Sharp inference rates at a power-law degenerate Darcy minimum

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Abstract

We determine sharp minimax rates for a linear conductivity functional at a power-law degenerate minimum of the Darcy potential on a ball. At this radial reference, the gradient vanishes to order $p-1$, and the critical dimension is $2p$ for a score nonzero at the minimum. Below it the risk is $N^{-(2s+d)/(4s+4p)}$, at equality it is $\sqrt{\log N/N}$, and above it the rate is parametric. The linearized result allows arbitrary conductivity perturbations. The same rates hold in the nonlinear experiment for radial conductivities at sufficiently large Sobolev smoothness. Moment corrections control the bias of a local adjoint cutoff, and annular alternatives give the critical logarithm. These rates answer the degenerate-critical-point part of Liu, Chang and Hu, Section 6, second bullet under 'Further open problems,' at an exact radial even-power minimum in the linearized experiment and for sufficiently smooth radial conductivities in the nonlinear experiment.

1 Experiment and results

The order of vanishing of the reference potential's gradient can determine whether a linear conductivity functional is estimable at the root-sample rate. At an even power-law minimum of order p, the transition takes place at dimension $2p$. The critical logarithm is forced by perturbations on disjoint annuli. The same annular construction supplies lower bounds in the nonlinear model. A reciprocal-conductivity reduction gives the matching nonlinear upper bound for radial conductivities.

This is a specified power-law case of the degenerate-critical-point problem posed by Liu, Chang and Hu [2, §6].

Let $\Omega = B_1(0) \subset \mathbb{R}^d$ with $d \geq 2$. Fix an even integer $p \geq 4$ and put

$$f_0 = 1, \qquad u_0(x) = \frac{|x|^p - 1}{p}, \qquad b(x) = \nabla u_0(x) = |x|^{p-2}x, \qquad g(x) = (d+p-2)|x|^{p-2}.$$

The conductivity equation is $\operatorname{div}(f\nabla u_f) = g$, with $u_f = 0$ on the boundary. The source vanishes at the center. The reference has a unique minimum there with zero Hessian and radial gradient of order $p-1$.

Fix $\psi \in C_c^\infty(\Omega)$ with $\psi(0) \neq 0$. In the linearized experiment the observation is Gaussian white noise in $L^2(\Omega)$ with mean

$$Ah = -\Delta_D^{-1} \operatorname{div}(hb)$$

and noise level $N^{-1/2}$. Here Δ_D^{-1} is the Dirichlet inverse of the Laplacian. The target is $\int_\Omega h\psi$, for $\|h\|_{H^s(\Omega)} \leq \eta$. The forward operator is bounded from L^2 to L^2 since b is bounded and the

Dirichlet energy estimate applies. In the nonlinear experiment the mean is u_f and the target is $\Phi(f) = \int_\Omega f\psi$. All risks below are minimax mean absolute errors. Write

$$\mathfrak{r}_N(s,d,p) = \begin{cases} N^{-(2s+d)/(4s+4p)}, & d < 2p, \\ \sqrt{\log N/N}, & d = 2p, \\ N^{-1/2}, & d > 2p. \end{cases}$$

Theorem 1. *For every $s \geq 0$ and fixed $\eta > 0$ the linearized risk is of order $\mathfrak{r}_N(s,d,p)$. The lower bounds hold even when h is radial and supported in any fixed ball about zero contained in Ω.*

Theorem 2. *For sufficiently large s and small fixed $\eta > 0$, the nonlinear risk over radial positive f with $\|f-1\|_{H^s} \leq \eta$ is of order $\mathfrak{r}_N(s,d,p)$. The nonlinear lower bounds hold also over the unrestricted ball of positive conductivities.*

Liu, Chang and Hu [2, §6] identify degenerate critical points as an open extension of their nondegenerate theory. The case above replaces the exponential flow at a nondegenerate minimum by an explicit power-law flow. The adjoint range condition for efficient Darcy inference is studied in [3]. Koers, Szabó and van der Vaart [1] use linear inverse methods in nonlinear PDE recovery and analyze characteristic reconstruction for Darcy. The reciprocal conductivity here makes the radial observation map exactly linear. The estimation result concerns a scalar functional and the extra singularity at this prescribed degenerate minimum.

2 An adjoint cutoff with small dual bias

Lemma 1. *Fix $s \geq 0$. Let β have bounded C^2 norm and support in a fixed compact subset of the unit ball. For all sufficiently small $a > 0$ there is a potential $w_a \in H^2 \cap H_0^1(\Omega)$ such that*

$$\left| \int h(\beta - b \cdot \nabla w_a) \right| \leq C a^{s+d/2} \|h\|_{H^s(\Omega)}, \tag{1}$$

$$\|\Delta w_a\|_2 \leq C \begin{cases} a^{d/2-p}, & d < 2p, \\ (1 + \log(1/a))^{1/2}, & d = 2p, \\ 1, & d > 2p. \end{cases} \tag{2}$$

The constants are uniform in β with the indicated support and norm bounds.

Proof. Write $\nu = p - 2 > 0$. In polar coordinates, the punctured-domain adjoint solution is

$$w(r\vartheta) = -\int_r^1 t^{-\nu-1} \beta(t\vartheta)\, dt.$$

It vanishes near the outer boundary. Its derivatives through order two satisfy $|D^j w(x)| \leq C|x|^{-\nu-j}$ for $0 \leq j \leq 2$. The radial derivatives follow by differentiation of the lower endpoint, and angular derivatives cost at most $|x|^{-1}$ per Euclidean derivative, with uniformly bounded angular derivatives of the integrand through order two. Multiplying w by a fixed radial cutoff dilated to equal zero for $r < a$ and one for $r > 2a$ gives a preliminary potential $\widetilde{w}_a$. Its defect $\beta - b \cdot \nabla \widetilde{w}_a$ is supported in $|x| \leq 2a$ and uniformly bounded, by the derivative and size bounds above. Integrating the bound on $D^2 w$ outside $2a$ and its cutoff analogue on $a \leq r \leq 2a$ gives precisely (2) for $\widetilde{w}_a$.

Choose an integer L with $L+1 > s - d/2$. We correct every moment of the defect of order at most L. For each multi-index α in that range,

$$\int y^\alpha |y|^\nu y \cdot \nabla V(y)\, dy = -(d + \nu + |\alpha|) \int |y|^\nu y^\alpha V(y)\, dy.$$

For the moment map take the fixed test functions $V_\gamma(y) = \chi(y)|y|^\nu y^\gamma$, where $\chi \geq 0$ is smooth, supported in a fixed annulus inside $|y| < 2$ and positive on an open subset. The matrix of moments of $|y|^\nu y \cdot \nabla V_\gamma$ differs from an invertible polynomial Gram matrix by multiplication by an invertible diagonal matrix. Now

$$b(x) \cdot \nabla \left(a^{-\nu} V_\gamma(x/a) \right) = |x/a|^\nu (x/a) \cdot \nabla V_\gamma(x/a).$$

Hence adding an appropriate bounded linear combination of these scaled potentials to $\widetilde{w}_a$ cancels the desired moments. The coefficients are uniformly bounded because the preliminary defect in y coordinates is uniformly bounded and compactly supported. The added Laplacian has L^2 norm at most $Ca^{d/2-p}$, preserving (2).

The corrected defect has the form $\rho_a(x/a)$ with ρ_a supported in the fixed ball of radius two, uniformly bounded and with all moments through degree L zero. Therefore $|\widehat{\rho_a}(\xi)| \leq C|\xi|^{L+1}$ for $|\xi| \leq 1$, and Plancherel bounds its L^2 norm at high frequencies. The homogeneous H^{-s} norm is uniformly bounded because $2(L+1) - 2s + d > 0$. Scaling gives a bound $Ca^{s+d/2}$ for the defect on $\mathbb{R}^d$. Testing against any bounded Sobolev extension of h gives (1). □

In the linearized model, $\langle Ah, \Delta w_a \rangle = \int h\, b \cdot \nabla w_a$ by the weak Dirichlet equation. Use the linear observation against Δw_a with $\beta = \psi$ and $a = N^{-1/(2s+2p)}$. The bias (1) and noise (2) give the upper bound of Theorem 1 in all three regimes.

3 Lower bounds on one scale and at the critical dimension

For a smooth radial template ζ supported in $\{1/2 < |x| < 1\}$, put

$$V(\rho) = -\int_\rho^\infty t^{p-1} \zeta(t)\, dt, \qquad \rho \geq 0,$$

where $\zeta(t)$ is its radial profile. The function V is smooth and radial, supported in the unit ball. It may be constant on the inner ball. For $h(x) = A_0 \zeta(x/a)$ with $0 < a < 1$, the linearized mean Ah equals $-A_0 a^p V(x/a)$, by its radial derivative and the boundary condition. In particular,

$$\|Ah\|_2 = |A_0| a^{p+d/2} \|V\|_2, \qquad \|h\|_{H^s(\Omega)} \leq C|A_0| a^{d/2-s}. \tag{3}$$

A template with nonzero spatial integral satisfies $|\int h\psi| \gtrsim |A_0| a^d$ as $a \downarrow 0$ because $\psi(0) \neq 0$.

If $d < 2p$, take $A_0 = \gamma a^{s-d/2}$ and $a = N^{-1/(2s+2p)}$, with $\gamma > 0$ fixed and sufficiently small. Then the Sobolev bound holds, the Gaussian Kullback–Leibler divergence from zero is bounded by a small constant, and the functional gap is of order $N^{-(2s+d)/(4s+4p)}$. For $d > 2p$ one fixed smooth radial supported direction of nonzero functional, at amplitude $\gamma/\sqrt{N}$, gives the ordinary parametric lower bound.

For $d = 2p$ choose the template so

$$\int_0^\infty t^{p-1} \zeta(t)\, dt = 0, \qquad \psi(0) \int_{\mathbb{R}^d} \zeta(x)\, dx > 0.$$

The powers $p-1$ and $d-1$ differ, so a linear combination of two disjoint radial bumps with opposite coefficients can satisfy the first condition and give the second after fixing the sign. Now V also vanishes on the inner ball. Let $a_j = a_0 4^{-j}$ with $a_0 > 0$ small and

$$H_j(x) = a_j^{-p-d/2} \zeta(x/a_j).$$

The means AH_j have disjoint annular supports and the same positive L^2 norm. The functions H_j satisfy $\int H_j \psi \geq c > 0$ by the template condition and $d = 2p$, uniformly in j after choosing a_0 small. Also $\int |H_j||\psi| \leq C$ and $\|H_j\|_{H^s} \leq C a_j^{-s-p}$. Set

$$J_N = \left\lfloor \frac{\log N}{4(s+p)\log 4} \right\rfloor, \qquad h_N = \frac{\gamma}{\sqrt{N J_N}} \sum_{j=1}^{J_N} H_j.$$

For large N, disjointness of the observation means gives $N\|Ah_N\|_2^2 \leq C\gamma^2$. The Sobolev triangle inequality and geometric summation give $\|h_N\|_{H^s} \leq C\gamma N^{-1/4}/\sqrt{J_N}$. The functional gap is at least $c\gamma\sqrt{J_N/N}$. Choosing γ small and applying the two-point Gaussian testing bound proves the critical lower bound. All these perturbations can be supported in any fixed interior ball about zero by choosing fixed templates or the largest scale accordingly.

4 The nonlinear radial experiment

For a positive radial conductivity put $\theta = 1/f$. The radial solution satisfies

$$\frac{d}{dr} u_f(r) = r^{p-1}\theta(r), \qquad u_f(1) = 0.$$

Indeed the flux $r^{d-1} f(r) u_f'(r)$ integrates the radial source to r^{d+p-2}. There is no center flux in the regular weak solution. Thus the Gaussian observation is exactly linear in θ. The target is the smooth nonlinear functional $G(\theta) = \int \psi/\theta$. The reciprocal map is bounded on the uniformly positive Sobolev class for $s > d/2$ and small fixed radius, by the Sobolev algebra and smooth composition bounds. It maps sufficiently small radial Sobolev balls at one into comparable balls. In particular the reciprocal class has uniformly bounded H^s norm, is uniformly positive, and its forward mean has bounded H^{s+1} norm. For the last assertion use $\nabla u_f = b\theta$ with smooth b and the Dirichlet trace.

Use two independent observations from Gaussian noise splitting. On the first, take an approximate minimum-distance prediction pilot q in the reciprocal class. We use the construction of Liu, Chang and Hu [2, Lemma S4.2], which only requires the uniform H^{s+1} bound on forward means. A bounded compactly supported Sobolev extension and a Fourier cutoff at $J \asymp N^{1/(2s+d+2)}$ give an L^2 prediction estimate with root mean square error $O(N^{-(s+1)/(2s+d+2)})$. The cutoff bias is $O(J^{-s-1})$ and the smoothed white noise has squared L^2 expectation $O(J^d/N)$ by the kernel's Hilbert–Schmidt norm. Its fixed moments have the same orders to their corresponding powers by Gaussian moment bounds. Choose a measurable approximate nearest forward mean in a fixed countable dense collection from the reciprocal class. The triangle inequality preserves all fixed moment bounds. Writing $v = u_f - u_{1/q}$ and $\delta = \|v\|_2$, they give

$$\left(\mathbb{E}_f \delta^\ell\right)^{1/\ell} \leq C_\ell N^{-(s+1)/(2s+d+2)} \qquad (\ell > 0). \tag{4}$$

We also need an L^2 bound for the parameter error $h = \theta - q$. If $k = \lfloor s \rfloor \geq 1$ and a is small, then

$$\|h\|_2 \leq C\left(a^k + a^{-(p-1)} \|\nabla v\|_2\right). \tag{5}$$

To see this, outside the ball of radius a, use $\nabla v = bh$ to control h by its weighted norm. Inside B_a, approximate h on B_{2a} by a polynomial of degree at most $k-1$, with L^2 error at most $Ca^k\|h\|_{H^k(B_{2a})}$. The approximation follows from the polynomial Poincaré inequality on the unit ball and a rescaling. The polynomial's L^2 norm on B_a is bounded by a fixed multiple of its norm on $B_{2a} \setminus B_a$ by equivalence of norms on the fixed polynomial space. This gives the same weighted bound inside the ball up to the approximation error. Since $\|h\|_{H^k}$ is uniformly bounded, (5) follows.

Interpolation from the uniform H^{s+1} bound gives $\|\nabla v\|_2 \le C\delta^{s/(s+1)}$. Optimizing a in (5) gives

$$\|h\|_2 \le C\delta^{sk/((s+1)(k+p-1))}.$$

This holds for small δ and extends to the remaining values by the uniform bound. Together with (4) it implies $\mathbb{E}_f\|h\|_2^2 = o(N^{-1/2})$ for sufficiently large s, since the exponent of N^{-1} in the bound is

$$\frac{2sk}{(k+p-1)(2s+d+2)},$$

which tends to one.

The exact expansion of G is

$$G(\theta) - G(q) = \int h\beta_q + O(\|h\|_2^2), \qquad \beta_q = -q^{-2}\psi,$$

uniformly on the reciprocal class. For sufficiently large s the functions β_q have uniform C^2 bounds and common compact support by Sobolev embedding. Use Lemma 1 with $\beta = \beta_q$ and $a = N^{-1/(2s+2p)}$. The L^2 representer for the cutoff term in the reciprocal experiment is $-\Delta w_a$ because $\nabla v = bh$ and both potentials have zero boundary trace. Starting at $G(q)$, add the second-sample residual observation against this representer. Countable pilot selection ensures measurability. Its conditional bias is bounded by $Ca^{s+d/2} + C\|h\|_2^2$, and its conditional noise has standard deviation at most $\sqrt{2/N}\|\Delta w_a\|_2$. These bounds give the asserted upper rate after taking expectation. Conditional expectation given the unsplit observation removes auxiliary randomization without increasing risk.

For the nonlinear lower bounds put $\theta = 1 + h$ for the same radial perturbations h as in the linearized proof, and $f = 1/(1+h)$. For fixed sufficiently large s and sufficiently small γ, these conductivities belong to the desired Sobolev ball by the composition bound. The observed mean difference is exactly $-Ah$. The linear functional gap in the reciprocal target has the same leading magnitude, since

$$G(1+h) - G(1) = -\int h\psi + \int \frac{h^2\psi}{1+h}.$$

In each of the three lower-bound constructions $\|h\|_\infty \to 0$ and $\int |h||\psi| \le C|\int h\psi|$. The latter follows by single-template scaling or disjointness and the common orientation of the critical annular templates. The nonlinear remainder is therefore negligible relative to the gap. The same Gaussian two-point bounds finish the proof.

References

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