Sharp all-times heat-content comparison and concentration on compact metric graphs

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Abstract

Bifulco and Täufer asked whether a Faber–Krahn inequality holds for the heat content of compact metric graphs at every time: is the heat content of a graph with Dirichlet conditions on a set of vertices bounded, at each time, by that of the interval of the same total length with one Dirichlet and one Neumann endpoint? They proved this for large times, and for small times when the edge lengths are commensurable (all ratios rational). We answer their question affirmatively at every positive time, for all edge lengths and every nonempty Dirichlet vertex set, with standard Kirchhoff conditions elsewhere. The proof is a sharp concentration comparison: for every nonnegative L^2 initial datum, the decreasing rearrangement of its heat evolution is dominated in integrated mass by the heat evolution of its rearrangement on that mixed interval. For initial datum one, equality at even one positive time occurs precisely for the mixed interval, up to non-Dirichlet degree-two subdivisions. An exact positive-kernel representation of the deficit separates the contributions of multiple level crossings and unequal crossing slopes. If every bridge has Dirichlet vertices on both sides, the comparison sharpens to the double-Dirichlet interval of the same length, and equality at one positive time is then equivalent to the known sharper torsion extremality, including balanced two-edge pumpkin chains. We also give a convolution formula for an asymmetric three-edge pitchfork. The proofs adapt established graph coarea, symmetrization and resolvent-to-semigroup methods.

Keywords. heat content; metric graph; quantum graph; Faber–Krahn inequality; symmetrization; Dirichlet Laplacian.
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1 Introduction and statements

The heat content of an absorbing metric graph measures the heat remaining from a constant initial temperature. Equivalently, after division by total length it is the survival probability for diffusion started uniformly on the graph. This quantity involves both eigenvalues and the integrals of eigenfunctions. An extremal eigenvalue comparison alone therefore does not give its ordering at every time.

Bifulco and Täufer [1] asked whether the heat content obeys a Faber–Krahn inequality at every time. For a graph whose Dirichlet-cut interior is connected, the question is whether its heat content at each time $t > 0$ is at most that of the interval of the same total length with a Dirichlet condition at one endpoint and a Neumann condition at the other. They proved this for all sufficiently large times, and for all sufficiently small times when the edge lengths are commensurable [1, Definition 2.5 and Theorem 2.6]. The all-times question is the last sentence of their abstract and the subject of [1, Remark 2.9], where they state their belief that the inequality holds at all times. Their [1, Remark 2.8] adds that the endpoint results

leave room for graphs violating the inequality at small times and singles out pitchfork-shaped graphs as the delicate candidates.

Let G be a finite compact connected metric graph of total length $L > 0$, and let D be a nonempty set of its vertices. We impose zero values at D, continuity and unweighted Kirchhoff conditions elsewhere. Write $T_G(t) = e^{-tH_G}$ for the corresponding heat semigroup. The interval model $I = [0, L]$ has Neumann condition at 0 and Dirichlet condition at L. For $f \geq 0$, its decreasing rearrangement is denoted by f^*, and $\mathcal{M}f(s) = \int_0^s f^*$. Definitions and normalizations are detailed in Section 2. The Dirichlet-cut interior $G \setminus D$ may be disconnected.

Theorem 1.1 (Heat concentration)**.** *Let $f \geq 0$ belong to $L^2(G)$. If $g \geq 0$ is nonincreasing in $L^2(0, L)$ and*

$$\mathcal{M}f(s) \leq \int_0^s g(x)\,\mathrm{d}x \qquad (0 \leq s \leq L),$$

then for every $t \geq 0$ and $s \in [0, L]$,

$$\mathcal{M}(T_G(t)f)(s) \leq \int_0^s T_I(t)g(x)\,\mathrm{d}x. \tag{1}$$

In particular one may take $g = f^*$.

This is integrated concentration order.

Theorem 1.2 (Sharp heat content and equality)**.** *For every $t \geq 0$,*

$$\mathcal{Q}_G(t) \leq \mathcal{Q}_I(t).$$

If (G, D) is not isometric to an interval with precisely one Dirichlet endpoint and one Neumann endpoint, after suppressing non-Dirichlet degree-two vertices, then for every $t > 0$ and every $s \in (0, L]$,

$$\mathcal{M}(T_G(t)\mathbf{1})(s) < \int_0^s T_I(t)\mathbf{1}(x)\,\mathrm{d}x.$$

Consequently equality in heat content at any one positive time holds if and only if (G, D) is this mixed interval.

Theorem 1.2 answers the question of Bifulco and Täufer affirmatively and in full: the inequality holds at every time, for every choice of edge lengths and every nonempty Dirichlet vertex set, and also when $G \setminus D$ is disconnected. Its equality statement is a one-time rigidity: equality at a single positive time already forces the mixed interval, whereas the extremal property in [1, Definition 2.5] concerns equality on a whole time interval. In particular every pitchfork satisfies the inequality at every time; Section 6 gives an explicit convolution formula for an asymmetric one. After division by L, the theorem concerns the diffusion started from the uniform distribution on G and absorbed at D. At every time its survival probability is largest on the mixed interval, and therefore so is every moment $\mathbb{E}\tau^p = p\int_0^\infty t^{p-1}\mathbb{P}(\tau > t)\,\mathrm{d}t$, $p > 0$, of its absorption time τ. The case $p = 1$ is the known torsional-rigidity inequality [4].

The exact defect formula in Theorem 4.2 explains this strictness through the actual level slopes of the heat solution. It also gives the source-dependent lower bound in Corollary 4.4.

For graphs in which every bridge separates two components each containing a Dirichlet vertex, the interval evolution in Theorem 1.1 improves to $T_I(4t)$, as proved in Theorem 5.1. For constant data this is the double-Dirichlet interval benchmark of length L. The associated defect and one-time equality transfer to the known torsion optimizer class are proved in the same section. The uniform mixed-interval rigidity above and this sharper class have different extremizers; balanced paired-edge chains attain the latter.

Relation to prior work

Bifulco and Täufer [1, Theorem 2.6 and Remarks 2.8–2.9] prove the fixed-length mixed-interval heat-content comparison in endpoint time regimes: small times under a commensurability assumption and large times in general. Their random-walk expansion also yields an all-times comparison formula for suitable pairs of equilateral graphs [1, Corollary 4.9]. Relative to that article, Theorem 1.2 gives the comparison and its rigidity at every positive time, with no commensurability condition. Bifulco and Mugnolo [2] develop heat-content formulas and graph surgery principles. Their Remark 2.6 (Remark 2.7 in arXiv:2502.09461v1) explains the decomposition at Dirichlet vertices, which also accommodates the disconnected $G \setminus D$ allowed here.

The underlying analytic methods have substantial precedent. Friedlander [3, Lemma 3 and Eq. (2.5)] proves the graph coarea and rearrangement estimate on which our level argument rests. The optimal torsional-rigidity inequality is due to earlier work, including [4]. The graph p-torsion preprint [5, Theorem 4.2] is another recent use of this level-set machinery. In the Euclidean setting, parabolic symmetrization and arbitrary-data concentration are classical [6, 7, 8]. In particular retaining the integrated forcing and passing from elliptic resolvents to heat flow follow that tradition. Abstract resolvent and semigroup domination is also established [9]. We give the graph-specific integrated differential inequality, its approximation argument for arbitrary L^2 data, and the continuous-time defect and topology argument explicitly.

The continuous-time defect also relates the heat comparison to the established torsion bound through Corollary 4.5. Täufer's 2024 seminar announcement [10] discussed how the optimal topology for heat retention depends on the timescale; Theorem 1.2 shows that, for the comparison with the mixed interval, the interval is optimal at every time.

This fourth edition states the question of Bifulco and Täufer, and how far Theorem 1.2 answers it, in the abstract and in this introduction; the theorems and proofs of the third edition are unchanged.

2 Graphs, concentration and the interval model

A finite compact metric graph G is obtained from finitely many intervals $[0, \ell_e]$, with $\ell_e > 0$, by identifying their endpoints. We assume that G is connected, has at least one edge, and has total length $L = \sum_e \ell_e > 0$. Integration is with respect to arclength on the edges; vertices have measure zero. Loops and multiple edges are allowed. Each loop has two incident germs at its vertex.

Let D be a nonempty subset of the vertex set. The closed form

$$a_G(\phi, \psi) = \sum_e \int_0^{\ell_e} \phi_e' \overline{\psi_e'} \, \mathrm{d}x, \qquad \operatorname{dom} a_G = \{\phi \in H^1(G) : \phi|_D = 0\}$$

defines a nonnegative self-adjoint operator H_G. Here $H^1(G)$ consists of continuous, edgewise H^1 functions. The operator acts as $-\phi''$ on each edge; outside D its domain satisfies continuity and the unweighted Kirchhoff condition. At a vertex in D there is no additional Kirchhoff condition. A degree-one vertex outside D is consequently Neumann. Write $T_G(t) = e^{-tH_G}$. The operator and its semigroup are unchanged by inserting or suppressing non-Dirichlet vertices of degree two.

The Dirichlet-cut interior $G \setminus D$ may be disconnected. Cutting all Dirichlet incidences decomposes the form into finitely many connected positive-length components, each with a Dirichlet endpoint. Indeed a path in the original connected graph from any point to D meets such an endpoint. On each component the form has trivial kernel. This decomposition is already recorded in [2, Remark 2.6] (Remark 2.7 in arXiv:2502.09461v1).

The form contraction property gives a positive sub-Markovian semigroup and positive resolvents. On each cut component the resolvent is positivity improving. To recall the elementary maximum-principle argument, if $u = (1 + hH_G)^{-1} f \geq 0$, $f \geq 0$, has an interior zero, then $u' = 0$ there and $u'' \leq u/h$. The integral form of this inequality, with zero initial value and slope, forces u to vanish on a short interval and then on the edge. At a zero non-Dirichlet vertex the outgoing derivatives are nonnegative and have zero sum, so all vanish and the same propagation applies on each germ. Connectedness forces $u = 0$, and hence $f = 0$, on that component. Therefore nonzero nonnegative forcing gives strict positivity off its Dirichlet endpoints. The standard equivalence between resolvent and semigroup positivity improvement [11, Theorem A.2] gives positivity improvement of the heat semigroup on each component. Here improvement means strict positivity almost everywhere; this is sufficient below.

The comparison operator H_I acts on $I = [0, L]$, with

$$H_I = -\frac{d^2}{dx^2}, \qquad \phi'(0) = 0, \quad \phi(L) = 0.$$

Thus decreasing data are placed with their largest values at the reflecting endpoint. Our diffusion generator is d^2/dx^2, without a factor $1/2$. The heat content means

$$\mathcal{Q}_G(t) = \langle \mathbf{1}, T_G(t)\mathbf{1} \rangle_{L^2(G)};$$

it is not the heat trace. In particular

$$\mathcal{Q}_I(t) = \frac{8L}{\pi^2} \sum_{n=0}^{\infty} \frac{\exp\bigl(-(2n+1)^2\pi^2 t/(4L^2)\bigr)}{(2n+1)^2}. \tag{2}$$

Definition 2.1. For a nonnegative function f on a nonatomic space of measure L, let f^* be its nonincreasing rearrangement on $(0, L)$, and set

$$\mathcal{M}f(s) = \int_0^s f^*(r)\,\mathrm{d}r = \sup_{|E|=s} \int_E f, \qquad 0 \leq s \leq L.$$

The order $f \preceq g$ means $\mathcal{M}f(s) \leq \mathcal{M}g(s)$ for all s.

The supremum permits measurable subsets of a level plateau. Two continuity facts will be useful:

$$\|\mathcal{M}f - \mathcal{M}k\|_\infty \leq \|f - k\|_1 \leq \sqrt{L}\,\|f - k\|_2, \tag{3}$$

$$\|f^* - k^*\|_2 \leq \|f - k\|_2. \tag{4}$$

The first follows directly from the supremum formula. For the second, equimeasurability preserves the squared norms and Hardy–Littlewood gives $\int fk \leq \int f^* k^*$. Expanding the squared distances proves the claim.

There is a second, reversed mixed operator in the proof. On the *mass coordinate* $s \in [0, L]$, let

$$\mathsf{A} = -\frac{d^2}{ds^2}, \qquad W(0) = 0, \quad W'(L) = 0, \qquad S_h = (1 + h\mathsf{A})^{-1}.$$

The mixed maximum principle shows that S_h is positive and satisfies

$$\|S_h F - S_h K\|_\infty \leq \|F - K\|_\infty \quad (F, K \in C[0, L]). \tag{5}$$

For example, subtract the constant $\|F - K\|_\infty$ from the difference and use its nonpositive value at 0 and zero derivative at L. The distinction between the physical conditions N at 0, D at L, and the integrated conditions D at 0, N at L, is essential.

Lemma 2.2 (The decreasing interval cone)**.** *If $g \geq 0$ is nonincreasing and belongs to $L^2(0, L)$, then $w = (1 + hH_I)^{-1} g$ is nonnegative and nonincreasing. Moreover*

$$\int_0^s w(x)\,\mathrm{d}x = S_h\left(\int_0^{\cdot} g(x)\,\mathrm{d}x\right)(s). \tag{6}$$

Proof. For smooth decreasing g, positivity gives $w \geq 0$. Its derivative $p = w'$ satisfies $p - hp'' = g' \leq 0$, $p(0) = 0$, and $p(L) \leq 0$, the latter because $w(L) = 0$ and $w \geq 0$. The scalar maximum principle gives $p \leq 0$. For general g, truncate, extend constantly past both endpoints, and mollify with a nonnegative kernel. This gives nonnegative smooth decreasing approximants in L^2. Resolvent continuity and closedness of the decreasing cone prove the assertion. Integrating $w - hw'' = g$, using $w'(0) = 0$ and $w(L) = 0$, proves (6), also for L^2 data. □

3 Resolvent concentration and the heat comparison

The graph coarea estimate in the next argument is the one used by Friedlander [3, Lemma 3, pp. 203–205]. We retain the integrated forcing term, as in classical Talenti and parabolic concentration arguments [7, 8]. An approximation step allows arbitrary data without assuming concavity or finite critical-level structure for their resolvents.

Lemma 3.1 (A dense analytic forcing class)**.** *The nonnegative cone of $L^2(G)$ has a dense subset consisting of functions that are strictly positive nonconstant real polynomials on every closed edge. If f belongs to this subset and $u = (1+hH_G)^{-1} f$, then u is positive off D, analytic and nonconstant on each closed edge, and has only finitely many vertex and critical values.*

Proof. On an edge of length ℓ, choose real polynomials a_j converging to $\sqrt{f}$ in L^4, and choose $\varepsilon_j \downarrow 0$. Put

$$f_j(x) = a_j(x)^2 + \varepsilon_j(1 + x/\ell).$$

Then $f_j \geq \varepsilon_j$, and

$$\|a_j^2 - f\|_2 \leq \|a_j - \sqrt{f}\|_4(\|a_j\|_4 + \|\sqrt{f}\|_4) \longrightarrow 0.$$

Each f_j is nonconstant: a nonconstant a_j^2 has leading even degree at least two, while a constant a_j leaves the added linear term. Apply the construction independently on the finitely many edges. Forcings need no vertex compatibility.

The weak maximum principle gives $u \geq 0$. An interior zero would contradict $u - hu'' = f > 0$ at a nonnegative minimum. At a non-Dirichlet zero vertex, all outgoing derivatives are nonnegative; Kirchhoff makes all zero, and the edge equation gives $u'' = -f/h < 0$ there, again contradicting nonnegativity immediately inside an edge. Thus $u > 0$ off D. On each edge the solution is a polynomial particular solution plus two exponentials. It extends analytically beyond the closed edge and cannot be constant, since its forcing is nonconstant. Its derivative therefore has finitely many zeros on that closed edge. The asserted finiteness follows. □

Lemma 3.2 (Inverse-distribution regularity)**.** *Suppose u has the regularity and positivity properties of Lemma 3.1, with $u|_D = 0$. Then its rearrangement $q = u^*$, extended by $q(0) = \max u$ and $q(L) = 0$, is Lipschitz. Consequently $V = \mathcal{M}u \in W^{2,\infty}(0, L)$, with $V(0) = 0$ and $V'(L) = 0$.*

Proof. Every level set has zero length, and $\mu(z) = |\{u > z\}|$ is continuous. Let $M = \max_e \|u_e'\|_\infty > 0$. Choose a simple arclength path from a point of D to a maximizer of u. For $0 \leq \alpha < \beta \leq \max u$, choose the *first* β crossing on this path, and the *last* preceding α

crossing. On the open segment between them, $\alpha < u < \beta$. The segment is counted only once in edge measure and has length at least $(\beta - \alpha)/M$. Hence

$$\mu(\alpha) - \mu(\beta) \geq \frac{\beta - \alpha}{M}. \tag{7}$$

In particular μ strictly decreases from L to 0, and its inverse q is M-Lipschitz, including at its endpoints. Integrating q proves the claim. Paths may pass through other Dirichlet vertices; connectedness of $G \setminus D$ has not been used. □

Proposition 3.3 (Shifted resolvent comparison)**.** *For every $h > 0$ and $f \geq 0$ in $L^2(G)$,*

$$\mathcal{M}\big((1 + hH_G)^{-1} f\big) \leq S_h \mathcal{M} f = \mathcal{M}\big((1 + hH_I)^{-1} f^*\big) \quad \text{on } [0, L]. \tag{8}$$

Proof. First take f from Lemma 3.1, set $u = (1 + hH_G)^{-1} f$ and $V = \mathcal{M}u$, and let $z \in (0, \max u)$ avoid all vertex and critical values. Write $s = \mu(z)$ and, over *all* crossings of the level z, define

$$A(z) = \sum_{u=z} |u'|, \qquad B(z) = \sum_{u=z} \frac{1}{|u'|}, \qquad N(z) = \#\{u = z\}.$$

There is at least one crossing by the path in Lemma 3.2. Coarea on the finitely many monotone edge pieces and inverse differentiation give

$$-\mu'(z) = B(z), \qquad V''(s) = -\frac{1}{B(z)}, \qquad A(z)B(z) \geq N(z)^2 \geq 1. \tag{9}$$

The last inequality is Cauchy–Schwarz, as in [3, Eq. (2.5)].

Integrate $u - hu'' = f$ over the superlevel set $E_z = \{u > z\}$. Every hot non-Dirichlet vertex includes all its germs and contributes zero by Kirchhoff. Dirichlet vertices do not occur in a positive hot set. Every boundary crossing has outward flux $-|u'|$. Thus

$$V(s) + hA(z) = \int_{E_z} f \leq \mathcal{M} f(s). \tag{10}$$

Disconnected hot sets cause no change to this identity. Combining (9)–(10) yields

$$V - hV'' \leq \mathcal{M} f \quad \text{almost everywhere}, \qquad V(0) = 0, \quad V'(L) = 0.$$

Lemma 3.2 justifies the weak equation across all exceptional mass coordinates. If $Z = V - S_h \mathcal{M} f$, testing $Z - hZ'' \leq 0$ with Z_+ gives

$$\int_0^L Z_+^2 + h \int_0^L |(Z_+)'|^2 \leq 0.$$

The boundary terms vanish, so $V \leq S_h \mathcal{M} f$.

For arbitrary $f \geq 0$, approximate by Lemma 3.1. The graph resolvent is an L^2 contraction; (3) gives uniform convergence of both mass profiles, and (5) permits the limit on the right. The proof uses the stated approximation of the resolvent. Finally Lemma 2.2 identifies $S_h \mathcal{M} f$ with the last expression in (8). □

Proof of Theorem 1.1. Fix $h > 0$. Set $u_m = (1 + hH_G)^{-m} f$ and $v_m = (1 + hH_I)^{-m} g$, where g is the decreasing interval datum in the theorem. Let $V_m = \mathcal{M}u_m$ and $F_m(s) = \int_0^s v_m$. Lemma 2.2 makes every v_m nonnegative and decreasing, and Proposition 3.3 gives

$$V_m \leq S_h V_{m-1}, \qquad F_m = S_h F_{m-1}.$$

Since $V_0 \le F_0$ and S_h is positive, induction yields $V_m \le F_m$ for every m.

For fixed $t > 0$, set $h = t/n$ and $m = n$. Spectral calculus gives, strongly in the respective L^2 spaces,

$$(1 + tH_G/n)^{-n} f \longrightarrow T_G(t)f, \qquad (1 + tH_I/n)^{-n} g \longrightarrow T_I(t)g.$$

Indeed the scalar spectral multipliers converge and are bounded by one. Equation (3) passes the comparison uniformly in s. The decreasing cone is closed in L^2, so the interval heat solution is still decreasing. At $t = 0$ the claimed inequality is the hypothesis; for $g = f^*$ it is equality. □

Remark 3.4. The argument proves concentration at every repeated-resolvent step before taking the heat limit. It does not infer timewise order from a scalar Laplace-transform inequality or from ordering two quadratic forms. General resolvent–semigroup domination frameworks, including decreasing rearrangements, are prior work [9]; the explicit graph comparison above verifies the structure needed here.

4 The exact defect and equality at a single time

In this section the initial datum is $\mathbf{1}$. Put

$$u(t) = T_G(t)\mathbf{1}, \quad v(t) = T_I(t)\mathbf{1}, \quad U(t,s) = \mathcal{M}u(t)(s), \quad F(t,s) = \int_0^s v(t,x)\,\mathrm{d}x, \quad W = F - U.$$

Theorem 1.1 gives $W \ge 0$. We will identify its source of strict positivity directly in continuous time.

Lemma 4.1 (Concavity of the survival profile)**.** *For $t > 0$, $u(t) > 0$ off D, and $u_{xx}(t,x) = u_t(t,x) < 0$ almost everywhere on every open edge. In particular each edge profile is strictly concave. The derivatives used here are bounded on each closed edge, locally uniformly for positive times.*

Proof. Sub-Markovianity and positivity imply $u(t+h) = T_G(t)T_G(h)\mathbf{1} \le T_G(t)\mathbf{1} = u(t)$. For positive time, spectral calculus puts $u(t)$ in every power domain of H_G; edgewise regularity gives the asserted smoothing and local bounds. In particular $w(t) = -u_t(t) = H_G u(t) \ge 0$.

Work on any component obtained by cutting at D. Its Dirichlet operator has trivial kernel, while $u(r)$ is nonzero there by injectivity of e^{-rH_G}. Thus $w(r)$ is nonzero on that component. For $0 < r < t$, $w(t) = T_G(t-r)w(r)$, so componentwise positivity improvement gives $w(t) > 0$ almost everywhere on each component. The edge heat equation $u_t = u_{xx} = -w$ implies strict concavity on every edge. Since $u \ge 0$, strict concavity gives positivity in every open edge. A zero value at a non-Dirichlet vertex would give a positive derivative into each incident edge, contrary to Kirchhoff. Thus $u(t) > 0$ at those vertices as well. □

Each positive-time edge profile is strictly concave, so there are finitely many critical and vertex levels. Lemma 3.2, with the same first-upper-crossing selection, applies at each time: its proof uses finite critical levels, positivity and a slope bound, not the resolvent equation. Thus $U(t,\cdot) \in W^{2,\infty}$ locally uniformly for $t > 0$, with $U(t,0) = 0$ and $U_s(t,L) = 0$.

At a regular level $z = u(t)^*(s)$, write $\alpha_1, \dots, \alpha_N > 0$ for the absolute crossing slopes and set

$$A = \sum_i \alpha_i, \qquad B = \sum_i \alpha_i^{-1}.$$

Coarea gives $U_{ss} = -1/B$. The unique top-s maximizing set is the hot set $E(t,s) = \{u(t) > u(t)^*(s)\}$, up to null sets. Its envelope derivative is

$$U_t(t,s) = \int_{E(t,s)} u_t(t,x)\,\mathrm{d}x = -A. \tag{11}$$

For clarity, the first equality does not omit a moving-boundary term. Bracket the time difference by evaluating at the old and new maximizing sets. Uniform convergence of u and continuity of its inverse distribution place their symmetric difference in a shrinking band around a null level set. Both difference quotients have the same limit. The second equality is the integrated heat equation, with the same Kirchhoff cancellation as in (10). Also $|U(t,s) - U(r,s)| \le \|u(t) - u(r)\|_1$ gives local Lipschitz regularity in time.

Theorem 4.2 (Exact nonnegative defect)**.** *There is a nonnegative measurable function $d(t,s)$, bounded on each compact positive-time cylinder, such that*

$$d(t,s) = A - \frac{1}{B} = \frac{N^2-1}{B} + \frac{1}{B}\sum_{i<j} \frac{(\alpha_i - \alpha_j)^2}{\alpha_i \alpha_j} \tag{12}$$

at almost every mass coordinate of every positive-time profile, and

$$W_t - W_{ss} = d, \qquad W(t,0) = 0, \quad W_s(t,L) = 0.$$

Let K be the heat kernel for A, with Dirichlet condition at the mass endpoint 0 and Neumann condition at L. Then

$$W(t,s) = \int_0^t \int_0^L K(t-r;s,y) d(r,y)\, \mathrm{d}y\, \mathrm{d}r, \qquad t > 0, \quad 0 \le s \le L. \tag{13}$$

The integral is the finite limit of nonnegative integrals starting at $\varepsilon > 0$; no separate integrability assumption at time zero is needed.

Proof. Equations (9) and (11) give $U_t - U_{ss} = -(A - 1/B)$. Moreover

$$AB - N^2 = \sum_{i<j} \left(\frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} - 2 \right),$$

which proves (12). In particular $d \ge 0$ and $d = 0$ precisely when $N = 1$ at a regular level. If $M(t) = \max_e \|u_x(t)\|_\infty$, then $d \le A \le 2|\mathsf{E}(G)| M(t)$. The weak derivatives define a jointly measurable version of d. For each time only finitely many mass values are exceptional, so the identities hold almost everywhere on positive-time cylinders. The established Sobolev and Lipschitz bounds exclude a singular residual.

The interval profile satisfies $F_t = F_{ss}$ with the displayed mixed mass conditions. On $[\varepsilon, t]$, variation of constants for the weak mixed heat problem gives

$$\begin{aligned} W(t,s) = &\int_0^L K(t-\varepsilon;s,y) W(\varepsilon,y)\, \mathrm{d}y \\ &+ \int_\varepsilon^t \int_0^L K(t-r;s,y) d(r,y)\, \mathrm{d}y\, \mathrm{d}r. \end{aligned} \tag{14}$$

This identity first holds in L^2; smoothing and a final-time-strip bound by the local L^∞ norm of d give continuous spatial representatives and equality at each s, including L. Both $U(\varepsilon,\cdot)$ and $F(\varepsilon,\cdot)$ tend uniformly to s by strong L^2 continuity and (3). The positive mixed semigroup is an L^∞ contraction. Hence the first term of (14) is nonnegative and tends uniformly to zero. The second increases as $\varepsilon \downarrow 0$. Monotone convergence proves (13) and its finiteness. □

Lemma 4.3 (A multiple-crossing band)**.** *Unless (G,D) is the DN interval up to non-Dirichlet degree-two subdivisions, every positive-time profile has an open interval of positive levels on which $N(z) \ge 2$ at every regular level.*

Proof. Let $m_D = \sum_{v \in D} \deg(v)$, counting germs. If $m_D \geq 2$, strict concavity and positivity give a positive derivative away from every Dirichlet incidence: it exceeds a positive secant slope. Choose disjoint short germ neighborhoods. They each cross every sufficiently small positive level, so $N(z) \geq m_D \geq 2$ there.

If $m_D = 1$, D is one leaf. A finite connected graph with a leaf and all degrees at most two is a path, giving precisely the DN interval. Otherwise there is a non-Dirichlet vertex v of degree $k \geq 3$. Put $c = u(t, v) > 0$ and let p_i be its outgoing derivatives. Kirchhoff gives $\sum_i p_i = 0$. A positive p_i gives a short increasing germ above c; a negative p_i gives a decreasing germ below c. A zero p_i also gives a strictly decreasing germ below c, by strict concavity. If all slopes vanish, all k germs cross a small band below c. Otherwise the numbers of upward and downward germs sum to k, and at least one of these two numbers is at least two. Choose the corresponding band above or below c, with a below band contained in $(0, c)$. Germ neighborhoods can be chosen disjoint even for loops. Omitting the finitely many exceptional levels does not remove the positive-width band. □

Proof of Theorem 1.2. For every positive time, Lemma 4.3 and (12) make $d(t, \cdot) > 0$ on a set of positive measure, unless (G, D) is the DN interval. Level bands correspond to mass bands of positive width by the strictly decreasing distribution. On every time interval $0 < a < b < t$, measurability and Fubini therefore give $\int_a^b \int_0^L d(r, y) \, \mathrm{d}y \, \mathrm{d}r > 0$.

The mixed kernel satisfies $K(\tau; s, y) > 0$ for $\tau > 0$ and $s, y \in (0, L]$. One way to see this is reflection about L:

$$K(\tau; s, y) = K_{DD,2L}(\tau; s, y) + K_{DD,2L}(\tau; s, 2L - y),$$

where $K_{DD,2L}$ is the positive Dirichlet kernel on $(0, 2L)$. Thus (13) gives $W(t, s) > 0$ for every $s > 0$. Conversely the stated interval and its non-Dirichlet subdivisions have the same operator as H_I, and $u(t)$ is already decreasing from its Neumann endpoint. Hence $W = 0$ there. At $s = L$ this proves the heat-content equality assertion. □

Corollary 4.4 (A source-dependent lower bound)**.** *Fix $t > 0$, $s > 0$, $0 < a < b < t$ and $0 < \delta < L$. Then*

$$W(t, s) \geq \kappa \int_a^b \int_\delta^L d(r, y) \, \mathrm{d}y \, \mathrm{d}r, \qquad \kappa = \min_{\substack{r \in [a,b] \\ y \in [\delta, L]}} K(t - r; s, y) > 0.$$

For a noninterval graph, some δ makes the integral positive. This is a graph-dependent defect estimate.

Corollary 4.5 (Integrated defect)**.** *The exact identity*

$$\int_0^\infty W(t, s) \, \mathrm{d}t = \int_0^\infty \int_0^L \min(s, y) d(r, y) \, \mathrm{d}y \, \mathrm{d}r$$

holds. In particular

$$\frac{L^3}{3} - \langle \mathbf{1}, H_G^{-1} \mathbf{1} \rangle = \int_0^\infty \int_0^L y \, d(r, y) \, \mathrm{d}y \, \mathrm{d}r.$$

Proof. Use Tonelli in (13) and $\int_0^\infty K(\tau; s, y) \, \mathrm{d}\tau = \min(s, y)$. The left side is finite since $0 \leq W \leq F$ and H_I has a positive spectral gap. At $s = L$, integrate the heat contents and use the interval torsion value $L^3/3$. □

The optimal torsion inequality is prior work [4]; Corollary 4.5 records its relation to this heat-level defect; it records the defect form of the torsion inequality of [4].

5 A two-crossing comparison

We say that (G, D) satisfies the two-sided bridge condition if every bridge of G separates it into two components each containing a vertex of D. Graphs without bridges satisfy this condition. It excludes any Dirichlet-free part attached through a single bridge. Mugnolo and Plümer [4, Theorem 4.6 and its proof] use this class, described there by double connectivity after identifying the Dirichlet vertices, and the two-preimage coarea factor to obtain the sharper torsion bound $L^3/12$, with caterpillar equality. Here we propagate that factor to arbitrary-data concentration at every time and its continuous-time defect.

Theorem 5.1 (Two-crossing concentration)**.** *Assume the two-sided bridge condition, and let f, g satisfy the hypotheses of Theorem 1.1. Then*

$$\mathcal{M}(T_G(t)f)(s) \le \int_0^s T_I(4t)g(x)\,\mathrm{d}x \qquad (t \ge 0,\ 0 \le s \le L).$$

In particular, if $J = [0, L]$ has Dirichlet conditions at both endpoints, then

$$\mathcal{Q}_G(t) \le \mathcal{Q}_J(t) = \mathcal{Q}_I(4t) \qquad (t \ge 0).$$

The comparison is sharp. Equality for datum one is attained by balanced two-edge pumpkin chains, described below. Corollary 5.3 identifies single-time heat equality with the established torsion extremizers.

Proof. First take a forcing in the dense class of Lemma 3.1 and write $u = (1 + hH_G)^{-1}f$. At a positive regular level z, the hot set E_z is nonempty, avoids D, and its boundary consists of the finitely many interior edge crossings. There is at least one such crossing. If there were exactly one, say x, then E_z would be a nonempty proper subset of $G \setminus \{x\}$ both open and closed there. Indeed its only boundary point in G would be x. Thus removing the interior point x disconnects G, and its edge is a bridge. The component on the hot side contains no vertex of D, since $u|_D = 0$. This contradicts the bridge condition. Hence $N(z) \ge 2$.

The hot-set identity and coarea formulas from Proposition 3.3 now give

$$V + hA \le \mathcal{M}f, \qquad AB \ge N^2 \ge 4, \qquad V'' = -1/B.$$

Consequently $V - 4hV'' \le \mathcal{M}f$, with $V(0) = 0$, $V'(L) = 0$. The same weak maximum-principle test gives

$$\mathcal{M}((1 + hH_G)^{-1}f) \le (1 + 4h\mathsf{A})^{-1}\mathcal{M}f.$$

The approximation in Proposition 3.3 passes this inequality to every nonnegative L^2 forcing. The decreasing interval cone is preserved by $(1 + 4hH_I)^{-1}$, and its integrated mass equals $(1 + 4h\mathsf{A})^{-1}\mathcal{M}g$. Induct exactly as in the proof of Theorem 1.1, using the graph step h and interval step $4h$. Taking $h = t/n$ gives the claimed comparison by strong Euler convergence and continuity of the mass profiles.

For decreasing interval data g, the function

$$v(t, x) = (T_I(4t)g)(2|x - L/2|) \quad (0 < x < L)$$

solves the Dirichlet heat equation on J. The reflecting condition of T_I at zero makes the two halves meet with zero derivative at $x = L/2$. The change of variables on the two halves preserves total mass. For $g = 1$ it proves $\mathcal{Q}_J(t) = \mathcal{Q}_I(4t)$.

For sharpness choose vertices $v_0, \dots, v_m$ and join each consecutive pair by exactly two parallel edges of the same length $a_j > 0$. Put $D = \{v_0\}$, impose Kirchhoff conditions elsewhere, and let $S = \sum_j a_j = L/2$. There are no bridges. The datum-one solution is identical on each pair by the edge-exchange symmetries. Its common profile on the concatenated coordinate

$r \in [0, S]$ solves the ordinary heat equation, with Dirichlet condition at 0 and Neumann condition at S: at an intermediate junction the Kirchhoff equation is twice the usual matching of derivatives. Thus its heat content is twice that of a DN interval of length S, equal to that of J by reflection about its midpoint. These examples include degree-four junctions. □

Corollary 5.2 (Two-crossing defect)**.** *For datum one put* $F_2(t,s) = \int_0^s T_I(4t)1(x)\,\mathrm{d}x$ *and* $W_2 = F_2 - U$*. Under the two-sided bridge condition its nonnegative defect source is*

$$d_2 = A - \frac{4}{B} = \frac{N^2-4}{B} + \frac{1}{B}\sum_{i<j}\frac{(\alpha_i-\alpha_j)^2}{\alpha_i\alpha_j}.$$

With the mass-coordinate kernel K *of Theorem 4.2,*

$$W_2(t,s) = \int_0^t \int_0^L K(4(t-r); s, y) d_2(r,y)\,\mathrm{d}y\,\mathrm{d}r.$$

In particular

$$\frac{L^3}{12} - \langle 1, H_G^{-1}1\rangle = \frac{1}{4}\int_0^\infty \int_0^L y\, d_2(r,y)\,\mathrm{d}y\,\mathrm{d}r.$$

Proof. All positive-time regularity and envelope identities used in Theorem 4.2 concern the same graph solution and remain unchanged. Since $N \geq 2$, the displayed source is nonnegative and $W_{2,t} - 4W_{2,ss} = d_2$, with the same mixed mass boundary conditions. The two-crossing comparison gives $W_2 \geq 0$. Variation of constants from time $\varepsilon > 0$, followed by the same monotone limit and uniform initial convergence to zero, proves the representation. Tonelli and $\int_0^\infty K(4t; s, y)\,\mathrm{d}t = \frac{1}{4}\min(s,y)$ give the last identity at $s = L$. This is a defect identity for the corresponding known torsion benchmark of [4]. □

Corollary 5.3 (One-time equality and torsion extremizers)**.** *Under the two-sided bridge condition, the following are equivalent:*

1. $\mathcal{Q}_G(t_0) = \mathcal{Q}_J(t_0)$ *at some* $t_0 > 0$*;*
2. $\mathcal{Q}_G(t) = \mathcal{Q}_J(t)$ *for every* $t > 0$*;*
3. $\langle 1, H_G^{-1}1\rangle = L^3/12$*.*

Thus the geometric equality cases are precisely the torsion extremizers in the corresponding graph class, with the Dirichlet identifications understood as in the established torsion classification.

Proof. If equality holds at t_0, the defect formula at $s = L$ integrates a nonnegative source against $K(4(t_0 - r); L, y) > 0$ for $0 < r < t_0$, $0 < y \leq L$. It follows that $d_2 = 0$ almost everywhere on $(0, t_0) \times (0, L)$. The same representation then gives $\mathcal{Q}_G(t) = \mathcal{Q}_J(t)$ for every $0 < t < t_0$.

For completeness, the heat content is real analytic for positive time. In an orthonormal eigenbasis it has the expansion

$$\mathcal{Q}_G(z) = \sum_n |\langle 1, \phi_n\rangle|^2 e^{-\lambda_n z}, \qquad \operatorname{Re} z > 0.$$

The coefficients are nonnegative and sum to L, so the series converges locally normally on that half-plane and defines a holomorphic function. The same holds for J. The identity theorem extends equality from $(0, t_0)$ to all positive times. Integrating yields the third statement. Conversely, the all-times heat comparison makes $\mathcal{Q}_J - \mathcal{Q}_G$ nonnegative and continuous. Under the third statement its integral over $(0, \infty)$ is zero, so it vanishes everywhere. □

To apply the stationary equality theorem precisely, cut every Dirichlet incidence into a separate Dirichlet leaf, obtaining components G_i of lengths $L_i > 0$. This leaves the quadratic form and torsion unchanged. Each component satisfies the two-sided bridge condition: a bridge with a Dirichlet-free side in a cut component would retain that side after the Dirichlet vertices were restored. Thus G_i becomes doubly connected after its Dirichlet leaves are identified. These are the boundary and connectivity conventions of [4, Assumption 2.1, Remark 2.3 and Theorem 4.6], which give

$$\langle \mathbf{1}, H_G^{-1}\mathbf{1}\rangle = \sum_i \langle \mathbf{1}, H_{G_i}^{-1}\mathbf{1}\rangle \le \frac{1}{12}\sum_i L_i^3 \le \frac{L^3}{12}.$$

Equality forces a single component and, by the equality part of that theorem, its Dirichlet-identification is a *caterpillar*. Here this established metric term means a chain of vertices $v_0, \dots, v_m$, with exactly two parallel edges of the same positive length between each consecutive pair; v_0 is Dirichlet and the other vertices have standard conditions. In particular the intermediate vertices have degree four and the other end has degree two. Insertion or suppression of non-Dirichlet degree-two vertices is immaterial. Equivalently, the original graph after identification of all vertices of D has this form. Conversely, the paired-edge heat solution constructed above attains the double-Dirichlet heat-content benchmark, and splitting the Dirichlet vertex leaves its operator unchanged. This proves the stated single-time equality characterization using the stationary classification of torsion optimizers in [4].

6 An independent formula for the asymmetric pitchfork

Consider a stem of length $a > 0$ joining a Dirichlet leaf to a Kirchhoff junction, with two further arms of lengths $b, c > 0$ ending at Neumann leaves. Write $P = P(a; b, c)$ and $L = a+b+c$. This example admits a separate positive convolution proof. It is a supporting exact calculation; general pendant-edge surgery principles are treated in [2, Section 6].

For $s > 0$ put $k = \sqrt{s}$, $A = \tanh(ka)$, $B = \tanh(kb)$ and $C = \tanh(kc)$. Let $Z_P(s)$ be the Laplace transform of the absorption time for diffusion on P started at the junction, and let $Z_I(s)$ be the corresponding transform on the mixed interval started at distance a from its absorbing endpoint. This point has coordinate $L - a$ in our Neumann-at-zero convention. Solving $Z'' = sZ$, with $Z = 1$ at the absorbing endpoint and the usual conditions elsewhere, gives

$$Z_P(s) = \frac{1}{\cosh(ka) + (B + C)\sinh(ka)}, \qquad Z_I(s) = \frac{\cosh(k(b + c))}{\cosh(kL)}. \tag{15}$$

The same calculation gives the heat-content transforms

$$\widehat{\mathcal{Q}}_P(s) = \frac{L}{s} - \frac{A + B + C}{sk(1 + A(B + C))}, \qquad \widehat{\mathcal{Q}}_I(s) = \frac{L}{s} - \frac{\tanh(kL)}{sk}. \tag{16}$$

For completeness, $\widehat{u} = (1 - Z)/s$ solves the heat resolvent with initial datum one. Integrating $Z'' = sZ$ over the graph cancels all Kirchhoff and Neumann fluxes. On the stem, with distance x from the absorbing endpoint, $Z(x) = \cosh(kx) + d\sinh(kx)$; the junction condition gives $d = -(A+B+C)/(1+A(B+C))$. Hence $\int_P Z = -Z'(0)/s = (A+B+C)/(k(1+A(B+C)))$, proving (16).

Proposition 6.1 (Pitchfork convolution identity)**.** *For $\ell > 0$ define the positive locally integrable function*

$$h_\ell(t) = \frac{2}{\ell}\sum_{n=0}^{\infty} \exp\left(-\frac{(n + 1/2)^2\pi^2 t}{\ell^2}\right), \qquad t > 0.$$

Let μ_P, μ_I be the absorption-time probability measures whose Laplace transforms are (15). *Then*

$$\mathcal{Q}_I(t) - \mathcal{Q}_P(t) = \big(h_b * h_c * h_{b+c} * \mu_P * \mu_I\big)(t), \qquad t > 0. \tag{17}$$

In particular the deficit is nonnegative, and

$$\int_0^\infty (\mathcal{Q}_I(t) - \mathcal{Q}_P(t))\,\mathrm{d}t = bc(b+c).$$

Proof. The interval mixed eigenfunction expansion, or the partial fraction expansion of tanh, gives

$$\int_0^\infty e^{-st} h_\ell(t)\,\mathrm{d}t = \frac{\tanh(\ell\sqrt{s})}{\sqrt{s}}, \qquad \int_0^\infty h_\ell(t)\,\mathrm{d}t = \ell.$$

The transforms Z_P, Z_I are those of probability measures: absorption is almost sure on these finite anchored graphs. This also follows by letting $s \downarrow 0$ in (15). Subtract (16), use $\tanh(k(b+c)) = (B+C)/(1+BC)$, and simplify to obtain

$$\widehat{\mathcal{Q}}_I(s) - \widehat{\mathcal{Q}}_P(s) = \frac{\tanh(kb)}{k}\frac{\tanh(kc)}{k}\frac{\tanh(k(b+c))}{k}\, Z_P(s) Z_I(s).$$

Tonelli and uniqueness of Laplace transforms prove the convolution identity almost everywhere. It holds everywhere by continuity: $h_\ell(t) = O(t^{-1/2})$ near zero and decays exponentially at infinity, so the triple convolution extends to a continuous bounded function on the real line when set to zero for $t \le 0$. Convolving with the finite measures preserves continuity. Finally integrate by Tonelli and multiply the five total masses. Strict positivity for each $t > 0$ also follows from Theorem 1.2. □

7 Scope and consequences

The comparison applies to finite compact connected unweighted metric graphs with positive edge lengths, a nonempty Dirichlet vertex set, and standard Kirchhoff conditions at every remaining vertex. Loops, multiple edges and disconnected Dirichlet-cut interiors are included. The bridge condition gives the sharper double-Dirichlet benchmark and transfers one-time equality to the established torsion extremizers. The concentration theorem permits arbitrary nonnegative L^2 data; the single-positive-time equality theorem concerns initial datum one.

The source-dependent bound in Corollary 4.4 quantifies the deficit through the heat solution's level crossings and slopes. The integrated identity in Corollary 4.5 connects this same source to the sharp torsion bound. The survival-probability interpretation uses the stated boundary conditions and generator d^2/dx^2.

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