THE SHARP CONSTANT IN THE MEAN-DISTANCE DIRICHLET EIGENVALUE INEQUALITY ON METRIC GRAPHS

SCOTT SCHUBR

Abstract. For a compact metric graph with Dirichlet conditions on a nonempty vertex set D, let ρ be the mean distance to D and λ_1 the first eigenvalue of the Dirichlet–Kirchhoff Laplacian. Mugnolo conjectured that $\lambda_1\rho^2$ is bounded by a universal constant smaller than one; in earlier work we proved this with $0.6410 < C_* \le 331/500$ for the optimal constant C_*. We determine C_* exactly: $C_* = z_*^4/4$, where z_* is the unique initial value for which the solution of the Abel equation $z^2 z' = \sigma z - 1$, $z(0) = z_*$, remains positive with $\sigma z \le 1$ for all $\sigma \ge 0$. Numerically $C_* = 0.6616687202\ldots$. The constant is not attained by any finite graph; it is approached by chains of bundles of parallel edges whose multiplicities follow an explicit profile with total mass two and mean one. The proof reduces the problem exactly to a weighted problem on an interval, whose Euler–Lagrange equation coincides with the equality case of a pointwise trial-function inequality. Exact rational and interval-arithmetic certificates give $0.6614 < C_* \le 0.66168$ independently of any numerical integration.

1. Introduction and statement

Let $\mathcal{G}$ be a finite connected compact metric graph with positive edge lengths; loops and multiple edges are allowed. Let $D \neq \emptyset$ be a set of vertices carrying Dirichlet conditions, while all other vertices carry continuity and Kirchhoff conditions (Neumann at unmarked leaves); see [2, 7] for background. With arclength measure write

$$L = |\mathcal{G}|, \qquad d(x) = \operatorname{dist}_{\mathcal{G}}(x, D), \qquad R = \max_{\mathcal{G}} d, \qquad \rho = \frac{1}{L}\int_{\mathcal{G}} d(x)\,dx,$$

and let $H^1_D(\mathcal{G})$ be the continuous edgewise H^1 functions vanishing on D. The first eigenvalue is

$$\lambda_1(\mathcal{G}, D) = \min_{0 \neq u \in H^1_D(\mathcal{G})} \frac{\int_{\mathcal{G}} |u'|^2}{\int_{\mathcal{G}} |u|^2}.$$

Define the optimal constant

$$C_* = \sup_{(\mathcal{G}, D)} \lambda_1(\mathcal{G}, D)\, \rho(\mathcal{G}, D)^2.$$

Sharp eigenvalue bounds under length, diameter and inradius constraints are studied in [5, 6, 3]. Mugnolo [8, Conj. 4.7] conjectured that $C_* < 1$, noting that no

Date: November 25, 2026.

2020 *Mathematics Subject Classification.* Primary 34B45; Secondary 34L15, 49R05, 81Q35.

Key words and phrases. Metric graph, quantum graph, Dirichlet eigenvalue, mean distance, spectral optimization, Abel equation, Barta inequality, computer-assisted proof.

universal bound holds with ρ replaced by the inradius R. In [10, Thm. 1.1] we proved

$$\left(\tfrac{5}{4}\arctan\tfrac{\sqrt{5}}{3}\right)^2 \le C_* \le \frac{331}{500},$$

the lower bound being approximately 0.641045, and left the value of C_* open. This paper determines it.

Definition 1.1. For $C > 0$ let (h_C, G_C) be the maximal solution, on the open region $\{G > 0\}$, of

$$h' = \sqrt{G}, \qquad G' = 2Ch\sqrt{G} - 1, \qquad h(0) = 0, \quad G(0) = 1. \tag{1}$$

We say that C *survives* if $G_C > 0$ on $[0, \infty)$, and *dies* otherwise. Put $C_0 = \inf\{C > 0 :\ C \text{ survives}\}$ and $q_C = 2Ch_C\sqrt{G_C}$.

Along (1), $\frac{d}{dt}(Ch^2 + 1 - t) = 2Chh' - 1 = G'$, and both sides equal 1 at $t = 0$. Hence, as long as the solution exists,

$$G = Ch^2 + 1 - t, \qquad \text{equivalently} \qquad h'^2 - Ch^2 = 1 - t. \tag{2}$$

Theorem 1.2. *$C_* = C_0$. Moreover:*

1. *$\lambda_1(\mathcal{G}, D)\rho(\mathcal{G}, D)^2 < C_*$ for every finite $(\mathcal{G}, D)$;*
2. *$C_0 = z_*^4/4$, where z_* is the unique $z_0 > 0$ such that the solution of*

$$z^2\frac{dz}{d\sigma} = \sigma z - 1, \qquad z(0) = z_0, \tag{3}$$

   exists and is positive on $[0, \infty)$ with $\sigma z(\sigma) \le 1$ throughout;
3. *$0.6614 < C_* \le 0.66168$;*
4. *numerically, $z_* = 1.275484393\ldots$ and $C_* = 0.6616687202\ldots$.*

Items (1)–(3) are proved. Item (3) rests on two finite computer-assisted certificates in exact rational and interval arithmetic (Section 5). The digits in item (4) come from two independent, non-validated numerical integrations, which agree to about 10^{-13}.

Strategy. Section 2 shows that C_* equals the supremum of $\lambda_1(w)\rho(w)^2$ over positive weights w on intervals. On general graphs the inequality $\le$ comes from distance-function test functions. Equality comes from "pumpkin chains", which realize every integer step weight. For this weighted problem, the first-order optimality condition on the support of w is $u'^2 - \lambda u^2 = K(\rho - t)$. This is exactly the equality case of the pointwise inequality [10, Lemma 2.1] recalled in Lemma 2.2. The ODE (1) is that equality case, and C_0 is the threshold for global existence (Section 3). The threshold solution serves both as the optimal trial function, giving the upper bound, and as the generator of an explicit extremal weight, giving the lower bound via Barta's inequality.

2. Reduction to weighted intervals

Let $T > 0$ and let w be a measurable weight on $[0, T]$ with $0 < m \le w \le M < \infty$. Define

$$\lambda_1(w) = \inf\Big\{\frac{\int_0^T u'^2 w}{\int_0^T u^2 w} :\ u \in H^1(0,T),\ u(0) = 0,\ u \ne 0\Big\}, \qquad \rho(w) = \frac{\int_0^T tw}{\int_0^T w},$$

and $J(w) = \lambda_1(w)\rho(w)^2$. Both ρ and J are invariant under $w \mapsto cw$, and J is invariant under dilations $w(t) \mapsto w(t/s)$. The infimum is attained by a positive eigenfunction of $-(wu')' = \lambda wu$ with $u(0) = 0$ and $(wu')(T) = 0$; see [12, Ch. 4, 10].

For a graph, let $n_{\mathcal{G}}(t) = \#\{x \in \mathcal{G} : \ d(x) = t\}$.

Lemma 2.1. *$n_{\mathcal{G}}$ is finite and piecewise constant away from finitely many t, $n_{\mathcal{G}} \geq 1$ on $(0, R)$, and $n_{\mathcal{G}} = 0$ for $t > R$. For every bounded Borel ϕ,*

$$\int_{\mathcal{G}} \phi(d(x))\, dx = \int_0^R \phi(t)\, n_{\mathcal{G}}(t)\, dt.$$

Proof. On an edge $e \cong [0, \ell]$ with endpoint distances a, b, one has $d(x) = \min(a + x,\, b + \ell - x)$, possibly after removing a loop's self-intersection, which the same formula covers. Hence d is piecewise affine with slopes ± 1. Each affine piece contributes the indicator of its image interval to $n_{\mathcal{G}}$. This gives the coarea formula with $|d'| = 1$. A shortest path from D to a point at distance R meets every level in $(0, R)$. □

Lemma 2.2 (Trial functions, [10, Lemma 2.1])**.** *Let $h : [0, \infty) \to \mathbb{R}$ be locally Lipschitz, $h(0) = 0$, with h not identically zero on any interval $[0, \varepsilon)$. If C, K satisfy $h'(t)^2 - Ch(t)^2 \leq K(1 - t)$ for a.e. $t \geq 0$, then $\lambda_1\rho^2 \leq C$ for every $(\mathcal{G}, D)$.*

Proof. Let $u = h(d/\rho) \in H^1_D(\mathcal{G})$. By Lemma 2.1, $\int_{\mathcal{G}} |u'|^2 = \rho^{-2} \int h'(t/\rho)^2 n_{\mathcal{G}}\, dt$ and $\int_{\mathcal{G}} u^2 = \int h(t/\rho)^2 n_{\mathcal{G}}\, dt$. With μ the law of d/ρ under normalized arclength, which has mean one, $\rho^2 \int |u'|^2 - C \int u^2 = L \int (h'^2 - Ch^2)\, d\mu \leq LK \int (1 - t)\, d\mu = 0$. Here $u \neq 0$ because μ charges every interval $[0, \varepsilon)$. □

Proposition 2.3. *For every $(\mathcal{G}, D)$, $\lambda_1(\mathcal{G}, D) \leq \lambda_1(n_{\mathcal{G}})$ and $\rho(\mathcal{G}, D) = \rho(n_{\mathcal{G}})$, the weight being taken on $[0, R]$.*

Proof. The second identity is Lemma 2.1 with $\phi(t) = t$ and $\phi \equiv 1$. For the first, every admissible u for the weighted problem gives the admissible $u \circ d$ on $\mathcal{G}$, with the same Rayleigh quotient by Lemma 2.1. □

Definition 2.4. Given levels $0 = t_0 < t_1 < \cdots < t_m$ and multiplicities $n_1, \ldots, n_m \in \mathbb{N}$, the *pumpkin chain* $\mathcal{P}$ has vertices $v_0, \ldots, v_m$, with n_i parallel edges of length $t_i - t_{i-1}$ joining v_{i-1} and v_i, and $D = \{v_0\}$. Subdividing every edge at its midpoint yields a simple graph with the same spectrum and the same d.

Proposition 2.5. *For a pumpkin chain, $d(x)$ equals the level of x, $n_{\mathcal{P}}$ is the step function equal to n_i on (t_{i-1}, t_i), and $\lambda_1(\mathcal{P}, D) = \lambda_1(n_{\mathcal{P}})$.*

Proof. The first two assertions are immediate. The ground state of a connected graph with nonempty Dirichlet set is simple and can be chosen positive off D [2, Thm. 1.5.6],[7]. A permutation of the edges in one bundle is an isometry fixing D. By simplicity the ground state is invariant under it, so it is a function $u(t)$ of the level alone. On each level interval u solves $-u'' = \lambda_1 u$, and the Kirchhoff condition at v_i reads $n_{i+1}u'(t_i^+) - n_i u'(t_i^-) = 0$. Hence u is a positive eigenfunction of the weighted problem with weight $n_{\mathcal{P}}$, Dirichlet at 0 and Neumann at t_m. A positive eigenfunction of a regular Sturm–Liouville problem belongs to the lowest eigenvalue [12, Thm. 4.3.1]. □

Corollary 2.6. *$C_* = \sup J(w)$, where the supremum is over continuous $w > 0$ on compact intervals $[0, T]$, or equivalently over positive step functions.*

Proof. The inequality $\le$ is Proposition 2.3, since $n_{\mathcal{G}}$ is a positive step function on $(0, R)$. Positive step functions are approximated by continuous positive weights (and conversely) in the sense below, so the two suprema agree.

For $\ge$, let w be continuous and positive on $[0, T]$, and let $\eta \in (0, \frac{1}{2})$. By uniform continuity there is a step function s with rational breakpoints and rational positive values such that $|s/w - 1| \le \eta$ on $[0, T]$. Each Rayleigh quotient changes by a factor in $[(1-\eta)/(1+\eta), (1+\eta)/(1-\eta)]$, so $\lambda_1(s)$ does too. Also $|\rho(s) - \rho(w)| \le 2T\eta/(1-\eta)$. Multiplying s by a common denominator does not change J and produces integer multiplicities. Proposition 2.5 then gives a graph with $\lambda_1\rho^2 = J(s) \to J(w)$ as $\eta \to 0$. □

3. The threshold solution

Write $F_C(h, G) = (\sqrt{G},\ 2Ch\sqrt{G} - 1)$. On $\{G > 0\}$, F_C is smooth, nondecreasing in C, and quasimonotone increasing: the off-diagonal partial derivatives $\partial_G F^{(1)} = 1/(2\sqrt{G})$ and $\partial_h F^{(2)} = 2C\sqrt{G}$ are nonnegative. The comparison theorem of Kamke and Müller [11, §10] therefore gives the following: if $C_1 < C_2$, then $h_{C_1} \le h_{C_2}$ and $G_{C_1} \le G_{C_2}$ on any interval on which both solutions exist.

Lemma 3.1. *If $q_C(t_1) > 1$ for some t_1, then C survives, and G_C and q_C increase on $[t_1, \infty)$.*

Proof. At t_1, $G' = q - 1 > 0$. As long as $G' > 0$, both h and G increase, so $q = 2Ch\sqrt{G}$ increases and G' stays positive. Hence $G \ge G(t_1) > 0$ on $[t_1, \infty)$, and F_C is Lipschitz there, so the solution is global. □

Lemma 3.2. *The set of dying C is open and down-closed, it contains $(0, \frac{1}{4}]$, and it does not contain 4. Consequently $0 < C_0 < 4$, the dying set is $(0, C_0)$, and C survives if and only if $C \ge C_0$.*

Proof. Down-closed. Let $C_1 < C_2$, and suppose C_2 dies at τ_2, so that $G_{C_2}(t) \to 0$ as $t \uparrow \tau_2$. If C_1 were alive on $[0, \tau_2]$, comparison on $[0, \tau_2)$ would give $G_{C_1}(\tau_2) \le 0$, which is absurd.

Open. Let C die at τ. Then $h \le h(\tau) =: \bar{h}$ on $[0, \tau)$, and $G' = q - 1 \to -1$ as $t \uparrow \tau$. Choose $\varepsilon > 0$ with $4C\bar{h}\sqrt{\varepsilon} < \frac{1}{2}$, and choose $t_0 < \tau$ with $G(t_0) < \varepsilon/2$. By continuous dependence on parameters on the compact interval $[0, t_0]$, every C' close to C has $G_{C'}(t_0) < \varepsilon$ and $h_{C'}(t_0) < \bar{h} + 1$. Afterwards, while $G_{C'} \le \varepsilon$ and $h_{C'} \le \bar{h} + 1$, we have $G'_{C'} \le -\frac{1}{2}$ and $h'_{C'} \le \sqrt{\varepsilon}$. So $G_{C'}$ reaches 0 within time 2ε, during which h grows by less than 1. Hence C' dies.

Small and large C. Let $C \le \frac{1}{4}$. As long as $G \le 1$ on $[0, t]$, we have $h \le t$ and therefore $G = Ch^2 + 1 - t \le Ct^2 + 1 - t$. The right side is at most 1 for $t \le 1/C$, so the bound $G \le 1$ persists up to $t = 2 \le 1/C$. There $G(2) \le 4C - 1 \le 0$. Hence C dies no later than $t = 2$.

For $C = 4$: $G' \ge -1$ gives $G \ge \frac{1}{2}$ on $[0, \frac{1}{2}]$, so $h(\frac{1}{2}) \ge \frac{1}{2\sqrt{2}}$ and $q(\frac{1}{2}) \ge 8 \cdot \frac{1}{2\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 2 > 1$. Lemma 3.1 then shows that 4 survives. □

Proposition 3.3. *Let $(h, G) = (h_{C_0}, G_{C_0})$ and $q = q_{C_0}$. Then:*

1. *$G > 0$ and $h' > 0$ on $[0, \infty)$; moreover $q \le 1$, G is nonincreasing, $G \to 0$, and $h^2 = (t - 1 + G)/C_0 \to \infty$;*

(2) *with* $I(t)=\int_0^t ds/(2G(s))$ *and* $w=e^{-I}/G=2I'e^{-I}$, *the weight* w *is smooth and positive,* $w(0)=1$, *and*

$$-(wh')'=C_0\,wh \quad \text{on } [0,\infty); \tag{4}$$

(3) $\int_0^\infty w=2$, $\int_0^\infty t\,w<\infty$, *and for every* $T>0$

$$\int_0^T (1-t)\,w\,dt=w(T)h(T)h'(T)=\frac{q(T)}{2C_0}\,w(T)\in\Big[0,\frac{w(T)}{2C_0}\Big]; \tag{5}$$

in particular $\int_0^\infty (1-t)\,w\,dt=0$.

Proof. (1) By Lemma 3.2, C_0 survives. If $q(t_1)>1$ for some t_1, continuous dependence on $[0,t_1]$ would give $q_C(t_1)>1$ for some $C<C_0$, and C would survive by Lemma 3.1, contradicting Lemma 3.2. Hence $q\le 1$ and $G'=q-1\le 0$, so $G\le 1$ and $G\downarrow g\ge 0$. The formula for h^2 is (2). If $g>0$, then $h\ge\sqrt{g}\,t$ and $q\ge 2C_0\sqrt{g}\,t\sqrt{g}\to\infty$, which is impossible.

(2) We have $h''=G'/(2\sqrt{G})=C_0h-1/(2h')$. Moreover

$$\frac{w'}{w}=-\frac{1}{2G}-\frac{G'}{G}=\frac{1-4C_0h\sqrt{G}}{2G}=-\frac{2C_0h-1/(2h')}{h'},$$

so $(wh')'=w'h'+wh''=-w(2C_0h-\frac{1}{2h'})+w(C_0h-\frac{1}{2h'})=-C_0wh$.

(3) Since $G\le 1$, $I(t)\ge t/2$ and $I(\infty)=\infty$. Therefore $\int_0^\infty w=\int_0^\infty 2I'e^{-I}=2$, and integrating by parts, $\int_0^\infty tw=2\int_0^\infty e^{-I}\le 4$. By (2) and (4), $(whh')'=(wh')'h+wh'^2=w(h'^2-C_0h^2)=(1-t)w$, which gives (5), using $h(0)=0$ and $whh'=wh\sqrt{G}=qw/(2C_0)$. Since w is integrable, there is a sequence $T_n\to\infty$ with $w(T_n)\to 0$, and $(1-t)w$ is integrable, so the last claim follows. □

4. Proof of Theorem 1.2

Upper bound and non-attainment. By Proposition 3.3(1), $h=h_{C_0}$ is C^1 with $h(0)=0$ and $h'>0$. It satisfies $h'^2-C_0h^2=1-t$ everywhere by (2). Lemma 2.2 with $K=1$ gives $\lambda_1\rho^2\le C_0$, hence $C_*\le C_0$.

Suppose $\lambda_1\rho^2=C_0$ for a finite $(\mathcal{G},D)$. The proof of Lemma 2.2 then shows that $u=h(d/\rho)$ attains the Rayleigh minimum, so it is a first eigenfunction, and in particular C^1 inside every edge and Kirchhoff at every unmarked vertex. Let x^* satisfy $d(x^*)=R$, and put $\beta=h'(R/\rho)/\rho>0$. If x^* is interior to an edge, then $d=R-|x-x^*|$ near x^*, and u has one-sided derivatives $\pm\beta$ there, a corner. If x^* is a vertex, it is not in D because $R>0$. It is a local maximum of d, so the outward derivative of u along every incident edge equals $-\beta$, and the Kirchhoff sum is $-\beta\deg x^*\neq 0$. Either way we have a contradiction, which proves (1).

Lower bound.

Lemma 4.1 (Barta–Picone [1])**.** *Let* $w>0$ *be continuous on* $[0,T]$, *and let* $\varphi\in C^1[0,T]$ *be piecewise* C^2 *with* $\varphi(0)=0$, $\varphi'(0)>0$, $\varphi>0$ *on* $(0,T]$, *and* $\varphi'(T)=0$. *Suppose* $w\varphi'$ *is piecewise* C^1. *Then* $\lambda_1(w)\ge \operatorname{ess\,inf}_{(0,T)}\big(-(w\varphi')'/(w\varphi)\big)$.

Proof. For $u\in H^1$ with $u(0)=0$, Picone's identity reads $wu'^2-w\varphi'(u^2/\varphi)'=w(u'-u\varphi'/\varphi)^2\ge 0$ on $(0,T]$. Integrate over $[\epsilon,T]$ and integrate by parts. The boundary term at T vanishes because $\varphi'(T)=0$. The one at ϵ is $O(u(\epsilon)^2/\epsilon)$, which tends to 0 because $u(\epsilon)^2\le\epsilon\int_0^\epsilon u'^2$. This yields $\int wu'^2\ge\int(-(w\varphi')'/\varphi)u^2$. □

Let h, w be as in Proposition 3.3. Fix $T > 1$ and $\delta \in (0,1)$, and set $\varphi = h$ on $[0, T-\delta]$ and

$$\varphi(t) = h(T-\delta) + h'(T-\delta)\Big((t-T+\delta) - \frac{(t-T+\delta)^2}{2\delta}\Big) \quad \text{on } [T-\delta, T],$$

so that $\varphi'(t) = h'(T-\delta)(T-t)/\delta$. Thus $\varphi \in C^1$, $\varphi'(T) = 0$, and $0 \le \varphi' \le h'(T-\delta)$. On $[0, T-\delta]$, (4) gives $-(w\varphi')'/(w\varphi) = C_0$. On $[T-\delta, T]$, with $M = \max_{[T-1,T]} |w'/w|$,

$$\frac{-(w\varphi')'}{w\varphi} = \frac{-\frac{w'}{w}\varphi' - \varphi''}{\varphi} \ge \frac{h'(T-\delta)\,(\delta^{-1} - M)}{h(T) + h'(T-\delta)},$$

using $\varphi \le h(T-\delta) + \delta h'(T-\delta) \le h(T) + h'(T-\delta)$. Since $h' > 0$ is continuous, the right side exceeds C_0 for δ small enough. Lemma 4.1 gives $\lambda_1(w|_{[0,T]}) \ge C_0$ for every $T > 1$.

Take $T = T_n$ from Proposition 3.3(3). By (5),

$$\rho(w|_{[0,T_n]}) = 1 - \frac{w(T_n)h(T_n)h'(T_n)}{\int_0^{T_n} w} \longrightarrow 1,$$

so $J(w|_{[0,T_n]}) \ge C_0 \rho(w|_{[0,T_n]})^2 \to C_0$. Corollary 2.6 gives $C_* \ge C_0$. Hence $C_* = C_0$.

The Abel form. Let $a = (2\sqrt{C})^{-1/2}$, and substitute

$$\sigma = \frac{\sqrt{C}}{a}\, h, \qquad z = \frac{\sqrt{G}}{a}.$$

Since $h' > 0$ while the solution exists, h is a valid independent variable. From (1), $\frac{d\sqrt{G}}{dh} = \frac{G'}{2\sqrt{G}\,h'} = \frac{Ch}{\sqrt{G}} - \frac{1}{2G}$. In the new variables this becomes (3) with $z(0) = 1/a = (2\sqrt{C})^{1/2}$, that is, $C = z(0)^4/4$. Moreover $z > 0 \iff G > 0$, and

$$\sigma z = \frac{\sqrt{C}\, h\sqrt{G}}{a^2} = 2Ch\sqrt{G} = q.$$

Thus "z positive on $[0,\infty)$ with $\sigma z \le 1$" is exactly "C survives and $q \le 1$ for all time". Note that $\sigma \to \infty$ as $t \to \infty$ when the solution is global, since $h \to \infty$. By Proposition 3.3(1), C_0 has this property.

Uniqueness: let $C > C_0$. By comparison, $h_C \ge h_{C_0}$, so (2) gives $G_C - G_{C_0} = Ch_C^2 - C_0 h_{C_0}^2 \ge (C - C_0)h_{C_0}^2 \to \infty$. If $q_C \le 1$ for all time, G_C would be nonincreasing, which is a contradiction. So $q_C > 1$ somewhere. For $C < C_0$, C dies. Since $C \mapsto z(0)$ is increasing, this proves item (2).

5. Finite certificates

Both certificates are independent of the numerical value of C_0 and can be rerun from the accompanying files.

Upper bound $C_* \le 66168/10^5$. Let $C = 66168/10^5$ and $T = 3936/1000$. The file `upper_certificate.json` lists 48 consecutive intervals $[t_{i-1}, t_i]$ with $t_i = iT/48$. On each it gives a polynomial $h(t) = \sum_{k=0}^{12} c_{i,k} s^k$, $s = (t - t_{i-1})/(t_i - t_{i-1})$, with rational coefficients in $10^{-12}\mathbb{Z}$, except for the constant terms, which are fixed by continuity. The script `certify_upper.py` checks in exact rational arithmetic that:

- the pieces are continuous and $h(0) = 0$;

- on each piece, every degree-22 Bernstein coefficient of $P = 1 - t + Ch^2 - h'^2$ is positive (the minimum exceeds $2.9 \cdot 10^{-7}$), so $P > 0$ on $[0, T]$;
- with $H = h(T)$ and $m = (2CH)^{-1}$, the tail $h = H + m(t - T)$ gives $P(T + s) = (1 - T + CH^2 - m^2) + Cm^2 s^2$, and the constant term exceeds 0.0127.

Lemma 2.2 with $K = 1$ then gives $C_* \le C$. The polynomial data were produced by a floating-point fit of the ODE solution, but only the exact rational data are used in the verification.

Lower bound $C_* > 0.6614$. Let $\mathcal{P}$ be the pumpkin chain with 46 levels of length $1/10$ and multiplicities

$$2049, 2136, 2207, 2260, 2294, 2305, 2294, 2261, 2205, 2128, 2032, 1919,$$
$$1793, 1656, 1513, 1366, 1219, 1076, 938, 808, 688, 579, 481, 395, 320,$$
$$256, 203, 159, 122, 93, 70, 52, 38, 28, 20, 14, 10, 7, 5, 3, 2, 1, 1, 1, 1, 1.$$

These are $\max(1, \mathrm{round}(2000\, w))$ evaluated at the level midpoints. The chain has 40,009 edges and $\rho = 800377/800180$ exactly. Let $\mu = 507932624/768344051$, so that $\mu\rho^2 \ge 0.6614$ exactly. The script `certify_lower.py` integrates the radial problem $-(nu')' = \mu n u$, $u(0) = 0$, $u'(0) = 1$, level by level from the closed-form sine–cosine solution. It uses 50-digit interval arithmetic, and all step endpoints are enclosures of exact rationals. It verifies that $u' > 0$ on each of 20 subintervals of every level, and that $u'(4.6) > 0$.

The flux nu' is continuous at the levels, so $u' > 0$ on $(0, 4.6]$. Hence $u > 0$, and the Prüfer angle $\theta = \arctan(u/(nu'))$ lies in $(0, \pi/2)$ at the free end. Since $\theta(4.6; \lambda)$ is strictly increasing in λ and equals $\pi/2$ at λ_1 [4, Ch. 8], we get $\mu < \lambda_1(n_{\mathcal{P}})$. By Proposition 2.5, $\lambda_1(\mathcal{P}, D)\rho^2 > \mu\rho^2 \ge 0.6614$.

Numerical value. The script `compute_C0.py` locates the threshold in two ways. It uses 30-digit RK4 on (1), where $C = 0.661668720$ dies and $C = 0.661668721$ survives (the decision is made by $t \approx 5.3$). It also bisects on (3) with an adaptive Runge–Kutta method, giving $z_* = 1.275484393011$ and $z_*^4/4 = 0.661668720218$. These computations are not interval-validated.

6. Remarks

Remark 6.1 (Shape of the extremal weight). Numerically, the extremal weight in Proposition 3.3 rises from $w(0) = 1$ to a maximum $w \approx 1.153$ at $t \approx 0.55$ and then decays rapidly. Formally, $q \to 1$ suggests $G \sim (4C_0^2 h^2)^{-1} \sim (4C_0 t)^{-1}$, and hence $\log w \sim -C_0 t^2$. This Gaussian-tail heuristic is not used in the proofs. The near-optimal weight found by direct numerical maximization of J over 200-cell step functions, $J \approx 0.66164$, has the same shape.

Remark 6.2 (Sharpness of the pointwise method). The polynomial trial function of [10] at 331/500 approximated the threshold profile. Theorem 1.2 shows that the pointwise method of Lemma 2.2 is sharp. Formally, the first variation of J in w on its support is proportional to $u'^2 - \lambda u^2 + 2\lambda \frac{\int u^2 w}{\rho \int w}(t - \rho)$, which vanishes exactly when $u'^2 - \lambda u^2 = K(\rho - t)$.

Remark 6.3. Abel equations of the second kind such as (3) are not, in general, integrable in closed form [9]. We do not know a closed form for z_*. A validated

enclosure of C_0 to more digits could be obtained with a rigorous Taylor-model ODE integrator.

AI assistance. AI tools (Anthropic Claude) assisted with deriving and checking the proofs, implementing and running the certificates, and preparing the manuscript. The author is responsible for the content.

References

[1] J. Barta, *Sur la vibration fondamentale d'une membrane*, C. R. Acad. Sci. Paris **204** (1937), 472–473.

[2] G. Berkolaiko and P. Kuchment, *Introduction to Quantum Graphs*, Math. Surveys Monogr., vol. 186, Amer. Math. Soc., Providence, RI, 2013.

[3] G. Berkolaiko, J. B. Kennedy, P. Kurasov, and D. Mugnolo, *Surgery principles for the spectral analysis of quantum graphs*, Trans. Amer. Math. Soc. **372** (2019), no. 7, 5153–5197.

[4] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw–Hill, New York, 1955.

[5] L. Friedlander, *Extremal properties of eigenvalues for a metric graph*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 199–211.

[6] J. B. Kennedy, *A family of diameter-based eigenvalue bounds for quantum graphs*, in *Discrete and Continuous Models in the Theory of Networks*, Oper. Theory Adv. Appl., vol. 281, Birkhäuser, Cham, 2020, pp. 213–240.

[7] P. Kurasov, *Spectral Geometry of Graphs*, Oper. Theory Adv. Appl., vol. 293, Birkhäuser, Cham, 2024.

[8] D. Mugnolo, *The role of expanders in the spectral geometry of metric graphs*, preprint, 2026, arXiv:2607.14312.

[9] A. D. Polyanin and V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations*, 2nd ed., Chapman & Hall/CRC, Boca Raton, FL, 2003.

[10] S. Schubr, *Mean distance and the first two Dirichlet eigenvalues of compact metric graphs*, Luminary Oddities Math Lab, November 24, 2026.

[11] W. Walter, *Differential and Integral Inequalities*, Ergeb. Math. Grenzgeb., vol. 55, Springer, Berlin, 1970.

[12] A. Zettl, *Sturm–Liouville Theory*, Math. Surveys Monogr., vol. 121, Amer. Math. Soc., Providence, RI, 2005.

Luminary Oddities Math Lab