Sharp collision criteria for frozen color histograms

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Abstract

Let a fixed probability law select an n-element subset of a finite ground set, and color every ground element independently once, with prescribed positive probabilities $q_1, \dots, q_m$. We compare the conditional histogram law after freezing the coloring with $\mathrm{Mult}(n, q)$. Under upper negative-cylinder inequalities, the normalized marginal collision sum $\tau = n^{-1} \sum_e \mathbb{P}(e \in S)^2$ determines expected chi-square convergence: the distance tends to zero if and only if $(m-1)\tau$ does, and is asymptotic to $(m-1)\tau$ in that regime. The statements are uniform over positive color probabilities, without a minimum-probability assumption. For fixed positive q and fixed $m \geq 2$, expected total variation tends to zero if and only if τ does. The necessary half follows from an explicit covariance statistic and holds for every fixed-size subset law. The proof combines the classical multinomial overlap kernel with negative-cylinder factorial moments. An explicit diffuse non-real-stable family separates the hypotheses from real stability; spanning-tree and fixed matroid-base-rounding laws give applications.

1 The frozen experiment and its collision criterion

Let E be finite, $n \geq 1$, and let μ be a probability law on $\binom{E}{n}$, the n-element subsets of E. Write

$$p_e = \mathbb{P}_\mu(e \in S), \qquad \tau = \frac{1}{n} \sum_{e \in E} p_e^2, \qquad \ell = \max_{e \in E} p_e. \tag{1}$$

Thus $\sum_e p_e = n$ and $0 < \tau \leq \ell \leq 1$. Our dependence hypothesis is

$$\mathbb{P}_\mu(I \subseteq S) \leq \prod_{e \in I} p_e \quad \text{for every } I \subseteq E. \tag{2}$$

We call (2) the upper negative-cylinder property.

Independently of S, assign independent colors $H_e \in \{1, \dots, m\}$ with $\mathbb{P}(H_e = i) = q_i > 0$ and $\sum_i q_i = 1$. This assignment is made once. For a fixed coloring h, let P_h be the law, over $S \sim \mu$ only, of

$$C_h(S) = \big(|S \cap h^{-1}(i)|\big)_{i=1}^m.$$

The reference is the prescribed law $M = \mathrm{Mult}(n, q)$ on $\mathcal{A}_{n,m} = \{a \in \mathbb{Z}_{\geq 0}^m : \sum_i a_i = n\}$, rather than a multinomial with a coloring-dependent mean. We use

$$\chi^2(P \| M) = \sum_{a \in \mathcal{A}_{n,m}} \frac{(P(a) - M(a))^2}{M(a)}, \qquad \mathrm{TV}(P, M) = \frac{1}{2} \sum_a |P(a) - M(a)|.$$

The source law μ is fixed independently of H. Averaging the histogram itself over H always gives M, for any fixed-size source law. The question here concerns the distance of its *conditional* law P_H from M.

Theorem 1 (Collision criterion)**.** *Assume* (2) *and* $m \geq 2$*, and put* $r = m - 1$*. Let* S, T *be independent with law* μ*, and* $J = |S \cap T|$*. Then*

$$\mathbb{E}_H \chi^2(P_H \| M) = \sum_{k=1}^{n} \binom{r+k-1}{k} \frac{\mathbb{E}(J)_k}{(n)_k}. \tag{3}$$

Here $(x)_k = x(x-1)\cdots(x-k+1)$ *is a falling factorial. If* $e\tau < 1$*,*

$$r\tau \;\leq\; \mathbb{E}_H \chi^2(P_H \| M) \leq r\tau + (1-e\tau)^{-r} - 1 - er\tau, \tag{4}$$

$$\mathbb{E}_H \chi^2(P_H \| M) \leq (1-e\tau)^{-r} - 1. \tag{5}$$

For arbitrary sequences of such experiments, allowing n, E, m, q, μ *to vary,*

$$\mathbb{E}_H \chi^2(P_H \| \operatorname{Mult}(n, q)) \longrightarrow 0 \quad \Longleftrightarrow \quad (m-1)\tau \longrightarrow 0. \tag{6}$$

In this regime,

$$\mathbb{E}_H \chi^2(P_H \| M) = (m-1)\tau(1 + o(1)), \tag{7}$$

uniformly over the stated class and all positive q*.*

The identity (3) requires no dependence hypothesis. Its kernel and orthogonal decomposition are classical multinomial reproducing-kernel and Hoeffding–Krawtchouk machinery: in particular, the shared-observation joint law is explicit in Diaconis and Griffiths [1, Remark 7, equation (30)]. We give a finite trace proof to make its normalization and use in this experiment explicit. The additional input is the upper-cylinder control of replica-overlap factorial moments, which yields (6)–(7).

Theorem 2 (Fixed-color total-variation criterion)**.** *For every fixed-size subset law, without assuming* (2)*, and every color* i *with* $0 < q_i < 1$*,*

$$\mathbb{E}_H \operatorname{TV}(P_H, \operatorname{Mult}(n, q)) \geq \frac{\tau^2}{2\{4 + 1/[q_i(1-q_i)]\}}. \tag{8}$$

Consequently, under (2)*, if* $m \geq 2$ *and the positive vector* q *are fixed along a sequence,*

$$\mathbb{E}_H \operatorname{TV}(P_H, \operatorname{Mult}(n, q)) \longrightarrow 0 \quad \Longleftrightarrow \quad \tau \longrightarrow 0. \tag{9}$$

The chi-square theorem permits arbitrary positive varying q and growing m. The TV equivalence has different quantifiers. Section 5 shows why varying rare colors prevent the same unrestricted TV necessity.

1.1 Relation to preceding approximation results

Multinomial approximation of dependent variables is classical; Loh [2] provides a general Stein-method treatment. Approximate de Finetti and permutation-mixture results compare exchangeable marginals or permutation mixtures with empirical or mean-matched product references [3, 4, 5, 6]. Their cancellation of first-degree terms is a different normalization from the prescribed-q experiment here. For example, a uniform n-subset of an L-element population has $\tau = n/L$. The exact first term of our expected chi-square is $(m-1)n/L$. Comparing instead to a multinomial with the frozen population's empirical color frequencies removes that first-marginal fluctuation and changes the approximation problem.

For real-stable fixed-total count laws, the author's earlier covariance certificate [9] gives a mean-matched multinomial TV bound. Applied after grouping coordinates, together with control of the coloring-dependent mean, it already supplies fixed-color TV sufficiency for real-stable subset laws. The present results instead need only (2), determine the expected chi-square criterion and its first coefficient without reciprocal-q_i losses, and give fixed-color TV necessity for arbitrary fixed-size laws. The nonstable example in Section 4 demonstrates that the broader source class is substantive. The classical kernel, negative-dependence inputs, and existing rounding algorithms are credited ingredients.

2 The classical overlap trace and factorial control

Lemma 3 (Overlap trace)**.** *Equation (3) holds for every probability law on* $\binom{E}{n}$.

Proof. All sums are finite and M is strictly positive. Expanding the square and using independent $S, T \sim \mu$ gives

$$1 + \mathbb{E}_H \chi^2(P_H \| M) = \mathbb{E}_{S,T} \sum_a \frac{\mathbb{P}_H(C_H(S) = a, C_H(T) = a)}{M(a)}. \tag{10}$$

For fixed S, T with overlap j, their common elements contribute j shared independent categorical observations; the other $n - j$ observations in each histogram are mutually independent. This joint histogram law is also obtained by starting with n independent q-colored coordinates, retaining a uniformly chosen j-subset of their positions, independently refreshing the rest with colors of law q, and forgetting the order. Indeed, conditional on the first histogram, any fixed j positions have the same color-count law as a uniform j-subset of positions. Let K_j be this histogram transition matrix. It is stationary for M, and its contribution to the sum in (10) is $\operatorname{Tr} K_j$.

Choose an orthonormal basis $1, \phi_1, \dots, \phi_r$ of $L^2(q)$, with the last $r = m - 1$ functions centered. Tensor products form an orthonormal basis on the ordered n coordinates. Uniform keep-j/refresh resampling acts on a product with k nonconstant coordinates by the scalar $(j)_k/(n)_k$: every one of those coordinates must be retained, or its refreshed centered factor integrates to zero. This scalar is zero when $k > j$.

Permutation-invariant functions identify isometrically with histogram functions in $L^2(M)$. In degree k, symmetrized tensor products are indexed by multisets of k labels from $\{1, \dots, r\}$. Distinct multisets have disjoint tensor-basis supports, so these products are orthogonal and can be normalized. There are $d_{r,k} = \binom{r+k-1}{k}$ such multisets. Together over $0 \le k \le n$ (with $d_{r,0} = 1$), their dimension is $\sum_{k=0}^{n} d_{r,k} = \binom{n+m-1}{n} = |\mathcal{A}_{n,m}|$; they exhaust the histogram space. Therefore

$$\operatorname{Tr} K_j = \sum_{k=0}^{n} d_{r,k} \frac{(j)_k}{(n)_k}.$$

Average over S, T in (10) and remove the constant term. □

For nonnegative numbers $x = (x_e)_{e \in E}$, let $e_k(x) = \sum_{|I|=k} \prod_{e \in I} x_e$ be the elementary symmetric polynomial, with an empty sum interpreted as zero.

Lemma 4 (Replica factorial moments)**.** *Under (2), for* $1 \le k \le n$,

$$a_k := \frac{\mathbb{E}(J)_k}{(n)_k} \le \min\left\{ \ell^k, \frac{k! e_k((p_e^2)_e)}{(n)_k}, \frac{(n\tau)^k}{(n)_k} \right\}. \tag{11}$$

Moreover $a_1 = \tau$ *exactly, and* $a_k \le (e\tau)^k$.

Proof. Writing $u_I = \mathbb{P}(I \subseteq S)$, independence of the two replicas gives

$$\mathbb{E}(J)_k = k! \sum_{|I|=k} u_I^2 \le k! e_k((p_e^2)_e) \le \left(\sum_e p_e^2\right)^k.$$

The last inequality retains just the ordered distinct-index terms in the nonnegative expansion of the power. Conditional on a realized S, the same factorial moment sums over ordered distinct k-tuples in S. The inclusion probability of each such tuple in T is at most ℓ^k, and there are $(n)_k$ tuples. This proves (11). For $k = 1$, $\mathbb{E}J = \sum_e p_e^2$ without any dependence assumption.

To pay the falling-factorial denominator uniformly, use the increasing function $f(x) = -\log(1 - x/n)$ on $[0, n)$. Its left Riemann sum gives

$$\begin{aligned}\log \frac{n^k}{(n)_k} &= \sum_{j=0}^{k-1} f(j) \le \int_0^k f(x)\,dx \\ &= k + (n-k)\log(1-k/n) \le k.\end{aligned}$$

At $k = n$ the integral is improper but finite, with $0 \log 0 = 0$. Hence $n^k/(n)_k \le e^k$, which proves the final bound. □

Proof of Theorem 1. Lemma 3 gives the identity. Each coefficient and each overlap moment is nonnegative, so retaining $k = 1$ gives the lower bound $r\tau$. With $x = e\tau < 1$, use $a_k \le x^k$ for $k \ge 2$ and

$$\sum_{k=0}^{\infty} \binom{r+k-1}{k} x^k = (1-x)^{-r}$$

to obtain (4). Applying the same bound also at $k = 1$ gives (5).

Suppose $r\tau \to 0$. Since $r \ge 1$, $\tau \to 0$ and $x \to 0$. The expansion $-\log(1-x) = x + O(x^2)$ gives, uniformly when $rx \to 0$,

$$(1-x)^{-r} - 1 - rx = O(r^2x^2 + rx^2).$$

Thus the upper remainder in (4), divided by $r\tau$, is $O(r\tau + \tau) = o(1)$. This proves (7), and its sufficient convergence direction. The lower bound $r\tau$ proves the necessary direction. □

Corollary 5 (Finite envelopes and typical colorings)**.** *The finite quantity*

$$U = \sum_{k=1}^{n} \binom{r+k-1}{k} \min\left\{\ell^k, \frac{k! e_k((p_e^2)_e)}{(n)_k}, \frac{(n\tau)^k}{(n)_k}\right\} \tag{12}$$

is a valid upper bound on $\mathbb{E}_H \chi^2(P_H \| M)$ for all τ. When $\ell < 1$, $(1-\ell)^{-r} - 1$ is another valid bound; when $e\tau < 1$, either upper bound in Theorem 1 is valid. For any such bound U,

$$\mathbb{E}_H \operatorname{TV}(P_H, M) \le \min\{1, \tfrac{1}{2}\sqrt{U}\}, \tag{13}$$

$$\mathbb{P}_H\{\operatorname{TV}(P_H, M) > \tfrac{1}{2}\sqrt{U/\delta}\} \le \delta, \quad 0 < \delta < 1. \tag{14}$$

In particular, if $r\tau \to 0$, expected TV is at most $(1/2 + o(1))\sqrt{r\tau}$ and tends to zero in probability over H.

Proof. The finite bound follows termwise from Lemma 4. The maximum-inclusion relaxation follows from the same binomial series. Cauchy–Schwarz gives $\operatorname{TV}(P, M) \le \frac{1}{2}\sqrt{\chi^2(P\|M)}$. Jensen gives (13); applying Markov to chi-square gives (14). If $U = 0$, the distance vanishes almost surely and the same conclusions hold directly. □

3 A total-variation lower bound with paid moments

The necessary direction of Theorem 2 cannot follow from a bare chi-square lower bound. We instead use a statistic whose moments are controlled under both the joint experiment and its independent reference.

Proof of Theorem 2. First take binary colors $H_e \sim \text{Bernoulli}(q)$, $0 < q < 1$, and set $s = q(1-q)$ and $Y_e = H_e - q$. Put

$$A(H) = \frac{\sum_e p_e Y_e}{\sqrt{ns}}, \qquad Z(H,S) = \frac{\sum_{e\in S} Y_e}{\sqrt{ns}}.$$

Let $\mathcal{J}$ be the joint law of (H, C) with $C = \sum_{e\in S} H_e$, where S is independent of H, and let $\mathcal{R} = \mathcal{L}(H) \otimes \text{Bin}(n, q)$. Finite conditioning gives

$$\text{TV}(\mathcal{J}, \mathcal{R}) = \mathbb{E}_H \,\text{TV}(P_H, \text{Bin}(n, q)) =: D. \tag{15}$$

The statistic $F(H, C) = A(H)(C - nq)/\sqrt{ns}$ is a function of this observed pair. Independence and centering of the Y_e give

$$\mathbb{E}_{\mathcal{J}} F = \frac{1}{n}\mathbb{E}_S \sum_{e\in S} p_e = \tau, \qquad \mathbb{E}_{\mathcal{R}} F = 0, \qquad \mathbb{E}_{\mathcal{R}} F^2 = \tau.$$

The last identity uses $\mathbb{E}A^2 = \tau$ and the unit variance of the standardized binomial under the product law.

For a centered Bernoulli variable, the fourth cumulant is $s(1 - 6s)$. Expanding independent centered summands therefore yields

$$\mathbb{E}A^4 = 3\tau^2 + \frac{1-6s}{n^2 s}\sum_e p_e^4 \le 3 + 1/s,$$
$$\mathbb{E}Z^4 = 3 + \frac{1-6s}{ns} \le 3 + 1/s.$$

The second formula holds after conditioning on any S, since its elements are distinct. To see the first upper bound, if $1 - 6s < 0$ drop its term; otherwise use $p_e \le 1$, $\sum_e p_e^4 \le \sum_e p_e = n$, $\tau \le 1$, and $n \ge 1$. The marginals of H and C under $\mathcal{J}$ are their original coloring law and $\text{Bin}(n, q)$. Consequently Cauchy–Schwarz, without assuming A and Z independent, gives $\mathbb{E}_{\mathcal{J}} F^2 \le 3 + 1/s$. Hence

$$\mathbb{E}_{\mathcal{J}} F^2 + \mathbb{E}_{\mathcal{R}} F^2 \le 4 + 1/s. \tag{16}$$

For two finite laws P, Q and real F, Cauchy–Schwarz against $P + Q$ implies

$$|\mathbb{E}_P F - \mathbb{E}_Q F|^2 \le (\mathbb{E}_P F^2 + \mathbb{E}_Q F^2)\sum_x \frac{(P(x) - Q(x))^2}{P(x) + Q(x)}$$
$$\le 2\,\text{TV}(P, Q)(\mathbb{E}_P F^2 + \mathbb{E}_Q F^2).$$

Terms with zero denominator are zero. Apply this to $\mathcal{J}, \mathcal{R}$ and use (15)–(16). It gives $D \ge \tau^2/[2(4 + 1/s)]$.

For m colors, project the histogram onto coordinate i. The indicator $\mathbf{1}_{\{H_e = i\}}$ is an independent Bernoulli coloring of parameter q_i. TV contracts under projection, proving (8). When m, q are fixed and positive with $m \ge 2$, this forces $\tau \to 0$ if expected full-histogram TV tends to zero. Conversely, under (2), $\tau \to 0$ implies $r\tau \to 0$ for fixed r; Corollary 5 supplies sufficiency. □

4 Source classes and applications

4.1 A diffuse non-real-stable upper-cylinder family

Real stability is sufficient for the usual fixed-size strongly Rayleigh negative-dependence inputs [7], but is not necessary for our theorem. Label five elements by $\mathbb{Z}/5\mathbb{Z}$. Assign probability $3/20$ to each cycle pair $\{i, i+1\}$ and $1/20$ to each of the five other pairs. All marginals are $p_i = 2/5$. Pair inclusion is at most $3/20 < 4/25$, singleton inclusion is equality, and inclusion of three or more elements is impossible. Thus (2) holds.

To verify nonstability directly, let $g(z) = \sum_{i<j} \mu(\{i,j\}) z_i z_j$ be its homogeneous quadratic generating polynomial and B its Hessian. The constant vector has eigenvalue $2/5$. Real Fourier vectors of frequency one have eigenvalue

$$\lambda = \frac{3}{10}\cos(2\pi/5) + \frac{1}{10}\cos(4\pi/5) = \frac{\sqrt{5}-2}{20} > 0.$$

Take a nonzero such real vector v with $\sum_i v_i = 0$. Then $\nabla g(\mathbf{1}) = (2/5)\mathbf{1}$, $g(\mathbf{1}) = 1$, and $g(v) = \lambda\|v\|^2/2 > 0$. Homogeneity gives

$$g(v + t\mathbf{1}) = g(v) + t\nabla g(\mathbf{1}) \cdot v + t^2 g(\mathbf{1}) = g(v) + t^2.$$

This has a root with $\Im t > 0$, at which every coordinate of $v + t\mathbf{1}$ has positive imaginary part. Hence g is not real stable.

For an arbitrarily diffuse sequence, replace each parent element by L clones. After selecting the parent pair, independently select one uniform clone from each chosen parent. Each clone has marginal $2/(5L)$, so $\tau = 2/(5L)$. Two clones of the same parent cannot both appear; for different parents, pair inclusion is the former pair weight divided by L^2. This proves all upper-cylinder inequalities. Identifying clones within each parent recovers g; such identification preserves real stability directly from its upper-half-plane definition. Thus every cloned law remains nonstable, while $\tau \to 0$.

4.2 Strongly Rayleigh samples and spanning trees

A multiaffine probability generating polynomial is real stable when it has no zero with all coordinates in the upper half-plane. Fixed-size strongly Rayleigh measures have negative association and therefore satisfy (2) [7]. In particular, a finite connected graph on at least two vertices with positive conductances has a strongly Rayleigh weighted spanning-tree law. Its edge inclusion probabilities are $p_e = w_e R_{\mathrm{eff}}(e)$, with the conductance convention for effective resistance. Our parameter is

$$\tau = \frac{\sum_e (w_e R_{\mathrm{eff}}(e))^2}{|V| - 1}.$$

On K_v, a uniform spanning tree has $n = v - 1$ and $p_e = 2/v$ by symmetry and fixed total, hence $\tau = 2/v$. For $m = o(v)$, Theorem 1 gives the sharp averaged chi-square asymptotic

$$\mathbb{E}_H \chi^2(P_H \| \operatorname{Mult}(v-1, q)) \sim \frac{2(m-1)}{v}.$$

Its TV upper consequence is $O(\sqrt{m/v})$. The balanced stable-law TV range was already accessible from the earlier covariance certificate; the sharp chi-square characterization is the assertion emphasized here. These are frozen independent *edge* colors, not the vertex-degree identity arising from Prüfer codes or an adversarial edge-color theorem.

4.3 Fixed matroid-base rounding

Let x be a fixed point in the base polytope of a matroid of rank $n \geq 1$. Fix, independently of the coloring, a randomized base-rounding law with marginal preservation and upper negative cylinders. Randomized swap and pipage rounding supply such laws under their usual hypotheses [8, Theorem 1.1 and Section 4]. Their output S has exactly n elements, $p_e = x_e$, and our parameter is

$$\tau = \frac{1}{n} \sum_e x_e^2.$$

All conclusions apply to its conditional color histogram. This is a consequence for the established rounding law. A point only in an independence polytope may produce a random output size; silently replacing that size by n would change the experiment.

4.4 Why a collision parameter can beat maximum inclusion

Select one deterministic element, and one uniform element from each of $n-1$ disjoint groups of size L. The generating polynomial is a product of a monomial and positive linear forms, so the law is real stable and satisfies (2). Here

$$\ell = 1, \qquad \tau = \frac{1 + (n-1)/L}{n}.$$

For $L = n^2$ and $m = \lfloor \sqrt{n} \rfloor$, the collision criterion holds although an infinite-series relaxation based on $\ell < 1$ cannot be used. This illustrates the difference between the two finite envelopes. The earlier covariance/good-mean argument also covers its TV convergence.

5 Sharpness and boundaries of the experiment

5.1 A classical square-root example

For $n = 1$, let μ be uniform on L singleton elements. Then $\tau = 1/L$, and P_H is an empirical categorical law. For uniform $q_i = 1/m$,

$$\mathbb{E}_H \operatorname{TV}(P_H, q) = \frac{m}{2L} \mathbb{E} \left| \operatorname{Bin}(L, 1/m) - L/m \right|.$$

For fixed $m \geq 2$, or for growing m with $L/m \to \infty$, the classical binomial central limit theorem gives

$$\mathbb{E}_H \operatorname{TV}(P_H, q) \sim \sqrt{\frac{m-1}{2\pi L}}.$$

Indeed the standardized binomial converges to a standard normal and its absolute value is uniformly integrable by its unit second moment. This old empirical-distribution example shows why a general $O(\sqrt{(m-1)\tau})$ TV order cannot be replaced by $O((m-1)\tau)$. The new criterion's chi-square coefficient one is exact already in degree one of the overlap expansion.

5.2 Rare colors and adaptive laws

For binary $q = (1-\varepsilon, \varepsilon)$, compare both histogram laws with the point mass on the all-first-color histogram. The union bound and the triangle inequality give, for every fixed-size source law,

$$\mathbb{E}_H \operatorname{TV}(P_H, \operatorname{Mult}(n, q)) \leq 2n\varepsilon.$$

For example, a deterministic selected subset has $\tau = 1$, but its expected TV tends to zero if $n\varepsilon \to 0$. Thus the unrestricted varying-q chi-square necessity is not an unrestricted TV necessity.

Independence of the source law from the coloring also matters. For example, color a $2n$-element ground set with binary fair colors, and then select deterministically n elements of a most numerous color. For each realized coloring this deterministic law has upper cylinders, but its histogram is a point mass at an endpoint and is far from $\mathrm{Bin}(n, 1/2)$. The iid coloring/independent-source joint calculation in Lemma 3 is absent. Our conclusions are for each fixed law, not simultaneous control over laws chosen after seeing H. They concern absolute distributional error.

6 Reproducibility and assistance

The accompanying standard-library Python programs perform finite exact rational controls. The first enumerates all colorings for twelve small fixtures, including spanning trees and a source without negative cylinders, and checks the overlap identity against direct histogram calculation. The second checks six negative-cylinder collision fixtures and four forced-element families against exact factorial envelopes. Their acceptance checks remain active under Python's optimization flag. These computations illustrate the finite identities and examples; the general proofs above do not depend on a finite test or a numerical limit. Build and execution instructions are in the source package.

AI assistance was used in mathematical exploration, drafting, calculation, and checking. The author is responsible for the statements, attribution, and final manuscript.

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