neumann interval. A shortest path of length $R = L$ exhausts every edge of $G$. Along that path no other Dirichlet mark can occur because its initial point is nearest to the terminal point. Thus $R = L$ reduces to the mixed interval.

In either model the implicit step $J_h f$ minimizes

$$\frac{1}{2} \int |u - f|^2 \, \mathrm{d}\mu + h\mathcal{E}_p(u), \qquad h > 0. \tag{73}$$

Here $\mu$ is arclength on $G$ and $\mu_*$ on $I_*$. We use superscripts $G, *$ to distinguish the two resolvents. The boundary relation on $I_*$ is $\phi(u'(R)) = \beta(f(R) - u(R))/h$. The dynamic condition is $\beta u_t(R) = -\phi(u'(R))$, read as the zero flux condition when $\beta = 0$.

The interval with an endpoint mass gives the exact all-time envelope for the linear datum-one flow (Section 13). The same geometry appears in the joint length and inradius first $p$-eigenvalue bound with $L^p$ normalization (Section 11). Here the $p$-energy generates a flow in $L^2$, and the stationary rescaling has a linear right-hand side. The coarea and implicit-step comparison are related to Vázquez's nonlinear mass symmetrization [23, Sections 4, 7 and 8]. We use the budget on one shortest path to compare the total masses at the endpoint of the mass interval.

**Theorem 20.1** (One implicit step)**.** *For $1 < p < \infty$ and $h > 0$, the sharp upper bound at fixed $L, R$ is*

$$\sup_G \int_G J_h^G 1 = \int J_h^* 1 \, \mathrm{d}\mu_*.$$

*For $1 < p \le 2$ and $R < L$, every finite graph gives strict inequality. For $p > 2$ and $R < L$, finite brooms attain equality for sufficiently small $h$. For every $p > 2$ and every $0 < R < L$, there are a finite broom and a step $h$ for which the graph's integral over its highest $s$ mass exceeds the corresponding interval integral for all sufficiently small $s > 0$.*

**Theorem 20.2** (Early heat content)**.** *Let $p > 2$ and define*

$$c_p = \frac{p-2}{2p(p-1)}, \qquad F(\eta) = \int_\eta^\infty [c_p(\eta_*^2 - \xi^2)]_+^{1/(p-2)} \, \mathrm{d}\xi, \qquad F(0) = 1,$$

*where the normalization determines $\eta_* > 0$. For every $0 < t < (R/\eta_*)^p$,*

$$\sup_G Q_G(t) = Q_*(t) = L - t^{1/p} \int_0^{\eta_*} F(\eta) \, \mathrm{d}\eta.$$

*At each such time the supremum is attained by a finite broom when $R < L$.*

**Theorem 20.3** (Late heat content)**.** *Let $p > 2$, $\alpha = (p-2)^{-1}$, and $C = \alpha L R^{p-1}$. There is a unique positive weak profile $g_G$ off $D$ satisfying $-\Delta_p g_G = \alpha g_G$ with the graph conditions. The interval profile $g_*$ is the unique positive solution on $(0, R]$ of*

$$-(\phi(g_*'))' = \alpha g_*, \qquad g_*(0) = 0, \qquad \phi(g_*'(R)) = \alpha\beta g_*(R).$$

*Set $M_G = \int_G g_G$ and $M_* = \int g_* \, \mathrm{d}\mu_*$. For every $t > 0$,*

$$(t + C)^{-\alpha} M_G \le Q_G(t) \le t^{-\alpha} M_G.$$

*The same bounds hold for $Q_*, M_*$. At fixed $L, R$, $\sup_G M_G = M_*$, with strict inequality for each finite graph if $R < L$. In particular,*

$$(1 + C/t)^{-\alpha} M_* \le t^\alpha \sup_G Q_G(t) \le M_*.$$

*The extremal late-time coefficient is approached by equal-arm brooms.*

The early-time result uses the constant-boundary similarity front of Pascal [22], also given in Hinton, Collis and Sader [20, Appendix A, (A3)–(A6)]. The separable large-time mechanism is classical in smooth Euclidean domains [21, Theorems 1.1 and 1.3, $m = 1$] and fits the abstract theory of homogeneous gradient flows [18]. The late-time geometric result optimizes the stationary profile mass over the graph class.

## 21 Coarea with a shortest-path budget

The energies are proper, convex, lower semicontinuous and densely defined in their $L^2$ spaces. For lower semicontinuity, use the Dirichlet anchor and the compact embedding of bounded edgewise $W^{1,p}$ families into continuous functions, including convergence of endpoint traces. Density on $I_*$ follows by prescribing the value at $R$ in a collar of arbitrarily small arclength. The convex subgradients are maximal monotone. In particular $S(t)f = \lim_{n\to\infty} J_{t/n}^n f$ in $L^2$ [19], including for the datum one. Testing a resolvent difference with its positive part shows order preservation. Testing with the negative part and the excess above one gives $0 \le J_h f \le 1$ for $0 \le f \le 1$. The datum-one iterates decrease by order preservation.

For a nonnegative function $u$, write $U(s)$ for the integral of its highest $s$ mass, counting a suitable fraction of a level set when it has positive mass. Thus $U'(s)$ is the decreasing quantile. Consider a graph step $u = J_h^G f$ in the datum-one sequence. On an oriented edge $j = \phi(u')$ satisfies $j' = (u - f)/h \le 0$. The flux is absolutely continuous and $u'$ is continuous on each edge. The profiles are edgewise concave and the vertex and critical values form a finite set. At a regular level $z$, put $s = |\{u > z\}|$. Integration over the superlevel set, using Kirchhoff flux at vertices, gives

$$A(s) = \int_{\{u>z\}} \frac{f-u}{h} = \sum_{u=z} |u'|^{p-1}, \qquad B(s) = \sum_{u=z} |u'|^{-1}.$$

Within a plateau mass interval use nested portions of that plateau to pad the top sets. Indeed $f - u = 0$ on each nontrivial edge plateau by the Euler equation. In this way $A$ is defined throughout, is nonnegative and nondecreasing, and satisfies

$$A = (s - U)/h \quad (f = 1), \qquad A \le (F_0 - U)/h \quad \text{in general,}$$

where $F_0$ is the concentration integral of $f$. The second bound follows by maximizing the integral of $f$ among sets of mass $s$.

Choose a maximizer of $u$ and a shortest path $\gamma$ to it from the nearest Dirichlet mark. Then $|\gamma| \le R$. At regular levels $B_\gamma$ is the inverse-slope sum along that path, and set $b(s) = B_\gamma/B$. On a plateau mass interval $b$ is the fraction of the total plateau mass on $\gamma$. Coarea and this plateau accounting give $0 \le b \le 1$ and $\int_0^L b(s)\,\mathrm{d}s = |\gamma| \le R$. The quantile is Lipschitz. In fact traversal of any intermediate level band along $\gamma$ requires arclength at least the level width divided by the largest edge slope. At regular levels $-U'' = 1/B$ and a path crossing gives $A^\nu \ge 1/B_\gamma$. On plateau mass intervals $U'' = 0$. We have therefore shown

$$-U'' \le bA^\nu, \qquad U(0) = 0, \quad U'(L) = 0. \tag{74}$$

These identities are in $W^{2,\infty}(0, L)$ and hold almost everywhere. They also hold for an edgewise concave stationary profile with nonnegative source $q$ and Kirchhoff flux, using $A(s) = \int_{\text{top } s} q$, provided $q$ vanishes on plateaus. If a regular level has an additional crossing off the path, $A^\nu > 1/B_\gamma$ and the inequality is strict.

## 22 The endpoint comparison for a single step

Let $U$ be the graph concentration integral for datum one and $V$ that of the interval with the atom. The interval profile is increasing in physical $x$ because its flux decreases and is nonnegative at $R$. Thus, writing $v = J_h^* 1$,

$$V(s) = \begin{cases} s\,v(R), & 0 \le s \le \beta, \\ \beta v(R) + \displaystyle\int_{L-s}^{R} v(x)\,\mathrm{d}x, & \beta \le s \le L. \end{cases}$$

Integration to $R$, including the endpoint flux, gives $-V'' = b_* A_*^\nu$, where $b_* = \mathbf{1}_{(\beta,L)}$ and $A_* = (s - V)/h$. Its derivatives match at $\beta$ and $V(0) = V'(L) = 0$.

Set $W = U - V$. Subtracting the equations gives $-W'' \le (b - b_*)A^\nu + b_*(A^\nu - A_*^\nu)$. For $\delta > 0$, use $\Phi_\delta(x) = (x + \delta)^\nu - \delta^\nu$. It approximates the power uniformly on the bounded range of the arguments. Its divided difference evaluated at $A, A_*$, divided by $h$, is bounded and nonnegative, denoted $c_\delta$. Since $A - A_* = -W/h$,

$$-W'' + b_* c_\delta W \le (b - b_*)A^\nu + o_\delta(1). \tag{75}$$

Let $K_\delta$ solve the homogeneous equation on the left with $K_\delta(0) = 0$, $K_\delta'(L) = 1$. Start the ODE with unit derivative at zero. The nonnegative potential makes the solution positive, increasing and convex up to $L$. Rescaling its derivative at $L$ gives $0 \le K_\delta(s) \le s$. Integration of (75) against $K_\delta$ gives $W(L)$ on the left. The weight $K_\delta A^\nu$ is nonnegative and nondecreasing. For any such weight $P$, $\int_0^L (b - b_*)P \le 0$, since one can bound the integrand using $P(\beta)$ on both sides of $\beta$ and use $\int b \le R$. Letting $\delta \downarrow 0$ proves the upper bound in Theorem 20.1.

For $1 < p \le 2$ the unregularized power has bounded divided differences. In that case the maximum of the graph solution is strictly below one. If it were one at a vertex, Kirchhoff would force every outward derivative to vanish. At an interior maximum the derivative already vanishes. The edge system $u' = \phi^{-1}(j)$, $j' = (u - 1)/h$ is locally Lipschitz for $\nu \ge 1$. The constant solution starting from $(u, j) = (1, 0)$ therefore propagates along a path to $D$, a contradiction. If $R < L$, there is an open edge segment off $\gamma$. It is strictly concave by $u < 1$, so there is a noncritical subsegment contributing crossings off $\gamma$ on a positive measure set of mass coordinates. The strict defect in (74), integrated against the positive unregularized Green section, gives strictness of the mass bound.

For sharpness when $\beta > 0$, take $n$ equal arms of length $\ell_n = \beta/(n - 1)$ beyond a stem of length $R - \ell_n$ rooted at the sole Dirichlet mark, for $n$ large enough. These graphs have length $L$ and inradius $R$. The minimized objectives in (73) are uniformly bounded by the zero trial. Stems extended constantly to $R$ are bounded in $W^{1,p}(0, R)$, hence compact in $C[0, R]$. Every arm differs from the junction value by at most a uniform multiple of $\ell_n^{1-1/p}$. Since $n\ell_n \to \beta$, their quadratic penalty tends to the atomic penalty. Lower semicontinuity bounds the limiting stem energy. Conversely, restrict the interval minimizer to the shortened stem and extend constantly over the arms to obtain the upper bound on the minimized objectives. Uniqueness identifies every such limit with $v$; the arm estimate also gives convergence of total masses.

## 22.1 Compact implicit fronts and loss of concentration order

Let $p > 2$ and $m = p/(p - 2)$. The deficit $w = 1 - J_h 1$ solves $(\phi(w'))' = w/h$ with value one at the Dirichlet marks. The half-line decreasing compact front and its length are

$$w(x) = (1 - x/d_h)_+^m, \qquad d_h = m[2h(p - 1)/p]^{1/p}.$$

This follows by integrating $(p - 1)|w'|^p/p = w^2/(2h)$ on its positive part. If $d_h$ is shorter than a broom's stem, extend this profile by zero before the junction. It also gives the deficit on $I_*$ with zero value and flux at $R$. The two deficits have equal integrals, proving finite-graph equality for small steps.

For the concentration claim fix a broom of stem length $a > 0$ and $n \ge 2$ arms of length $b > 0$ with $a + b = R$ and $a + nb = L$. Put $q = p/(p - 1)$. For a junction deficit $r \in (0, 1)$, the compact arm fronts penetrate by $d_h r^{1/m}$. Kirchhoff at the branch multiplies their individual flux by $n$ on the stem, so the stem first integral is

$$\frac{p - 1}{p}|w'|^p = \frac{w^2 + (n^q - 1)r^2}{2h}.$$

The corresponding stem length is the continuous strictly decreasing function

$$a_n(r, h) = \frac{d_h}{m} \int_r^1 [w^2 + (n^q - 1)r^2]^{-1/p} \, \mathrm{d}w$$

with endpoint values $d_h, 0$. At $d_h = R$, put $r_b = (b/R)^m$. Then $a_n(r_b, h) < a_1(r_b, h) = R - b = a$, whereas $a_n(0, h) > a$. Thus the unique solution of $a_n(r, h) = a$ has penetration strictly less than $b$. This persists for $d_h > R$ sufficiently close to $R$. The joined stem and compact arm

profiles satisfy the weak resolvent equation and are identified by uniqueness. In particular $J_h^G 1 = 1$ on positive arclength.

At these same steps the interval deficit at $R$ is positive. A zero endpoint deficit would force zero flux there and hence the zero-constant first integral on the interval, with a nonzero front of length $d_h > R$. Since $J_h^* 1$ is increasing, its maximum is below one. This yields the reversal of the concentration order at small positive masses, while the sharp total-mass comparison holds.

# 23 The early heat envelope

First compare any graph to the mixed interval of length $L$. For the respective datum-one implicit iterates denote the concentration integrals by $U_i, V_i$. The interval iterates are increasing in physical $x$ because their flux decreases to zero at the Neumann endpoint. Integration to that endpoint and (74) on the graph give

$$-V_i'' = [(V_{i-1} - V_i)/h]^\nu, \qquad -U_i'' \le [(U_{i-1} - U_i)/h]^\nu.$$

The iterates decrease, so the bracketed quantities are nonnegative. If $U_{i-1} \le V_{i-1}$, the graph right-hand side is no larger where $U_i > V_i$. Testing the difference with $(U_i - V_i)_+$ and using the mixed traces in (74) yields $U_i \le V_i$. Initially both concentration integrals are $s$. Concentration integrals are uniformly Lipschitz in the $L^1$ norm of their profiles, by the supremum characterization. The exponential formula passes their order to the heat flow.

For $p > 2$ let $F, \eta_*$ be as in Theorem 20.2 and put $v = -F'$. The deficit $w(t,x) = F(x/t^{1/p})$ has boundary value one and compact support in $x \le \eta_* t^{1/p}$. On its positive part the identity $(v^{p-1})' = -\eta v/p$ gives $w_t = \Delta_p w$. The deficit and flux vanish at the moving front, giving a weak solution after extension by zero. Its initial deficit is zero in $L^2$ on any bounded interval. In the power-law fluid notation of [20, Appendix A], the power index is $n = p - 1$; the exact normalized profile is the same [22]. For example, $p = 3$ gives $\eta_*^3 = 18$.

Until the front reaches $R$, the complement $1 - w$ satisfies the mixed condition on the length-$L$ interval and the dynamic endpoint condition on $I_*$. These solutions are identified by uniqueness. To see this within the gradient flow, the trial has $L^2$ time derivative on compact positive time intervals, satisfies the Euler identities there, and is continuous to the datum at time zero. Both solutions have deficit integral $t^{1/p} \int_0^{\eta_*} F$. The mixed-interval concentration bound thus bounds $Q_G(t)$ by $Q_*(t)$ on the stated time window. At any fixed time in this window choose a broom in the preceding sequence with stem longer than the front. On the entire preceding time interval the complement of the same deficit, extended by one past the front, satisfies every vertex condition. It has the same heat content, proving attainment.

# 24 The sharp late-time coefficient

## 24.1 Positive profiles and a common time shift

For $p > 2$, minimize the functional

$$\mathcal{E}_p(f) - \frac{\alpha}{2} \int f^2 \, \mathrm{d}\mu$$

on each domain of the energy. The Dirichlet anchor bounds $\|f\|_\infty$ by $R^{1-1/p} \|f'\|_p$, which gives coercivity since $p > 2$. Compactness and lower semicontinuity give a nonnegative minimizer. The minimum is negative by a small nonzero trial. If a component separated by Dirichlet marks were zero, a small trial supported inside it would strictly decrease the functional. The Euler equation gives edgewise concavity; an interior zero forces zero throughout its edge. At

a nonmarked zero vertex all outward derivatives are nonnegative, and Kirchhoff forces all to vanish. Concavity then propagates zero on every incident edge. Together these facts give positivity off $D$. On $I_*$ the same variational argument gives a positive profile on $(0, R]$: a zero at $R$ would also have zero flux and force zero throughout by concavity. Variation gives the specified stationary equations including the atom.

Let $f$ be any of these positive profiles, or any positive weak solution of the same equations. Testing with $f$ gives $\int |f'|^p = \alpha \int f^2 \, \mathrm{d}\mu$. Write $m_f = \|f\|_\infty$. By a shortest path to a maximizer and Hölder's inequality,

$$m_f^p \le R^{p-1} \int |f'|^p \le \alpha L R^{p-1} m_f^2, \qquad m_f^{p-2} \le C.$$

The same argument on $I_*$ uses the interval length $R$ and the total mass $L$. By homogeneity, $(t+c)^{-\alpha} f$ is an exact heat solution for $c > 0$. With $c = C$ its initial value is at most one. In the other direction use the initial datum $\min(1, nf)$. Its flow is bounded above by $(t+n^{-(p-2)})^{-\alpha} f$. It converges to the datum-one flow in $L^2$ by the positive profile and contraction. The resulting squeeze is

$$(t + C)^{-\alpha} f \le S(t)1 \le t^{-\alpha} f. \tag{76}$$

For a fixed graph or interval, $t^\alpha S(t)1$ converges in $L^2$ to every positive weak profile by (76). This proves uniqueness in that full class. Integration gives the two time-shift bounds in Theorem 20.3.

### 24.2 Optimization in mass coordinates

Assume $\beta > 0$. Let $U, V$ be the concentration integrals of the graph and interval stationary profiles. On graph edges the flux is strictly decreasing and the profile has no nontrivial plateaus. In (74) use $q = \alpha g_G$ to get $A = \alpha U$. On the interval the positive endpoint flux and the decreasing flux give a physically increasing profile. Its concentration integral starts with the linear part $[0, \beta]$ and integration to the endpoint, including the atom, gives the precise equation

$$-U'' \le b(\alpha U)^\nu, \qquad -V'' = b_*(\alpha V)^\nu, \qquad U(0) = V(0) = U'(L) = V'(L) = 0.$$

The graph inequality is strict on a set of positive measure because a noncritical open subsegment off $\gamma$ gives additional crossings.

Set $W = U - V$ and

$$a = b_* \alpha^\nu \frac{U^\nu - V^\nu}{U - V},$$

using the derivative at coincidence. On $(\beta, L)$ both $U, V$ are separated from zero, so $a$ is bounded. For $x > 0$ the divided difference of $x^\nu$ with its value at 1 is less than 1, also at coincidence, because $0 < \nu < 1$. Hence

$$-W'' - aW \le (b - b_*)(\alpha U)^\nu, \qquad -V'' - aV > 0 \quad \text{on } (\beta, L).$$

Although the potential in the linearized operator has negative sign, its mixed Green section is positive and increasing. To prove this, solve $Y'' = -aY$, $Y(0) = 0$, $Y'(0) = 1$. On $(0, \beta)$ both $Y$ and $V$ are positive linear functions. Until any first positive zero of $Y$, the Wronskian $Y'V - YV'$ has derivative $Y(-V'' - aV) \ge 0$. Thus $Y/V$ is nondecreasing and bounded below away from zero by its value on $(0, \beta)$, ruling out a first zero. The strictly positive Wronskian derivative on $(\beta, L)$ and $V'(L) = 0$ give $Y'(L) > 0$. Since $Y'' \le 0$, $Y' \ge Y'(L) > 0$. Now $K = Y/Y'(L)$ has $K'(L) = 1$ and is increasing and positive for $s > 0$.

Multiplying the inequality for $W$ by $K$ and integrating by parts yields $W(L)$ on the left. The nonnegative weight $K(\alpha U)^\nu$ is increasing, so the path budget gives $\int_0^L K(b - b_*)(\alpha U)^\nu \le 0$

as in the one-step case. The strict graph defect gives $W(L) < 0$. This proves the strict profile mass bound when $R < L$.

For sharpness use the equal-arm brooms. The uniform supremum bound and testing identity give uniformly bounded energies for their stationary profiles. The stem compactness and arm estimate of the resolvent argument apply to the stationary functionals. In particular the arm integrals of $f$ and $f^2$ tend to $\beta f(R)$ and $\beta f(R)^2$ for every subsequential stem limit. The kinetic energy has the lower limit bound on the stem, while restriction of the interval minimizer to the shortened stem, with constant recovery on the arms, supplies the upper bound on the minimized objectives. The limiting function minimizes the interval functional and is nonzero since the minimum is negative. Positivity and uniqueness identify it as $g_*$. It follows that $M_G \to M_*$ on the broom sequence. Taking the supremum in (76) proves the two bounds on $t^\alpha \sup_G Q_G(t)$. In particular its relative deficit from $M_*$ is at most $\alpha C/t$.

## 25 Finite-complexity bounds for torsion and resolvents

Let $G$ be a finite connected compact metric graph with strictly positive edge lengths, total length $L$, and a nonempty Dirichlet vertex set $D$. Functions are continuous at all vertices. The natural condition at each unmarked vertex is conservation of outward flux. Write

$$R = \max_{x \in G} \operatorname{dist}(x, D), \qquad 0 < R < L, \qquad \delta = L - R, \qquad k_G = 2b_1(G) + \ell(G),$$

where $b_1(G)$ is the cycle rank and $\ell(G)$ counts all degree-one vertices, including Dirichlet ones. Loops and parallel edges are allowed. All integrals on $G$ use arclength.

For $p > 1$, write $q = p/(p-1)$ and let $u_p$ be the weak $p$-torsion function, with edge equation $-(|u_p'|^{p-2}u_p')' = 1$. We use its unpowered mass $S_p(G) = \int_G u_p$. For the linear Laplacian $H$ with the same marked vertices, put $u_z = (H + z)^{-1}1$ and $J_G(z) = \int_G u_z$, where $z \geq 0$. At $z = 0$ this is $S_2(G)$.

For $0 < r \leq R$, the model interval has length $r$, a Dirichlet condition at zero and speed measure $dx + (L - r)\delta_r$. The sharp qualitative bounds Sections 2–3 are

$$S_p(G) \leq A_p(L, R), \qquad J_G(z) \leq F_{L,R}(z), \qquad A_p(L, r) = \int_0^r (L - s)^q \, \mathrm{d}s.$$

Here $F_{L,r}(0) = A_2(L, r)$. For $z > 0$, with $s = \sqrt{z}$,

$$F_{L,r}(z) = \frac{L}{z} - \frac{\tanh(sr) + s(L - r)}{z^{3/2}[1 + s(L - r)\tanh(sr)]}.$$

Both envelopes are approached by equal-arm brooms. The linear length-only Saint–Venant comparison and related explicit stower calculations appear in Mugnolo and Plümer [24, Theorem 4.6 and Example 3.10]. Özcan [5, Theorem 4.2] proves the nonlinear length-only bound with degree-one Dirichlet vertices, using the powered mass $S_p^{p-1}$. Here both $L$ and $R$ stay fixed while the complexity varies.

**Theorem 25.1** (Geometric deficit and complexity)**.** *For $u = u_p$ set $e = A_p(L, R) - S_p(G)$. There are constants $c, C > 0$ depending only on $p, L, R$ such that every maximizer $x$ of $u$ and every shortest path $\gamma$ from $D$ to $x$ satisfy*

$$R - |\gamma| \leq Ce, \qquad \int_{G \setminus \gamma} \operatorname{dist}(y, x) \, \mathrm{d}y \leq Ce, \qquad e \geq \frac{c}{k_G}.$$

*The same conclusions hold for $u = u_z$, $e = F_{L,R}(z) - J_G(z)$, with constants depending only on $z, L, R$. At fixed parameters the power 1 of the inverse complexity is sharp. The order $e$ in the moment estimate is attained up to a constant.*

A broom below has a positive stem from its only Dirichlet mark to a junction and a finite number of positive arms with Neumann tips. An arm cap imposes no lower bound on arm length.

**Theorem 25.2** (Exact broom optimizer)**.** *Let $n \geq 2$ satisfy $\delta/(n-1) < R$. At fixed $L, R$, among brooms with at most $n$ arms both $S_p$ and $J_G(z)$ are maximized uniquely, up to a marked isometry, by the equal $n$-arm broom $G_n$. Its arms have length $b = \delta/(n-1)$ and its stem has length $R - b$. It has $k_{G_n} = n + 1$, and as $n \to \infty$,*

$$A_p(L,R) - S_p(G_n) = \frac{\delta^{q+1}}{n} + O(n^{-\min(2,q)}),$$
$$F_{L,R}(z) - J_{G_n}(z) = \frac{\delta\, v_R'(R)^2}{n} + O(n^{-2}),$$

*where $v_R$ is the endpoint-mass resolvent profile. Its endpoint slope is $\delta$ at $z = 0$, and for $z > 0$ it is $\delta\ \mathrm{sech}(sR)/[1 + s\delta \tanh(sR)]$, $s = \sqrt{z}$.*

# 26 Variational deficit identities

Write $x$ for a maximum point of the solution in question, $r = |\gamma| \leq R$ for the length of a shortest path from $D$ to $x$, and $M = u(x)$. Subdivision at $x$ is harmless. Use arclength increasing from $D$ on $\gamma$ and write $g = u|_\gamma$.

## 26.1 Torsion

The admissible functions are continuous, edgewise $W^{1,p}$, and zero on $D$. Testing the weak equation with $u_p$ gives $S_p(G) = \int_G |u_p'|^p$. Equivalently $S_p(G)$ is the maximum of

$$\Phi(f) = q \int_G f - \frac{1}{p-1} \int_G |f'|^p.$$

This is the normalization of Section 2. For an anchored function on $[0, r]$, Fubini gives $\int_0^r g + (L-r)g(r) = \int_0^r (L-s)g'(s)\,\mathrm{d}s$. The maximizing slope in the resulting path functional is $v_r'(s) = (L-s)^{q-1}$. Set

$$B_p(a,b) = |a|^p - |b|^p - p|b|^{p-2}b(a-b).$$

Since $|v_r'|^{p-2}v_r' = L - s$, retaining all terms in the path comparison gives

$$\begin{aligned} e = &\int_r^R (L-s)^q\,\mathrm{d}s + \frac{1}{p-1}\int_0^r B_p(g', v_r')\,\mathrm{d}s \\ &+ \frac{1}{p-1}\int_{G\setminus\gamma} |u'|^p + q\int_{G\setminus\gamma}(M-u). \end{aligned} \tag{77}$$

Every term is nonnegative. The complement has positive length, so $e = 0$ would force a constant on an open edge interval there. This contradicts the torsion equation. In particular $e > 0$.

The first term gives $R - r \leq \delta^{-q} e$. Put $d = \delta^{q-1}$. Where $|g'| < d/2$, the Bregman term is at least $m_p d^p$, where $m_p = \min_{|t|\leq 1/2}(|t|^p - 1 - p(t-1)) > 0$. Thus the low-speed subset of $\gamma$ has length at most $C_0 e$.

For any $y \notin D$, take a shortest path $\sigma$ to $y$ from $D$. Use the actual endpoint $u(y)$ in the path functional on $\sigma$. Its value is at most $A_p(L, |\sigma|) \leq A_p(L, R)$ by the same Fubini identity and convexity calculation. Adding back the graph complement gives

$$\int_{G\setminus\sigma} |u'|^p \leq (p-1)e + pL(M - u(y)). \tag{78}$$

### 26.2 Positive linear resolvent

It suffices to consider $z > 0$. The weak maximum principle gives $0 \le u_z \le 1/z$. With $w(t) = 2t - zt^2$, the variational value is $J_G(z) = \int_G w(u) - \int_G |u'|^2$. For the path functional the integral of $w$ includes the endpoint term $(L-r)w(g(r))$. Let $v_r$ be its maximizer. Completing the quadratic form gives

$$e = F_{L,R}(z) - F_{L,r}(z) + \int_\gamma |u' - v_r'|^2 + z\int_\gamma |u - v_r|^2 + z(L-r)|M - v_r(r)|^2 + \int_{G\setminus\gamma} |u'|^2 + \int_{G\setminus\gamma} [w(M) - w(u)]. \tag{79}$$

The last term is nonnegative since $w$ is strictly increasing on $[0, 1/z]$. The explicit formula in Section 3 yields

$$\partial_r F_{L,r}(z) = v_r'(r)^2, \qquad v_r'(r) = \frac{(L-r)\operatorname{sech}(sr)}{1 + s(L-r)\tanh(sr)}.$$

The positive endpoint slope is bounded below on $0 \le r \le R$. Also $v_r'' = zv_r - 1 \le 0$ by the interval maximum principle. Thus (79) gives $R - r \le Ce$ and, for a fixed $d > 0$, bounds the length of $\{|g'| < d/2\}$ by $C_0 e$. If $e = 0$ then $r = R$, $M = v_R(R) < 1/z$, and $u = M$ on the complement. This contradicts the resolvent equation on an open edge interval there.

On a shortest path $\sigma$ to $y$, place $u(y)$ at its endpoint atom. Its functional is at most $F_{L,|\sigma|}(z) \le F_{L,R}(z)$, so

$$\int_{G\setminus\sigma} |u'|^2 \le e + L[w(M) - w(u(y))]. \tag{80}$$

## 27 A second path and arclength growth

Consider the two cases together. Let $m = p$ for torsion and $m = 2$ for the resolvent, and use $h_y = M - u(y)$ or $h_y = w(M) - w(u(y))$, respectively. In both cases the first deficit identity gives $R - r \le Ce$, $\int_{G\setminus\gamma} h_y \, \mathrm{d}y \le Ce$, and a low-speed set on $\gamma$ of length at most $C_0 e$ at fixed threshold $d/2 > 0$. The second path bound is $\int_{G\setminus\sigma} |u'|^m \le C_1(e + h_y)$.

For small enough $e$ one has $r \ge R/2$. Choose a fixed $B > C_0$ with $(d/2)^m(B - C_0) > C_1$. If $B(e + h_y) \le R/2$, the last $B(e + h_y)$ of $\gamma$ must intersect $\sigma$ in positive arclength. Otherwise its high-speed portion alone would cost at least

$$(d/2)^m[B(e + h_y) - C_0 e] > C_1(e + h_y)$$

in derivative energy off $\sigma$. At a shared point the path coordinate on each geodesic is its distance to $D$, since a shorter anchored prefix would shorten the whole path. The two terminal paths then give $\operatorname{dist}(y, x) \le 2B(e + h_y) + (R - r)$. For larger $e$ or larger $e + h_y$, the diameter bound $L$ gives $\operatorname{dist}(y, x) \le C(e + h_y)$ as well. Integration off $\gamma$ proves the moment estimate. This adapts the second-geodesic argument for first eigenvalues in Section 19.

For any center $x$, the ball of radius $t$ has arclength at most $k_G t$. This is a local form of the radius bound of [17, Theorem 1]. To see it directly, cut at a regular positive distance level before the ball covers the graph. The closed ball is connected. Each outside component has $N_i \ge 1$ cutpoint leaves. If $N_i \ge 2$, its frontier contribution satisfies $N_i \le 2(N_i - 1)$. If $N_i = 1$ and the component is a tree, it contains an original leaf. Otherwise such a component has a cycle. Euler characteristic under gluing gives

$$b_1(G) = b_1(\text{ball}) + \sum_i b_1(\text{outside component } i) + \sum_i (N_i - 1).$$

Therefore the frontier count $\sum_i N_i$ is at most $k_G$. On each edge the distance function has finitely many affine pieces of slope 1 or $-1$, after subdividing at $x$ if needed. Integrating the frontier count proves the growth bound. In particular $k_G \le 2|E|$ for any finite edge presentation, since $b_1 = |E| - |V| + 1$ and $\ell(G) \le 2(|V| - 1)$.

The complement of $\gamma$ has length at least $\delta$. Layer cake and the growth bound now give

$$\int_{G\setminus\gamma} \operatorname{dist}(y, x)\,\mathrm{d}y \ge \int_0^{\delta/k_G} (\delta - k_G t)\,\mathrm{d}t = \frac{\delta^2}{2k_G}.$$

Combining the two moment bounds proves $e \ge c/k_G$. The equal-arm calculations below show sharpness.

# 28 Brooms under an arm cap

The exact equal-arm values are given in Proposition 2.3 and Section 3; their expansions give the first coefficients below. We compare arbitrary arm lengths under the cap. Let the stem length be $a > 0$ and the $m \le n$ arm lengths be $b_i > 0$, with $b = \max_i b_i$. Then $a = R - b$, $\sum_i b_i = \delta + b$, and $b \ge \delta/(n-1)$. Equality in the last inequality forces $m = n$ and all arms equal.

## 28.1 Torsion

The stem flux at arclength $s$ from its mark is $L - s$. The flux in arm $i$ is $b_i - t$ at arclength $t$ from the junction. Their integrated slopes solve the weak torsion problem by the Neumann and Kirchhoff conditions. Integrating the $q$th powers of the fluxes gives

$$\begin{aligned} A_p(L, R) - S_p(G) &= \int_\delta^{\delta+b} t^q\,\mathrm{d}t - \frac{\sum_i b_i^{q+1}}{q+1} \\ &\ge G(b) := \int_\delta^{\delta+b} t^q\,\mathrm{d}t - \frac{b^q(\delta+b)}{q+1}. \end{aligned}$$

The inequality follows from $b_i^q \le b^q$, with equality precisely for equal arms. The convex tangent inequality for $t^q$ at $b > 0$ gives

$$G'(b) = (\delta + b)^q - b^q - \frac{q}{q+1}\delta b^{q-1} > 0.$$

Thus $b = \delta/(n-1)$ gives the exact unique optimum. For that broom, $G(b) = \delta^q b + O(b^{\min(2,q)})$, which proves the stated coefficient. At $p = 2$ the exact deficit is $\delta^2 b + \frac{2}{3}\delta b^2$.

## 28.2 Positive linear resolvent

Take $z > 0$, $s = \sqrt{z}$. An arm with junction value $c$ has graph resolvent profile

$$u_i(t) = \frac{1}{z} + \left(c - \frac{1}{z}\right)\frac{\cosh(s(b_i - t))}{\cosh(sb_i)}.$$

Its outgoing slope there is $(1 - zc)\tanh(sb_i)/s$. Set $\alpha = \sum_i \tanh(sb_i)$ and $t = \tanh(sa)$. Solving the stem equation with the Kirchhoff flux at the junction gives $u'_{\mathrm{stem}}(0) = A/s$ for $A = (t + \alpha)/(1 + t\alpha)$. Integration of the equation over the graph gives

$$J_G(z) = \frac{L}{z} - \frac{A}{z^{3/2}}.$$

For fixed $t$, $A$ is strictly increasing in $\alpha$. Concavity of tanh gives

$$\alpha \geq (\delta + b)\,\frac{\tanh(sb)}{b},$$

with equality precisely when all positive arms have length $b$.

Put $x = sb$, $D_0 = s\delta$, and let $\alpha_*(x) = (1 + D_0/x)\tanh x$ be this lower value, with $t = \tanh(sR - x)$. Differentiation shows that $A$ along this curve has derivative with the sign of

$$\alpha_*' + \alpha_*^2 - 1 = \frac{D_0}{x^2}\left[x(1 + \tanh^2 x) - \tanh x\right] + D_0^2\left(\frac{\tanh x}{x}\right)^2 > 0.$$

Here $x > \tanh x$ for $x > 0$. The minimum of $A$ therefore occurs at the smallest feasible $b = \delta/(n-1)$, and both equalities show uniqueness. For the equal family,

$$n\tanh(sb) = s(\delta + b) + O(b^2), \qquad nb = \delta + b.$$

Substituting $s(\delta+b)$ for $\alpha$ in the stem formula gives exactly $F_{L,R-b}(z)$. The derivative of that model then gives $F_{L,R}(z) - J_{G_n}(z) = v_R'(R)^2 b + O(b^2)$. This proves the coefficient for $z > 0$. The torsion calculation at $p = 2$ gives the result at $z = 0$.

In the equal family the maximizers are the arm tips. For torsion this follows from the flux profiles. For $z > 0$ the stem formula gives $1 - zc = \operatorname{sech}(sa)/[1 + \alpha\tanh(sa)] > 0$. Thus the arms are increasing and their positive junction slopes make the concave stem increasing. For a tip $x$, the distance from $x$ to an off-path arm point at depth $t$ from the junction is $b + t$. The off-path moment equals $(n-1)\int_0^b (b+t)\,\mathrm{d}t = \frac{3}{2}\delta b$. This proves the moment order in Theorem 25.1. The two nonzero leading deficits with $k_{G_n} = n + 1$ prove the optimal inverse-complexity power. The exact coefficient of the optimal deficit over unrestricted graph topologies remains open.

## 29 Exact finite-cap torsional rigidity

Fix $p > 1$, $q = p/(p-1)$, and $0 < R < L$, writing $\delta = L - R$. For a finite compact connected metric graph $G$ with nonempty Dirichlet mark set $D$, let $u$ solve $-\Delta_p u = 1$ with the natural Kirchhoff flux at vertices outside $D$. The unpowered torsional rigidity is $S_p(G) = \int_G u = \int_G |u'|^p$. We use arclength, and the inradius is the maximum distance to $D$. The unrestricted sharp supremum at fixed $L, R$ is

$$A_p(L,R) = \int_0^R (L - s)^q\,ds$$

by Section 2. The earlier finite-complexity result Sections 25–28 gives the sharp inverse power of complexity and an exact arm-cap comparison within brooms. We prove a comparison across all rooted tree topologies and then solve both general capped problems. Length-only torsion and surgery comparisons appear in [24, 5].

**Theorem 29.1.** *Let a finite metric tree of length $L$ and height $R$ have only its root marked, and at most $n$ nonroot leaves. Then $L \leq nR$ and $n \geq 2$. Put $b = (L-R)/(n-1)$ and $a = R - b$. The unique maximum up to rooted isometry is the broom $B_n$ with stem length $a$ and $n$ arms of length $b$, allowing $a = 0$. Its value is*

$$T_n(L,R) = S_p(B_n) = \frac{L^{q+1} - (nb)^{q+1} + nb^{q+1}}{q+1}. \tag{81}$$

*Proof.* Orient every edge away from the root. At an interior point $x$ of an edge let $M(x)$ be the arclength of the descendant tree. Integrate the equation over that tree, using zero flux at its leaves and Kirchhoff at internal vertices. The forward flux is $M(x)$ and $u' = M(x)^{1/(p-1)}$. In particular $S_p = \int M(x)^q\,dx$. Set $\mu(t) = |\{x : M(x) > t\}|$. Then

$$(L - nt)_+ \le \mu(t) \le (L - t)_+, \qquad t \ge 0. \tag{82}$$

For the lower bound the inactive region $M \le t$ is a union of at most $n$ maximal descendant branches, each with at least one distinct leaf and arclength at most $t$. If the active set is nonempty in arclength, follow an active ray to a distal end of the active region. If it ends in the interior of an edge, its following branch has length $t$ and is inactive. If it ends at a vertex, all outgoing branches there are inactive and their combined mass is at least $t$, the limit of the incoming active mass. Thus at least $t$ length is inactive, giving the upper bound. Vertex thresholds can be omitted from the integrals.

For the proposed broom its distribution $\mu_b$ equals $L - nt$ on $0 \le t \le b$, is constant at $a$ on $b \le t \le nb$, and equals $(L - t)_+$ thereafter. By (82), $\mu_b - \mu$ is nonpositive in the first range and nonnegative in the last. On the middle range it is nondecreasing, since $\mu$ is nonincreasing. Thus it can change sign only from negative to positive.

Fubini on root paths gives

$$\int_0^\infty \mu(t)\,dt = \int_G M(x)\,dx = \int_G \operatorname{dist}(y, D)\,dy.$$

At each regular depth $s \in (0, R)$ the number $N(s)$ of edge points there satisfies $1 \le N(s) \le n$, and $\int_0^R N(s)\,ds = L$. This gives feasibility. The distance integral $\int_0^R sN(s)\,ds$ is maximized by the step allocation of $N(s)$ for the broom, namely one on $(0, a)$ and $n$ on $(a, R)$. This follows by moving the excess count above one to the largest depths, or by integrating the signed difference against $s - a$. Thus $\int(\mu_b - \mu) \ge 0$. The sign change and the increasing weight $qt^{q-1}$ give $\int qt^{q-1}(\mu_b - \mu) \ge 0$: subtract the weight at the sign-change point and use the unweighted integral for the constant term. The same applies if the difference has one sign. Layer cake proves the torsion comparison. The weight is strictly increasing, so equality requires the distance comparison to be exact. The step allocation for $N(s)$ is unique. With one ray below $a$ and $n$ above $a$, the leaf cap prohibits any further branching or early termination above $a$. This identifies $B_n$ up to rooted isometry. Formula (81) is the direct integration of the stem and arm fluxes. □

## 30 Distance-preserving graph cuts

Let $\rho(x) = \operatorname{dist}(x, D)$. For an edge of length $\ell$ with endpoints $v, w$, its profile is $\min\{\rho(v) + s, \rho(w) + \ell - s\}$, because any route to its interior enters through an endpoint. Split every interior peak into two separate endpoints. Orient the resulting segments by increasing $\rho$. Each original unmarked vertex has at least one incoming edge by a shortest path from $D$. Keep one incoming attachment there and detach the other incoming endpoints to form leaves. All resulting nonroot vertices have unique predecessors of smaller $\rho$. Hence the pieces form a rooted forest with roots $D$. The retained predecessors preserve distance to the root, arclength and inradius. This is a rooted distance construction of the type used in [17, Theorem 1] for the radius–length inequality.

Every original torsion trial transfers to the cut forest, with only continuity conditions relaxed. Thus its total torsion is at least that of $G$. Split each Dirichlet root into a separate copy for every outgoing branch. The value is unchanged since all the copies have trace zero. The resulting separate branches are rooted trees with degree-one roots.

Write $k_G = 2b_1(G) + \ell(G)$, where $\ell(G)$ is the original number of degree-one vertices. Each separate branch has no more than $k_G - 1$ nonroot leaves: its root has degree one, so its nonroot leaf count is one plus the sum of excess degrees $(\deg(v) - 2)_+$ of its nonroot vertices. Nonzero terms come from distinct original vertices with degrees that did not increase. The total original excess is $k_G - 2$, by the degree identity. When $G$ has $E \geq 2$ edges, the branch has at most $2E - 2$ nonroot leaves as well. In fact for $V \geq 2$ the original graph has at least one nonleaf, so $\ell(G) \leq V - 1$ and $k_G - 2 = 2E - 2V + \ell(G) \leq 2E - 3$. For $V = 1$, only the original mark exists and the separated branches are paths. A single-edge graph also has only path branches and satisfies the stated bound for any ambient cap of at least two edges.

## 31 Global complexity and edge caps

Write $c_n = (n^{q+1} - n)/(n-1)^{q+1}$, and let $L_n^* \in (R, nR)$ be the unique solution of

$$c_n \left(1 - \frac{R}{L_n^*}\right)^q \left(q + \frac{R}{L_n^*}\right) = q. \tag{83}$$

**Theorem 31.1.** *For $n = K - 1 \geq 2$ and $R < L \leq L_n^*$, the exact maximum of $S_p(G)$ under $k_G \leq K$ is $T_{K-1}(L, R)$, for arbitrary graph topologies and nonempty $D$. All extremizers are obtained from $B_{K-1}$ by identifying disjoint pairs of its arm tips, retaining only the rooted stem end as a mark. For $n = 2J - 2 \geq 2$ and $R < L \leq L_n^*$, the corresponding maximum over graphs admitting a presentation with at most $J$ edges is $T_{2J-2}(L, R)$. The unique maximizer up to marked metric isometry is the rooted stem with $J - 1$ equal loops at its other endpoint.*

*Moreover $L_n^*/(nR) \to (q-1)/q$. In particular, at fixed $L, R, p$ these formulas hold for all sufficiently large caps. The deficits of the optimal values from $A_p(L, R)$ satisfy*

$$\lim_{K\to\infty} K\Delta_K^{\mathrm{top}} = \delta^{q+1}, \qquad \lim_{J\to\infty} J\Delta_J^{\mathrm{edge}} = \tfrac{1}{2}\delta^{q+1}.$$

*Proof.* For $0 < x \leq nR$, set

$$H_n(x) = \begin{cases} x^{q+1}/(q+1), & x \leq R, \\ T_n(x, R), & x > R. \end{cases}$$

This is the sharp bound for a branch of length $x$, height at most $R$ and at most $n$ leaves away from its root. For $x \leq R$ use the upper bound in (82) and the interval witness. For $x > R$, apply Theorem 29.1 at the branch's actual height and use $\partial_R T_n(x, R) = c_n(x - R)^q > 0$ at fixed length.

The ratio $H_n(x)/x$ is strictly increasing on $(0, R]$. For $x > R$ the sign of its derivative is the sign of

$$q - c_n y^q (q + 1 - y), \qquad y = 1 - R/x.$$

The subtracted term strictly increases with $y$. At $x = nR$ the left derivative of $H_n$ is $R^q(n - n^q)/(n-1) < 0$, so the derivative of the ratio is negative there. Its sign thus changes exactly once, at the unique $L_n^*$ in (83). In particular the ratio strictly increases on $(0, L_n^*]$.

Make the cuts and split the marked roots as in Section 2. The branch lengths $x_i > 0$ sum to $L$. Each branch has height at most $R$ and at most $n$ leaves away from its root under the corresponding cap, so

$$S_p(G) \leq \sum_i H_n(x_i) \leq \sum_i \frac{x_i}{L} H_n(L) = T_n(L, R).$$

The last inequality is strict when there is more than one branch, by $L \leq L_n^*$.

The rooted broom attains the topological cap. Pairing tips makes loops without changing the torsion: the symmetric profile descends and satisfies Kirchhoff at the identified tips, and distances to the root still equal depth. Pairing every tip gives $J$ edges in the edge-capped case. All these graphs attain the respective upper bounds.

For equality there must be exactly one rooted branch. The original marks had their degree preserved by cutting, so there is only one and it has degree one. The cut tree is the exact broom by Theorem 29.1. All reidentifications are among its tips at depth $R$. Indeed a cut at an edge peak produces two leaves. At an original vertex with detached endpoints, those endpoints are leaves and so have depth $R$. The retained vertex has the same depth and cannot have an outgoing segment, so it too is a tip.

Identifying a group of $m \geq 2$ tips removes $m$ leaves and adds $m-1$ cycles. Thus it increases $k_G$ by $m-2$, and the topological cap allows only pairs. A pair saves one edge by suppressing its unmarked degree-two join, while a group of three or more tips saves none. The edge cap of $J$ requires saving $J-1$ of the $2J-1$ broom edges, so every tip is paired. This proves both equality classes. To evaluate (83), set $s_n = R/L_n^*$. Since $c_n = 1 + (q+1)/n + o(n^{-1})$ and $(1-s)^q(q+s) < q$ for $0 < s < 1$, we have $s_n \to 0$. The derivative of this last function at zero is $1-q^2$, so its expansion in (83) gives $ns_n \to q/(q-1)$. This proves the asserted cap range at fixed $L, R, p$. Expanding (81) gives the two deficit limits. □

**High-density obstruction.** Define $\lambda_n = c_n^{1/q}/(c_n^{1/q}-1)$, which lies in $(1,n)$ by the opposite signs of $\partial_L T_n$ at $R$ and $nR$. For $\lambda_n R \leq L \leq nR$ take an $n$-arm broom of length $L-\varepsilon$ and height $R$, and attach a short interval of length $\varepsilon$ at its only mark. The mark now has degree two and the tree still has $n+1$ leaves. Its inradius is $R$ and its rigidity is

$$T_n(L-\varepsilon, R) + \frac{\varepsilon^{q+1}}{q+1}.$$

This exceeds $T_n(L,R)$ for small positive $\varepsilon$: the first variation is positive for $L > \lambda_n R$, and at equality it is zero with $\partial_L^2 T_n = -qRL^{q-1}/(L-R) < 0$. The construction thus beats the single-branch value at topological cap $n+1$. Since $\lambda_n/n \to q/(q+1)$, the explicit range in Theorem 31.1 and this obstruction both occur at length proportional to the cap.

# 32 Sharp finite-complexity coefficients for eigenvalues and resolvents

Let $G$ be a finite compact connected metric graph with a nonempty set $D$ of Dirichlet vertices, length $L$, and inradius $R = \max_{x \in G} \operatorname{dist}(x, D)$, where $0 < R < L$. Set $\delta = L - R$ and $q = p/(p-1)$. The reference model is the interval $[0, R]$ with measure $ds + \delta\, \delta_R$ and Dirichlet condition at zero. Its energy is $\int_0^R |u'|^p\, ds$. Write $\lambda_*$ for its first $p$-eigenvalue, $U_*$ for its positive model-$L^p$-normalized first eigenfunction, and $m_* = U_*(R)$. For $z > 0$ put $J_G(z) = \langle 1, (A_G + z)^{-1} 1 \rangle$, where $A_G$ is the nonnegative Dirichlet–Kirchhoff Laplacian. Let $F(z)$ and $v_R$ be the corresponding reference quadratic resolvent value and solution. Set

$$c_z = 1 - z v_R(R) = \frac{\operatorname{sech}(\sqrt{z}\, R)}{1 + \sqrt{z}\, \delta \tanh(\sqrt{z}\, R)}.$$

The sharp unrestricted eigenvalue infimum is $\lambda_*$ and the sharp unrestricted resolvent supremum is $F(z)$ Sections 3, 11–12 and 25–28. The exact equal-broom coefficients and the folded eigenvalue witnesses are known Sections 17–19 and 25–28. They give the sharp inverse power, and the exact global coefficients require a matching bound over all graph topologies. Kennedy treats linear extremal sequences and suggests stowers as an alternative to stars [12].

For a topological cap $K$, require $k_G = 2b_1(G) + \ell(G) \le K$, where $\ell(G)$ counts degree-one vertices. For an edge cap $J$, require a presentation with at most $J$ edges. Let $\Lambda$ and $\Delta(z)$ respectively denote the eigenvalue infimum and the deficit of the resolvent supremum from $F(z)$ under the chosen cap. Set $n = K - 1$ or $n = 2J - 2$, respectively.

**Theorem 32.1.** *As either cap tends to infinity at fixed $L, R, p, z$,*

$$\Lambda - \lambda_* = \frac{(p-1)\lambda_*^q m_*^p \delta^{q+1}}{n} + o(n^{-1}),$$
$$\Delta(z) = \frac{c_z^2 \delta^3}{n} + O(n^{-3/2}).$$

*The equal $n$-arm broom and its full pair fold attain the leading terms for the topological and edge caps, respectively.*

## 33 Cut trees and the torsion input

We use two facts from Sections 29–31. Distance-preserving cuts relax the form to a rooted forest of the same length and inradius. After splitting marked roots into one copy per outgoing branch, each branch has at most $n$ nonroot leaves under the chosen cap. Identifying all copies at one common marked root gives a tree $T$ with the same energy and norm as the direct sum. Its first eigenvalue can only decrease and its resolvent value can only increase in this comparison. The geometric shortest-path reduction has an antecedent in [17, Theorem 1].

For a rooted tree of length $X$, height $h$ and at most $n$ nonroot leaves, the exact upper bound for unit-forcing $p$-torsion is $T_n(X, h)$, attained by the equal broom. Its arm length is $b = (X - h)/(n - 1)$. In the interior range $0 < h < X$, write

$$A_p(X, h) = \frac{X^{q+1} - (X-h)^{q+1}}{q+1}, \qquad T_n(X, h) = \frac{X^{q+1} - (nb)^{q+1} + nb^{q+1}}{q+1}.$$

Thus $n(A_p(X, h) - T_n(X, h)) \to (X - h)^{q+1}$, uniformly on compact subsets of $0 < h < X$. At $p = 2$ the error in the unscaled gap is $O(n^{-2})$. The same source proves the global bound $J_T(0) \le T_n(L, R)$ for sufficiently large $n$ using only the branchwise leaf cap. This form applies to the reidentified tree even if its total leaf count exceeds $n$.

## 34 Weighted descendant losses and the first eigenvalue

Choose any rooted path $\gamma$ of length $R$ in $T$. Assign $a(x)$ to be depth on $\gamma$ and $R$ elsewhere. Composition with $a$ embeds the endpoint-mass model isometrically into $L^p(T)$. For the true rooted Dirichlet Sobolev embedding, equipped with energy norm $\|u'\|_p$, and for the model embedding, respectively, write

$$\mathcal{S}_T(f) = \int_T \left( \int_{\mathrm{desc}(x)} f \right)^q dx, \qquad \mathcal{S}_*(f) = \int_0^R \left( \int_{\{a > t\}} f \right)^q dt \quad (f \ge 0).$$

These are the $q$-powers of the dual norms of the forcing $f$. The integration operator is the unweighted rooted Hardy operator [25]. Fubini expresses $\int_T fu$ as the pairing of $u'$ and its descendant flux. Since any $L^p$ derivative is realized by a root integral, Hölder's inequality gives the exact norm. The model is proved in the same way with its assigned coordinates and endpoint atom.

At regular depth $t$ the actual descendant regions $D_i(t)$ partition the portion deeper than $t$, which is contained in $\{a > t\}$. By edge coarea the loss is

$$\mathcal{S}_*(f) - \mathcal{S}_T(f) = \int_0^R \left[ \left( \int_{\{a>t\}} f \right)^q - \sum_i \left( \int_{D_i(t)} f \right)^q \right] dt. \tag{84}$$

At each depth the integrand is nonnegative and nondecreasing in $f \geq 0$. Indeed, write the first mass as $b + \sum a_i$, where $a_i$ are the actual masses and $b$ is the extra assigned mass. Every partial derivative of $(b + \sum a_i)^q - \sum a_i^q$ in these nonnegative coordinates is nonnegative.

For a nonnegative normalized first eigenfunction $u_T$ of the tree put $f_T = u_T^{p-1}$. It has unit $L^q$ norm and attains the dual embedding norm by pairing against $u_T$. Likewise $f_* = U_*(a(\cdot))^{p-1}$ attains the model dual norm. Thus, writing $\lambda_T$ for the tree eigenvalue, we have

$$\mathcal{S}_*(f_T) - \mathcal{S}_T(f_T) \leq \lambda_*^{-(q-1)} - \lambda_T^{-(q-1)} \leq \mathcal{S}_*(f_*) - \mathcal{S}_T(f_*). \tag{85}$$

For the equal broom, $f_* \leq m_*^{p-1}$ everywhere, so monotonicity of (84) bounds its right side by $m_*^p(A_p(L,R) - T_n(L,R))$. Its positive first eigenfunction is unique up to scaling [13, Theorem 3.3], since the marked stem end leaves a connected interior. Therefore it is invariant under arm permutations. Identifying all disjoint pairs of arm tips preserves its norm and energy. This trial and the cut comparison give the same eigenvalue for the fold. The topological witness has $k_G = n + 1$; the fully folded witness has $1 + n/2$ edges.

For the lower bound it suffices to take cap sequences with bounded scaled deficit. Cutting gives $\lambda_T \to \lambda_*$. Take $x$ attaining the maximum $M$ of a nonnegative normalized $u_T$, let $r$ be its depth, and write $g$ for the restriction to its root path with constant extension to $R$. The extended model mass is

$$B = \int_0^r g^p + (L - r)M^p \geq 1, \qquad \lambda_* B \leq \int_0^r |g'|^p \leq \lambda_T.$$

It follows that $B \to 1$ and the energy off this path is $o(1)$. Compactness in $C[0,R]$ and uniqueness of the positive normalized model minimizer give $g \to U_*$ uniformly and $M \to m_*$. Indeed every convergent subsequence has model mass one and energy at most $\lambda_*$ by lower semicontinuity. The model is strictly increasing, since its endpoint flux is positive and the flux decreases along the interval. It follows that $r \to R$.

Fix a small $\eta > 0$ and put $s = R - \eta$. Let $C$ be the full descendant tree rooted at the point of depth $s$ on the path to $x$. Off the path, the difference between $u_T$ and its value at the path attachment is uniformly $o(1)$, by Hölder's inequality and the vanishing off-path energy. Attachments below $s$ have a fixed gap below $M$ by strict increase of $U_*$. Their off-path arclength therefore tends to zero, since $\int_{\text{off path}} (M^p - u_T^p) = B - 1 \to 0$. It follows that the length and height of $C$ satisfy $L_C \to \delta + \eta$ and $h_C \to \eta$, using $r - s \leq h_C \leq \eta$. Each point of $C$ is within distance $2\eta$ of $x$. By the global Hölder estimate, there is a constant $v_\eta \geq 0$ with $v_\eta \to m_*$ as $\eta \downarrow 0$ such that $u_T \geq v_\eta$ on $C$ eventually for each fixed sufficiently small $\eta$.

In (84) restrict to $t \geq s$ and replace $f_T$ by $v_\eta^{p-1}\mathbf{1}_C$. The actual integral is $v_\eta^p S_p(C)$. The model integral is at least $v_\eta^p A_p(L_C, h_C)$. If $\gamma$ enters $C$, it contains a longest path of $C$, and the model term above $s$ is exactly the local endpoint-mass formula. If $\gamma$ stays outside $C$ except possibly at its root, all its assigned depths there are $R$, which dominate the local model coordinates after shifting by $s$. All nonroot leaves of $C$ are leaves of one original root branch. Hence

$$\mathcal{S}_*(f_T) - \mathcal{S}_T(f_T) \geq v_\eta^p \big(A_p(L_C, h_C) - T_n(L_C, h_C)\big).$$

Multiplying by $n$, taking the lower limit, and then taking $\eta \downarrow 0$ gives the sharp dual coefficient $m_*^p \delta^{q+1}$. The comparison to the original graph preserves this lower bound. The derivative of $\lambda^{-(q-1)}$ now gives the factor $(p-1)\lambda_*^q$ in Theorem 32.1.

## 35 Positive resolvent and the Green difference

Use the same cut tree $T$ and assigned coordinates $a$. On $L^2(T)$ let $Q$ denote the true Green operator. Its kernel is the shared length of the root paths to $x, y$, by descendant flux integration. The model has kernel $Q_*(x,y) = \min\{a(x), a(y)\}$. This kernel is nonnegative as an operator, by the Gram representation using the functions $\mathbf{1}_{[0,a(x)]}$, and its action on functions of $a$ is the true Green operator with the endpoint measure. We have $Q_*(x,y) - Q(x,y) \geq 0$ pointwise, since each assigned coordinate is at least the actual depth.

For $B$ a bounded nonnegative self-adjoint operator put $\Phi_z(B) = B(I+zB)^{-1}$. The inverse difference identity gives

$$\Phi_z(Q_*) - \Phi_z(Q) = (I + zQ_*)^{-1}(Q_* - Q)(I + zQ)^{-1}.$$

Let $f_* = (I + zQ_*)^{-1}1$, $f = (I + zQ)^{-1}1$, and $e_T = F(z) - J_T(z)$. Then

$$e_T = \langle f_*, (Q_* - Q)f \rangle.$$

Here $f_* \geq c_z$ along the model, with equality at the endpoint. Also $f = 1 - zu_T \geq 0$ for the tree solution $u_T$ by the maximum principle.

At a tree maximizer of $u_T$, write $r$ for its depth and $M$ for its value. The exact path deficit identity in Sections 25–27 gives $R - r = O(e_T)$ and $z(L-r)|M - v_r(r)|^2 \leq e_T$, where $v_r$ is the model resolvent for length $r$ and atom $L-r$. The smooth endpoint profile and $L - r \geq \delta$ imply $M \leq v_R(R) + O(\sqrt{e_T})$. Thus $f \geq c_z - O(\sqrt{e_T})$ everywhere. By the pointwise nonnegative kernel, for small deficits,

$$e_T \geq (c_z^2 - O(\sqrt{e_T}))(A_2(L,R) - J_T(0)).$$

The branchwise torsion comparison bounds the last factor below by $\delta^3/n + O(n^{-2})$. The exact equal-arm resolvent witness in Section 28 has deficit $c_z^2\delta^3/n + O(n^{-2})$, using $v_R'(R) = c_z\delta$. Its solution descends under pair folding by equal tip values and zero tip flux, so the fully folded graph supplies the edge witness. For a uniform lower bound it suffices to consider original graph deficits $e_G \leq C/n$ for a fixed $C > c_z^2\delta^3$. Since $e_T \leq e_G$, the preceding estimate yields $e_G \geq c_z^2\delta^3/n - O(n^{-3/2})$, as claimed.

## 36 Sharp finite-complexity coefficients for heat content

Let $G$ be a finite compact connected metric graph with a nonempty Dirichlet vertex set $D$, length $L$, and inradius $R = \max_{x \in G} \operatorname{dist}(x, D)$, with $0 < R < L$. Write $A_G$ for the nonnegative Dirichlet–Kirchhoff Laplacian, $H_G(t) = \langle 1, e^{-tA_G}1 \rangle$, and $\delta = L - R$. The reference interval has length $R$, is Dirichlet at zero, and carries arclength and an atom of mass $\delta$ at $R$. Its heat content $H_*(t)$ is the sharp unrestricted supremum Sections 13–15. Write $h_*(a,t)$ for its absorption-time density from $a > 0$. Set

$$\mathcal{C}(t) = \int_0^t h_*(R, t-s)h_*(R,s)\,ds.$$

It is strictly positive. In Proposition 7.2, the derivative of the reference heat content in $R$ at fixed $L$ is $\delta^2\mathcal{C}(t)$.

**Theorem 36.1.** *The optimal deficit from $H_*(t)$ at each fixed $t > 0$ is $\delta^3\mathcal{C}(t)/n + O(n^{-3/2})$, as the cap tends to infinity, uniformly on compact positive time intervals. Here $n = K - 1$ for the topological cap $2b_1(G) + \ell(G) \leq K$, with $\ell(G)$ the number of degree-one vertices. Here $n = 2J - 2$ for graphs admitting a presentation with at most $J$ edges. Equal-arm brooms and their fully folded versions attain the respective leading coefficients.*

*Moreover, if $e_t = H_*(t) - H_G(t)$ and $e = A_2(L,R) - \langle 1, A_G^{-1}1\rangle$, then $ce \le e_t \le Ce$ for all graphs in the stated class and all $t$ in any fixed compact positive time interval. Here $A_2(L,R)$ is the reference zero-mass torsion.*

The interpretation of graph heat content in terms of hitting times is classical. The equilateral-graph comparison [1, Eq. (4.12)] uses a discrete return-time expansion, and [2] gives small-time expansions and surgery rules. The reference convolutions in Section 7 already describe the parameter derivative on a fixed endpoint model. The comparison below integrates convolutions against an explicit nonnegative kernel for a general graph. The corresponding positive-resolvent pairing on a cut tree appears in Section 35.

## 37 A positive time-domain comparison

Choose a shortest marked path $\gamma$ of length $R$. Assign $a(x) = \operatorname{dist}(x, D)$ on $\gamma$ and $a(x) = R$ off it. The Green kernel of the endpoint model, planted on the same arclength space, is $Q_*(x,y) = \min(a(x), a(y))$. Let $Q_G$ be the true Dirichlet Green kernel. Its diagonal is the squared norm of point evaluation in the energy space, so

$$Q_G(y,y) = \sup_{u \ne 0} \frac{|u(y)|^2}{\int_G |u'|^2} \le \operatorname{dist}(y, D).$$

For a fixed source $y$, its point evaluation representer is harmonic away from $y$ and zero on $D$. The maximum principle and symmetry give $0 \le Q_G(x,y) \le \min(Q_G(x,x), Q_G(y,y))$. Therefore $B(x,y) = Q_*(x,y) - Q_G(x,y) \ge 0$ pointwise.

Let $\nu_y$ be the probability law of absorption times from $y$, specified by survival $e^{-tA_G}1(y)$. This function decreases from one to zero for interior points away from $D$. The model laws $\nu_a^*$ are defined analogously. Their Laplace transforms at $z > 0$ are $f_z = (I + zQ_G)^{-1}1$ and $f_{*,z} = (I + zQ_*)^{-1}1$ in the planted coordinates. Indeed integration of survival gives $f_z = 1 - z(A_G + z)^{-1}1$. On $L^2(G)$ the resolvent difference of the two Green operators satisfies

$$Q_*(I + zQ_*)^{-1} - Q_G(I + zQ_G)^{-1} = (I + zQ_*)^{-1}(Q_* - Q_G)(I + zQ_G)^{-1}.$$

Pair both sides with one and invert the scalar Laplace transform. Uniqueness for finite measures gives

$$(H_*(t) - H_G(t))\, dt = \iint_G B(x,y)\, (\nu^*_{a(x)} * \nu_y)(dt)\, dx\, dy. \tag{86}$$

Both heat contents are integrable, with torsion as their integrals. The model inverse has a zero action on the orthogonal complement of functions of $a$. The constant function lies in the model subspace, so this complement does not enter the equation.

For every $\epsilon > 0$, $h_*(a,t)$ extends jointly continuously to $a \in [\epsilon, R]$ and all $t \in \mathbb{R}$ by zero for $t \le 0$. Indeed the transform at $\Re z > 0$ is, by the endpoint ODE,

$$f_*(z,a) = \frac{\cosh(\sqrt{z}(R-a)) + \sqrt{z}\,\delta \sinh(\sqrt{z}(R-a))}{\cosh(\sqrt{z}R) + \sqrt{z}\,\delta \sinh(\sqrt{z}R)}.$$

On a fixed positive vertical line the quotient decays exponentially in $\sqrt{|\operatorname{Im} z|}$, uniformly in $a \ge \epsilon$. The denominator has no zeros there by the positive reference operator. Fourier inversion of the exponentially weighted probability gives a jointly continuous density, and unweighting preserves continuity. Positivity at every $a > 0, t > 0$ follows by evolving the nonnegative, nonzero time derivative of the absorption probability from $t/2$ to $t$. It is nonzero by the spectral derivative of total survival. Interior positivity follows from the strong maximum

principle. At the atom, vanishing at positive time would force its time derivative to vanish by nonnegativity, and the atom boundary condition would force zero spatial derivative, against the Hopf lemma.

In particular, restricting the right side of (86) to a product set of $x, y$ where $a(x) \geq \epsilon$ gives a continuous density

$$\iint_{\text{restriction}} B(x,y) \int h_*(a(x), t-s)\, \nu_y(ds)\, dx\, dy. \tag{87}$$

By positivity and continuity this is a lower bound on the heat difference at each $t > 0$, with equality when $B$ vanishes outside the restriction.

## 38 A quantitative torsion comparison

Let $e_t = H_*(t) - H_G(t)$ and $e = \iint_G B = A_2(L,R) - \langle 1, Q_G 1\rangle$, where $A_2(L,R) = (L^3 - (L - R)^3)/3$. Fix a compact positive time interval for $t$. The length-only concentration comparison [A, Theorem 1.1] gives $\nu_y([0, t/2]) \geq 1 - \theta$ with a uniform $\theta < 1$. For $x$ off the planted path, $a(x) = R$, and the minimum of $h_*(R, t-s)$ for $0 \leq s \leq t/2$ is positive uniformly on the chosen time interval. Thus (87) gives

$$\int_{G\setminus\gamma} \int_G B(x,y)\, dy\, dx \leq C e_t. \tag{88}$$

Each unit Green column is 1-Lipschitz on the graph. Indeed it is piecewise affine and its aggregate flux out of any regular positive superlevel is one: integrate its equation with a single unit source over that set. All outgoing flux magnitudes are nonnegative, so every nonzero edge slope has absolute value at most one. For path points of depths $a \leq b$ it follows that $B(a,b) \leq B(b,b)$, since the Green column at $b$ changes by at most $b - a$. For every $x$ off the path the maximum principle gives $B(x,b) \geq B(b,b)$. Integrating first over $a \leq b$ and then using symmetry therefore gives

$$\iint_{\gamma^2} B \leq \frac{2R}{\delta} \int_{G\setminus\gamma} \int_\gamma B.$$

With (88) and symmetry this proves $e \leq C e_t$.

For the reverse comparison, on any time strip $s \geq s_0 > 0$ the graph absorption density is uniformly bounded. Indeed the energy point evaluation bound and spectral calculus give

$$|A_G e^{-sA_G} 1(y)| \leq \sqrt{L}\, \|A_G^{3/2} e^{-sA_G} 1\|_2 \leq C_L s^{-3/2}.$$

The same argument works on the interval with the atom. For a convolution density at time $t$, split according to whether the graph absorption time is at most $t/2$. On that portion the model density is uniformly bounded, while on the other portion the graph density is uniformly bounded. The opposite factor has total mass at most one. The continuous heat difference in (86) is therefore bounded above by $C \iint B = Ce$, as required.

## 39 The terminal part of a near-maximizer

Make the distance-preserving cuts of Section 30, keeping the planted path intact by choosing its incoming predecessors. Split marks with several outgoing branches and identify all the resulting roots at one Dirichlet point. The cut tree $T$ has the original arclength and inradius. Each of its root branches has at most $n$ nonroot leaves. Form inclusion gives $\langle v, Q_G v\rangle \leq \langle v, Q_T v\rangle$ for all $v$. Put $e_0 = A_2(L,R) - \langle 1, Q_T 1\rangle \leq e$.

Let $x_T$ maximize the tree torsion solution at depth $r$. The geodesic moment estimate of Sections 25–27 gives $R-r=O(e_0)$ and an off-path distance moment about $x_T$ of order $O(e_0)$. For small $e$, choose $\eta=\sqrt{e+n^{-2}}$. For $s=R-\eta<r$, let $C$ be the full descendant tree rooted at the point of depth $s$ on the path to $x_T$. Every off-path point outside $C$ lies at distance at least $r-s$ from $x_T$, so the length of these points is $O(e_0/\eta)$. The local length $L_C$, height $h_C$, and diameter consequently satisfy

$$L_C-h_C=\delta+O(e/\eta+e),\qquad r-s\le h_C\le\eta,\qquad \operatorname{diam} C\le 2\eta.$$

The diameter bound also holds in $G$.

The local exact torsion bound gives

$$\iint_{C^2} B\ge\langle\mathbf{1}_C,(Q_*-Q_T)\mathbf{1}_C\rangle\ge A_2(L_C,h_C)-T_n(L_C,h_C). \tag{89}$$

To verify the second inequality, the kernel of $Q_T$ is the shared root-path length. At regular depth the square of the reference assigned mass is at least the sum of the squares of actual descendant masses for forcing $\mathbf{1}_C$. Restrict their difference to depths above $s$. If the planted path enters $C$, its remaining portion is a longest local path, giving the exact local model there. If it misses $C$, the assigned depths of all points in $C$ equal $R$, and dominate the local assigned coordinates. The actual integral on the restricted depths is the tree torsion of $C$. Its number of nonroot leaves is at most $n$, so Theorem 29.1 of Sections 7 and 13 bounds it by $T_n$, the equal-arm value. With reserve $\delta_C=L_C-h_C$ and $b_C=\delta_C/(n-1)$, its exact gap is $\delta_C^2b_C+2\delta_Cb_C^2/3$. Thus the right side of (89) is at least $\delta^3/n-O((e/\eta+e)/n+n^{-2})$.

Fix a vertical line $\Re z=\sigma>0$. The complex absorption transforms are bounded by one. The inverse difference identity gives

$$\|f_z-f_{*,z}\|_1\le|z|(1+|z|/\sigma)\,e.$$

In fact the resolvent factor on the left is $(I+zQ_G)^{-1}=I-z(A_G+z)^{-1}$. The graph semigroup is an $L^1$ contraction, so its Laplace transform has norm at most $1/\sigma$. The remaining forcing is $Bf_{*,z}$, of $L^1$ norm at most $e$.

The identity $f_z=1-zQ_Gf_z$ and the Green Lipschitz bound give a Lipschitz constant at most $L|z|$. The same bound holds in the reference coordinate by its minimum kernel. Since $L_C$ is bounded below, averaging the $L^1$ estimate over $C$ and using its diameter at most $2\eta$ yields

$$\sup_{y\in C}|f_z(y)-f_*(z,R)|\le C|z|(1+|z|/\sigma)e+C|z|\eta.$$

On $C^2$ the convolution in (87) is the vertical-line inversion of $f_*(z,a(x))f_z(y)$. Subtract $f_*(z,R)^2$. In each resulting term a model factor decays exponentially in $\sqrt{|\operatorname{Im} z|}$, since $a(x)\ge R-\eta$. The other factor is bounded by the last display or $L|z|\eta$. Integration on the line gives

$$\sup_{x,y\in C}\left|\int h_*(a(x),t-s)\nu_y(ds)-\mathcal{C}(t)\right|\le C(e+\eta).$$

For the lower bound under the cap, it suffices to consider $e_t\le C_0/n$ for a fixed $C_0$ larger than the claimed coefficient on the chosen time interval. Then $e=O(n^{-1})$, $\eta=O(n^{-1/2})$ and $e/\eta=O(n^{-1/2})$. Inserting the positive convolution lower bound and (89) into (87) gives $e_t\ge\delta^3\mathcal{C}(t)/n-O(n^{-3/2})$.

## 40 The attaining sequences

For the equal $n$-arm broom, $B$ vanishes whenever either point is on the stem. Its total mass is the exact torsion gap $\delta^3/n + O(n^{-2})$. The arms lie in a terminal region of diameter $2\delta/(n-1)$ and length bounded below. Its assigned coordinates are within $\delta/(n-1)$ of $R$. The same resolvent and density argument gives a uniform $O(n^{-1})$ error in the convolution on those pairs. Equality in (87) gives the matching coefficient with $O(n^{-2})$ remainder. Pairing equal tips preserves the symmetric heat profile and its zero tip flux, hence the heat content, as in the midpoint loop-cutting comparison of [2, Proposition 6.4]. The broom has $2b_1 + \ell = n + 1$, and its full pair fold has $1 + n/2$ edges. These examples approach the envelope, so the lower-bound argument applies to near-optimal sequences under either cap. This proves Theorem 36.1.

## 41 Exact capped optimizers for resolvents

Let $G$ be a finite compact connected metric graph of length $L$ with nonempty Dirichlet vertex set $D$. Set $R = \max_{x \in G} \operatorname{dist}(x, D)$, where $0 < R < L$, and $J_G(z) = \langle 1, (A_G + z)^{-1} 1 \rangle$ for $z \geq 0$, with $A_G$ the Dirichlet–Kirchhoff Laplacian. The sharp unrestricted envelope, exact comparison within brooms and leading cap coefficients are known Sections 3 and 25–35. We give exact finite-cap values.

**Theorem 41.1.** *For a rooted tree of length $L$, height $R$, sole mark at the root, and at most $n$ nonroot leaves, feasibility requires $n \geq 2$ and $L \leq nR$. For every $z \geq 0$ the unique maximizer is the $n$-arm broom with arm length $b = (L - R)/(n - 1)$ and stem length $R - b$. The stem may vanish when $L = nR$.*

For arbitrary topologies, set $n = K - 1$ under the cap $2b_1(G) + \ell(G) \leq K$, where $\ell(G)$ counts degree-one vertices. For a presentation of at most $J$ edges use $n = 2J - 2$. Let $n \geq 2$ and put $c_n = (n^3 - n)/(n-1)^3$. The zero-mass threshold $L_n^* \in (R, nR)$ from Section 31 is the unique solution of $c_n(1 - R/L_n^*)^2(2 + R/L_n^*) = 2$.

**Theorem 41.2.** *If $L \leq L_n^*$, the optimal $J_G(z)$, under either cap, is the value of the equal $n$-arm broom for all $z \geq 0$. Under the topological cap the equality cases are exactly the disjoint tip-pair quotients of that broom. Under the edge cap every tip must be paired, giving the unique rooted stem with $J - 1$ equal loops.*

## 42 Shape improvements on rooted trees

After suppressing unmarked degree-two vertices, a fixed leaf cap bounds the number of edges, so there are finitely many rooted skeletons. Nonnegative edge lengths of sum $L$ compactify each type by contracting zero lengths. Heights converge because they are maxima of root path lengths. The variational formula

$$J_T(z) = \sup_u \left( 2 \int_T u - z \int_T u^2 - \int_T |u'|^2 \right)$$

gives continuity under these contractions. In fact $|u(x)|^2 \leq L \int_T |u'|^2$ bounds maximizing trials in energy and sup norm, so their oscillations on shrinking edges vanish. On surviving edges affine reparameterization gives subsequential uniform and weak $W^{1,2}$ convergence, with coincident traces on contracted vertices and no increase of energy in the limit. Conversely a limiting trial pulls back with constant values on shrinking edges and all objective terms converge. This is a special case of form convergence on shrinking graphs [26]. Thus an optimizer exists. The leaf count cannot increase under contraction in a tree.

Orient edges away from the root, let $u = (A_T + z)^{-1}1$, and put $V(u) = 2u - zu^2$. The outward proximal flux is positive on each edge with positive descendant length, by integrating $-u'' = 1 - zu > 0$ over its descendants. For $z > 0$, positivity of $1 - zu$ follows from its homogeneous equation and marked boundary value one. Hence $u$ and $V(u)$ increase along root paths. Differentiation in unit edge coordinates at the stationary trial gives the derivative with respect to length: the edge mean of $u'^2 + V(u)$, which is constant on that edge. The torsion version of this derivative is in [24, Proposition 3.11].

We will use a virtual stem of length zero if the marked root itself branches. The same derivative holds at zero length by the edge transfer. For $z > 0$, with $w = 1 - zu$, use normalized outward admittance $q = -w'/(zw)$. Vertex inputs add, leaves have zero input, and an edge of length $\ell$ maps a distal input $q$ to

$$\frac{q + \tanh(\sqrt{z}\,\ell)/\sqrt{z}}{1 + zq\,\tanh(\sqrt{z}\,\ell)/\sqrt{z}}.$$

The root input is $L - zJ_T(z)$. The transfer is analytic in edge lengths including zero and in $z$ near zero. At $z = 0$ the root input is identically $L$, so the equation extends the resolvent analytically there. The collapsed edge has limiting trace at the common vertex and flux equal to its downstream sum. This justifies the endpoint length derivative used below.

Write $r_i$ for leaf depths, $s_v$ for branch depths, and $k_v \geq 2$ for outgoing degrees at branching vertices. If there is a virtual stem, the first branch depth is its length. The length identity is $L = \sum_i r_i - \sum_v (k_v - 1)s_v$. Let $F_{v,j}$ be the outgoing proximal derivatives at $v$. The derivative with respect to $s_v$, at fixed other depths and divided by $k_v - 1$, is

$$Q_v = -V(u(v)) + \frac{2}{k_v - 1} \sum_{i<j} F_{v,i} F_{v,j}. \tag{90}$$

Indeed increasing $s_v$ lengthens its incoming edge and shortens its outgoing edges, and the incoming flux is the sum of the outgoing fluxes. The leaf-depth derivative at fixed branches is $V(u(\text{leaf}))$ by the Neumann condition.

If a leaf is shallower than $R$, raise its depth by $\varepsilon$ and raise its parent branch depth by $\varepsilon/(k_v - 1)$. Length and height are preserved, all positive edges stay positive for small $\varepsilon$, and a virtual stem may become positive. The derivative of the objective is $V(u(\text{leaf})) + Q_v > 0$, since the potential increases along paths and the flux cross term is positive. Thus every optimizer has all leaves at $R$.

If there are two or more branching vertices, choose a deepest one $v$ with parent branching vertex $p$. Each arm from $v$ goes directly to a leaf and has the same proximal flux

$$B_v = (1 - zu(v))\,\tanh(\sqrt{z}(R - s_v))/\sqrt{z}$$

for $z > 0$, with value $R - s_v$ at zero. The flux at $p$ into the edge leading to $v$ is at least $k_v B_v$ because flux decreases along that edge. The initial flux of each other child at $p$ is at least $B_v$: its descendant tree contains a length-$(R - s_p)$ reflecting path, and input admittance increases on adding outgoing branches. Also $1 - zu(p) \geq 1 - zu(v)$. The zero-mass bound follows directly from descendant length. Hence (90) gives

$$Q_p \geq -V(u(p)) + 2k_v B_v^2 > -V(u(v)) + k_v B_v^2 = Q_v.$$

Raise $s_p$ by $\varepsilon/(k_p - 1)$ and lower $s_v$ by $\varepsilon/(k_v - 1)$, preserving length and leaf depths. This feasible change strictly improves the objective. Therefore all branching occurs at one junction and the arms are equal.

Using all $n$ arms strictly improves over a smaller feasible count. At $z = 0$ this is the exact torsion formula. At $z > 0$ the prior within-broom argument Section 28 applies, including

a zero stem by the transfer formula. In detail write $x = \sqrt{z}\,b$, $D_0 = \sqrt{z}(L - R)$, and $\alpha = (1 + D_0/x)\tanh x$. The derivative of the dimensionless root input as $x$ increases has the sign of

$$\alpha' - 1 + \alpha^2 = \frac{D_0}{x^2}\,[x(1 + \tanh^2 x) - \tanh x] + \frac{D_0^2}{x^2}\tanh^2 x > 0.$$

Smaller $b$ therefore gives larger $J_T(z)$. These comparisons prove Theorem 41.1.

## 43 Combining marked branches at every mass

The sharp value $H_{n,z}(X)$ for a rooted branch of length $X$, height at most $R$ and at most $n$ leaves is the mixed interval for $X \le R$, and the equal-arm broom of height $R$ for $R < X \le nR$. Indeed the preceding theorem optimizes at the actual height, and the equal-broom value strictly increases with height at fixed length. If $b$ is its arm length, raising height lengthens the stem by $n/(n-1)$ times the change and shortens each arm by $1/(n-1)$ times the change. The derivative is $n/(n-1)$ times the incoming-minus-arm Hamiltonian at the hub, which is positive by their fluxes $n\sigma, \sigma$. At height $X$ the mixed interval is the limiting broom.

The quotient $H_{n,z}(X)/X$ is strictly increasing up to $L_n^*$. For $X \le R$ this follows from the mixed-interval formula. For $X > R$, take $z > 0$, put $s = \sqrt{z}$, $\alpha = s(R - b)$, $\beta = sb$, $b = (X - R)/(n-1)$, $t = \tanh\alpha$, and $v = \tanh\beta$. The dimensionless root admittance is $A = (t + nv)/(1 + ntv)$ and $H_{n,z} = X/z - A/z^{3/2}$. Differentiating in $X$ at fixed height gives

$$A'(X) = s\,(1 - t^2)(1 + nv^2)/(1 + ntv)^2.$$

The sign of the normalized value derivative is the sign of $A - XA'$. Multiply the latter by $(1 + ntv)^2 \cosh^2\alpha \cosh^2\beta$, and call the result $G$. With $r = \alpha/\beta$ and

$$U_r(\beta) = \frac{\sinh(2r\beta)/(2\beta) - r}{\sinh^2\beta},$$

the double-angle identities give

$$\frac{G}{\beta \sinh^2\beta} = U_r(\beta) + nU_1(\beta) + (n^2+1)\frac{\sinh(2r\beta)}{2\beta} + 2n\frac{\sinh^2(r\beta)\coth\beta}{\beta} - (r+n)(n+1).$$

For example the numerator before this division is

$$\tfrac{1}{2}\sinh(2\alpha)(\cosh^2\beta + n^2\sinh^2\beta) + \tfrac{n}{2}\cosh(2\alpha)\sinh(2\beta) - (\alpha + n\beta)(\cosh^2\beta + n\sinh^2\beta).$$

We have $\beta^2/\sinh^2\beta \ge 1 - \beta^2/3$. Indeed in $(1 - \beta^2/3)(\sinh\beta/\beta)^2$ the quadratic coefficient vanishes and all higher coefficients are negative, by the consecutive coefficient ratios of $(\cosh(2\beta) - 1)/(2\beta^2)$. Using the cubic lower bound for sinh, $\sinh(r\beta) \ge r\beta$, and $\beta\coth\beta \ge 1$, we obtain

$$\frac{G}{\beta\sinh^2\beta} \ge g_0(r) + \frac{2}{9}[(3n^2+2)r^3 - n]\beta^2, \qquad g_0(r) = \tfrac{2}{3}r^3 + 2nr^2 + n(n-1)r - n^2 - \tfrac{n}{3}.$$

The expansion at zero mass gives $XH'_{n,0}(X) - H_{n,0}(X) = b^3 g_0(r)$. The exact torsion comparison Section 31 gives $g_0(r) \ge 0$ when $X \le L_n^*$. Since $g_0$ increases on $r \ge 0$ and $g_0(2/3) = 16/81 - n/9 - n^2/3 < 0$, we have $r > 2/3$, so the coefficient of $\beta^2$ is positive for $n \ge 2$. Thus the branch ratio increases strictly for $z > 0$ up to and including $L_n^*$. At zero mass the strict ratio increase on that interval is the known torsion result. The two branch formulas meet at $X = R$.

Apply the distance-preserving cuts of Section 30 and split each marked root into one copy per branch. The cutting relaxes the resolvent variational space and preserves length and inradius. Each resulting root branch has at most $n$ nonroot leaves for the corresponding cap. If their lengths are $X_i$, then

$$J_G(z) \le \sum_i H_{n,z}(X_i) \le \sum_i \frac{X_i}{L} H_{n,z}(L) = H_{n,z}(L).$$

Strictness forces a single equal broom at equality. Restoring the graph can identify only equal-depth arm tips. A group of $m \ge 2$ tips increases $2b_1 + \ell$ by $m - 2$ relative to the broom. Consequently only pairs fit the topological cap, and the edge cap requires all tips to be paired to reach $J$ edges, as in Section 31. Such folds preserve the symmetric solution by equal traces and zero tip flux. This proves Theorem 41.2.

The length-range restriction in Theorem 41.2 cannot be removed for the topological cap. At fixed height the equal-broom length derivative is $V(u_0) - n\sigma^2$, where $u_0$ is the junction value and $\sigma$ an arm flux. This follows by transferring length between stem and arms. At $L = nR$ the stem is zero, $u_0 = 0$, and the left derivative is negative. For lengths sufficiently close to $nR$, shortening the broom slightly and appending a short interval at the same marked root increases the objective. The interval is separated by the Dirichlet mark. The shortened broom's root leaf is replaced by the new tip, preserving the topological cap and the inradius.

## 44 Two edges at every mass

For $z > 0$ let $s = \sqrt{z}$ and put

$$N(b) = \frac{b}{z} - \frac{\tanh(sb)}{z^{3/2}}, \qquad D(a) = \frac{a}{z} - \frac{2\tanh(sa/2)}{z^{3/2}} = 2N(a/2).$$

These are the mixed and doubly marked interval values. At $z = 0$ they mean $b^3/3$ and $a^3/12$.

**Theorem 44.1.** *At edge cap two the feasible inradii are $L/4 \le R < L$. For every $z \ge 0$, the maximum and equality types up to marked metric isometry are as follows.*

1. *If $R > L/2$, the unique maximizer is the one-loop stem with only its tip marked. The stem and loop lengths are $a = 2R - L$ and $2b = 2(L - R)$. Its value at $z > 0$ is*

$$\frac{L}{z} - \frac{\tanh(sa) + 2\tanh(sb)}{z^{3/2}(1 + 2\tanh(sa)\tanh(sb))}.$$

   *At zero mass use the continuity limit.*

2. *If $R = L/2$, the maximum is $D(L) = 2N(R)$. The equality graphs are the interval with both ends marked, the circle with one mark, and the interval with only its midpoint marked.*

3. *If $L/3 \le R < L/2$, the maximum is $D(2R) + N(L - 2R)$. Equality occurs for the path combining these two pieces at a marked vertex, and for the stem with a loop of length $2R$ whose hub alone is marked.*

4. *If $L/4 \le R < L/3$, every feasible graph gives $D(2R) + D(L - 2R)$. These are the two doubly marked pieces, realized by an all-marked path, a two-mark circle, a stem and loop with both vertices marked, or two loops at one marked vertex.*

*Proof.* Each edge midpoint is at distance at least half that edge's length from the marks, so $L \le 4R$. A two-loop bouquet with both loops based at a mark is feasible for $L/4 \le R < L/2$. At $R = L/2$ use the doubly marked interval, and above $L/2$ use the one-loop stem.

The connected shapes with at most two edges are paths, circles, a stem and one loop, and two loops at a common vertex. A marked vertex separates the form into its incident pieces. Therefore the interval models are a single mixed or doubly marked interval, two mixed intervals, one of each, or two doubly marked intervals. A loop at a marked hub is a doubly marked interval. The remaining case is the loop and stem with only the tip marked. The midpoint cut [2, Proposition 6.4] gives a broom of stem length $a$ and arm lengths $b, b$, with $R = a + b$. For the deficit profile $1 - zu$ the hub's outgoing scaled flux-to-value ratio is $2\tanh(sb)$. The stem transports a ratio $v$ to $(\tanh(sa) + v)/(1 + v\tanh(sa))$. The resulting ratio at the marked tip equals the deficit from $L/z$ times $z^{3/2}$, giving the value in the theorem.

For $R > L/2$ a mixed and doubly marked pair must have mixed length $R$. It is strictly below $N(R) + N(L - R)$. The two mixed intervals are then the only competitor to the loop. With $t = \tanh(sa)$ and $v = \tanh(sb)$, their loss minus the loop loss, before division by $z^{3/2}$, is

$$\frac{t + v}{1 + tv} + v - \frac{t + 2v}{1 + 2tv}.$$

Its numerator over the positive common denominator is $(t+2v)tv + tv^2(1+2tv) > 0$. At zero mass the gap in values is $ab(a + 3b) > 0$. At $R = L/2$, the single doubly marked interval and the two equal mixed intervals tie. The only positive-length mixed plus doubly marked pair then has both lengths $R$ and is strictly worse.

For $R < L/2$ two mixed intervals are infeasible. One of each type is feasible precisely for $R \ge L/3$. The possibilities are mixed length $R$ with doubly marked length $L - R$, or mixed length $L - 2R$ with doubly marked length $2R$. The second wins by concavity of tanh:

$$2\tanh(s(L - R)/2) \ge \tanh(sR) + \tanh(s(L - 2R)).$$

Equality occurs at $R = L/3$, when the lengths coincide. At zero mass the gap is $(L-R)(3R-L)^2/4$. Two doubly marked intervals have lengths $2R, L - 2R$. When a mixed interval of the shorter length is feasible, it improves the value since $N(b) > D(b)$ for $b > 0$ at every mass. Otherwise only the doubly marked pair remains. The listed equality types follow by restoring the marked presentations. □

## 45 Exact capped optimizers for nonlinear $p$-resolvents

Let $p > 1$, $q = p/(p-1)$, $z \ge 0$, and let $u$ be the nonnegative solution on a finite metric graph of

$$-\Delta_p u + zu^{p-1} = 1$$

with nonempty Dirichlet marks and Kirchhoff flux elsewhere on a connected graph. Put $I_{p,z}(G) = \int_G u$. The exact variational value

$$J_{p,z}(G) = \max_v \int_G (pv - z|v|^p - |v'|^p) = (p - 1)I_{p,z}(G)$$

uses the marked Sobolev space. The maximizer is unique by strict convexity of the energy, and can be taken nonnegative. At zero reaction this is the $p$-torsion problem [5] and Section 2. The linear length derivative at zero reaction is known [24]. For $p = 2$, the exact all-mass range from Sections 41–44 is stronger than the general sufficient bound below. The implicit $L^2$ step in Sections 20–22 has a linear reaction term. Here we study the homogeneous reaction $zu^{p-1}$. Below we extend the homogeneous comparison to general absorption and sharp quantitative thresholds.

**Theorem 45.1.** *For a rooted finite tree of length $L$, height $0 < R < L$, sole mark at its root, and at most $n$ nonroot leaves, an equal $n$-arm broom is a maximizer at every $z \geq 0$, whenever $L \leq nR$. Its stem and arm lengths are $R - b, b$ with $b = (L - R)/(n - 1)$. Unless the hub is saturated, the maximizing rooted isometry type is unique. When the hub is saturated, the equality class is determined by the penetration length below.*

**Theorem 45.2.** *For every fixed $p, L, R$ the equal-broom value is the exact arbitrary-graph optimum at every $z \geq 0$ under either cap once $n - 1 > [p(L - R) + (p - 1)L]/R$, with $n = K - 1$ for $2b_1(G) + \ell(G) \leq K$, where $\ell(G)$ counts degree-one vertices, and $n = 2J - 2$ for a presentation with at most $J$ edges. A sharper sufficient cap inequality is given below. If the hub is unsaturated, the topological equality types are precisely disjoint tip-pair quotients, and the edge-cap equality type is the fully folded equal-loop stem. If it is saturated, the path equality criterion below applies.*

## 46 Shape variations and the rooted optimizer

On rooted trees use orientation away from the root, flux $F = |u'|^{p-2}u'$, and $V(u) = pu - zu^p$. For $z > 0$, the maximum principle gives $0 \leq u \leq c = z^{-1/(p-1)}$. The source $1 - zu^{p-1}$ is nonnegative, so integration on descendant trees gives $0 \leq F \leq L$ and $u' \geq 0$. Thus $V$ is nondecreasing along each root path. The same holds at zero mass. The bound $u \leq RL^{q-1}$ follows by integrating $u' = F^{q-1}$ along such a path.

The optimal value is continuous under vanishing-edge contractions with a fixed rooted skeleton, by the same variational compactification as in the linear case. The path bound $\|v\|_\infty \leq L^{1/q}\|v'\|_p$ bounds maximizing trials; all oscillations on collapsing edges vanish. Pull-back of limiting trials along surviving edges gives the reverse bound. When varying positive edge lengths, the stationary variational value has edge derivatives equal to the constant on each edge

$$\mathcal{H} = (p - 1)|u'|^p + V(u).$$

To justify the envelope calculation, write each edge on its unit interval. For positive edge lengths the objectives and derivatives with respect to length vary continuously under strong $W^{1,p}$ convergence. Uniqueness, compactness, objective convergence and uniform convexity give that strong convergence for maximizers: their energy norms converge as the potential integrals do. The upper and lower derivative comparisons using old and new maximizers give the stated derivative, initially as the edge average and then as the constant by the edge equation. The same argument works for a positive multiplicity weight on an arm.

Write the terminal depths $r_i$ and junction depths $s_v$, with branching outgoing degrees $k_v \geq 2$. If the root branches, allow a virtual zero stem joining it to the branch vertex at depth zero. The length identity is $L = \sum_i r_i - \sum_v (k_v - 1)s_v$. For each branching vertex let $F_{v,j}$ be its outgoing proximal fluxes. The derivative in branch depth divided by $k_v - 1$ is

$$Q_v = -V(u(v)) + \frac{p - 1}{k_v - 1}\left[\left(\sum_j F_{v,j}\right)^q - \sum_j F_{v,j}^q\right]. \tag{1}$$

The terminal depth derivative is $V(u(\text{leaf}))$. If a terminal is short, raise its depth by $\varepsilon$ and its parent depth by $\varepsilon/(k_v - 1)$. Length and height are preserved. The derivative is nonnegative by (1) and is positive when the parent is unsaturated, since all outgoing fluxes there are positive. If the virtual stem starts at zero, use this inequality on the open positive part of the change and continuity at its start.

This deformation can bring all terminal depths to $R$ without decreasing the value. Follow each change until that leaf reaches $R$ or an edge collapses. In the latter case contract and

suppress unmarked degree-two vertices. A leaf already at $R$ stays there and cannot collapse before reaching the intended depth, because the moving leaf remains strictly deeper than its parent. Each event either increases the number of finished leaves or decreases the finite skeleton size; insertion of a virtual stem is needed at most once because subsequent parent moves at the first branch only deepen it. At least one leaf stays at $R$ throughout.

Now take a deepest branching vertex $v$ with branching parent $P$ if there is more than one junction. All its arms have common proximal flux $B$ since all end at depth $R$. The flux from $P$ into $v$'s subtree is at least $k_v B$. Every other child flux at $P$ is at least $B$. For this last fact, keep a path in that subtree to a depth-$R$ leaf. The restricted solution on the path is a supersolution of the corresponding mixed interval equation: the jumps of its outgoing flux at side junctions account for nonnegative discarded branch fluxes. With the same root value, comparison makes its proximal flux at least that of the path solution. The mixed interval input flux is nondecreasing in length, by the same restriction argument at a nonnegative terminal flux, and nonincreasing in root value, by the pointwise comparison and integration of $1 - zu^{p-1}$. Since the path is longer and $u(P) \le u(v)$, its flux is at least $B$.

The bracket in (1) is nondecreasing in every nonnegative flux. For $k = k_v$, the parent cross term per branching excess is therefore at least

$$B^q\big((k+1)^q - k^q - 1\big).$$

This follows by adding its other fluxes, each at least $B$, to the one at least $kB$, and using increasing increments of the convex power. The child's corresponding term is $B^q(k^q - k)/(k-1)$, which is smaller if $B > 0$, by strict convexity. Also $V(u(P)) \le V(u(v))$, so $Q_P \ge Q_v$. Move $P$ deeper and $v$ shallower at reciprocal branching-excess rates, holding all leaf depths fixed. This preserves length and height and does not decrease the objective. Follow to the first merger, which decreases the number of internal junctions, and repeat. Thus an equal-arm broom with some feasible leaf count $m \le n$ dominates the original tree.

To compare arm counts, use a single arm with real multiplicity $m = 1 + (L - R)/b$ and stem $R - b$; the functional and length weight each arm integral by $m$. This is a continuous family through the integer arm brooms. Let $F_0$ be the proximal arm flux and $\mathcal{H}_a$ the arm Hamiltonian. Along the arm,

$$\int_0^b (V(u) - |u'|^p) = b\mathcal{H}_a - p\int_0^b |u'|^p.$$

Differentiating in $b$ gives $\mathcal{H}_a - \mathcal{H}_{\text{stem}} + (p(m-1)/b)\int |u'|^p$. The stem-to-arm Hamiltonian difference is $(p-1)(m^q - 1)F_0^q$, and $|u'|^p \le F_0^q$ along the arm. Thus the derivative is at most $[p(m-1) - (p-1)(m^q - 1)]F_0^q \le 0$, strictly if $F_0 > 0$. Contraction continuity covers a zero stem. Increasing arm count decreases $b$, proving the rooted optimum.

For $1 < p \le 2$, saturation cannot occur at finite distance from the root: an incoming edge reaching $c$ with zero flux has first integral $(p-1)|u'|^p = V(c) - V(u) \asymp (c-u)^2$ near equilibrium, so reaching $c$ takes infinite length unless the edge is constant. Propagate a constant backwards to the Dirichlet root to get a contradiction. For sufficiently small mass the direct bound $u \le RL^{q-1}$ also precludes saturation. Thus all necessary shape improvements in these regimes are strict, giving uniqueness. For $p > 2$ the same first integral gives the finite penetration length from the classical one-dimensional reaction time map [27, §5]

$$\rho_{p,z} = (p-1)^{1/p} z^{-1/p} \int_0^1 [(p-1) - pv + v^p]^{-1/p}\, dv.$$

Any rooted tree with a stem longer than $\rho_{p,z}$ has the same boundary layer on that stem, and is constant $c$ thereafter, by uniqueness of the variational solution. For a positive-stem broom

and high enough $z$, a small unequal-arm change preserving $L, R$ still has such a stem. This gives nonunique equality.

An unsaturated final hub gives uniqueness even if other rooted trees can have saturated subtrees. In an equality trajectory the leaf count stays at (n), since contraction never increases it and the arm-count comparison is strict near an unsaturated final hub. Consider the last nontrivial shape move, if there was one. During a constant-value leaf-depth move the moving parent is saturated. During a constant-value junction merger both moving junctions are saturated, by the strict flux gap when (B>0) and the nonnegative potential difference. Uniform convergence of maximizing profiles at edge contractions carries saturation to the end of that last move. The saturated junction remains a junction after contraction: a branched vertex can disappear only by losing a leaf, by merging into another branch vertex, or by reaching the marked root, which has value zero. Thus the sole final hub would be saturated, a contradiction.

## 47 Global graph comparison uniformly in mass

Let $H(X)$ be the sharp rooted value with volume $X \le L$, height at most $R$, and at most $n$ leaves. For fixed $X$, the rooted comparison gives an equal broom at each feasible height $h$. Along these brooms the stem length has derivative $n/(n-1)$ in $h$, and each arm length has derivative $-1/(n-1)$. Thus the derivative in $h$ of the value is $n(\mathcal{H}_{\mathrm{stem}} - \mathcal{H}_a)/(n-1) \ge 0$. Continuity covers the zero stem and the limiting path at $h = X$. Consequently $H(X)$ is the mixed interval value for $X \le R$ and the $n$-arm broom value at height $R$ for $X > R$, provided $nR \ge L$.

For $X \le R$, $H(X)/X$ strictly increases. Indeed, rescaling the interval to unit length leaves the potential term in the normalized objective unchanged and replaces the energy coefficient by $X^{-p}$. The optimal profile has nonzero derivative on a positive-measure set.

For $X > R$, put $b = (X-R)/(n-1)$ and write $u_0, F_0$ for the hub value and the proximal flux on one arm. Differentiation at fixed $R$ gives

$$H' = V(u_0) - \frac{(p-1)(n^q - n)}{n-1} F_0^q.$$

It follows that

$$XH' - H = \int (V(u_0) - V(u)) + \int |u'|^p - X\frac{(p-1)(n^q-n)}{n-1} F_0^q.$$

The stem's contribution to the first integral is nonnegative, and its energy is at least $(R-b)n^q F_0^q$ because the flux is nonincreasing along the stem. The constant arm Hamiltonian gives $V(u) - V(u_0) = (p-1)(F_0^q - F(x)^q)$, so the negative potential contribution from the $n$ arms has magnitude at most $(p-1)nbF_0^q$. Dropping their nonnegative energy gives

$$XH' - H \;\ge\; F_0^q \left[ (R-b)n^q - (p-1)nb - X\frac{(p-1)(n^q-n)}{n-1} \right].$$

Write $b_L = (L-R)/(n-1)$. The bracket is smallest at $X = L$, giving a sufficient condition

$$(R - b_L)n^q > (p-1)nb_L + L\frac{(p-1)(n^q-n)}{n-1}. \tag{2}$$

This implies $R - b_L > 0$, and follows from the simpler cap inequality in the statement by dividing by $n^q$ and using $n^{1-q} \le 1$. It gives strict positivity whenever $F_0 > 0$. If $F_0 = 0$, the arms are constant and the stem still has strictly positive energy because it joins the Dirichlet

value to the positive hub value. Thus $H(X)/X$ strictly increases on the whole range. The two formulas for $H$ meet continuously at $R$.

Apply the distance-preserving rooted-tree cut of Section 30 to a graph with the same topological or edge cap. The cut relaxes the variational maximum by splitting equality constraints, and each rooted component has at most $n$ leaves and height at most $R$. With positive component volumes $X_j$, one obtains $\sum_j H(X_j) \le H(L)$ from the strict increase of $H(X)/X$, with equality possible only for a single component. In every unsaturated uniqueness regime the rooted extremizer is the equal broom. The original graph must be a permitted leaf folding of it, since no cut identifies unequal depths or nonterminal points of the resulting rooted tree. Conversely, such foldings retain the symmetric profile, and the identified terminal fluxes are all zero. They therefore attain the optimum also in the saturated case, where further equality shapes can occur.

## 48 Exact high-mass range for $p > 2$

Let $z > 0$. If $R \ge \rho_{p,z}$, a shortest Dirichlet-to-inradius path is an embedded interval of length $R$. For any nonnegative admissible profile, the deficit from $LV(c)$ is $\int [V(c) - V(u) + |u'|^p]$. Each term is nonnegative, so it dominates the deficit on this path. With its initial Dirichlet value the least possible path deficit is the mixed interval value

$$D_{p,z} = p(p-1)^{-(p-1)/p} \int_0^c [V(c) - V(s)]^{(p-1)/p}\, ds.$$

Indeed its unique optimizing profile reaches $c$ at $\rho_{p,z}$ by the first integral and stays constant. The displayed cost follows by change of variables on this boundary layer. Equality in the graph bound holds exactly when this path has no positive-length side edge incident before $\rho_{p,z}$ and there is no other Dirichlet point. Necessity follows from continuity and strict positivity of the potential deficit below $c$, and sufficiency follows by extending the mixed profile by $c$ on the rest of the graph. Saturation at the broom hub occurs exactly when its stem is at least $\rho_{p,z}$: in that case the compact layer pastes into the broom, and conversely a saturated hub has zero flux and the first integral forces that much stem. It follows that equality in the saturated regime consists precisely of the admissible graphs with an unbranched initial segment of length $\rho_{p,z}$ from their sole Dirichlet point.

For $R > \rho_{p,z}$, a broom with sufficiently many arms attains the bound, even over graphs without complexity constraints. If $L > R = \rho_{p,z}$, the bound is an unattained supremum. Indeed, any positive length off the initial path would have to attach at its endpoint or later, forcing an inradius larger than $R$. For convergence, take brooms with arm length $b \to 0$ and trial profiles equal to the mixed interval layer on their stems $[0, R-b]$ and constant with the stem endpoint value on each arm. The added potential deficit is at most $L[V(c) - V(u(R-b))] \to 0$, and the omitted path deficit also tends to zero.

At the precise cap transition $R - b = \rho_{p,z}$, this equality class still reduces to the broom and its permitted foldings. After the distance-preserving cut, deleting the unbranched initial segment leaves a rooted tree of length $L - \rho_{p,z} = n(R - \rho_{p,z})$, height at most $R - \rho_{p,z}$, and at most $n$ nonroot leaves. Its length is at most the sum of its root-to-leaf lengths, which is at most $n(R - \rho_{p,z})$. Equality forces all $n$ leaf paths to have full height and to share no positive-length edge, giving the equal star attached at $\rho_{p,z}$.

## 49 General continuous absorption laws

Let $p > 1$, $q = p/(p-1)$, and let $f : [0, \infty) \to \mathbb{R}$ be continuous and nonincreasing with $f(0) > 0$. Suppose either $f > 0$ on $[0, \infty)$, or $f > 0$ on $[0, c)$ and $f(c) = 0$ for a finite $c > 0$.

Define $V(s) = p\int_0^s f(t)\,dt$ for nonnegative $s$. On a finite connected metric graph marked at a nonempty set of vertices let

$$J_f(G) = \max_{v\ge 0,\ v|_D=0} \int_G (V(v) - |v'|^p).$$

The maximizer is unique, solves $-\Delta_p u = f(u)$, and may be restricted to $u \le c$ when $c$ exists. This uses concavity of $V$, strict convexity of the derivative energy, the marked Poincaré bound, and truncation at $c$. One may extend $f$ by its positive value at zero for negative arguments to obtain the full Euler equation, since the unrestricted maximizer of that extended functional is nonnegative.

For rooted trees of fixed length $L$, height $0 < R < L$, sole Dirichlet mark at the root, and at most $n$ nonroot leaves, the equal $n$-arm broom is optimal whenever $L \le nR$. For arbitrary metric graphs with either of the two caps of Theorem 45.2 it is optimal whenever, with $b = (L-R)/(n-1)$,

$$(R-b)n^q > (p-1)nb + L\frac{(p-1)(n^q-n)}{n-1}.$$

The inequality is independent of the entire function $f$. When the broom hub has $f(u) > 0$, the tree optimum is unique and the graph equality cases are precisely the allowed tip-pair quotients.

To prove the extension, each outward flux satisfies $F(x) = \int_{\mathrm{desc}(x)} f(u)$, so $u$ and $V(u)$ are nondecreasing. The edge Hamiltonian is still $(p-1)F^q + V(u)$, and the same contraction and envelope limits apply since the maximizing profiles are uniformly bounded by the linear-growth bound $V(s) \le pf(0)s$. In the deepest-junction comparison, the path restriction retains a nonnegative measure of discarded outward fluxes. The weak comparison of that restriction with a mixed interval solution follows by testing their difference with its positive part: the gradient term is monotone and the source $f$ is nonincreasing. Input flux order at the proximal endpoint follows from right derivative order at common boundary values. The same argument with its longer path restricted gives monotonicity in length, and the integrated equation gives monotonicity in root value. Every other inequality in the leaf-depth, deepest-junction, weighted arm-count and normalized graph proofs uses only the power of the flux, the Hamiltonian and the order of $V$. For strictness, $f(u) > 0$ at a junction forces positive flux on each of its positive outgoing edges by continuity. The last-move proof gives uniqueness whenever the final broom junction is unsaturated.

When $c < \infty$ let

$$\rho_f = (p-1)^{1/p}\int_0^c [V(c) - V(s)]^{-1/p}\,ds, \qquad D_f = p(p-1)^{-1/q}\int_0^c [V(c) - V(s)]^{1/q}\,ds.$$

If $\rho_f = \infty$, saturation cannot occur. If it is finite, the first integral gives a unique mixed profile reaching $c$ exactly at $\rho_f$ and constant thereafter. Every graph of length $L$ and inradius $R \ge \rho_f$ has $J_f(G) \le LV(c) - D_f$ by comparing the nonnegative objective deficit along an embedded shortest path of length $R$. Equality holds exactly for graphs with sole Dirichlet point and an unbranched initial segment of length $\rho_f$ from it. At $R > \rho_f$ this value is attainable by finite brooms, and for $L > R = \rho_f$ it is a sharp unattained supremum. For a fixed leaf cap or the two graph caps, equality in the transition $R - b = \rho_f$ forces all allowed equal arms by length saturation, with the usual permitted endpoint quotients.

For example, suppose that $f(s) \sim A(c-s)^\alpha$ at the finite equilibrium for $A, \alpha > 0$, then $V(c) - V(s) \sim pA(c-s)^{1+\alpha}/(1+\alpha)$. The penetration length is finite exactly when $p > 1+\alpha$. Thus a continuous non-Lipschitz equilibrium reaction can give compact layers even at $p = 2$. For the model $f(s) = A(c-s)_+^\alpha$, put $B = pA/(1+\alpha)$. In the finite-length regime both

constants are explicit,

$$\rho_f = \left(\frac{p-1}{B}\right)^{1/p} \frac{c^{1-(1+\alpha)/p}}{1-(1+\alpha)/p}, \qquad D_f = p(p-1)^{-1/q} B^{1/q} \frac{c^{1+(1+\alpha)/q}}{1+(1+\alpha)/q}.$$

The finiteness condition is the one-dimensional compact-support criterion. The corresponding spatial free-boundary exponent is studied in [28].

# 50 Sharp cap rates and the crossover

Continue with the general absorbing-potential setting. Let $f$ have finite equilibrium $c$ and put $W(w) = V(c) - V(c-w)$. Assume

$$W(w) \sim C w^{\sigma} \quad (w \downarrow 0), \qquad C > 0, \quad 1 < \sigma < p.$$

The penetration radius is $R = (p-1)^{1/p} \int_0^c W(w)^{-1/p}\, dw$. Fix $L > R$ and $\delta = L - R$. The exact unrestricted supremum is $J_* = LV(c) - D_f$, and no finite graph attains it at this critical radius.

Put

$$\eta = \frac{1}{\sigma} - \frac{1}{p}, \qquad Q = (p-1)^{1/p} C^{-1/\sigma} \int_0^{\infty} \left[t^{-\sigma/p} - (1+t^{\sigma})^{-1/p}\right] dt.$$

The integral is positive and finite, since $\sigma/p < 1$ at zero and the difference is $O(t^{-\sigma/p-\sigma})$ at infinity. It equals $\mathrm{B}(1/\sigma, 1-\eta)/(p\sigma\eta)$: set $y = t^{\sigma}$, write the difference of powers as $\frac{1}{p}\int_0^1 (y+s)^{-1-1/p}\, ds$, and apply Tonelli.

**Theorem 50.1** (Critical radius)**.** *Let $J_n$ be the best value under the rooted leaf cap, or under either graph cap with effective index $n$. Then*

$$J_* - J_n \;\sim\; \frac{Q^{-1/\eta}}{1+1/\eta} \left(\frac{\delta}{n-1}\right)^{1+1/\eta}. \tag{3}$$

In particular, for homogeneous absorption $f(s) = 1 - zs^{p-1}$ with $p > 2$, the exponent is $(3p-2)/(p-2)$. Here $\sigma = 2$ and $C = p(p-1)zc^{p-2}/2$. For the objective $\int u$ in that model, divide the constant in (3) by $p-1$.

## 50.1 Critical rate

For sufficiently large $n$ the optimizer is the equal-arm broom, by the mass-uniform cap inequality. Use the real arm multiplicity $m = 1 + \delta/b$, so the stem is $a = R - b$ and $b \downarrow 0$. Write $J(b)$ for its exact weighted variational value. It converges to $J_*$ by the compact-profile trial and the path upper bound, which apply also with a real arm multiplicity $m \geq 1$.

We first construct the exact solution for small $b$. For $w_h \in (0, c)$ put $\Delta = (m^q - 1)W(w_h)$, and solve

$$R - b = (p-1)^{1/p} \int_{w_h}^{c} \left[W(w) + \Delta\right]^{-1/p} dw. \tag{4}$$

The right side decreases continuously and strictly from $R$ to zero, giving a unique $w_h$. Equation (4) constructs the stem solution, whose Hamiltonian is $V(c) + \Delta$. On each arm tentatively use the compact profile starting at junction deficit $w_h$. Its initial flux satisfies $(p-1)F_0^q = W(w_h)$. The stem flux at the hub is exactly $mF_0$ by (4). It remains to check that each compact arm reaches equilibrium before its tip.

As $b \to 0$, (4) implies $\Delta \to 0$. Otherwise the integral would stay strictly below the unrestricted penetration integral. Also $w_h \to 0$, and $w_h/(\Delta/C)^{1/\sigma} \to 0$ since $W(w_h)/\Delta = 1/(m^q - 1)$. Subtract (4) from the defining integral for $R$:

$$\frac{b}{(p-1)^{1/p}} = \int_0^{w_h} W(w)^{-1/p}\,dw + \int_{w_h}^{c} [W(w)^{-1/p} - (W(w)+\Delta)^{-1/p}]\,dw.$$

The first integral is $o(\Delta^\eta)$. The second is asymptotic to $C^{-1/\sigma}\Delta^\eta$ times the integral in $Q$, by rescaling $w = (\Delta/C)^{1/\sigma}t$. For a fixed small $\varepsilon > 0$, use two-sided comparability $W(w) \asymp w^\sigma$ on $0 < w < \varepsilon$ and the bounds by a constant multiple of $t^{-\sigma/p}$ at zero and $t^{-\sigma/p-\sigma}$ at infinity. The part $w \geq \varepsilon$ is $O(\Delta) = o(\Delta^\eta)$. Thus $b \sim Q\Delta^\eta$. Each tentative arm penetrates length

$$\ell_h = (p-1)^{1/p}\int_0^{w_h} W(w)^{-1/p}\,dw = O(\Delta^\eta m^{-q\eta}) = o(b).$$

The compact arm followed by the constant equilibrium is therefore admissible for all sufficiently small $b$, and the stem and arms match in value and Kirchhoff flux. Uniqueness identifies this candidate as the true solution.

The exact stationary derivative along the real-multiplicity family is

$$J'(b) = \mathcal{H}_{\text{arm}} - \mathcal{H}_{\text{stem}} + \frac{p(m-1)}{b}\int_0^b |u'|^p\,dx.$$

The Hamiltonian difference equals $-\Delta$. The arm energy is at most $\ell_h W(w_h)/(p-1)$, hence the last term divided by $\Delta$ is $O(m^{1-q}\ell_h/b) = o(1)$. Consequently $-J'(b) \sim Q^{-1/\eta}b^{1/\eta}$. Integrate from zero, using $J(0+) = J_*$, to obtain (3).

## 50.2 Unsaturated comparison

At a noncritical height where the collapsed broom endpoint is unsaturated, the exact gap instead has order $1/(n-1)$. Its limiting functional has a stem of length $R$ with an endpoint term $\delta V(u(R))$. Uniform variational convergence follows by contracting the equal arms and taking fixed constant arm extensions for reverse bounds. In particular the hub value tends to the unique limiting endpoint $u_*$, and the proximal arm flux satisfies $F_0/b \to f(u_*)$ by continuity and the vanishing arm oscillation. The same derivative formula then gives $-J'(b) \to (p-1)\delta^q f(u_*)^q$: the stem-to-arm Hamiltonian difference is $(p-1)(m^q-1)F_0^q$, and the energy correction is $O(b^{q-1})$. This yields the leading gap $(p-1)\delta^{q+1}f(u_*)^q/(n-1)$. Above the penetration radius the optimal gap vanishes exactly for sufficiently large $n$, because the entire compact layer fits on the stem.

## 50.3 Uniform crossover on the scale $1/n$

**Theorem 50.2** (Crossover)**.** *Write the penetration radius as $\rho$, keep $L > \rho$ fixed, and put $d_n = (L-\rho)/n$, $r = 1/\eta$, $K = Q^{-r}/(1+r)$. For $R = \rho + \theta d_n$, let $J_n(R)$ and $J_\infty(R)$ denote the optimal capped value and the unrestricted supremum. Then uniformly for $\theta$ in any fixed compact real interval,*

$$\frac{J_\infty(R) - J_n(R)}{K d_n^{1+r}} \;\longrightarrow\; (1-\theta)_+^{1+r} - (-\theta)_+^{1+r}. \tag{5}$$

*Convergence holds also in $C^1$ on every such interval. The limiting derivative is $(1+r)[(-\theta)_+^r - (1-\theta)_+^r]$.*

The fixed-level reference is $J_{\mathrm{sat}} = LV(c) - D_f$. In the exact finite-$n$ family saturation starts at $R_n = \rho + d_n$, where the stem is exactly $\rho$. For $R < R_n$ put $d = \rho - [R - (L - R)/(n-1)] = n(R_n - R)/(n-1)$. The same scalar construction (4) with $R - b$ replaced by $\rho - d$ gives $d \sim Q\Delta^{\eta}$, uniformly for bounded $\theta < 1$. Uniformity follows by applying the rescaled dominated convergence along any sequence of positive $d \to 0$ and $n \to \infty$. It also gives $\ell_h = O(d\, n^{-q\eta}) = o(b)$, because $d/b$ is uniformly bounded. Consequently the solution at each of these heights has compact arms. The exact height derivative at fixed $L, n$ is $J_n'(R) = n\Delta/(n-1)$. Since $d' = -n/(n-1)$, integration gives $J_{\mathrm{sat}} - J_n(R) \sim Kd^{1+r}$ on the side $\theta < 1$, with uniform relative error over that side of any compact $\theta$-interval. On $\theta \geq 1$ the deficit is exactly zero.

For the unrestricted side, the limiting endpoint functional described above gives precisely $J_\infty(R)$: every finite graph falls under a sufficiently large cap and the cap optimizers themselves form a sequence of graphs. For $R < \rho$ tending to $\rho$, compactness of the endpoint functional in unit coordinates gives endpoint gap $w_h = c - u(R) \to 0$. It is positive, since otherwise zero terminal flux and the first integral force stem length at least $\rho$. Its terminal flux is $F = (L - R)f(c - w_h)$. The monotone density estimate $pf(c - w) \sim \sigma C w^{\sigma - 1}$ follows from bounding $W'(w)$ by left and right secant slopes of the convex $W$ at $(1 \pm \varepsilon)w$ and then sending $\varepsilon \downarrow 0$. Thus $\Delta_\infty = (p-1)F^q - W(w_h)$ is asymptotic to $(p-1)F^q$, since $q(\sigma - 1) < \sigma$, and $w_h/(\Delta_\infty/C)^{1/\sigma} \to 0$. The same stem length rescaling yields $\rho - R \sim Q\Delta_\infty^{\eta}$. The endpoint-weighted envelope derivative in height is $J_\infty'(R) = \mathcal{H}_{\mathrm{stem}} - V(u(R)) = (p-1)F^q$; the second term accounts for the derivative of the endpoint mass $L - R$. Integrating to $\rho$ gives $J_{\mathrm{sat}} - J_\infty(R) \sim K(\rho - R)^{1+r}$. For $R \geq \rho$ its deficit is zero. Subtracting gives (5).

The same uniform Hamiltonian asymptotics before integration give

$$d_n^{-r}\big[J_\infty'(R) - J_n'(R)\big] \;\longrightarrow\; Q^{-r}\big[(-\theta)_+^r - (1-\theta)_+^r\big].$$

The derivatives extend continuously through their respective saturation points with value zero, by the same formulas and $\Delta \to 0$. Passing from $R$ to $\theta$ establishes the stated $C^1$ convergence.

## 51 Sharp normalized cap comparison

Let $q = p/(p-1)$, $n \geq 2$, and write $H(X)$ for the $n$-arm broom objective at fixed height $R$, arm length $b = (X - R)/(n-1)$ and stem $a = R - b$. Let $F_0 > 0$ be the arm input flux. Using the edge Hamiltonians in both the length derivative and the value gives the exact identity

$$\frac{XH' - H}{(p-1)F_0^q} = q \int_{\mathrm{broom}} \left(\frac{F}{F_0}\right)^q - R\frac{n^{q+1} - n}{n-1}. \tag{6}$$

Indeed the stem contribution relative to the hub potential is $(p-1)\int_{\mathrm{stem}}(F^q - n^q F_0^q)$; the corresponding arm contribution is $(p-1)\int_{\mathrm{arms}}(F^q - F_0^q)$. The Hamiltonian penalty from the derivative is $X(p-1)(n^q - n)F_0^q/(n-1)$, and the combined coefficient is independent of $b$.

Choose $b = 1$, $a = s$, so $X = n + s$ and $R = 1 + s$. At constant source one the right side of (6) is

$$B_n(s) = \frac{q}{q+1}\big[(n+s)^{q+1} - n^{q+1} + n\big] - (1+s)\frac{n^{q+1} - n}{n-1}.$$

Since $B_n(s)/n^q \to (q-1)s - 1$ locally uniformly for $s > 0$, a zero $s_n$ exists with $s_n \to 1/(q-1) = p - 1$. This zero is the first and only one in $s > 0$: $B_n(0) < 0$, $B_n$ is strictly convex, and it tends to positive infinity. At fixed $R = 1 + s_n$ it corresponds to the exact zero-reaction turning point in $X$.

Consider first the concave potential $V(s) = pG\min(s, c)$ for $s \geq 0$, with $G > 1$, and choose $c$ to be the value reached along a stem of length $s_n$ and the initial flux-carrying portion of an

arm of length $1/G$. Its exact candidate flux is

$$F_{\mathrm{stem}}(x) = n + G(s_n - x), \quad 0 \le x \le s_n, \qquad F_{\mathrm{arm}}(y) = (1 - Gy)_+, \quad 0 \le y \le 1.$$

The profile is obtained by integrating $u' = F^{q-1}$ with root value zero and stays at $c$ after the short transition. Kirchhoff matches $n$ arm inputs to the stem output. The profile satisfies the variational equation with subgradient $pG$ below $c$ and zero on the plateau, so strict convexity of the derivative energy makes it the maximizer.

The flux integral in (6) is now

$$I_n(G) = \int_0^{s_n} (n + Gx)^q \, dx + \frac{n}{G(q+1)}.$$

At $G = 1$ it agrees with the constant source integral. For $p > 2$ we have $1 < q < 2$, so

$$I_n'(1) = q \int_0^{s_n} x(n + x)^{q-1} \, dx - \frac{n}{q+1} < 0$$

for all sufficiently large $n$. The first term is $O(n^{q-1}) = o(n)$. Choosing $G > 1$ close enough to one therefore makes (6) strictly negative at the constant-source turning point.

Finally approximate the step source by continuous nonincreasing $f_\varepsilon$ equal to $G$ below $c - \varepsilon$, falling linearly to zero at $c$, and staying zero thereafter. Their potentials converge uniformly on bounded profiles. Variational compactness on the fixed broom gives strong $W^{1,p}$ convergence of the unique maximizing profiles. In particular the integrals of $F^q$ converge. At the hub $u_h < c$ by construction, so on a fixed short initial arm interval the source is identically $G$ for small $\varepsilon$; convergence of the arm gradients there and the affine flux formula give $F_{0,\varepsilon} \to 1$. Formula (6) remains strictly negative for small $\varepsilon$. The strict negative derivative persists just inside the zero-reaction range by continuity in $X$ for a fixed such smoothened law.

### 51.1 Sharp criterion across all monotone absorption laws

For arbitrary positive $a, b$, put $G = f(u_h)/F_0$ when the hub is unsaturated. Since $f(u)$ decreases along the arm, $G \ge 1/b$ and

$$\frac{F_{\mathrm{stem}}(x)}{F_0} \ge n + G(a - x), \qquad \frac{F_{\mathrm{arm}}(y)}{F_0} \ge (1 - Gy)_+.$$

The first inequality comes from the reaction along the stem being at least the hub reaction. The second comes from the arm flux decreasing at rate at most $f(u_h)$. Hence

$$\inf_f \int_{\mathrm{broom}} (F/F_0)^q = \min_{G \ge 1/b} \left[ \int_0^a (n + Gx)^q \, dx + \frac{n}{G(q+1)} \right]. \tag{7}$$

The step-law construction and its continuous approximations above prove the reverse inequality for each $G > 1/b$, and constant reaction attains the endpoint $G = 1/b$. The minimum is attained in the scalar problem because the displayed function is strictly convex and coercive. In particular, the constant reaction achieves this minimum exactly when

$$q \int_0^a x(n + x/b)^{q-1} \, dx \;\ge\; \frac{nb^2}{q+1}.$$

Together (6) and (7) give a necessary and sufficient scalar test for nonnegativity of the normalized branch derivative at the fixed broom shape across all continuous nonincreasing absorption laws with unsaturated hub. A saturated hub gives strict positivity from the stem energy.

When $1 < q < 2$, the worst-case flux integral at $b = 1$ and $a = s$ in a fixed compact subset of $(0, \infty)$ is $sn^q + O(n^{q/2})$. The lower bound drops the arm term. For the upper bound choose

$G = n^{(2-q)/2}$ in (7) and estimate the stem increment by the mean-value theorem. Thus in the normalized derivative at $s$ near $s_* = 1/(q-1)$, the worst-case correction to $(q-1)s-1$ is $O(n^{-q/2}+n^{-1}) = o(n^{1-q})$. The constant-source turning point is

$$s_n = s_* - \frac{q}{(q+1)(q-1)}\, n^{1-q} + o(n^{1-q}),$$

by expansion of $B_n(s)/n^q$. This identifies the leading separation between the constant-source range and the range uniform over all monotone reactions for this derivative test.

### 51.2 Exact uniform turning point and the full range for $1 < p \le 2$

At scale $b = 1$, define

$$E_n(s) = q \inf_{G \ge 1} \left[ \int_0^s (n+Gx)^q \, dx + \frac{n}{G(q+1)} \right] - (1+s)\frac{n^{q+1}-n}{n-1}.$$

Each function before taking the infimum has derivative at least $qn^q - (n^{q+1}-n)/(n-1) > 0$. The inequality follows from $1 - n^{1-q} < (q-1)(n-1)$. Hence $E_n$ is strictly increasing with at least this slope, continuous for $s > 0$ by coercivity of the minimization locally uniformly in $s$, negative near zero and positive for large $s$. The negative limit at zero follows by taking fixed $G$ large before sending $s \downarrow 0$. Write $s_n^{\mathrm{univ}}$ for its unique zero. Since the aspect ratio $s = a/b$ decreases with volume at fixed height, this gives nondecreasing normalized branch value up to

$$L = R\frac{n + s_n^{\mathrm{univ}}}{1 + s_n^{\mathrm{univ}}}$$

for every reaction in the class, with strict increase below the endpoint. The distance-preserving cut gives the exact arbitrary-graph capped optimum throughout this range, with the equality criteria already established for the nonlinear broom. The scalar zero is sharp for the normalized comparison across all laws by (7). For $1 < q < 2$ the preceding estimates give $s_n^{\mathrm{univ}} = 1/(q-1) + o(n^{1-q})$.

For $q \ge 2$, this uniform range is precisely the full constant-source range. Define

$$\beta = \left( \frac{2n^{2-q}}{q(q+1)} \right)^{1/2}.$$

For $s \ge \beta$ the scalar minimum is attained at $G = 1$, because

$$q \int_0^s x(n+x)^{q-1} \, dx \ge qn^{q-1}s^2/2 \ge n/(q+1).$$

It remains to check $B_n(\beta) < 0$, which implies $s_n > \beta$. Set $t = q\beta$, $h = t/(qn)$ and $\lambda = t'/t = \frac{1}{2}(1/(q(q+1)) - \log n)$, with derivative in $q$ at fixed $n$. Bounding the constant-source integral by its upper endpoint shows

$$\frac{B_n(\beta)}{(1+\beta)n^q} \le A_1 + A_2 - \frac{n - n^{1-q}}{n-1}, \qquad A_1 = \frac{t(1+h)^q}{1+t/q}, \quad A_2 = \frac{qn^{1-q}}{(q+1)(1+t/q)}.$$

Here $t \le 2/\sqrt{3} < 7/6$, $h < 3/10$ and $\lambda < -1/4$. The last ratio increases in $q$. The logarithmic derivative of $A_2$ is $[\lambda(2+\beta)+\beta/q]/(1+\beta) < 0$. For $A_1$, the logarithmic derivative of $t/(1+t/q)$ is $(\lambda + t/q^2)/(1+t/q)$, and that of $(1+h)^q$ is at most $h^2/2 + \lambda q h/(1+h) < -h/5$. If $n \ge 3$, then $\lambda < -5/12$ and the first term is negative. If $n = 2$ and the first term is positive, then $t/q^2 \le h$, $h > 1/4$, and the first term is less than $7/24 - 1/4 = 1/24$. The second is then less than $-1/20$. Thus $A_1$ decreases in all cases.

At $q = 2$, write $\beta_0 = 1/\sqrt{3}$. The difference between $(1+\beta_0)(1+1/n)$ and $2\beta_0(1+\beta_0/n)^2 + 2/(3n)$ is $(1-\beta_0)(1-1/n) - 2\beta_0/(3n^2) > 0$ for $n \ge 2$. The displayed upper bound is therefore strictly negative at $q = 2$ and hence at every $q \ge 2$. This proves $E_n(s_n) = 0$ and $s_n^{\mathrm{univ}} = s_n$.

### 51.3 Homogeneous absorption for $p > 3$

Fix $p > 3$, so $1 < q < 3/2$, and use the homogeneous law $1 - zu^{p-1}$. At the constant-source turning point take $b = 1$, $a = s_n \to s_* = p - 1$. Let $I(z) = \int (F_z/F_{0,z})^q$ in (6), with $F_{0,z}$ the proximal flux of one arm. In the constant-source profile,

$$u_s(x) = n^{q-1}(x + O(n^{-1})), \qquad u_a(y) = u_h + A(y), \qquad A(y) = \frac{1 - (1-y)^q}{q}.$$

The estimates are uniform over the respective edges. Set $g_s = u_s^{p-1}$, $g_a = u_a^{p-1}$, and $\bar{g}_a = \int_0^1 g_a$. Variational convergence $u_z \to u_0$ as $z \downarrow 0$ and the exact descendant-flux formula give

$$\frac{d}{dz}\frac{F_{s,z}(x)}{F_{0,z}}\bigg|_{0+} = \int_x^{s_n} (\bar{g}_a - g_s(t))\,dt, \qquad \frac{d}{dz}\frac{F_{a,z}(y)}{F_{0,z}}\bigg|_{0+} = (1-y)\bar{g}_a - \int_y^1 g_a(t)\,dt.$$

The factor $z$ in the flux identity means that uniform convergence of $u_z$ suffices. The stem contribution to $I'(0+)$ is $O(n^q)$, since $g_s, \bar{g}_a = O(n)$, the length stays bounded and the baseline power derivative is $O(n^{q-1})$.

On the arms,

$$g_a(y) = u_h^{p-1} + (p-1)u_h^{p-2}A(y) + O(u_h^{p-3})$$

uniformly. The constant term cancels and

$$\int_y^1 A(t)\,dt - (1-y)\int_0^1 A(t)\,dt = \frac{(1-y)[1-(1-y)^q]}{q(q+1)}.$$

Integrating against $q(1-y)^{q-1}$ and accounting for $n$ arms, their contribution to $I'(0+)$ is

$$-\frac{q(p-1)s_*^{p-2}}{(q+1)^2(2q+1)}n^{3-q}(1 + o(1)),$$

since $u_h^{p-2} \sim s_*^{p-2}n^{2-q}$. As $3 - q > q$, the arm deficit dominates the stem term. Formula (6) becomes negative for sufficiently small $z > 0$ at the zero-reaction turning point for each sufficiently large $n$. Continuity gives failure just inside the constant-source range. This is a limit of the normalized comparison for the fixed homogeneous law.

## 52 Exact first eigenvalues under small caps

Use the quotient $\int_G |u'|^p / \int_G |u|^p$ on continuous edgewise Sobolev functions vanishing at the Dirichlet marks. Write $b_1$ for the cycle rank and $\ell$ for the number of degree-one vertices. The Dirichlet inradius is the greatest distance to the marked set. Kennedy's length–diameter argument uses one-ended Dirichlet stars [12]. Del Pezzo and Rossi develop the mixed first $p$-eigenvalue and its length derivatives [13]. The length–inradius bounds, finite brooms and leading cap asymptotics in Sections 17–19 and 32–34 provide the fixed-inradius setting. We determine the exact optimization at small caps.

**Theorem 52.1** (Three leaves)**.** *Let $T$ be a finite metric tree of length $L$, rooted height $R$, and at most three nonroot leaves. Impose Dirichlet condition at the sole marked root and the usual Kirchhoff condition elsewhere. Assume $0 < R < L \le 3R$. Then the unique minimizer of the first linear eigenvalue is the equal three-arm broom, with stem $a = (3R - L)/2$ and arm $b = (L - R)/2$. The zero-stem case is the equal star.*

## 53 Leaf alignment and leaf count

For a tree with positive stem, choose the positive first eigenfunction $u$ normalized in $L^2$, and write its eigenvalue as $\lambda$. Away from the root the outward flux is nonnegative, and is positive at the start of every edge of positive descendant length, by integrating $-u'' = \lambda u$ over descendants. Thus $u$ increases along each root path. Write $V(u) = \lambda u^2$. The derivative of $-\lambda$ in one edge length is the constant edge Hamiltonian $u'^2 + V(u)$. Scale edges to unit intervals, differentiate the stationary Rayleigh quotient, and use continuity of the normalized positive first eigenfunction under small changes of positive lengths. Compactness and the Rayleigh principle also give eigenvalue continuity at zero-length contractions, including when a contraction makes the first eigenvalue multiple at a branching Dirichlet root.

Use terminal and junction depths as in the exact resolvent theorem Sections 41–43. Increasing a short leaf depth and its parent's depth in the ratio $1 : 1/(k-1)$, with $k$ the number of children of that parent, preserves length and height. The derivative of $-\lambda$ is the leaf potential minus the parent potential plus the nonnegative flux cross term. It is strictly positive by the positive descendant fluxes. Follow this move until alignment or contraction. From a positive stem the same finite-event argument as in the resolvent proof Section 42 aligns all remaining leaves at height $R$, with strict improvement whenever a move is needed.

At a branching Dirichlet root the components split. If a direct root leaf is short, varying its depth with the parent as above creates a positive virtual stem for every positive deformation parameter. The strict derivative on that open interval and continuity at the starting tree give strict improvement. If a branching root has no short direct leaf, the only unaligned shape with at most three leaves has a direct arm of length $R$ and a separate binary branch. Align the latter within its own component, keeping its length and its own height fixed. This does not increase the minimum of the two component eigenvalues. If that height is smaller than $R$, promote it at fixed component length by increasing the height of its equal two-arm broom, until it equals $R$ or the branch becomes a single path. The derivative of its negative eigenvalue with respect to height is $2(\mathcal{H}_{\text{stem}} - \mathcal{H}_{\text{arm}}) > 0$ while both arms are positive. A remaining short direct path can then be aligned by the virtual-stem move. Otherwise the aligned tree retains a branching root. Its strict comparison with the equal three-arm broom is given below.

For equal brooms at fixed $L, R$, the optimized first eigenvalue strictly decreases with feasible leaf count. To see this, let a single arm of length $b$ carry real multiplicity $m = 1 + (L - R)/b$, with stem $R - b$, and normalize the weighted eigenfunction in weighted $L^2$. At the current value of $\lambda$, take the partial length and multiplicity derivative of the stationary quadratic action $\int (V(u) - u'^2)$ with $\lambda$ held fixed. Rayleigh stationarity and normalization identify this partial derivative with $-\lambda'(b)$. Since the arm flux decreases from $F_0 > 0$ to zero, the same weighted calculation as for the datum-one resolvent gives

$$-\lambda'(b) = \mathcal{H}_{\text{arm}} - \mathcal{H}_{\text{stem}} + \frac{2(m-1)}{b} \int_{\text{arm}} u'^2 \le -(m-1)^2 F_0^2 < 0.$$

Here the Hamiltonian difference is $-(m^2 - 1)F_0^2$ by flux matching. Thus shorter equal arms at higher multiplicity strictly improve the minimum. A zero stem follows by continuity. Since $L > R$ excludes a path, it suffices to compare the two possible aligned three-leaf branching types.

## 54 Exact two-junction comparison

With three leaves, an aligned tree is an equal three-arm broom or has two binary junctions. In the latter case write their depths $a - \delta, a + \delta$ with $\delta > 0$. The length relation forces the same $a, b$ as for the three-arm broom, because $L = 3R - 2a$. The inter-junction edge has

length $2\delta$, each deeper arm has length $b-\delta$, and the remaining shallow arm has length $b+\delta$. Feasibility gives $\delta \le \min(a,b)$. The star at $L = 3R$ is forced by equality between total length and the sum of the three root-to-leaf lengths.

For $a, b > 0$ let $k$ be the first positive wavenumber of the equal three-arm broom. Write $A = \tan(ka)$, $B = \tan(kb)$, and $C = \tan(k\delta)$. Then $3AB = 1$ and $kR < \pi/2$. In particular $C \le B$ and $3BC \le 1$. A mixed arm of length $\ell$ has dimensionless admittance $\tan(k\ell)$. Series transfer through a segment of tangent $t$ changes distal admittance $U$ to $(U+t)/(1-Ut)$, and parallel admittances add.

At the shallow junction, the branch through the deeper junction has admittance

$$\frac{2(B+C^3)}{D}, \qquad D = 1 - 3BC + 3C^2 - BC^3,$$

by transferring the sum of the two deeper arm admittances through $2\delta$. The other branch has $(B+C)/(1-BC)$. Both denominators are positive because $BC \le 1/3$ and $D = 1 - 3BC + (3-BC)C^2 > 0$. The stem tangent is $(1-3BC)/(3B+C)$.

The secular quantity at the broom wavenumber is one minus the product of the stem tangent and the sum of the two branch admittances. After clearing its positive denominator $(3B+C)(1-BC)D$, its numerator equals

$$2C\left[B^2 + (4B - 3B^3)C - (4B^2+1)C^2 + 7BC^3 - 4B^2C^4\right].$$

This is strictly positive. Indeed $B^2 - 3B^3C \ge 0$, $C^2 \le BC$, $4B^2C^2 \le (4/3)BC$ and $4B^2C^4 \le (4/3)BC^3$. The bracket is at least $(5/3)BC + (17/3)BC^3 > 0$. There is no pole below this test wavenumber, since all branch tangents and their inputs are nondecreasing up to their first poles and the displayed denominators at the test point stay positive. Therefore the two-junction tree's first eigenvalue is strictly larger. If its stem is zero, the root splits the two branches and the same positive denominators exclude a branch pole at or below $k$.

This proves the rooted three-leaf theorem. It uses the existing length-preserving alignment and weighted arm comparisons, then an exact spectral comparison instead of the deepest-junction inequality used for decreasing reaction.

## 55 Exact topological caps three and four

**Theorem 55.1** (Linear topological caps)**.** *Let $G$ be a finite connected metric graph with at least one Dirichlet mark, length $L > R > 0$, and Dirichlet inradius $R$. Under the cap $2b_1(G) + \ell(G) \le n+1$ for $n = 2, 3$, the optimal first linear eigenvalue is the $n$-arm broom eigenvalue at length $\min(L, (n+1)R/2)$ and height $R$. In particular for $L \ge (n+1)R/2$ its exact value is*

$$\frac{4}{R^2}\arctan^2\!\left(\frac{1}{\sqrt{n}}\right).$$

*For $L \le (n+1)R/2$, the equality types are the equal $n$-arm broom and its permitted disjoint tip-pair quotients.*

Apply the distance-preserving cut of Section 30 and split a Dirichlet root into copies for its branches. The cut relaxes the eigenvalue variational constraints. Each rooted component has at most $n$ leaves, height at most $R$ and volume $X \le L$, and the cut eigenvalue is the minimum of the component eigenvalues. For $n = 2$ the aligned two-arm proof follows directly from the leaf-depth move. For $n = 3$ apply the theorem above. At any fixed $X$, promote the optimized component height to $\min(R, X)$, using

$$\frac{d(-\lambda)}{dh} = \frac{n}{n-1}(\mathcal{H}_{\text{stem}} - \mathcal{H}_{\text{arm}}) > 0$$

when both lengths are positive. The endpoint cases follow by continuity. If $X \le R$, the resulting path has first eigenvalue at least $\pi^2/(4R^2)$.

For $X > R$, the first wavenumber is given by

$$n \tan(ka) \tan(kb) = 1, \quad a = R - b, \quad b = \frac{X - R}{n - 1}.$$

At fixed $kR < \pi/2$, $\tan(ka)\tan(kb)$ is strictly increasing as $|a - b|$ decreases, by its cosine quotient. Hence the eigenvalue decreases strictly until $a = b = R/2$ at $X = (n+1)R/2$ and increases thereafter. Minimization over component lengths $X \le L$ gives the asserted lower bound. In the strictly decreasing range, and also at its endpoint, an equality component must use all volume. The rooted equality theorem forces one broom before folding, and restoring the graph can identify only equal-depth tip groups. A group of $k \ge 2$ tips spends $k - 2$ units of the topological cap, as in the exact resolvent proof. Only pairs fit the saturated cap. The symmetric no-flux tips allow all these pair quotients.

For arbitrary $L > (n+1)R/2$, attach the residual length as a path at the broom's marked root. Put Dirichlet marks on its far endpoint and at enough intermediate points that every subinterval of the added path is at most $2R$. This path has inradius at most $R$ and first eigenvalue at least $\pi^2/(4R^2)$. It is separated from the broom by the marked junction, so the full graph has exactly the broom's smaller first eigenvalue. The number of degree-one vertices is $n + 1$, there are no cycles, and the broom retains inradius $R$. Thus the theorem holds for all $L > R$, without a bound on the number of Dirichlet marks.

## 56 Topological cap three for every $p > 1$

Use the standard Rayleigh quotient $\int |u'|^p / \int |u|^p$, with eigenvalue $\lambda$. Under a rooted cap of two leaves the equal two-arm broom is uniquely optimal for every $p > 1$. The leaf-alignment proof applies verbatim with flux $F = |u'|^{p-2}u'$ and Hamiltonian $(p-1)|u'|^p + \lambda|u|^p$. Simplicity and positivity hold on a rooted tree with positive stem. If its Dirichlet root branches into two paths, the strict virtual-stem move repairs a short leaf, and the fully aligned zero-stem case is the equal star. A path cannot have $L > R$.

We use the within-broom comparison of Section 17 in the following spectral phase coordinate. For a symmetric $m$-arm broom with integer $m \ge 2$, put

$$A_p(t) = \int_0^t \frac{dr}{1 + r^p}, \qquad \alpha_p = A_p(\infty), \qquad \Gamma = m^{1/(p-1)}.$$

For $t > 0$ set

$$C_p(t) = \alpha_p - A_p(\Gamma t), \qquad k(t) = \frac{A_p(t) + C_p(t)}{R}, \qquad a(t) = \frac{A_p(t)}{k(t)}, \quad b(t) = \frac{C_p(t)}{k(t)}.$$

The first eigenvalue of this broom is $(p-1)k(t)^p$. To verify this, normalize the increasing $p$-sine by $S(0) = 0$, $S' = (1 - S^p)^{1/p}$ until $S = 1$. Its tangent $S/S'$ satisfies $T' = 1 + T^p$ and therefore $T^{-1} = A_p$. The ratio of outward flux to $u^{p-1}$ is $k^{p-1}/T^{p-1}$. At the hub the stem phase is $ka = A_p(t)$; the arm phase there is $\alpha_p - kb = A_p(\Gamma t)$. The $m$ equal arm fluxes match the stem. The constructed profile is positive on the non-Dirichlet tree, so it is the first mode.

The volume $X(t) = a(t) + mb(t) = R + (m-1)b(t)$ decreases strictly from $mR$ to $R$. The quantity $k(t)$ is minimized at the unique maximizer of $A_p(\Gamma t) - A_p(t)$, namely

$$t_* = \left(\frac{\Gamma - 1}{\Gamma^p - \Gamma}\right)^{1/p}.$$

This follows by differentiating the two integrals and clearing their positive denominators. Thus the broom eigenvalue strictly decreases as $X$ grows from $R$ to $X_* = X(t_*)$ and increases thereafter.

For $m = 2$ the cut and height-promotion argument now yields the exact first $p$-eigenvalue under the topological cap $2b_1 + \ell \le 3$: it is the two-arm broom value at volume $\min(L, X_*)$ and height $R$. In the initial range the equality types are the rooted broom and its tip-pair folding. Above $X_*$ the residual marked path construction gives attainment without increasing the cap, since its component eigenvalues are bounded below by the mixed interval value $(p-1)(\alpha_p/R)^p$, strictly larger than the balanced optimal broom value. For $p = 2$ this reduces to $X_* = (m+1)R/2$.

## 57 Exact two-edge $p$-eigenvalue phase diagram

**Theorem 57.1** (Two edges)**.** *For a presentation with at most two edges the feasible inradii are $L/4 \le R < L$, as in Section 44. If $R \le L/2$, every feasible marked graph has first eigenvalue $(p-1)(\alpha_p/R)^p$: its marked presentation splits into mixed intervals and doubly marked intervals, and the first eigenvalue of each is the mixed interval eigenvalue at its own inradius. The minimum comes from a piece of inradius $R$.*

*If $R > L/2$, the unique minimizing marked type is the loop with a stem and only the outer tip of the stem marked. Its value is the two-arm broom eigenvalue with $a = 2R - L$, $b = L - R$.*

The classification of the marked presentations in Section 44 leaves the tip-marked loop and stem as the only noninterval case. Its loop midpoint cut gives the equal two-arm broom with the stated lengths. To see the strict improvement over their value, take the mixed $p$-eigenfunction on a path of length $R$ and copy its terminal segment $[a, R]$ onto the extra arm. At the mixed path eigenvalue $\lambda$, integration by parts on that segment gives

$$\int_a^R (|u'|^p - \lambda u^p) = -|u'(a)|^{p-2}u'(a)u(a) < 0.$$

The extra arm therefore strictly lowers the Rayleigh quotient. Equal arms of the optimizing broom have equal terminal traces, so folding those tips attains its eigenvalue. Its $p$-eigenvalue may be read from the preceding symmetric-broom equation.

## 58 Local stability at an interior volume

**Theorem 58.1** (Local stability)**.** *Fix $p > 1$, $n \ge 2$, and $0 < R < L < nR$. In the space of rooted metric trees of length $L$, height $R$, sole Dirichlet root, and at most $n$ nonroot leaves, the equal $n$-arm broom is a strict local minimizer of the first $p$-eigenvalue. More precisely, its deficit grows linearly with the sum of all new internal branch lengths and the losses of terminal depth. This holds for every $n, p$, including the regime where the global optimizer changes near maximal volume.*

Let $B_n$ have stem $a > 0$ and arms $b > 0$. Suppress unmarked degree-two vertices and use the topology from edge lengths on the finitely many rooted skeletons, allowing zero edges to contract. Any tree sufficiently close to $B_n$ has $n$ positive terminal arms, a positive initial stem, and may refine the hub by short edges between branching vertices. A smaller leaf count cannot converge to the broom, and no new positive branch is available at the root or an arm interior without using an extra leaf. For each new internal edge $e$, let $\ell_e$ be its length and $m_e$ the number of its descendant leaves, so $2 \le m_e < n$. Let $r_i \le R$ be the terminal depths and put $d = \sum_i (R - r_i)$.

Write $\Delta a$ for the change in initial stem length. Counting each edge in the sum of leaf depths, and subtracting the total length constraint, gives the exact identity

$$(n-1)\Delta a = -d - \sum_e (m_e - 1)\ell_e.$$

At the broom let $u$ be its positive first mode normalized by $\int u^p = 1$, let $F > 0$ be the arm input flux, and set $q = p/(p-1)$. The edge length derivative of the negative eigenvalue is the Hamiltonian $\mathcal{H} = (p-1)|u'|^p + \lambda u^p$. It follows from the stationary Rayleigh quotient in unit-interval coordinates. At the broom limit the initial stem Hamiltonian is $(p-1)n^q F^q + V_h$, the limit on a new edge $e$ is $(p-1)m_e^q F^q + V_h$, and the arm Hamiltonian is $(p-1)F^q + V_h$, where $V_h = \lambda u(\text{hub})^p$.

For these limits use normalized variational convergence under edge contraction. The nonnegative modes converge uniformly to the unique positive broom mode, by compactness and simplicity. The outward flux on any edge is $\lambda \int_{\text{desc}} u^{p-1}$; this converges to $m_e F$ along a collapsing internal edge. The derivative argument can be made on the open line segment interpolating between the broom and the nearby positive-edge lengths. The total length is constant there. Leaf depths interpolate linearly, with at least one leaf at $R$ throughout, so the height is also constant. Integrate the derivatives on this segment and pass to the collapsed endpoint using continuity. The uncertainty is $o(\|\Delta\ell\|_1)$, uniformly across the finitely many refinements.

Using the exact length constraint to eliminate the sum of arm length changes, the first-order change in the negative eigenvalue equals

$$(p-1)F^q \left[ (n^q - 1)\Delta a + \sum_e (m_e^q - 1)\ell_e \right].$$

Equivalently the eigenvalue deficit has the expansion

$$\lambda(T) - \lambda(B_n) = (p-1)F^q \left[ \frac{n^q - 1}{n-1}\, d + \sum_e \left\{ \frac{n^q - 1}{n-1}(m_e - 1) - (m_e^q - 1) \right\} \ell_e \right] + o\left( d + \sum_e \ell_e \right).$$

Here $\|\Delta\ell\|_1 = O(d + \sum_e \ell_e)$, since each arm length change is its terminal depth change minus the changes along the preceding stem and new internal edges. The coefficients in braces are positive by the strict increase of $(m^q - 1)/(m - 1)$ for $m > 1$. This gives the strict local minimizer and a uniform positive linear lower bound in a neighborhood of $B_n$.

The statement is local in rooted tree geometry at fixed finite leaf cap. It supplies a first-order certificate for every possible refinement of the broom junction.

## 59 Exact optimizers near path volume

**Theorem 59.1** (Near-path values)**.** *Fix $p > 1$, $n \geq 2$ and height $R > 0$. For all $L > R$ sufficiently close to $R$, the equal $n$-arm broom is the unique rooted minimizer of the first $p$-eigenvalue among trees with sole Dirichlet root and at most $n$ nonroot leaves. The length neighborhood may depend on $p, n, R$.*

Use the Rayleigh quotient with eigenfunction normalized in $L^p$, flux $F = |u'|^{p-2}u'$ directed away from the root, $q = p/(p-1)$, and edge Hamiltonian $\mathcal{H} = (p-1)F^q + \lambda u^p$. On a positive-stem tree the ground state is positive and simple. Its flux is the integral of $\lambda u^{p-1}$ on its descendants. In particular it is nonnegative and decreases along an edge. The variation of $-\lambda$ in an edge length is $\mathcal{H}$, by the standard Rayleigh shape derivative.

First consider trees with positive stem. The short-leaf variation from Section 42 and the preceding eigenvalue proof strictly improves the eigenvalue until all terminal depths are $R$,

unless the tree is already aligned. It moves the short leaf deeper and its parent deeper by reciprocal branching excess, preserving length and height. Discard zero edges by continuity and continue through the finitely many alignment or contraction events. A positive stem remains positive.

Put $\delta = L - R$. In an aligned tree the length identity is

$$\delta = \sum_v (k_v - 1)(R - s_v),$$

with $s_v$ the junction depth and $k_v$ the number of its children. Every junction lies at depth at least $R - \delta$, and their descendants lie within distance $\delta$ of the first junction. The first eigenvalue is at most the mixed path value $\lambda_R$, by extending the mixed path trial constantly along side branches. The total length beyond the first junction is at most $2\delta$. On that part of the tree the descendant flux estimate gives $F \le 2\lambda_R \delta\, u_{\max}^{p-1}$. Integrating $u' = F^{1/(p-1)}$ along a path of length at most $\delta$ to a maximizing terminal yields

$$\frac{u(v)}{u_{\max}} \ge 1 - (2\lambda_R \delta^p)^{1/(p-1)}$$

at every junction, uniformly through the aligned deformations.

Take a deepest junction $v$ with branching parent $P$. Its $k = k_v$ arms are equal, and by symmetry their input fluxes have one common value $B > 0$. The child subtree contributes at least $kB$ to its incoming flux at $P$. For every other child of $P$, keep a path from $P$ to an aligned leaf. Its actual outgoing flux at $P$ is bounded below by $\lambda u(P)^{p-1}(R - s_P)$, because the path is in its descendant set and $u$ increases. On the other hand $B \le \lambda u_{\max}^{p-1}(R - s_v)$. Since $s_P \le s_v$, every sibling flux at $P$ is at least $(1 - O(\delta))B$.

For a junction of $k_w$ children with fluxes $F_j$, the depth derivative of $-\lambda$, divided by branching excess, is

$$Q_w = -\lambda u(w)^p + \frac{p-1}{k_w - 1}\left[(\sum_j F_j)^q - \sum_j F_j^q\right].$$

The power bracket is coordinatewise nondecreasing. With entries $kB, B, \dots, B$ at $P$, the bracket per excess is at least $B^q[(k+1)^q - k^q - 1]$ by increasing convex increments. At $v$ it is exactly $B^q(k^q - k)/(k-1)$, strictly smaller. By continuity the strict comparison remains valid with entries $kB, (1 - O(\delta))B, \dots$, uniformly for the finite set of possible branch counts under leaf cap $n$. The potential difference also gives $Q_P \ge Q_v$ in the favorable direction, because $u(P) \le u(v)$. Consequently $Q_P > Q_v$ for sufficiently small $\delta$.

Move $P$ deeper and $v$ shallower at reciprocal branching-excess rates. This preserves all aligned leaf depths and the length. The two derivatives give strict improvement of $\lambda$ until the first collision of junctions. The uniform comparison applies throughout the move because the length identity keeps every junction within $\delta$ of terminal depth. Contract and repeat until there is one junction. This gives the equal broom at some feasible count $m \le n$.

The first eigenvalue strictly decreases with feasible arm count at fixed $L, R$. In the weighted one-arm problem use multiplicity $m = 1 + (L - R)/b$ and stem $R - b$. Differentiating the stationary Rayleigh action $\int(\lambda u^p - |u'|^p)$ with $\lambda$ held at the current eigenvalue identifies its partial derivative with $-\lambda'(b)$. As in the weighted resolvent calculation,

$$-\lambda'(b) = \mathcal{H}_{\text{arm}} - \mathcal{H}_{\text{stem}} + \frac{p(m-1)}{b}\int_{\text{arm}} |u'|^p \le [p(m-1) - (p-1)(m^q - 1)]F_0^q < 0,$$

where the $m$ equal input fluxes sum to the stem flux, and the arm flux is bounded by $F_0$. The last inequality is strict by convexity. Decreasing $b$ increases the count. Thus among positive-stem trees equality forces the equal $n$-arm broom.

For a tree with branching Dirichlet root, there is only one branch reaching height $R$ when $L < 2R$, and all other root branches have total length at most $\delta$. Their eigenvalues are at least $\delta^{-p}$, by the rooted path Sobolev bound, so for small $\delta$ the long branch alone carries the first eigenvalue. Its rooted volume $X \in [R, L)$ is strictly less than $L$, its height is $R$, and it has positive local stem. The positive-stem result gives its eigenvalue at least that of the $n$-arm broom of volume $X$ and height $R$, with the path at $X = R$. The exact symmetric-broom phase formula shows this value is strictly decreasing in volume up to its minimizing volume $X_*$. Hence its eigenvalue is strictly greater than that of the equal $n$-arm broom of volume $L < X_*$. This completes the proof.

For an explicit sufficient range, let $c_*$ be the maximum, over integers $2 \le k < n$, of the unique roots $c_k \in (0, 1)$ of

$$(k + c_k)^q - k^q - c_k^q = \frac{k^q - k}{k - 1},$$

and take $c_* = 0$ if $n = 2$. The left side is strictly increasing in $c_k$. Increasing convex increments show that the parent power bracket per excess is at least the left side with $c_k$ replaced by the actual sibling flux lower ratio. Therefore the proof applies whenever $L < 2R,\ L < X_*,\ \delta^{-p} > \lambda_R$, and

$$1 - (2\lambda_R \delta^p)^{1/(p-1)} > c_*^{1/(p-1)}.$$

In particular, at $p = 2$ it applies for every $n \ge 2$ throughout $0 < (L - R)/R < 1/\pi$.

### 59.1 Arbitrary graph caps near path volume

For fixed $p, R$ and a fixed cap the same conclusion gives the exact global first $p$-eigenvalue over arbitrary connected graphs for all sufficiently small $L - R > 0$, under either the topological cap $2b_1 + \ell \le K$ with $n = K - 1 \ge 2$, or the edge cap $J$ with $n = 2J - 2 \ge 2$. The exact value is the equal $n$-arm broom eigenvalue. Under the topological cap the equality types are its allowed disjoint tip-pair quotients. Under the edge cap every tip must pair, giving the equal-loop stem.

Indeed the distance-preserving marked cuts in Section 30 give a collection of rooted components of height at most $R$, at most $n$ leaves each, and total length $L$. The cut Rayleigh minimum bounds the original eigenvalue from below. For $L < 2R$ exactly one component reaches $R$, and every other component has total length at most $L - R$. The steep eigenvalue lower bound on these short components leaves the first cut eigenvalue on the height-$R$ component, since its mixed-path trial gives a value at most $\lambda_R$. Apply the rooted theorem at that component volume and then use the strict near-path volume monotonicity. Equality forces the component volume to be $L$, so the tree is the sole rooted broom after cutting. Restoring the graph identifies only equal-depth tips. The same exact topological and edge count accounting as for the capped resolvent Section 43 gives the stated pairings. The symmetric eigenfunction has equal traces and zero flux at the tips, so those pairings attain equality for the first $p$-eigenvalue.

## 60 Exact rooted optimizers near maximal volume

Fix $p > 1$, $n \ge 2$ and height $R > 0$. Write

$$q = \frac{p}{p - 1}, \qquad \alpha_p = \int_0^\infty \frac{dt}{1 + t^p}, \qquad \lambda_R = (p - 1)(\alpha_p / R)^p,$$

so that $\lambda_R$ is the first mixed interval eigenvalue at length $R$. Consider rooted trees with sole Dirichlet root, at most $n$ nonroot leaves, and total length $L = nR - \varepsilon$. As $\varepsilon \downarrow 0$, the sharp

infimum of the first $p$-eigenvalue is

$$\lambda_R - \frac{p\lambda_R}{R}\left[\max_{2\le m\le n}\frac{m^{\,q-1}-1}{m-1}\right]\varepsilon + o(\varepsilon). \tag{1}$$

For $1 < p < 2$ the maximum is at $m = n$ and the equal $n$-arm broom gives the sharp first-order term. For $p = 2$ the ratios are all one. For $p > 2$ the maximum is at $m = 2$. In the last case, if $n \ge 3$, the equal $n$-arm broom is strictly suboptimal for every sufficiently small positive $\varepsilon$, even among rooted trees with positive stem.

## 60.1 Upper bound on the possible eigenvalue gain

For $\varepsilon < R$ the tree must have exactly $n$ leaves, because total length is at most the sum of terminal depths. Suppress unmarked degree-two vertices. Write $r_i = R-\delta_i$ for the leaf depths, let $\ell_e$ be the lengths of the core edges terminating at a junction, including the initial stem if it is positive, and let $D_e$ be their descendant leaf sets with $m_e = |D_e| \ge 2$. Counting edges in the depths yields the exact identity

$$\varepsilon = \sum_i \delta_i + \sum_e (m_e - 1)\ell_e. \tag{2}$$

Each core edge and each leaf-depth deficit is $O(\varepsilon)$. The terminal arms have lengths $R - \delta_i - \sum_{e:\, i\in D_e} \ell_e$.

It suffices first to work with positive stem. A tree with a branching Dirichlet root is approximated at the same $L, R$ by positive-stem trees as follows. Add a stem of length $\eta$, multiply the original core edge lengths and all $\delta_i$ by $1 - (n-1)\eta/\varepsilon$, and choose the terminal arms from the new path lengths. Equation (2) gives the same volume and a zero $\delta_i$ keeps the height $R$. Taking $\eta \downarrow 0$ and using Rayleigh continuity gives the desired extension.

Interpolate the core edge lengths from zero to their target lengths, and each terminal arm length from $R$ to its target, with parameter $t \in [0, 1]$. At $t = 0$ the tree is the $n$-arm star marked at the hub. At each $t > 0$ the stem is positive and the first mode is simple and positive. Normalize that mode in $L^p$. As $\varepsilon \downarrow 0$, every subsequential limit of the modes on the terminal arms lies in the normalized star first-mode set, of the form $c_i\phi$ on arm $i$, where $\phi \ge 0$ is the unit-normalized mixed interval mode and $\sum_i c_i^p = 1$. This follows from uniform compactness on edges, small oscillation on the shrinking core, convergence of the Rayleigh minimum under contraction, and the disjoint variational problem at the star. It holds uniformly in $t \in (0, 1]$ and over the finitely many skeletons in the sense of distance to that compact set of limiting modes.

Let $f_i = c_i^{p-1}|\phi'(0)|^{p-1}$ for any such limiting amplitudes. The core flux converges to $\sum_{i\in D_e} f_i$ because the flux equals the integral of $\lambda u^{p-1}$ on its descendants, and the shrinking core contributes $o(1)$. Its Hamiltonian therefore approaches $(p-1)(\sum_{i\in D_e} f_i)^q$, since the state tends to zero at the collapsed root. The terminal arm Hamiltonian approaches $(p-1)f_i^q$. In these limits the coefficients can vary with the tree and $t$, but every such vector satisfies

$$\sum_i f_i^q = |\phi'(0)|^p = \frac{p\lambda_R}{(p-1)R}. \tag{3}$$

The last equality follows by integrating the constant mixed-interval Hamiltonian $(p-1)|\phi'|^p + \lambda_R\phi^p$.

By the edge-length derivative of the stationary Rayleigh quotient, the $t$-derivative of the negative eigenvalue is bounded above, up to a uniform $o(\varepsilon)$, by

$$(p-1)\left[\sum_e \ell_e\left\{\left(\sum_{i\in D_e} f_i\right)^q - \sum_{i\in D_e} f_i^q\right\} - \sum_i \delta_i f_i^q\right].$$

Here the subtraction accounts for precisely the lengths removed from the terminal arms. Hölder gives

$$\left(\sum_{i\in D_e} f_i\right)^q - \sum_{i\in D_e} f_i^q \le (m_e^{q-1}-1)\sum_i f_i^q.$$

Drop the nonpositive shortfall term and apply (2)–(3). This proves that the $t$-derivative is at most the coefficient of the gain in (1), uniformly up to $o(\varepsilon)$. Integrate using continuity at the star. The uniform error follows by contradiction using the stated compactness, since the total absolute length variation is $O_n(\varepsilon)$.

### 60.2 Sharp witnesses and counterexamples

For any $2 \le m \le n$, take an $m$-arm broom with stem $a = \varepsilon/(m-1)$ and $m$ arms of length $R - a$. Add $n - m$ paths of length $R$ at its marked root. The total length is $nR - \varepsilon$. The paths and broom decouple at the Dirichlet root. The first eigenvalue is the broom eigenvalue, which is strictly smaller than $\lambda_R$.

To compute its leading gain, use the exact symmetric-broom phase equation with $\Gamma = m^{1/(p-1)}$ and $A_p(x) = \int_0^x (1+r^p)^{-1}dr$:

$$ka = A_p(x), \qquad kR = \alpha_p - [A_p(\Gamma x) - A_p(x)], \qquad \lambda = (p-1)k^p.$$

Since $A_p(x) = x + O(x^{p+1})$ and $x \sim \alpha_p a/R$, this gives

$$\lambda = \lambda_R - \frac{p\lambda_R}{R}(\Gamma - 1)a + o(a).$$

Choosing the $m$ that maximizes the ratio in (1) gives equality to first order. The location of the maximum follows from strict concavity of $m^{q-1}$ for $p > 2$ and strict convexity for $p < 2$.

For $p > 2, n \ge 3$, the strict gap between the ratio at $m = 2$ and at $m = n$ proves that the binary broom plus the remaining paths beats the equal $n$-arm broom for sufficiently small $\varepsilon > 0$. The fixed-volume positive-stem approximation above preserves that strict inequality for each such $\varepsilon$. Thus the global eigenvalue optimizer differs from the equal broom in this range, although the latter is a strict local minimizer at every fixed $L < nR$.

### 60.3 First-order geometry of near-optimal trees

For any sequence attaining the coefficient in (1), the total terminal-depth loss is $o(\varepsilon)$. So is the weighted core length on edges whose descendant number is outside the set maximizing the ratio in (1), by the strict gap in the derivative bound for those numbers. In particular if $1 < p < 2$, the sole initial stem carries all the leading length deficit, with $(n-1)a = \varepsilon + o(\varepsilon)$.

If $p > 2$ and $n \ge 3$, there is a single binary core edge of length $\varepsilon + o(\varepsilon)$, and all other core lengths total $o(\varepsilon)$. Indeed, the descendant pairs of two distinct edges with $m_e = 2$ are disjoint after suppressing degree-two vertices. If the sets were nested and equal, there would be a branching vertex between them with no additional descendant leaf. Retain the restricted $\sum_{i\in D_e} f_i^q$ in Hölder for these edges. Their weighted sum over disjoint pairs is at most $\max_{m_e=2} \ell_e \sum_i f_i^q$. Therefore first-order optimality forces one binary edge to carry the entire leading deficit. A positive common stem has $m_e = n$ and hence length $o(\varepsilon)$.

### 60.4 Exact near-star geometry away from $p = 2$

For each fixed $1 < p < 2$ and $n \ge 2$ the equal $n$-arm broom uniquely minimizes the rooted first $p$-eigenvalue for all sufficiently small $\varepsilon > 0$. For $p > 2, n = 2$ the earlier two-leaf theorem applies. For each fixed $p > 2$ and $n \ge 3$, the unique rooted minimizing shape for all sufficiently

small $\varepsilon > 0$ is a binary broom of stem length $\varepsilon$ and arm length $R - \varepsilon$, with $n - 2$ paths of length $R$ attached at the same marked root. Its value is the exact first mode of the binary broom. Among trees required to have positive stem, the same infimum is approached and is unattained in this last regime.

Existence in the unrestricted rooted class follows from the finite skeleton compactification and Rayleigh continuity. Consider any sequence of minimizers with $\varepsilon \downarrow 0$. The geometry of the preceding section holds along this sequence. Let $u \geq 0$ be an $L^p$-normalized first mode on each tree and put $k = (\lambda/(p-1))^{1/p}$. The sharp positive gain in (1) implies $\alpha_p - kR \asymp \varepsilon$. On a terminal arm of length $b_i$, the Neumann endpoint value $M_i$ and the initial value obey

$$u(P_i) = M_i S(\alpha_p - k b_i),$$

where $S$ is the increasing $p$-sine with $S(t) \sim t$ at zero. The phase $\alpha_p - k b_i$ is comparable to $\varepsilon$ for every arm. The descendant formula bounds all derivatives uniformly on the shrinking core.

If $1 < p < 2$, the initial stem is positive and all remaining core lengths total $o(\varepsilon)$. The values $u(P_i)$ differ from one another by $o(\varepsilon)$, and the phases differ by $o(\varepsilon)$, since the terminal-depth losses total $o(\varepsilon)$. The terminal amplitudes $M_i$ therefore differ by $o(1)$, and their common limiting value is positive by normalization. If $p > 2$, the two arms after the long binary edge share one parent and have asymptotically equal phases, so their amplitudes differ by $o(1)$. Every other $P_i$ is at depth $o(\varepsilon)$, so $u(P_i) = o(\varepsilon)$ and those amplitudes tend to zero. Thus the two active arms have the same positive limiting amplitude.

The eigenvalue used in the shape derivative is simple. It is simple whenever the initial stem is positive. Otherwise, when $p > 2$, each root branch outside the long binary edge has only $o(\varepsilon)$ terminal-depth loss and core-length budget. Its first eigenvalue is at least $\lambda_R - o(\varepsilon)$ by the proved uniform near-star bound after rescaling to its own height. A branch with one leaf has the mixed interval lower bound $\lambda_R$. Since the minimizer has $\lambda_R - c\varepsilon + o(\varepsilon)$ with $c > 0$, only the root branch containing the binary edge carries the first eigenfunction. Its own root has degree one, so that first mode is simple.

At any such minimizer, treat the $\delta_i$ and core lengths $\ell_e$ as independent coordinates subject to the budget (2) and $\min_i \delta_i = 0$. The derivative of $-\lambda$ with respect to $\delta_i$ is $-\mathcal{H}_i \leq 0$, where $\mathcal{H}_i$ is the Hamiltonian on arm $i$. The gain per unit of volume deficit on a core edge $e$ is

$$G_e = \frac{\mathcal{H}_e - \sum_{i \in D_e} \mathcal{H}_i}{m_e - 1}.$$

Indeed, increasing a core length shortens each descendant terminal arm by that length. The same contraction limits of the Hamiltonians apply as before. If $1 < p < 2$, equal limiting amplitudes give the limiting $G_e$ proportional to $(m_e^q - m_e)/(m_e - 1)$, with a common positive factor. This ratio is strictly increasing in $m_e$, so the initial stem has a strictly higher gain than every other core edge. If $p > 2$, the main binary edge has a strictly positive limiting gain. An ancestor edge has the same limiting numerator but denominator at least two, and all disjoint core edges have zero limiting numerator. A core edge cannot contain just one of the active pair below their binary junction, where both children are terminal arms. Therefore the long binary edge has a strict gain gap in this regime as well.

If a minimizer retains a positive deficit on a short terminal depth or on any core edge other than the winning edge, transfer a sufficiently small amount of that deficit to the winning edge. The budget and height stay fixed, and every edge remains positive when needed for its derivative. The strict gain inequalities make this transfer decrease the first eigenvalue. This contradicts optimality. The argument holds eventually along every sequence of minimizers, so for all sufficiently small $\varepsilon$ the winning edge carries the entire budget. Equation (2) then gives exactly the claimed rooted shapes and uniqueness. Positive-stem approximation proves the stated infimum when the unique optimum branches at the root.

## 60.5 The linear case selected by the cubic term

For $p = 2$ and every $n \geq 2$, the equal $n$-arm broom is also the unique rooted optimizer for all sufficiently small $\varepsilon > 0$. The linear $O(\varepsilon)$ gain in (1) ties across descendant counts. The first separating term is cubic.

Consider minimizers as $\varepsilon \downarrow 0$, put $k = \sqrt{\lambda}$, $k_R = \pi/(2R)$, and write $D = \pi/(2k) - R$. By (1), $D \sim \varepsilon$. Let $P_i$ be the parent of terminal arm $i$, $d_i$ its depth, and $F_i$ the outward flux into that arm. Normalize the vector $F$ so $\sum_i F_i^2 = 1$. For points $x, y$ in the core let $w(x, y)$ be the length of their common initial root path. Direct integration of the core equation and Kirchhoff flux gives

$$u(x) = \sum_i w(x, P_i)F_i + k^2 \int_{\text{core}} w(x, y)u(y)\, dy.$$

Indeed the outward flux at a core point is the sum of the distal arm fluxes and the integral of $k^2 u$ in its distal core. This identity applies separately to components at a branching Dirichlet root. Write $W(x) = \sum_i w(x, P_i)F_i$. Since the core has length at most $\varepsilon$ and $w \leq \varepsilon$, it gives $u = W + O(\varepsilon^3)$ uniformly on the core.

At $P_i$ the mixed-arm transfer is

$$u(P_i) = \frac{F_i}{k} \tan\big(k(D + \delta_i + d_i)\big).$$

Multiply by $F_i$, sum, and cancel the diagonal contribution $d_i F_i^2$ from the shared paths. Expanding tan to order three gives

$$D = \langle F, MF\rangle + k^2 \left[\int_{\text{core}} W(x)^2\, dx - \frac{1}{3}\sum_i F_i^2 (D + \delta_i + d_i)^3\right] + O(\varepsilon^5), \tag{4}$$

where

$$\langle F, MF\rangle = \sum_e \ell_e \left[\left(\sum_{i \in D_e} F_i\right)^2 - \sum_{i \in D_e} F_i^2\right] - \sum_i \delta_i F_i^2 \leq \varepsilon.$$

The inequality follows by Cauchy–Schwarz and the budget (2).

Because $D/\varepsilon \to 1$ and the bracket in (4) is $O(\varepsilon^3)$, every subsequential limit saturates this first-order bound. It has $\sum_i \delta_i = o(\varepsilon)$. Every edge with a positive limiting fraction of the budget must have $F$ uniform on its descendants and zero outside them, by equality in Cauchy–Schwarz and $\|F\|_2 = 1$. Two different core edges cannot have identical descendant sets after degree-two suppression. Therefore there is one dominant edge, with $m$ descendants, length $a = \varepsilon/(m-1) + o(\varepsilon)$, and $F_i \to 1/\sqrt{m}$ on those descendants. All other core lengths are $o(\varepsilon)$.

The bracket in (4) then equals

$$\frac{m - m^3}{3(m-1)^3}\varepsilon^3 + o(\varepsilon^3).$$

The core integral comes from the dominant edge, on which $W(x)$ is its along-edge distance times $\sqrt{m} + o(1)$, up to $o(\varepsilon)$ from its initial depth. For the term over arms, each active arm has $D + \delta_i + d_i = m\varepsilon/(m-1) + o(\varepsilon)$. Thus

$$D \leq \varepsilon - \frac{k_R^2}{3}\frac{m^3 - m}{(m-1)^3}\varepsilon^3 + o(\varepsilon^3).$$

The equal $m$-arm broom with its remaining paths gives equality through the cubic term, by the same equation or by expanding its phase formula. The ratio $(m^3 - m)/(m-1)^3 =$

$m(m+1)/(m-1)^2$ decreases strictly in $m \geq 2$. Comparing an optimum to the $n$-arm broom therefore forces $m = n$ in every limiting geometry.

Consequently the common stem is positive, its length is $\varepsilon/(n-1) + o(\varepsilon)$, and the arm amplitudes are asymptotically equal. The marginal gain argument from the preceding section applies, because the limiting core gain is proportional to $(m_e^2 - m_e)/(m_e - 1) = m_e$. The common stem has strictly the largest gain per unit deficit. A transfer from any other positive edge or any short terminal would improve the eigenvalue, so the entire budget is on the common stem. This gives exact uniqueness of the equal broom near maximal volume.

# 61 The near-star crossover at $p = 2$

Normalize the rooted height to one by metric scaling. Fix a leaf cap $n \geq 2$, write $L = n - \varepsilon$, and let

$$p = 2 + \theta\varepsilon^2$$

for $\theta$ in a fixed compact real interval. At a first eigenvalue $\lambda = (p-1)k^p$, define the effective length gain $D = \alpha_p/k - 1$, where $\alpha_p = \int_0^\infty (1+t^p)^{-1}\,dt$. Minimizing $\lambda$ at fixed $p$ is equivalent to maximizing $D$.

For each $m \in \{2, \dots, n\}$, let $T_m$ be the equal $m$-arm broom with stem $\varepsilon/(m-1)$, with $n-m$ full-height paths attached at its marked root. For all sufficiently small $\varepsilon > 0$, uniformly over the specified compact $\theta$-interval, every rooted minimizing tree belongs to this finite family. Its optimal gain satisfies

$$D_{\max} = \varepsilon - \varepsilon^3 \min_{2 \leq m \leq n} \left[ \theta \frac{m \log m}{m-1} + \frac{\pi^2}{12} \frac{m(m+1)}{(m-1)^2} \right] + o(\varepsilon^3). \tag{5}$$

The coefficient selects the exact winning member for fixed $\theta$ away from ties. At the limiting ties the exact broom secular equations compare the two candidates. The formula describes the transition from the common $n$-arm stem to a binary stem as $p$ increases through the window of order $\varepsilon^2$ about two.

## 61.1 A uniform core expansion

Use the core edge lengths $\ell_e$, terminal deficits $\delta_i$, and descendant sets $D_e$ from the near-star argument. With $m_e = |D_e|$, the budget is $\varepsilon = \sum_i \delta_i + \sum_e (m_e - 1)\ell_e$. The parent of arm $i$ is $P_i$, its depth is $d_i$, and its length is $b_i = 1 - d_i - \delta_i$. Normalize the outward arm fluxes $F_i \geq 0$ by $\sum_i F_i^q = 1$, where $q = p/(p-1)$. The formulas hold also for a first mode supported on one component at a branching Dirichlet root.

For an optimum, the common broom trial bounds $\lambda$ above by the mixed interval value. The rooted Poincaré bound bounds $\lambda$, and hence $k$, below by a positive constant uniformly in this $p$-window. To see $D = O(\varepsilon)$, let $M = \|u\|_\infty$ on the shrinking core. Descendant flux integration gives

$$M \leq \varepsilon \big( \textstyle\sum_i F_i + \lambda\varepsilon M^{p-1} \big)^{1/(p-1)},$$

so $M = O(\varepsilon)$. At one terminal arm $F_i$ is bounded below, and the Neumann-arm relation

$$u(P_i) = F_i^{q-1}\, k^{-1} T_p(k(D + \delta_i + d_i))$$

then gives $D = O(\varepsilon)$, since $T_p'(t) = 1 + T_p(t)^p$ and $T_p(t) \geq t$.

Let $w(x,y)$ be shared root-path length for core points and $W(x) = \sum_i w(x, P_i) F_i$. On a core edge the outgoing flux equals $F_e^0 = \sum_{i \in D_e} F_i$ plus the integral of $\lambda u^{p-1}$ on its distal core. The integral term is $O(\varepsilon^2)$ since $u = O(\varepsilon)$ and $p - 2 = O(\varepsilon^2)$. Uniformly for bounded $x \geq 0$,

$$x^{q-1} = x + (q-2)x\log x + O((q-2)^2),$$

with $x \log x = 0$ at zero. Thus the integral term enters the slope to first order with coefficient one up to $o(\varepsilon^2)$, even at zero limiting flux. Integrating the slope gives $u = W + O(\varepsilon^3)$ on the core. Since $k \to \pi/2$, $\lambda u^{p-1} = (\pi^2/4)W + o(\varepsilon)$ uniformly.

Integrate the core slopes to each $P_i$, multiply the matching equation by $F_i$ and sum. The leading core slope powers are $(F_e^0)^{q-1}$ and the distal source correction is the shared-path integral as in the linear case. Use $T_p(t) = t + t^{p+1}/(p+1) + O(t^{2p+1})$ uniformly for $p$ near two, from integrating its differential equation. On phases of size $O(\varepsilon)$, its cubic term agrees with $t^3/3$ up to $o(\varepsilon^3)$. Cancel the diagonal shared-depth terms to obtain

$$\begin{aligned} D = \sum_e \ell_e \left[ (\sum_{i \in D_e} F_i)^q - \sum_{i \in D_e} F_i^q \right] - \sum_i \delta_i F_i^q \\ + \frac{\pi^2}{4} \left[ \int_{\text{core}} W(x)^2 \, dx - \frac{1}{3} \sum_i F_i^2 (D + \delta_i + d_i)^3 \right] + o(\varepsilon^3). \end{aligned} \tag{6}$$

For the cubic terms one may replace $F_i^q$ by $F_i^2 + o(1)$. All errors are uniform on the fixed compact $\theta$-interval and finitely many skeletons.

## 61.2 Selection of a single descendant group

The common broom trial gives $D \geq \varepsilon - O(\varepsilon^3)$, by its exact phase equation. Hölder bounds the leading expression in (6) by $\varepsilon + O(\varepsilon^3)$. Hence $D/\varepsilon \to 1$. As $q \to 2$, first-order equality forces all depth loss to be $o(\varepsilon)$ and one dominant core edge with $m$ descendants to carry the full leading budget. Any core edge carrying positive limiting budget would require the limiting flux vector to be uniform on its descendants and zero elsewhere. Two distinct branching edges have distinct descendant sets. The limiting flux vector is therefore $1/\sqrt{m}$ on the dominant descendants and zero outside, while the dominant length is $\varepsilon/(m-1) + o(\varepsilon)$.

The cubic bracket in (6) is then $(m - m^3)\varepsilon^3/[3(m-1)^3] + o(\varepsilon^3)$. For the first line of (6), retain the exact estimate by the budget and $m_e^{q-1} - 1$. The excess of this coefficient over its linear value $m_e - 1$ is

$$m_e^{q-1} - m_e = -\theta m_e \log(m_e) \, \varepsilon^2 + o(\varepsilon^2).$$

Only the dominant edge contributes to this excess at order $\varepsilon^3$. This bounds $D$ from above by the right side of (5) for the selected $m$. Each $T_m$ attains its coefficient through this order by its exact single-broom phase equation, or directly by (6) with equality in the leading Hölder comparison. Compactness of $\theta$ makes the maximum expansion uniform.

## 61.3 Exact reduction to the finite family

An optimum near the limiting geometry has simple first eigenvalue. If its root splits branches, all components outside the dominant edge have height $1 - o(\varepsilon)$ and only $o(\varepsilon)$ volume-deficit budget. Their gains are $O(o(\varepsilon))$, by the same core bound $D = O(\text{budget})$ rescaled to their own height, while the winning eigenvalue has gain $\varepsilon + o(\varepsilon)$. The active branch has positive local stem and simple ground state. If the common stem is positive, simplicity holds directly.

At a minimizer consider marginal gains per unit deficit as in the exact near-star analysis. In the limiting $p = 2$ flux vector uniform on the dominant $m$ descendants, its edge gives gain proportional to one. An ancestor edge strictly loses because its descendant count is larger and its numerator is the same. A proper descendant edge with $j < m$ active leaves gives gain proportional to $j/m < 1$. A disjoint edge gives zero. A terminal-depth loss has nonpositive gain. These strict gaps persist along every subsequence with the corresponding limiting $m$. A small transfer from any positive nonwinning budget to the dominant edge would therefore improve $\lambda$, holding fixed length and height. It follows that an exact minimizer has only that one core edge and no terminal losses. This is precisely $T_m$.

Finally write $a(m) = m \log m/(m-1)$ and $b(m) = \pi^2 m(m+1)/(12(m-1)^2)$. Then $a' > 0$, $b' < 0$, and

$$-\frac{b'(m)}{a'(m)} = \frac{\pi^2}{12} \frac{3m+1}{(m-1)(m-1-\log m)}$$

is strictly decreasing for $m \geq 2$. Thus $b$ is strictly convex as a function of $a$. The limiting phase boundaries occur between consecutive counts $m, m+1$ at $\theta_m = [b(m)-b(m+1)]/[a(m+1)-a(m)] > 0$, and these thresholds decrease strictly with $m$. Every count from $n$ down to two occurs on a nonempty interval of the limiting scaled parameter.

### 61.4 Exact adjacent thresholds at small positive deficit

For small fixed $\varepsilon > 0$, the adjacent pair $T_m, T_{m+1}$ has exactly one crossing in the scaling window near $\theta_m$. Its location is

$$p_m(\varepsilon) = 2 + \theta_m \varepsilon^2 + o(\varepsilon^2).$$

If the fixed compact window contains all these limiting thresholds in its interior, the exact sequence of minimizing counts is $n, n-1, \ldots, 2$ as $p$ increases across it. At each crossing just the two adjacent optimizing shapes tie.

To prove uniqueness of the crossing, write $r_m(p) = (m^{1/(p-1)} - 1)/(m-1)$ and denote the gain of the individual broom by $D_m$. Its exact secular equation is

$$D_m = \varepsilon r_m(p) + h_m(p, D_m, \varepsilon), \quad h_m = \frac{1+D_m}{\alpha_p} \left[A_p(\Gamma x) - \Gamma A_p(x)\right], \quad A_p(x) = \frac{\alpha_p \varepsilon}{(m-1)(1+D_m)}.$$

For $p$ in a neighborhood of two and $D_m = O(\varepsilon)$ we have $x \asymp \varepsilon$. The integral defining $A_p$ gives $h_m = O(\varepsilon^{p+1})$, $\partial_{D_m} h_m = O(\varepsilon^{p+1})$, and $\partial_p h_m = O(\varepsilon^{p+1} |\log \varepsilon|)$. In detail $A_p(x) - x$ and its fixed-$x$ $p$-derivative are bounded by these powers at scale $x$, the derivative of its remainder in $x$ is $O(x^p)$, and implicit differentiation of the last equation gives $\partial_{D_m} x, \partial_p x = O(\varepsilon)$. The bounded and positive $\alpha_p$ is differentiable here by its integral. It follows that

$$\frac{\partial D_m}{\partial p} = \varepsilon r_m'(p) + O(\varepsilon^{p+1} |\log \varepsilon|).$$

Since $r_m'(2) = -a(m)$, the difference $D_m - D_{m+1}$ is strictly increasing in $p$ in the window. Its opposite signs on the two sides of the limiting crossing follow from (5) applied to the individual witnesses. Thus there is a unique exact crossing. Strict convexity of the limiting polygon leaves all other candidates separated at each crossing. Together with the exact finite-family reduction and uniform expansion, this gives the ordered phases.

## 62 Scope

The results apply to finite compact connected unweighted metric graphs with positive edge lengths, a nonempty Dirichlet vertex set and standard Kirchhoff conditions elsewhere; loops, multiple edges and a disconnected Dirichlet-cut interior are included. The stationary envelope holds for every fixed $1 < p < \infty$, and the resolvent, heat and stability results concern the linear case $p = 2$ with initial datum one. On this whole class the endpoint-mass interval is the exact heat envelope at every positive time (Section 13), and the heat deficit at a fixed positive time controls the marked measured distance to the mixed interval with the exponent $1/3$ (Section 16), which the pitchfork profile shows is sharp; Section 8 gives explicit constants for every smaller exponent. Section 11 gives the sharp first $p$-eigenvalue bound at fixed length and inradius for every $1 < p < \infty$, and Section 17 its sharp inverse-edge rate. Sections 20–24 treat

the $p$-Laplacian heat flow from datum one. On rooted trees and on graphs with exact root-preserving tree quotients, the sharp heat envelope holds at every positive time and the cubic reserve and the exponent $1/3$ itself hold uniformly; on quotients the resistance height refines the envelope. Weighted trees with matching multiplicities $c_e \geq 1$ obey the same heat envelope in terms of their total mass and their resistance or physical height. The positive response comparison behind these results is preserved by parallel merging, common-stem attachment and ancestor-to-descendant mass relocation.

**Editions and the companion paper.** This is the twelfth edition of this article. The twelfth edition revises Sections 52–61, first added in the eleventh edition, with exact rooted optimizers near maximal and path volume and the near-star crossover at $p = 2$. The tenth edition extended Sections 45–51 with sharp cap rates, the crossover on the scale $1/n$ and general absorption laws. The ninth edition added the exact capped optimizers of the nonlinear $p$-resolvent, and the eighth edition added Sections 41–44, which determine the exact capped optimizers of the resolvent at every mass. The seventh edition added Sections 36–40, which give the sharp first-order heat-content deficit under edge and topological caps. The sixth edition added Sections 32–35, which give the sharp first-order coefficients of the finite-complexity gaps for eigenvalues and resolvents. The fifth edition added Sections 29–31, which determine the exact capped maxima of $p$-torsional rigidity. The fourth edition added Sections 25–28, which quantify the torsion and resolvent deficits by graph complexity. The third edition added Sections 17–24 on finite edge counts and the nonlinear heat flow. The second edition added Sections 11–16, which extend the heat envelope and the cubic stability exponent from rooted trees to all finite connected graphs and add the sharp first $p$-eigenvalue bound. It is a separate work from the companion paper [A], Archive edition 3 of 1 November 2026, which proves the length-only all-times heat-content comparison together with general-data concentration, single-time equality and a double-Dirichlet comparison under a bridge condition. The overlapping material of this article, namely the constant-data source argument of Section 5 and the pitchfork identity of Proposition 9.1, is identified where it occurs. The length–inradius envelopes, the rooted-tree heat envelope, the mass relocation and quotient results, the effective stability estimates and the cubic pitchfork profile are the contributions of this article beyond [A].

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